Neighbourhoods of Randomness and Independence

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Augment information geometric measures in spaces of distributions, via explicit geometric representations of neighbourhoods for these important states for stochastic processes:

- randomness,
- uniformity,
- independence.

Significant theoretically because very general, and practically because topological so stable under perturbations.

Gamma manifold and randomness

- Gamma manifold $\mathcal{G}$
- Fisher information
- α-Connection
- α-Curvatures
- Randomness
- Log-gamma manifold and uniformity

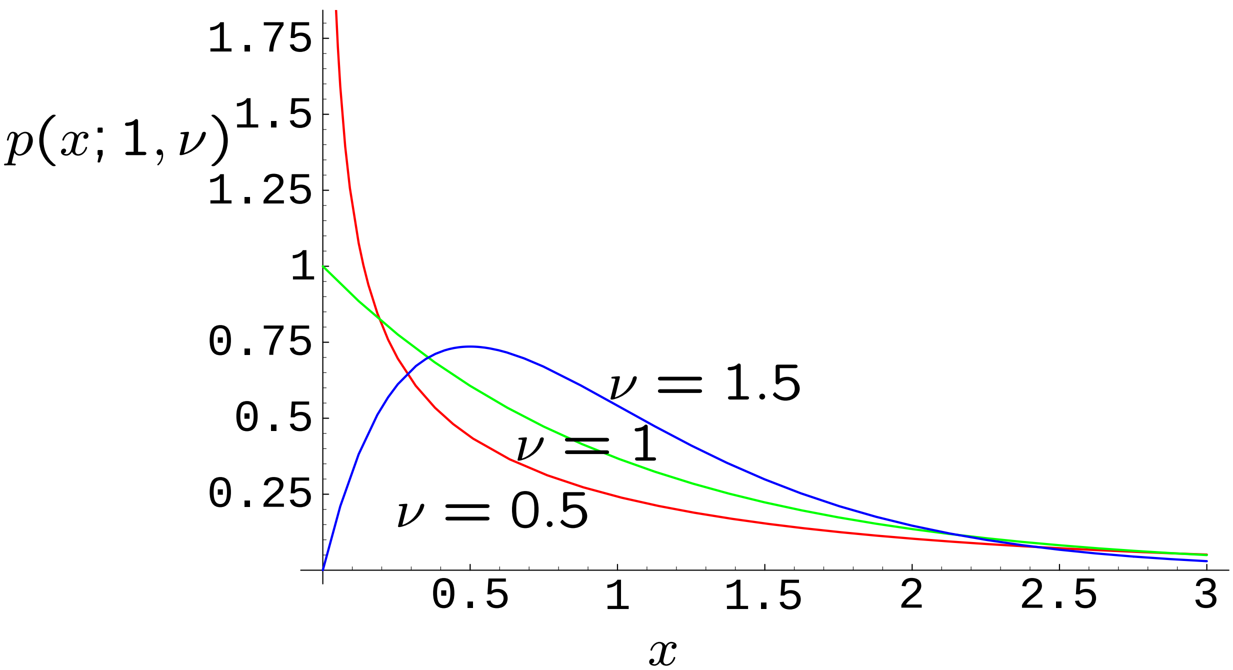
Gamma manifold $\mathcal{G}$

$$p(x;\beta,\nu) = \left(\frac{\nu}{\beta}\right)^{\nu} \frac{x^{\nu-1}}{\Gamma(\nu)} e^{-x\nu/\beta} \quad \text{for} \quad \beta,\nu \in \mathbb{R}^{+}$$

where β is the scale parameter, ν is the shape parameter.

Then $\bar{x} = \beta$ and $Var(x) = \beta^2/\nu$.

The special case $\nu = 1$ corresponds to the situation of the random or Poisson process with mean inter-event interval β.



Gamma probability density functions $p(x;\beta,\nu)$ with unit mean $\beta = 1$, and $\nu = 0.5, 1, 1.5$. The case $\nu = 1$ corresponds to an exponential distribution from an underlying Poisson process; $\nu < 1$ corresponds to clustering and conversely $\nu > 1$ corresponds to smoothing.

Gamma Fisher information metric:

$$ds_g^2 = \frac{\nu}{\beta^2}\,d\beta^2 + \left(\psi'(\nu) - \frac{1}{\nu}\right)\,d\nu^2 \quad \text{for} \quad \beta, \nu \in \mathbb{R}^+ \,.$$

where $\psi(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)}$ is the digamma function.

α-**Connection**

The independent components of the $\nabla^{(\alpha)}$-connection are given by:

$$\begin{aligned}
\Gamma^{(\alpha)1}_{11} &= -\frac{\alpha+1}{\beta}\,, \\
\Gamma^{(\alpha)1}_{12} &= \frac{\alpha+1}{2\,\nu}\,, \\
\Gamma^{(\alpha)2}_{11} &= \frac{(\alpha-1)\,\nu}{2\,\beta^2\,(\nu\,\psi'(\nu)-1)}\,, \\
\Gamma^{(\alpha)2}_{22} &= \frac{(1-\alpha)\,\left(1+\nu^2\,\psi''(\nu)\right)}{2\,\nu\,(\nu\,\psi'(\nu)-1)}\,.
\end{aligned}$$

α-**Curvatures**

The non zero components of the α-**Ricci tensor**, are:

$$R_{11}^{(\alpha)} = \frac{-\left(\alpha^2 - 1\right)\, \nu\, \left(\psi'(\nu) + \nu\, \psi''(\nu)\right)}{4\, \beta^2 \left(\nu\, \psi'(\nu) - 1\right)^2}$$

$$R_{22}^{(\alpha)} = \frac{-\left(\alpha^2 - 1\right)\, \left(\psi'(\nu) + \nu\, \psi''(\nu)\right)}{4\, \nu\, \left(\nu\, \psi'(\nu) - 1\right)}$$

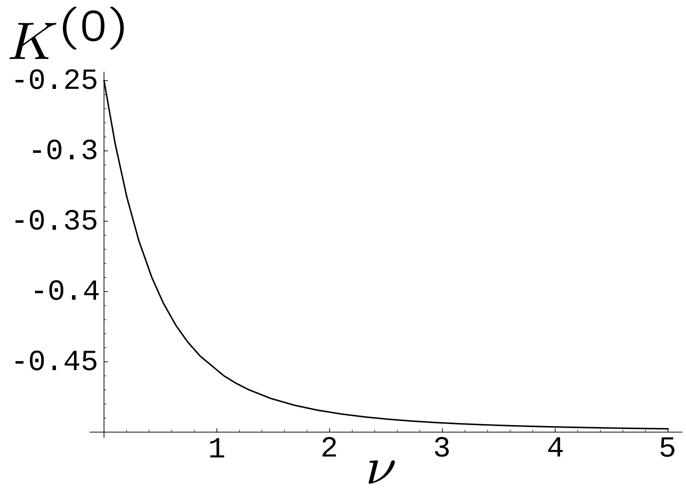
The α-**eigenvalues** and the α-**eigenvectors** for the α-Ricci tensor:

$$\left(\alpha^2 - 1\right) \begin{pmatrix} \frac{-\nu\left(\psi'(\nu)+\nu\,\psi''(\nu)\right)}{4\,\beta^2\left(\nu\,\psi'(\nu)-1\right)^2} \\ \frac{-\left(\psi'(\nu)+\nu\,\psi''(\nu)\right)}{4\,\nu\left(\nu\,\psi'(\nu)-1\right)} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The α-**Gaussian curvature**:

$$K^{(\alpha)} = \frac{-\left(\alpha^2 - 1\right)\left(\psi'(\nu) + \nu\,\psi''(\nu)\right)}{4\left(\nu\,\psi'(\nu) - 1\right)^2}$$



Note that $K^{(0)}(\beta,\nu) \to -\frac{1}{4}$ as $\nu \to 0$ and $K^{(0)}(\beta,\nu) \to -\frac{1}{2}$ as $\nu \to \infty$.

Randomness

Proposition

Every neighbourhood of a random process contains a neighbourhood of stochastic processes subordinate to gamma distributions.

Proof

Dodson and Matsuzoe provided affine immersion in $\mathbb{R}^3$ for $\mathcal{G}$, manifold of gamma pdfs.

Natural coordinates $(\mu = \frac{\nu}{\beta}, \nu)$ Then $\mathcal{G}$ is the graph of the affine immersion $\{h, \xi\}$ where ξ is a transversal vector field along h

$$h : \mathcal{G} \to \mathbb{R}^3 : (\mu, \nu) \mapsto (\mu, \nu, \varphi),\ \xi = (0, 0, 1).$$

where $\varphi = \log \Gamma(\nu) - \nu \log \mu$.

Submanifold of exponential pdfs is curve

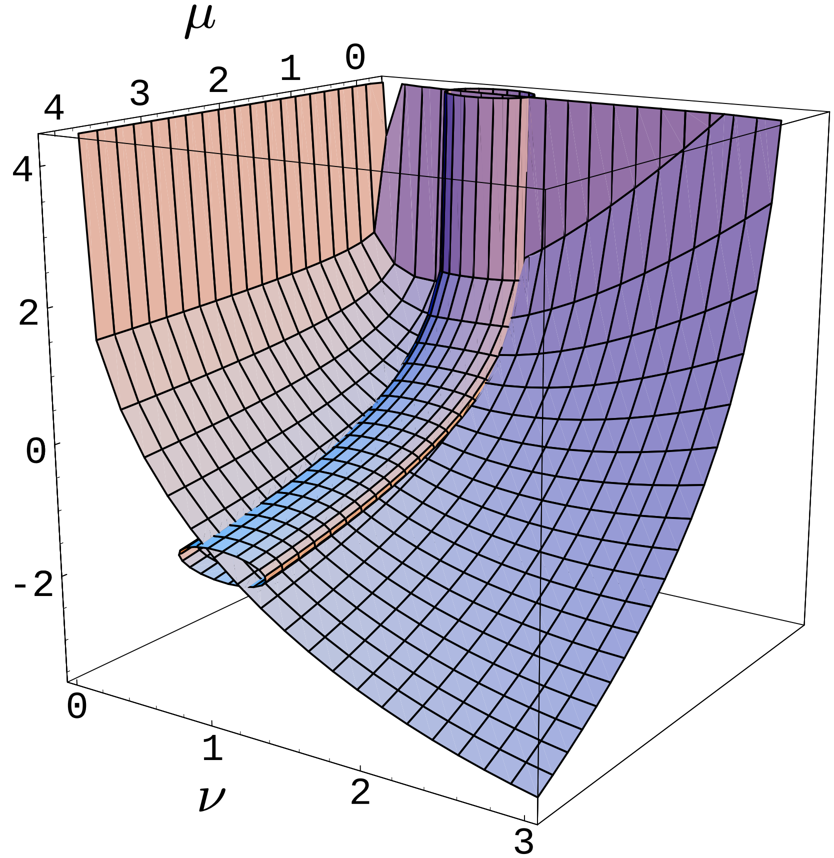
$$(0, \infty) \to \mathbb{R}^3 : \mu \mapsto \{\mu, 1, \log \frac{1}{\mu}\}$$

Tubular neighbourhood in $\mathbb{R}^3$

$$\left\{\mu - \frac{0.6\cos\theta}{\sqrt{1+\mu^2}}, 1 - 0.6\sin\theta, \frac{-0.6\mu\cos\theta}{\sqrt{1+\mu^2}} - \log\mu\right\}$$

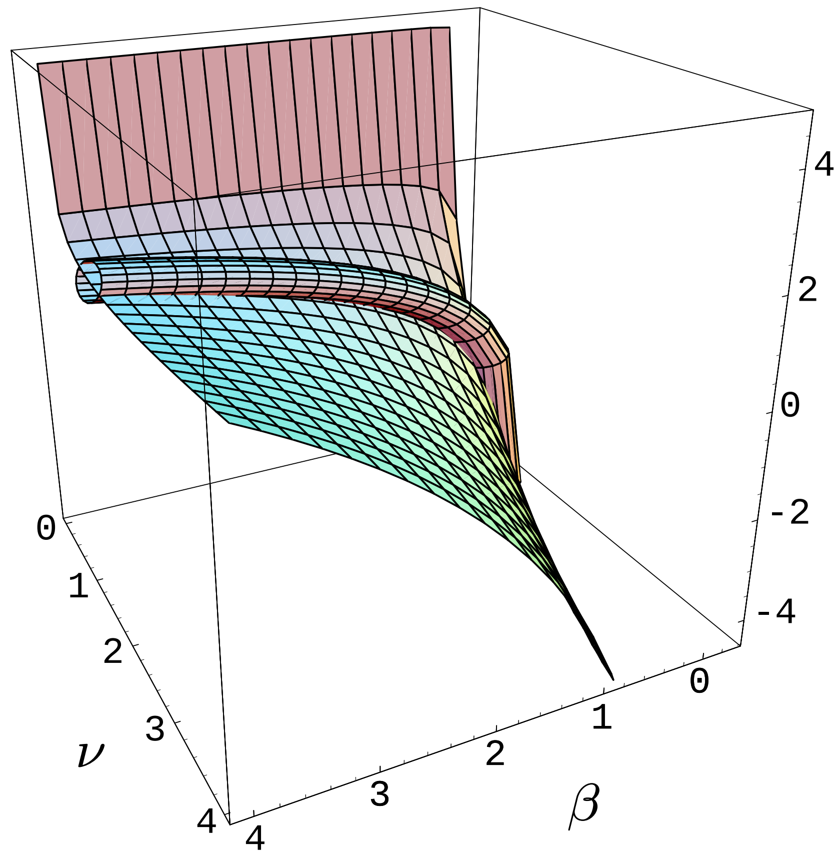
$\theta \in [0, 2\pi)$ contains all immersions for small enough perturbations of exponential distributions.

This tubular neighbourhood intersects with the gamma manifold immersion to yield the required neighbourhood of gamma distributions. $\square$



Gamma manifold affine immersion in natural coordinates $\mu = \nu/\beta, \nu$ as a surface in $\mathbb{R}^3$.

Tubular neighbourhood surrounds all exponential distributions—these lie on the curve $\nu = 1$ in the surface.



Continuous image of gamma manifold affine immersion as surface in $\mathbb{R}^3$ using standard coordinates.

Tubular neighbourhood surrounds all exponential distributions—these lie on the curve $\nu = 1$ in the surface.

Log-gamma manifold $\mathcal{L}$ and uniformity

The pdfs for random variable $N \in [0,1]$ given by

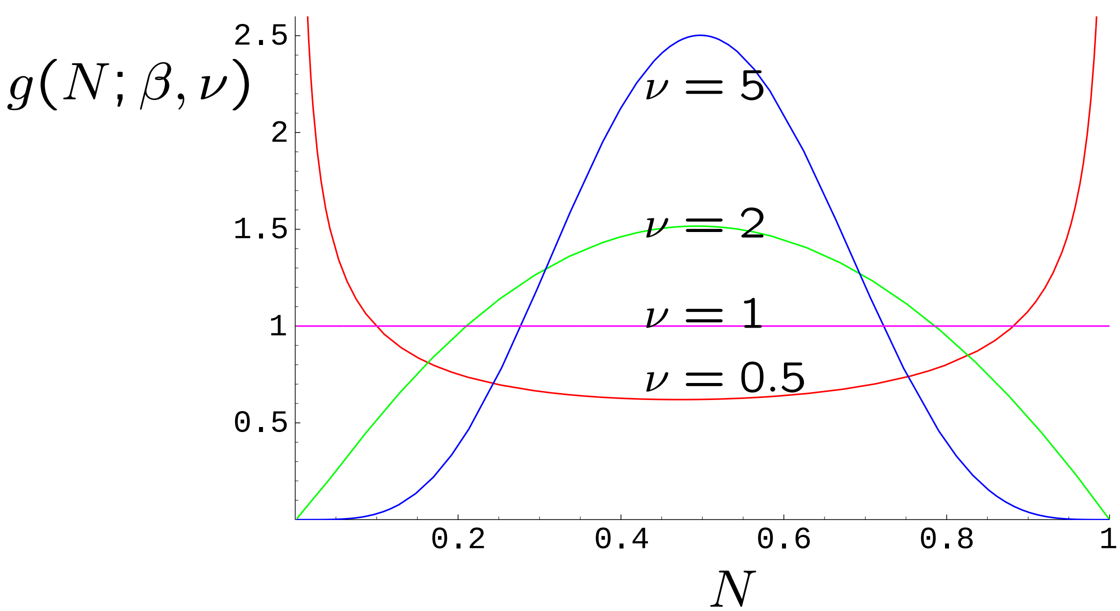
$$g(N,\beta,\nu) = \frac{\frac{1}{N}^{1-\frac{\nu}{\beta}} \, (\frac{\nu}{\beta})^{\nu} \, (\log \frac{1}{N})^{\nu-1}}{\Gamma(\nu)}$$

for $\beta, \nu \in \mathbb{R}^{+}$.

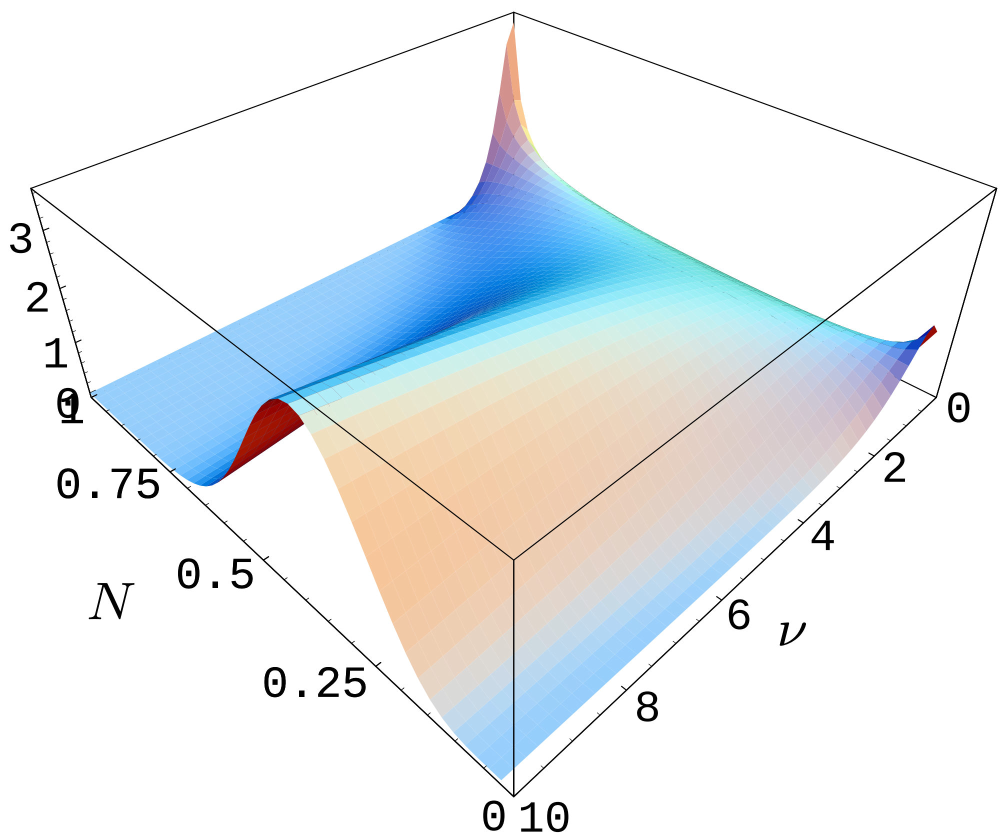
Uniform distribution

$$\lim_{\beta \to 1} g(N,\beta,1) = g(N,1,1) = 1 \ .$$

Log-gamma pdfs $g(N; \beta, \nu)$, $N \in [0, 1]$, **with central mean** $< N >= 0.5$, **and** $\nu = 0.5, 1,\ 2,\ 5.$ **Cases** $\nu < 1$ **correspond in gamma distributions to clustering in an underlying spatial process; conversely,** $\nu > 1$ **corresponds to dispersion and greater evenness than random.**



Family of log-gamma pdfs with mean $\frac{1}{2}$



Neighbourhood of the uniform distribution

$\mathcal{L}$ isometric with gamma manifold, $\mathcal{G}$. Hence, immersion of $\mathcal{G}$ in $\mathbb{R}^3$ above, represents also the log-gamma manifold $\mathcal{L}$. Then, since the isometry sends the exponential distribution to the uniform distribution on $[0, 1]$, we obtain a general deduction

Proposition (Dodson)
Every neighbourhood of the uniform distribution contains a neighbourhood of log-gamma distributions. $\square$

Equivalently,

Every neighbourhood of a uniform stochastic process contains a neighbourhood of stochastic processes subordinate to log-gamma distributions. $\square$

Freund bivariate exponential 4-manifold F and independence

- Freund bivariate exponential distributions and correlation
- Freund Fisher information
- Natural coordinate system and potential function
- α-Connection
- α-Curvatures
- Freund submanifold F_1
- F_1 and independence
- Neighbourhood of independent

Freund distributions

Freund's joint density function of X and Y is

$$f(x,y) = \begin{cases} \alpha_1 \beta_2 e^{-\beta_2 y - (\alpha_1+\alpha_2-\beta_2)x} & \text{for } 0 < x < y, \\ \alpha_2 \beta_1 e^{-\beta_1 x - (\alpha_1+\alpha_2-\beta_1)y} & \text{for } 0 < y < x \end{cases}$$

where $\alpha_i, \beta_i > 0 \quad (i = 1, 2)$.

Provided that $\alpha_1 + \alpha_2 \neq \beta_1$, the marginal density function of X is

$$f_X(x) = \left(\frac{\alpha_2}{\alpha_1+\alpha_2-\beta_1}\right) \beta_1 \, e^{-\beta_1 x} +$$

$$\left(\frac{\alpha_1-\beta_1}{\alpha_1+\alpha_2-\beta_1}\right) (\alpha_1+\alpha_2) \, e^{-(\alpha_1+\alpha_2)x}, \; x \geq 0$$

and provided that $\alpha_1+\alpha_2 \neq \beta_2$, The marginal density function of Y is

$$f_Y(y) = \left(\frac{\alpha_1}{\alpha_1+\alpha_2-\beta_2}\right) \beta_2 \, e^{-\beta_2 y} +$$

$$\left(\frac{\alpha_2-\beta_2}{\alpha_1+\alpha_2-\beta_2}\right) (\alpha_1+\alpha_2) \, e^{-(\alpha_1+\alpha_2)y}, \; y \geq 0$$

This system of distributions termed bivariate mixture exponential distributions rather than simply bivariate exponential distributions.

The marginal density functions $f_X(x)$ and $f_Y(y)$ are exponential distributions only in the special case $\alpha_i = \beta_i \quad (i = 1, 2)$.

Covariance and correlation coefficient

$$Cov(X,Y) = \frac{\beta_1\,\beta_2 - \alpha_1\,\alpha_2}{\beta_1\,\beta_2\,(\alpha_1+\alpha_2)^2}$$

$$\rho(X,Y) =$$

$$\frac{\beta_1\,\beta_2 - \alpha_1\,\alpha_2}{\sqrt{{\alpha_2}^2 + 2\,\alpha_1\,\alpha_2 + {\beta_1}^2}\,\sqrt{{\alpha_1}^2 + 2\,\alpha_1\,\alpha_2 + {\beta_2}^2}}$$

Note that $-\frac{1}{3} < \rho(X,Y) < 1$.

$\rho(X,Y) \to 1$ as $\beta_1,\,\beta_2 \to \infty$

$\rho(X,Y) \to -\frac{1}{3}$ as $\alpha_1 = \alpha_2$ and $\beta_1,\,\beta_2 \to 0$.

Independent case:

$\rho(X,Y) = 0$ iff $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$.

Fisher information metric

F the set of Freund bivariate mixture exponential distributions becomes a 4-manifold, metric

$$\begin{bmatrix} \frac{1}{{\alpha_1}^2+\alpha_1\,\alpha_2} & 0 & 0 & 0 \\ 0 & \frac{\alpha_2}{{\beta_1}^2\,(\alpha_1+\alpha_2)} & 0 & 0 \\ 0 & 0 & \frac{1}{{\alpha_2}^2+\alpha_1\,\alpha_2} & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{{\beta_2}^2\,(\alpha_1+\alpha_2)} \end{bmatrix}$$

with respect to the coordinates $(\alpha_1, \beta_1, \alpha_2, \beta_2)$.

Natural coordinate system

It was noted by Leurgans, Tsai, and Crowley that the family of Freund distributions forms an exponential family, with natural parameters

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha_1 + \beta_1, \alpha_2, \log\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right), \beta_2)$$

and its potential function

$$\varphi(\theta) = -\log(\frac{\theta_1\,\theta_2\,\theta_4}{e^{\theta_3}\,\theta_2 + \theta_4}) = -\log(\alpha_2\,\beta_1).$$

α-Connection

The nonzero independent components of the $\nabla^{(\alpha)}$-connection are given by:

$$\begin{aligned}
\Gamma^{(\alpha)1}_{11} &= -\frac{1+\alpha}{2\,\alpha_1}+\frac{-1+3\,\alpha}{2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)1}_{13} &= \Gamma^{(\alpha)2}_{12}=\Gamma^{(\alpha)3}_{13}=\Gamma^{(\alpha)4}_{34}=\frac{\alpha-1}{2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)1}_{22} &= -\Gamma^{(\alpha)3}_{22}=\frac{(1+\alpha)\,\alpha_1\,\alpha_2}{2\,(\alpha_1+\alpha_2)\,{\beta_1}^2}\,,\\
\Gamma^{(\alpha)1}_{33} &= \Gamma^{(\alpha)2}_{23}=\frac{(1+\alpha)\,\alpha_1}{2\,\alpha_2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)1}_{44} &= -\Gamma^{(\alpha)3}_{44}=\frac{-(1+\alpha)\,\alpha_1\,\alpha_2}{2\,(\alpha_1+\alpha_2)\,{\beta_2}^2}\,,\\
\Gamma^{(\alpha)3}_{11} &= \Gamma^{(\alpha)4}_{14}=\frac{(1+\alpha)\,\alpha_2}{2\,\alpha_1\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)3}_{33} &= -\frac{1+\alpha}{2\,\alpha_2}+\frac{-1+3\,\alpha}{2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)4}_{44} &= \frac{\alpha-1}{\beta_2}\,. \qquad \square
\end{aligned}$$

α-**Curvatures**

The α-**Ricci tensor**:

$$[R_{ij}^{(\alpha)}] = \left(\alpha^2 - 1\right).$$

$$\begin{bmatrix} \frac{-\alpha_2}{2\alpha_1(\alpha_1+\alpha_2)^2} & 0 & \frac{1}{2(\alpha_1+\alpha_2)^2} & 0 \\ 0 & \frac{-\alpha_2}{2(\alpha_1+\alpha_2)\beta_1{}^2} & 0 & 0 \\ \frac{1}{2(\alpha_1+\alpha_2)^2} & 0 & \frac{-\alpha_1}{2\alpha_2(\alpha_1+\alpha_2)^2} & 0 \\ 0 & 0 & 0 & \frac{-\alpha_1}{2(\alpha_1+\alpha_2)\beta_2{}^2} \end{bmatrix}$$

The α-**eigenvalues** and the α-**eigenvectors** of the α-Ricci tensor are given by:

$$\left(\alpha^2 - 1\right) \begin{pmatrix} 0 \\ \frac{1}{(\alpha_1+\alpha_2)^2} - \frac{1}{2\,\alpha_1\,\alpha_2} \\ \frac{-\alpha_2}{2\,(\alpha_1+\alpha_2)\,\beta_1{}^2} \\ \frac{-\alpha_1}{2\,(\alpha_1+\alpha_2)\,\beta_2{}^2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\alpha_1}{\alpha_2} & 0 & 1 & 0 \\ -\frac{\alpha_2}{\alpha_1} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The α-**scalar curvature**

$$R^{(\alpha)} = \frac{-3\left(\alpha^2 - 1\right)}{2}$$

Note that the Freund manifold has a constant scalar curvature.

The α-**sectional curvatures**

$$\begin{aligned}
\varrho^{(\alpha)}(1,2) &= \varrho^{(\alpha)}(1,4) = \frac{\left(1-\alpha^2\right)\,\alpha_2}{4\,(\alpha_1+\alpha_2)}\,, \\
\varrho^{(\alpha)}(1,3) &= 0, \\
\varrho^{(\alpha)}(2,3) &= \varrho^{(\alpha)}(3,4) = \frac{\left(1-\alpha^2\right)\,\alpha_1}{4\,(\alpha_1+\alpha_2)}\,, \\
\varrho^{(\alpha)}(2,4) &= \frac{1-\alpha^2}{4}\,.
\end{aligned}$$

The α-**mean curvatures**

$$\begin{aligned}
\varrho^{(\alpha)}(1) &= \frac{\left(1-\alpha^2\right)\,\alpha_2}{6\,(\alpha_1+\alpha_2)}\,, \\
\varrho^{(\alpha)}(2) &= \varrho(4) = \frac{1-\alpha^2}{6}\,, \\
\varrho^{(\alpha)}(3) &= \frac{\left(1-\alpha^2\right)\,\alpha_1}{6\,(\alpha_1+\alpha_2)}\,.
\end{aligned}$$

Submanifold $F_1 \subset F$: $\alpha_1 = \alpha_2,\ \beta_1 = \beta_2$

The pdfs are of form :

$$f(x, y; \alpha_1, \beta_1) =$$

$$\begin{cases} \alpha_1 \beta_1 \, e^{-\beta_1 y - (2\,\alpha_1 - \beta_1) x} & \text{for } 0 < x < y \\ \alpha_1 \beta_1 \, e^{-\beta_1 x - (2\,\alpha_1 - \beta_1) y} & \text{for } 0 < y < x \end{cases}$$

Parameters $\alpha_1, \beta_1 > 0$. The marginal pdfs, for X and Y are given by :

$$f_X(x) =$$

$$\left(\frac{\alpha_1}{2\,\alpha_1 - \beta_1} \right) \beta_1 \, e^{-\beta_1 x} + \left(\frac{\alpha_1 - \beta_1}{2\,\alpha_1 - \beta_1} \right) (2\,\alpha_1) \, e^{-2\,\alpha_1 x}$$

$$f_Y(y) =$$

$$\left(\frac{\alpha_1}{2\,\alpha_1 - \beta_1} \right) \beta_1 \, e^{-\beta_1 y} + \left(\frac{\alpha_1 - \beta_1}{2\,\alpha_1 - \beta_1} \right) (2\,\alpha_1) \, e^{-2\,\alpha_1 y}$$

with $x, y \geq 0$

F_1 an exponential family with natural coordinates (α_1, β_1) and potential function

$$\psi = -\log(\alpha_1\,\beta_1)$$

Covariance: $Cov(X,Y) = \frac{1}{4}\left(\frac{1}{{\alpha_1}^2} - \frac{1}{{\beta_1}^2}\right),$

Correlation: $\rho(X,Y) = 1 - \frac{4\,{\alpha_1}^2}{3\,{\alpha_1}^2 + {\beta_1}^2},$

Independence case: $\alpha_1 = \beta_1$

F_1 has Fisher metric: $[g_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\beta_1^2} \end{bmatrix}$.

The α-**curvature tensor**, α-**Ricci tensor**, and α-**scalar curvature** of F_1 are zero.

Freund manifold and neighbourhoods of independence

Proposition

In the affine embedding of the Freund submanifold F_1 in $\mathbb{R}^3$, a tubular neighbourhood of the curve $\alpha_1 = \beta_1$ will contain all immersions of bivariate exponential processes sufficiently close to the case of independence.

The submanifold F_1 can be realized in $\mathbb{R}^3$ by the graph of a potential function, the affine immersion $\{f, \xi\}$:

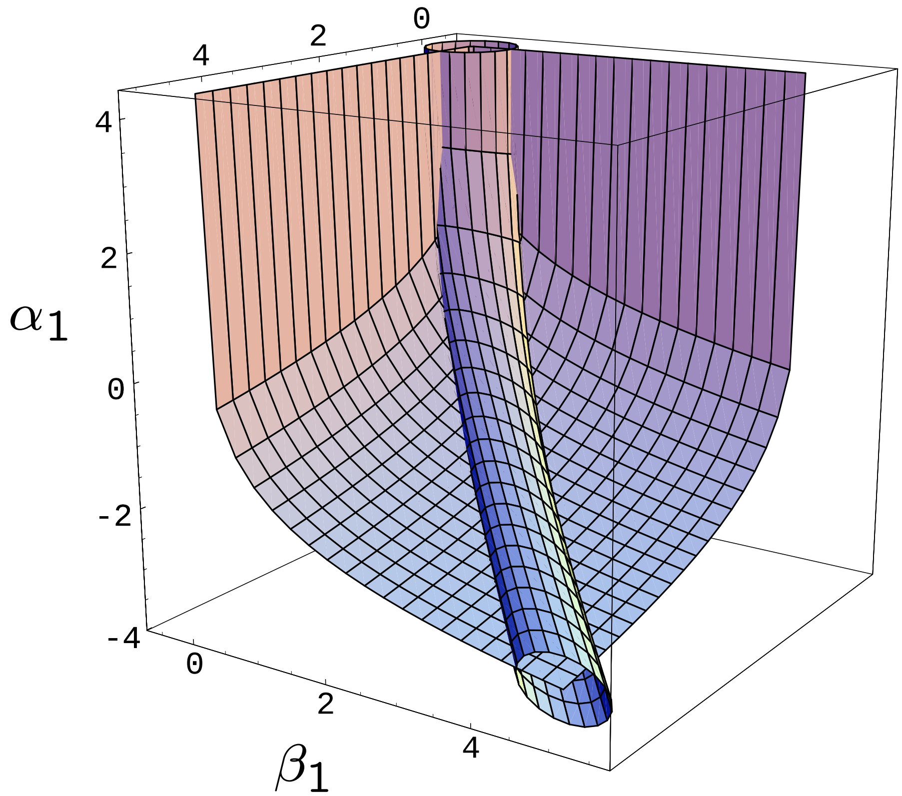
$$f \;:\; \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^3 : (\alpha_1, \beta_1) \mapsto (\alpha_1, \beta_1, \psi)\,,$$
$$\xi = (0, 0, 1)\,.$$

where $\psi = -\log(\alpha_1\,\beta_1)$.

The submanifold $W \subset F_1$ consisting of the independent case ($\alpha_1 = \beta_1$) is represented by the curve:

$$(0, \infty) \to \mathbb{R}^3 : (\alpha_1) \mapsto (\alpha_1, \alpha_1, -2 \log \alpha_1) .$$

This curve represents all bivariate distributions having identical exponential marginals and zero covariance; its tubular neighbourhood represents departures from independence and will contain all immersions of bivariate exponential processes sufficiently close to the case of independence. $\square$



Affine immersion in natural coordinates (α_1, β_1) as a surface in $\mathbb{R}^3$ for the Freund submanifold F_1.

Tubular neighbourhood surrounds the curve ($\alpha_1 = \beta_1$ in the surface) consisting of all bivariate distributions having identical exponential marginals and zero covariance.

Bivariate Gaussian 5-Manifold N and independence

- Bivariate Gaussian distributions and correlation
- Bivariate Gaussian Fisher information
- Natural coordinate system and potential function
- α-Curvatures
- Submanifold N_1 with zero means and identical standard deviation
- N_1 and independence
- Neighbourhood of independent

Bivariate Gaussian distribution

The pdf for the two-dimensional Gaussian distribution has the form:

$$f(x,y;\mu_1,\mu_2,\sigma_1,\sigma_{12},\sigma_2) =$$

$$\frac{e^{-\frac{1}{2\triangle}\left(\sigma_2(x-\mu_1)^2-2\,\sigma_{12}\,(x-\mu_1)\,(y-\mu_2)+\sigma_1(y-\mu_2)^2\right)}}{2\,\pi\,\sqrt{\triangle}}$$

where $-\infty < x_1 < x_2 < \infty$,
$-\infty < \mu_1 < \mu_2 < \infty$, $0 < \sigma_1, \sigma_2 < \infty$, and $\triangle = \sigma_1\,\sigma_2 - {\sigma_{12}}^2; \sigma_{12}$ is covaraince

Correlation coefficient : $\rho = \frac{\sigma_{12}}{\sqrt{\sigma_1\,\sigma_2}}$

Marginal distributions are Gaussian with parameters (μ_1, σ_1) and (μ_2, σ_2).

Fisher information metric

The Fisher information metric with respect to the parameters $(\mu_1, \mu_2, \sigma_1, \sigma_{12}, \sigma_2)$ is:

$$\begin{bmatrix} \frac{\sigma_2}{\triangle} & -\frac{\sigma_{12}}{\triangle} & 0 & 0 & 0 \\ -\frac{\sigma_{12}}{\triangle} & \frac{\sigma_1}{\triangle} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_2^2}{2\,\triangle^2} & -\frac{\sigma_{12}\,\sigma_2}{\triangle^2} & \frac{\sigma_{12}^2}{2\,\triangle^2} \\ 0 & 0 & -\frac{\sigma_{12}\,\sigma_2}{\triangle^2} & \frac{\sigma_1\,\sigma_2+\sigma_{12}^2}{\triangle^2} & -\frac{\sigma_1\,\sigma_{12}}{\triangle^2} \\ 0 & 0 & \frac{\sigma_{12}^2}{2\,\triangle^2} & -\frac{\sigma_1\,\sigma_{12}}{\triangle^2} & \frac{\sigma_1^2}{2\,\triangle^2} \end{bmatrix},$$

$$\triangle = \sigma_1\,\sigma_2 - {\sigma_{12}}^2 .$$

Natural coordinate system

$$\theta_1 = \frac{\mu_1\sigma_2 - \mu_2\sigma_{12}}{\sigma_1\sigma_2 - {\sigma_{12}}^2},$$

$$\theta_2 = \frac{\mu_2\sigma_1 - \mu_1\sigma_{12}}{\sigma_1\sigma_2 - {\sigma_{12}}^2},$$

$$\theta_3 = \frac{-\sigma_2}{2\left(\sigma_1\sigma_2 - {\sigma_{12}}^2\right)},$$

$$\theta_4 = \frac{\sigma_{12}}{\sigma_1\sigma_2 - {\sigma_{12}}^2},$$

$$\theta_5 = \frac{-\sigma_1}{2\left(\sigma_1\sigma_2 - {\sigma_{12}}^2\right)}$$

and its potential function

$$\varphi(\theta) = \log(2\pi\sqrt{\triangle}) + \frac{{\mu_2}^2\sigma_1 + {\mu_1}^2\sigma_2 - 2\mu_1\mu_2\sigma_{12}}{2\triangle}.$$

α-Curvatures

The α-**Ricci tensor**:

$$R^{(\alpha)} = \left(\alpha^2 - 1\right).$$

$$\left[\begin{array}{ccccc} \frac{\sigma_2}{2\triangle} & -\frac{\sigma_{12}}{2\triangle} & 0 & 0 & 0 \\ -\frac{\sigma_{12}}{2\triangle} & \frac{\sigma_1}{2\triangle} & 0 & 0 & 0 \\ 0 & 0 & \frac{{\sigma_2}^2}{2\triangle^2} & -\frac{\sigma_2\sigma_{12}}{\triangle^2} & \frac{3{\sigma_{12}}^2-\sigma_1\sigma_2}{4\triangle^2} \\ 0 & 0 & -\frac{\sigma_2\sigma_{12}}{\triangle^2} & \frac{3\sigma_1\sigma_2+{\sigma_{12}}^2}{2\triangle^2} & -\frac{\sigma_1\sigma_{12}}{\triangle^2} \\ 0 & 0 & \frac{3{\sigma_{12}}^2-\sigma_1\sigma_2}{4\triangle^2} & -\frac{\sigma_1\sigma_{12}}{\triangle^2} & \frac{{\sigma_1}^2}{2\triangle^2} \end{array}\right]$$

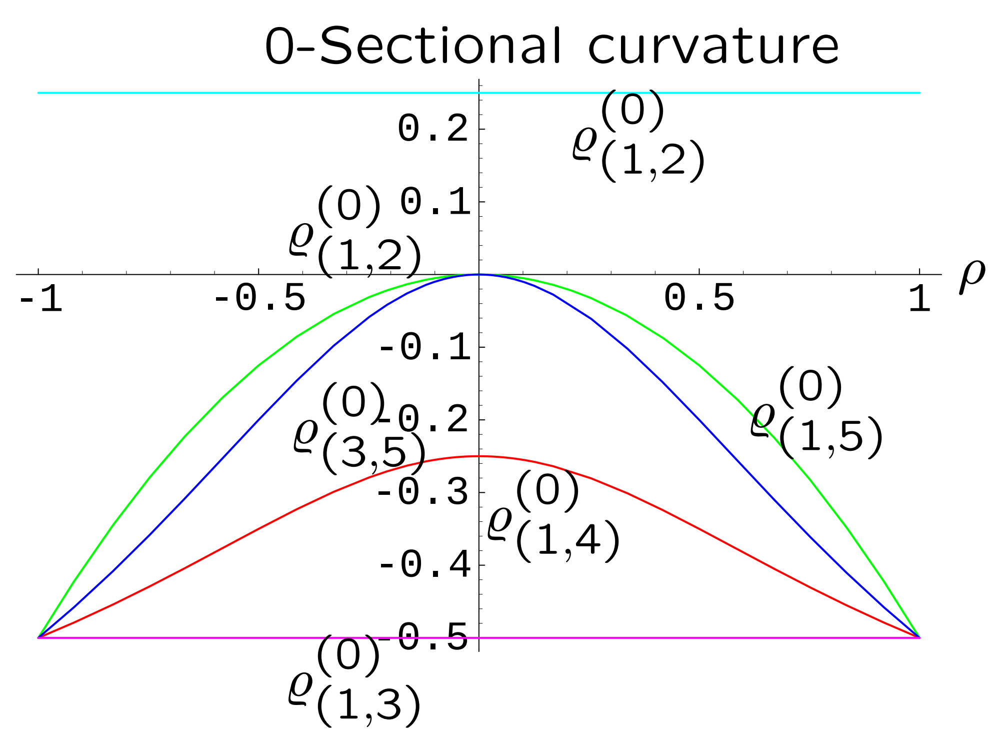
The α-**scalar curvature**:

$$R^{(\alpha)} = \frac{9\,\left(\alpha^2 - 1\right)}{2}$$

The α-**sectional curvatures** as a function only of the correlation ρ:

$$\varrho^{(\alpha)} = \left(\alpha^2 - 1\right).$$

$$\left[\begin{array}{ccccc} 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{\rho^2}{2} \\ -\frac{1}{4} & 0 & \frac{\rho^2}{2} & \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{1}{2} \\ \frac{1}{2} & \frac{\rho^2}{2} & 0 & \frac{1}{2} & \frac{\rho^2}{1+\rho^2} \\ \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\rho^2}{2} & \frac{1}{2} & \frac{\rho^2}{1+\rho^2} & \frac{1}{2} & 0 \end{array}\right].$$

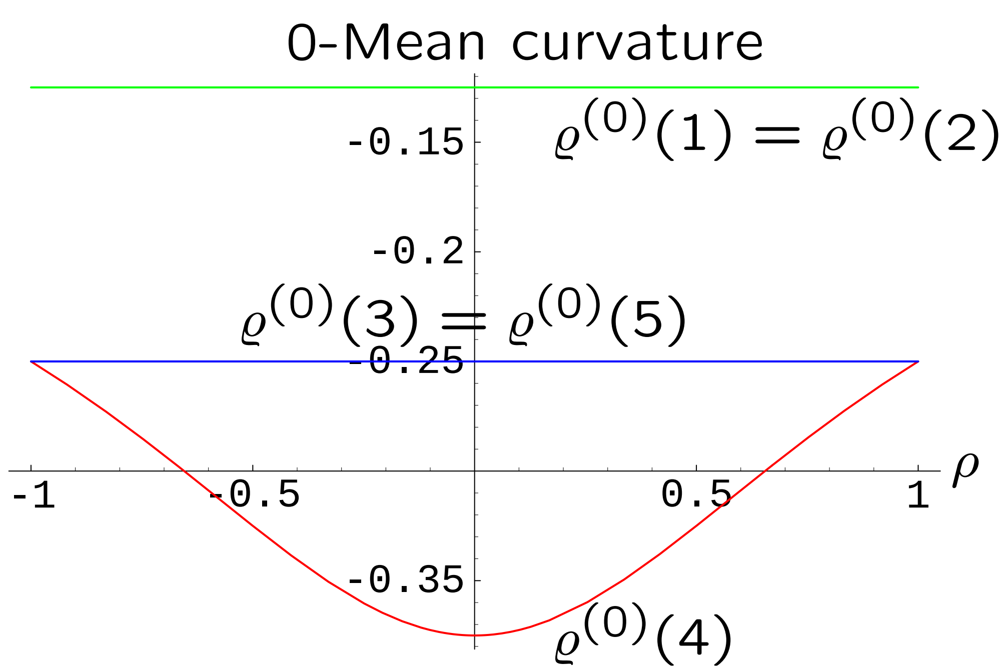


The α-**mean curvatures** as a function only of correlation:

$$\varrho^{(\alpha)}(1) = \varrho^{(\alpha)}(2) = \frac{\alpha^2 - 1}{8},$$

$$\varrho^{(\alpha)}(3) = \varrho^{(\alpha)}(5) = \frac{\alpha^2 - 1}{4},$$

$$\varrho^{(\alpha)}(4) = \frac{\left(\alpha^2 - 1\right)\left(3 + \rho^2\right)}{8\left(1 + \rho^2\right)}.$$



Submanifold with zero means and identical standard deviation

$N_1 \subset N$: $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = \sigma$

The distributions are of form:

$$f(x, y; \sigma, \sigma_{12}) = \frac{1}{2\pi\sqrt{\triangle}}\, e^{-\frac{1}{2\triangle}\left(\sigma x^2 - 2\sigma_{12} x\, y + \sigma y^2\right)}$$

where $\triangle = \sigma^2 - {\sigma_{12}}^2$.

The marginal functions are $f_X = f_Y \equiv N(0, \sigma)$; with correlation coefficient $\rho = \frac{\sigma_{12}}{\sigma}$.

The Fisher metric: $[g_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\beta_1^2} \end{bmatrix}$.

The α-**curvature tensor**, α-**Ricci tensor**, and α-**scalar curvature** of F_1 are zero.

N_1 and neighbourhoods of independence

Proposition

The bivariate Gaussian 5-manifold admits a 2-dimensional submanifold through which can be provided a neighbourhood of bivariate distributions having identical Gaussian marginals and zero covariance containing the independent case.

Outline proof

Bivariate Gaussians with zero means ($\mu_1 = \mu_2 = 0$) and identical standard deviation $\sigma_1 = \sigma_2 = \sigma$ is represented by the surface in $\mathbb{R}^3$ by the affine immersion $\{f, \xi\}$

$$f : N_1 \to \mathbb{R}^3 : (\theta_1, \theta_2) \mapsto (\theta_1, \theta_2, \varphi(\theta)),\ \xi = (0, 0, 1).$$

where $(\theta_1, \theta_2) = (\frac{-\sigma}{2\triangle}, \frac{\sigma_{12}}{\triangle})$, $\triangle = \sigma^2 - {\sigma_{12}}^2$

and potential function $\varphi(\theta) = \log(2\pi\sigma)$.

Independent case

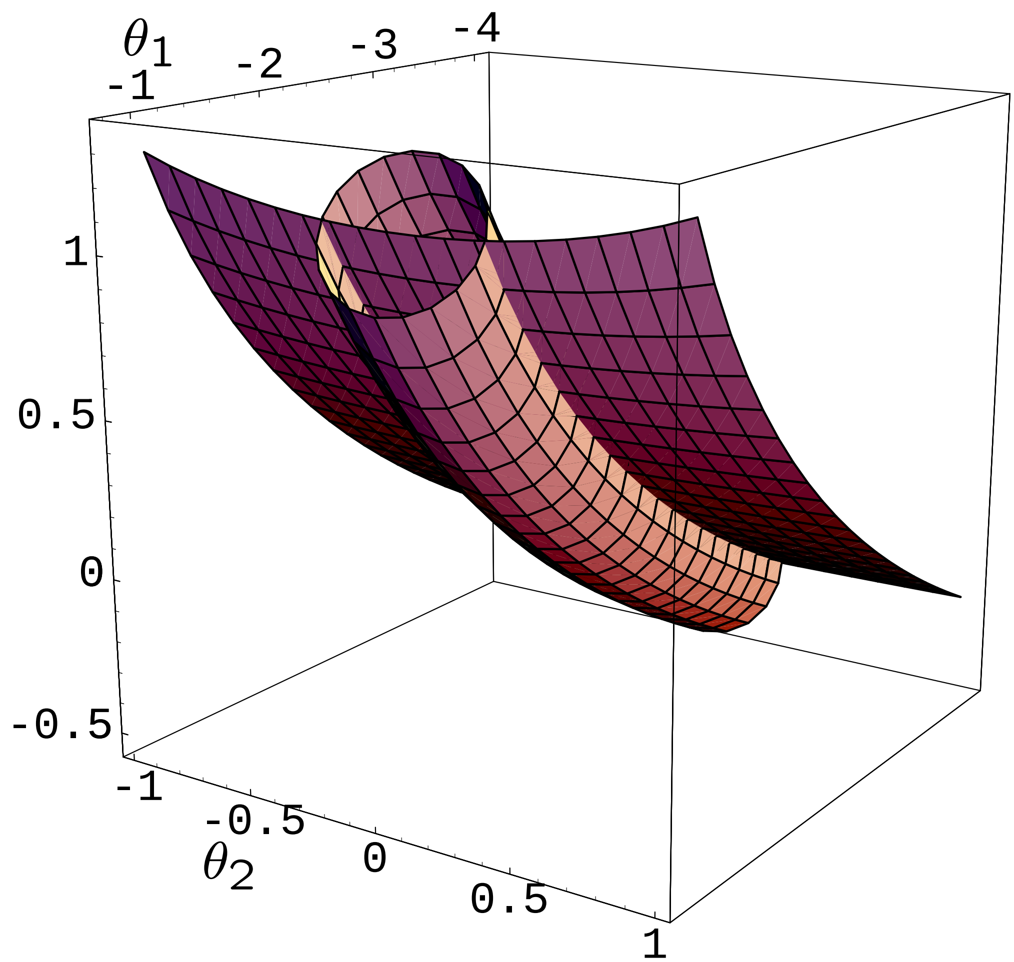
The Submanifold of the independent case with zero means and identical standard deviation is curve

$$(-\infty, 0) \to \mathbb{R}^3 : (-\frac{1}{2\,\sigma}) \mapsto (-\frac{1}{2\,\sigma}, 0, \log(2\,\pi\,\sigma))$$

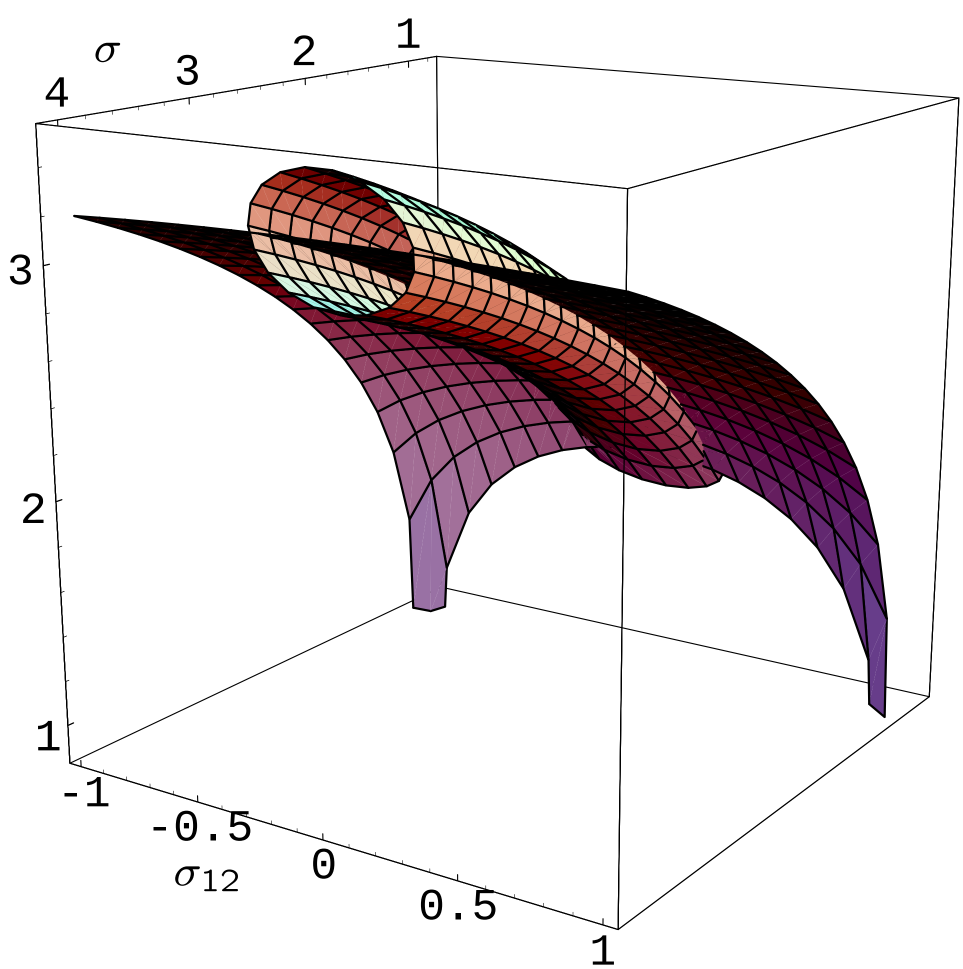
A Tubular neighbourhood represents departures from independence and will contain all immersions of bivariate Gaussian processes sufficiently close to the independence case.
□

Corollary (Dodson)
Via the Central Limit Theorem, the tubular neighbourhood of the case of zero covariance will contain all immersions of limiting bivariate processes sufficiently close to the independence case for all processes with marginals that converge in distribution to Gaussians.

Affine immersion in natural coordinates $(\theta_1, \theta_2) = (\frac{-\sigma}{2\triangle}, \frac{\sigma_{12}}{\triangle})$ as surface in $\mathbb{R}^3$ for bivariate Gaussian distributions with zero means and identical standard deviation σ. Tubular neighbourhood surrounds curve ($\sigma_{12} = 0$ in the surface) representing all bivariate distributions having identical Gaussian marginals and zero covariance. Tubular neighourhood contains all small departures from independence.



Continuous image of the affine immersion as a surface in $\mathbb{R}^3$ using standard coordinates for the bivariate Gaussian distributions with zero means and identical standard deviation σ. Tubular neighbourhood surrounds curve of independent states, ($\sigma_{12} = 0$,) and contains all small departures from independence.



Results

Explicit representations in $\mathbb{R}^3$ of tubular neighbourhoods that provide the following:

A. All processes sufficiently close[1] to a Poisson[2] process

B. All processes sufficiently close to a uniform process

C. All bivariate processes sufficiently close to the independent bivariate Poisson[3] process

D. All bivariate processes sufficiently close to the independent bivariate Gaussian[4] process

[1] *In an information geometric immersion*
[2] *Via unique associated exponential process*
[3] *Marginals having same mean*
[4] *Marginals both* $N(0, \sigma)$

References:

1) Khadiga Arwini and C.T.J. Dodson. Information geometric neighbourhoods of randomness and geometry of the McKay bivariate gamma 3-manifold. Sankhya: Indian Journal of Statistics 66, 2 (2004).

2) Khadiga Arwini and C.T.J. Dodson. Neighbourhoods of independence and associated geometry. Preprint (2004).