The growth rate of vertex-transitive planar graphs

László Babai *

Abstract

A graph is *vertex-transitive* if all of its vertices are equivalent under automorphisms. Confirming a conjecture of Jon Kleinberg and Éva Tardos, we prove the following trichotomy theorem concerning locally finite vertex-transitive planar graphs: the *rate of growth of a graph with these properties is either linear or quadratic or exponential.* The same result holds more generally for locally finite, *almost* vertex-transitive planar graphs (the automorphism group has a finite number of orbits). The proof uses the elements of hyperbolic plane geometry.

1 Introduction

Graphs with symmetry provide a fertile testing ground for combinatorial theories of algorithmic relevance (cf. the Appendix for examples). In this paper we consider the effect of symmetry on the *growth rate*, a central characteristic of infinite, locally finite graphs. We give a positive answer to a question of Jon Kleinberg and Éva Tardos which arose in connection with their disjoint path approximation algorithm. Along the way, we develop a structure theory of infinite, locally finite planar *almost vertex-transitive* graphs. The last requirement means that there are only a finite number of inequivalent vertices under the automorphism group (see the definition below).

1.1 Symmetry. The permutations of a set V form the group $\mathrm{Sym}(V)$ under composition. The subgroups of this group are the permutation groups acting on V. If G is such a subgroup then G splits V into *orbits:* two elements $x, y \in V$ are equivalent under the action of G if some member of G maps x to y. The equivalence classes are the orbits. The group is *transitive* if it has a single orbit. It is *almost transitive* if it has a finite number of orbits.

The *automorphisms* of a graph $X = (V, E)$ are its self-isomorphisms. They form the group $\mathrm{Aut}(X)$, a permutation group acting on the vertex set V. The graph X is said to be *vertex-transitive* if $\mathrm{Aut}(X)$ acts transitively on V, and *almost vertex-transitive* if $\mathrm{Aut}(X)$ has a finite number of orbits on V.

A graph is *locally finite* if its vertices have finite degree. We define an infinite graph to be *planar* if all of its finite subgraphs are planar.

In this paper we shall be concerned with infinite, locally finite, connected graphs. Through a limit construction (see Section 5.2) such graphs can be used to analyse local properties of families of finite graphs. In particular, infinite locally finite vertex-transitive connected *planar* graphs arise as covers of large finite vertex-transitive graphs embedded on surfaces. The geometry associated with the infinite graphs has been successfully applied to a classification of the finite graphs in question [Ba2].

1.2 Uniform growth. Let $X = (V, E)$ be a connected graph and $r \geq 0$ an integer. For a vertex $v \in V$, let $B_X(v, r)$ denote the *ball of radius* r about v, i. e., the set of vertices at distance $\leq r$ from v. (The subscript X will usually be omitted.) The *growth function* of X at v is the function $g_v(r) := |B(v, r)|$.

Let $\mathbf{R}^+$ denote the set of positive reals. Two monotone non-decreasing functions $f_1, f_2 : \mathbf{R}^+ \to \mathbf{R}^+$ will be said to be *equivalent* if there exist constants $c_1, c_2, C_1, C_2, r_0 > 0$ such that for all $r \geq r_0$, $C_1 f_1(c_1 r) \leq f_2(r) \leq C_2 f_1(c_2 r)$. This is clearly an equivalence relation.

The *rate of growth* of X about v is the equivalence class of g_v (extended monotonically to $\mathbf{R}^+$). Thus *linear, quadratic, polynomial,* and *exponential* rates of growth are defined by the respective growth functions $\Theta(r), \Theta(r^2), r^{O(1)}$, and $2^{\Theta(r)}$.

It is clear that a connected, locally finite graph has the same growth rate at each vertex.

We say that X has *uniform* growth rate f if the constants c_1, c_2, C_1, C_2, r_0 can be chosen independently of v. In other words, X has uniform growth rate f if there exist positive constants c_1, c_2, C_1, C_2, r_0 such that for every vertex v and all $r \geq r_0$,

$$(1.1) \qquad C_1 f(c_1 r) \leq |B(v, r)| \leq C_2 f(c_2 r).$$

It is clear that *for connected, almost vertex-transitive graphs, the growth rate is uniform.*

*Department of Computer Science, University of Chicago, 1100 E 58th St, Chicago IL 60637-1504. E-mail: laci@cs.uchicago.edu

1.3 Digression: growth of groups. The *Cayley graph* of the group G with respect to the set S of generators is a graph with vertex set G; the edges are the pairs $\{g, gs\}$ where $g \in G, s \in S$. The rate of growth of a *finitely generated group* is defined as the rate of growth of any of its locally finite Cayley graphs (the rate does not depend on the choice of the generators). Questions regarding the rate of growth of finitely generated groups have been initiated in differential geometry with the discovery by J. Milnor of a connection between curvature and the growth rate of the fundamental group [Mi]. Milnor observed that groups with a nilpotent subgroup of finite index have polynomial growth rate. A major result of recent mathematics was Gromov's proof of the converse (only these groups have polynomial growth rate) [Gro], confirming Milnor's conjecture. A characterization of all vertex-transitive graphs of polynomial growth was later given by Trofimov [Tr]. Groups with intermediate (fractionally exponential) growth rates were constructed by Grigorchuk [Gri].

1.4 The main result. In connection with their recent *disjoint path approximation algorithm,* [KT1, KT2], Jon Kleinberg and Éva Tardos raised the question, what rates of "uniform growth functions" would be admitted by families of finite *planar* graphs. (For the definitions and the precise question, see Section 5.) That problem led them to "what seemed like a cleaner question of the same general flavor" [KT3], namely the question of the growth rate of infinite, locally finite, connected vertex-transitive planar graphs.

The two-way infinite path has linear growth rate, the planar grid has quadratic, the infinite regular trivalent tree has exponential growth rate. Noting these and similar examples, Kleinberg and Tardos formulated the conjecture that these are the only possible growth rates. In this paper we confirm this conjecture in a more general form, extending it to *almost vertex-transitive* graphs.

Theorem 1.1. *The growth rate of an infinite, locally finite, connected, almost vertex-transitive, planar graph is linear, quadratic, or exponential.*

We state a problem which is equivalent to the stronger conjecture of Kleinberg and Tardos. (For the original formulation of the Kleinberg-Tardos question and its equivalence to Problem 1.1, see Section 5.)

Problem 1.1. Suppose an infinite, locally finite, connected planar graph has uniform growth rate f. Does it follow that f is linear, quadratic, or exponential?

A positive answer would generalize Theorem 1.1 in view of the observation that almost vertex-transitive graphs have *uniform* growth rates.

2 Preliminary results

2.1 Graphs with two or more ends. "Ends" are another important graphic notion which first arose in connection with finitely generated infinite groups. By means of ends, we achieve a critical initial classification of the objects under consideration.

The space of *ends* of a connected, locally connected, locally compact Hausdorff space X is defined as the inverse limit of the directed family of the set of components of $X \setminus C$ for all compact subsets C (Hopf [Ho2]). The analogous concept for connected graphs was developed by Halin [Hal].

The space of ends of a connected, locally finite graph X can be defined as the inverse limit of the sets of infinite components obtained by deleting finite subsets C of the edge set of the graph X. However, the reader unfamiliar with inverse limits need not worry; we give equivalent definitions below which do not require inverse limits.

We say that a graph X has *at least k ends* if there exists a finite set C of edges such that $X \setminus C$ has at least k infinite components. X has exactly k ends if it has at least k but not at least $k+1$ ends. X has infinitely many ends if it has at least k ends for every integer k.

The ends themselves can be defined as equivalence classes of rays (one-way infinite paths): two rays are equivalent if the deletion of no finite set of edges separates their infinite components (Halin [Hal]).

Note that the two-way infinite path has two ends, the infinite plane grid has one end, and the infinite ternary tree has infinitely many ends; the number of ends of a finite graph is zero.

The following observation goes back to Hopf [Ho2] who proved it for Cayley graphs.

Proposition 2.1. *An infinite, locally finite, connected, almost vertex-transitive graph has either one or two or infinitely many ends.*

The idea is to prove that such graphs are "tree-like" in some sense. Indeed, let X be such a graph and C a finite connected subgraph such that $X \setminus C$ has $k \geq 3$ infinite components, $Y_1, \ldots, Y_k$. Let d be the diameter of C and s the greatest distance between orbits of $\mathrm{Aut}(X)$. Then there is a copy C_i of C (under some automorphism) inside each Y_i, at distance $\leq d+s+1$ from C. Now C_i splits X into k infinite components, one of which contains C and therefore all the Y_j except Y_i. Therefore $X \setminus (C \cup \bigcup_{i=1}^k C_i)$ has $\geq k(k-1)$ infinite components. For $k \geq 3$ this is a strict increase. ∎

The proof yields the following by-product:

PROPOSITION 2.2. *If an infinite locally finite almost vertex-transitive graph has more than two ends then it has exponential growth rate.*

Indeed, continuing the above process we see that the removal of a ball of radius $t(d+s+1)$ will result in $\geq k(k-1)^{t-1}$ infinite components, and therefore any ball of radius $t(d+s+1)+1$ contains $\geq k(k-1)^{t-1}$ vertices. ∎

The case of two ends is similarly simple.

PROPOSITION 2.3. *If an infinite, locally finite, almost vertex-transitive graph has two ends then it has linear growth rate.*

We leave the easy proof to the reader. ∎

We should mention that a great deal is known about the structure of the groups acting on graphs with ≥ 2 ends (Dunwoody [Du]). Dunwoody's work represents far reaching generalization of earlier work on ends of groups. The *ends* of a finitely generated group are defined as the ends of its Cayley graphs. Different choices of finite sets of generators result in topologically equivalent sets of ends. Freudenthal characterized groups with two ends [Fr], and Stallings [St] the groups with infinitely many ends.

It should be clear from this brief history that the case of one end is usually the difficult one. Our context is no exception. Only in the case of one end shall we require the assumption of planarity.

For more information and literature about ends of vertex-transitive graphs we refer to [Ba4, p. 1480].

2.2 Plane maps. A *rotation system* of the locally finite graph X is a cyclic ordering ρ_v of the edges incident with vertex v for every $v \in V(X)$. A rotation system defines faces (cf. [GT, Chapter 3.2]). Each face-boundary is either a two-way infinite walk or a closed walk in X; the latter will be filled in with disks to obtain a 2-cell embedding. Let Δ denote the 2-dimensional cell complex obtained. Δ defines a 2-dimensional manifold M, and Δ is called a *map* on M.

The manifold M is *plane* if every cycle splits it into two parts. A *plane embedding* of the graph X is a map on a plane manifold. Observe that we require the faces to be disks (2-cell embedding).

If X is 2-connected then every face of a plane embedding is either an infinite *path* or a *cycle* (no repeated vertices).

In this section we consider some properties of plane embeddings of graphs with one end.

LEMMA 2.1. *Let Δ be the map corresponding to a plane embedding of an infinite, locally finite, 2-connected planar graph X with one end. Then Δ has at most one infinite face.*

(Note that no symmetry assumption is made in this lemma.)

Proof. Assume ℓ_1 and ℓ_2 are two infinite boundary-lines. Let P be a shortest path connecting them. Let H denote the "H-shape" formed by $\ell_1 \cup \ell_2 \cup P$. P cuts each ℓ_i into two pieces ℓ_{ij} $(1 \leq i,j \leq 2)$. (The ℓ_{ij} are rays, i. e., one-way infinite paths.) With appropriate numbering we may assume that in clockwise order around P, the order of the four rays is ℓ_{11}, ℓ_{12}, ℓ_{22}, ℓ_{21}.

Let Q be a shortest path in $X \setminus P$ connecting $\ell_{11} \cup \ell_{21}$ to $\ell_{12} \cup \ell_{22}$. Then (except for its endpoints) Q is disjoint from H. Moreover, at its endpoints, Q must leave ℓ_i on the same side as P. Let C denote the cycle formed by Q and the unique path in H connecting the endpoints of Q. C cuts the manifold into two parts. It is now easy to see that both parts of C will contain infinitely many vertices of H, contrary to the assumption of one end. ∎

The following result is due to Bonnington, Imrich, and Watkins [BIW]. For completeness, we sketch our independently found proof.

LEMMA 2.2. *Let Δ be the map corresponding to a plane embedding of an infinite, locally finite, 2-connected planar graph X with one end. Assume that the group of map-automorphisms of Δ is almost vertex-transitive. Then Δ has no infinite faces.*

Proof. By Lemma 2.1, we may assume there is only one infinite face; let ℓ be its boundary. We label the vertices of ℓ as $\{v_i : i \in \mathbf{Z}\}$, where v_i and v_{i+1} are adjacent along ℓ.

ℓ is invariant under $\mathrm{Aut}(\Delta)$. Let α be an automorphism which preserves the direction of ℓ and acts nontrivially on ℓ. So the restriction of α to ℓ is a shift: $v_i^\alpha = v_{i+t}$ for some fixed $t \neq 0$. We may assume $t > 0$.

Let s be the maximum distance between pairs of orbits of $\mathrm{Aut}(\Delta)$ on vertices.

Let us choose a positive integer $r \geq t$ and set $L_1 = \{v_r, v_{r+1}, \dots\}$ and $L_2 = \{v_{-r}, v_{-r-1}, \dots\}$.

Set $B_r = B_X(v_0, r)$ and $B = B_{r+s}$. Since X has one end, there exists a path P in $X \setminus B$ between $L_1 \setminus B$ and $L_2 \setminus B$. Each vertex of P is within distance s of some vertex of ℓ; such a vertex of ℓ cannot belong to B_r and therefore it must belong to $L_1 \cup L_2$. Therefore there exist two adjacent vertices x_i $(i = 1, 2)$ on P such that the distance of x_i to L_i is $\leq s$. It follows that the distance of L_1 and L_2 (in X) is $\leq 2s+1$; let us say $\mathrm{dist}_X(y_1, y_2) \leq 2s+1$, where $y_i \in L_i$. Now an appropriate power β of α will move y_2 within $t/2$ of v_0 along ℓ: $y_2^\beta = v_j$, $|j| \le t/2$. On the other hand, y_1 moves further away from v_0 along ℓ, therefore $y_1^\beta \in L_1$. Since β preserves distances in X, it follows that the distance between v_0 and L_1 is $\le t/2 + 2s + 1$. Since the choice of r was arbitrary, this means that infinitely many vertices of ℓ belong to $B(v_0, t/2 + 2s + 1)$, a contradiction. ∎

Recall that a manifold M is *simply connected* if all loops are contractible in M (there are no "holes").

LEMMA 2.3. *Let Δ be the map corresponding to a plane embedding of an infinite, locally finite, 2-connected planar graph X with one end. Then the corresponding manifold M is simply connected.*

Proof. Let us consider a loop C in the manifold. By continuous deformation (homotopy) we may assume that C is a cycle in X ("simplicial approximation"). C splits M into two parts; one of the parts, say P_1, must contain a finite number of vertices (because X has one end). But this means P_1 consists of a finite number of faces and is bounded by C. By induction on the number of faces in P_1 it follows that C is contractible (removing the faces one-by-one). ∎

Finally, we shall need the following result from [BW].

LEMMA 2.4. *Let X be a connected vertex-transitive graph of finite degree d. Assume X has one end. Then the connectivity of X is $\ge 3(d+1)/4$.* ∎

3 Geometry for vertex-transitive planar graphs

In this section we discuss the case of vertex-transitive graphs with one end.

3.1 Euclidean and hyperbolic tilings. The spheres, the Euclidean plane, and the hyperbolic planes will be referred to as the *natural geometries*. The natural geometries are two-dimensional Riemannian manifolds and possess a group of isometries. Each natural geometry is characterized by its Gauss curvature κ. The curvature is positive for the spheres, zero for the Euclidean plane, and negative for the hyperbolic planes.

An *Archimedean tiling* of a two-dimensional Riemannian manifold is a tiling by regular polygons such that the group of isometries of the tiling acts transitively on the vertices of the tiling.

We characterize one-ended vertex-transitive planar graphs as tilings of natural geometries. This characterization is one of the main tools in establishing the growth result.

THEOREM 3.1. *Let X be a locally finite connected vertex-transitive planar graph with at most one end. Then X has an embedding on a natural geometry as an Archimedean tiling; all automorphisms of X extend to automorphisms of the tiling and are induced by isometries of the geometry.*

REMARK 3.1. This result was obtained by the author in February 1992 [Ba6] but, due to procrastination, it remained unpublished until now. The result was announced as Theorem 3.61 in [Ba4, p. 1494].

If we set the length of each edge to 1, this uniquely determines κ. The sphere being compact, it corresponds to finite graphs (zero ends); for infinite graphs we obtain Euclidean or hyperbolic tilings.

The vertex-transitive case of Theorem 1.1 is an immediate consequence. Indeed, by Propositions 2.2 and 2.3, we only need to consider the case when X has one end. Then, by Theorem 3.1, we have two cases. In each case, the growth rate is the same as the growth rate of the area of discs as a function of the radius, i. e., quadratic in the Euclidean plane and exponential in the hyperbolic planes.

3.2 Proof sketch for vertex-transitive graphs. We sketch the proof of Theorem 3.1.

Let the vertices of X have degree d. If $d = 2$ then the graph is a two-way infinite path which has two ends. Therefore we have $d \ge 3$. From Lemma 2.4 we infer that X is 3-connected.

Planar 3-connected graphs have a unique plane embedding (i. e., the corresponding rotation system is unique up to simultaneous inversion at every vertex). This was proved by Whitney [Wh] for finite graphs and extended to infinite by Imrich [Im] and Thomassen [Th].

Let Δ and M be the corresponding cell complex and manifold, repectively. It follows that $\mathrm{Aut}(X)$ extends to $\mathrm{Aut}(\Delta)$ and to a group of *homeomorphisms* of M.

We note that M has no boundary (because Δ has no infinite faces by Lemma 2.2).

The *star* of a vertex v in Δ is the set of faces incident with v. Suppose the faces are n_i-gons, $i = 1, \ldots, d$ (d is the degree of each vertex).

In the Euclidean plane, the angles of such regular polygons are $\alpha_i = \pi(1 - 2/n_i)$.

Next we endow M with the stucture of a Riemannian manifold.

Our aim is to make all faces regular with unit sidelength. To this effect, we shall endow each star with a natural geometry of curvature κ. We shall use a familiar continuity argument to determine the right value of κ.

Let us draw these regular polygons with unit sidelength on the natural geometry of curvature κ; and let $\beta_i(\kappa)$ be the angle of the regular n_i-gon. Note that $\beta_i(\kappa)$ is a continuous, strictly increasing function of κ, and $\beta_i(0) = \alpha_i$.

If $\sum \alpha_i = 2\pi$, the star fits on the Euclidean plane, and we choose $\kappa = 0$.

If $\sum \alpha_i > 2\pi$, let κ be the unique negative number such that $\sum \beta_i(\kappa) = 2\pi$. Such a κ exists since $\lim_{\kappa\to-\infty} \beta_i(\kappa) = 0$. This defines a unique hyperbolic plane.

If $\sum \alpha_i < 2\pi$, a similar argument gives a unique positive κ (sphere) in most cases; a small number of cases need to be eliminated separately. (The sphere being compact, the graphs falling under this case will be finite.)

We thus introduced a Riemannian manifold structure on all stars in an invariant way, therefore the overlapping portions of the stars receive the same geometry. This transforms M into a Riemannian manifold of constant curvature.

The Riemannian metric is invariant under the automorphisms of the graph X, extended to M, and all the faces are regular polygons.

Finally, we require a classical result from Riemannian geometry (cf. [Ch, p. 180, Theorem 4.5]).

Theorem 3.2. **(Killing (1893), Hopf (1925))** *A complete simply connected 2-dimensional Riemannian manifold of constant Gauss curvature κ is uniquely determined up to isometry by κ.*

In other words, such a manifold is either a sphere ($\kappa > 0$), or the Euclidean plane ($\kappa = 0$), or a hyperbolic plane ($\kappa < 0$).

Completeness of a Riemannian manifold means that the geodesic lines can be indefinitely continued (their domain is the entire real line).

Claim 3.1. *M, as a Riemannian manifold, is complete.*

Proof of the claim: Consider the Voronoi diagram of $V(X)$ on M, i. e., associate with every point of M the nearest vertex of the graph. A cell of the Voronoi diagram consists of the points of M nearest to a particular vertex; the nearest vertex is ambigouos on the boundary lines of the Voronoi diagram. Each Voronoi cell is inside the corresponding star and is a convex polygon in the corresponding local natural geometry. All Voronoi cells are isometric (by vertex-transitivity).

Let δ be the shortest distance between a boundary point of a Voronoi cell and a boundary point of the corresponding star.

We claim that any geodesic segment can be continued by δ. This will prove completeness. Indeed, let x be the endpoint of our geodesic segment, and let v be the nearest vertex to x. Continue the geodesic beyond x as a straight line segment within the local geometry of the star about v. This takes us by at least δ further. ∎

Claim 3.1 and Lemma 2.3, together with Theorem 3.2, complete the proof of Theorem 3.1. ∎

4 Almost vertex-transitive graphs

Now we turn to the general case. Assume X is an infinite, locally finite, connected, *almost vertex-transitive,* planar graph. By Propositions 2.2 and 2.3 we may assume that X has *one end.*

4.1 Reduction to 3-connected graphs. Techniques of [HT] and [Ba1] can be used to reduce the problem to the 3-connected case: a 3-connected "skeleton" of X can be found with the same growth rate and with closely related automorphism group.

A *subdivison* of a graph Y is a graph obtained from Y by adding a finite number of new vertices of degree 2 on each edge of Y. (An edge receiving k new vertices becomes a path of length $k+1$.) A *topological subgraph* of a graph X is a graph Y such that some subdivision of Y is a subgraph of X.

Theorem 4.1. *Let X be an infinite, locally finite, connected, almost vertex-transitive graph with one end. Then X has a topological subgraph Y such that*

(i) Y is 3-connected;

(ii) Y has one end;

(iii) X and Y has the same growth rate;

(iv) $\mathrm{Aut}(X)$ *acts almost vertex-transitively on $V(Y)$.*

(Note: planarity is not assumed for this result.)

For the proof, we need some definitions from connectivity theory. For a subset A of the vertex set of the graph X, let ∂A denote the *boundary* of A, i. e., the set of vertices outside A which are adjacent to some vertex in A. Let X be a graph of connectivity κ. For any nonempty proper subset $F \subset V(X)$ we have $|\partial F| \geq \kappa$. If equality holds here and F induces a connected subgraph then we call F a *fragment* and the set $C = \partial F$ a *minimum cutset.* If X has one end then all fragments are finite or co-finite. Fragments of minimum size are called *atoms.* Atoms are *disjoint* [Wa, Ma]; moreover, if A is an atom and C is a minimum cutset then either $A \cap C = \emptyset$ or $A \subseteq C$ [Ma]. We note that a fragment is never a cutset: if F is a fragment, then let F' be another fragment corresponding to the same minimum cutset $C = \partial F = \partial F'$. Then $C \cup F'$ induces a connected subgraph, so $X \setminus F$ is connected.

Now we sketch the proof of Theorem 4.1. We follow the ideas of [Ba1]. Y will be created through the following process. If X is 3-connected, then set $Y := X$.

Assume now that the connectivity of X is $\kappa = 2$. Let us consider an atom A of X.

Assume first that $|A| \geq 2$. We claim in this case that for any atom A and any minimum cutset C, $A \cap C = \emptyset$. Indeed, otherwise, by Mader's quoted result, $A \subseteq C$ and therefore $A = C$ (since $|A| \geq 2 = |C|$). This is impossible since fragments cannot be cutsets.

Now, replace A by an edge connecting the two vertices which comprise ∂A. Do this simultaneously to all atoms. Since the atoms are disjoint and they do not intersect the minimum cutsets, performing this operation with different atoms will not interfere with each other. If multiple edges arise through this operation, replace each set of parallel edges by a single edge.

The result is a topological subgraph X_1 of X. It is easy to see that X_1 is locally finite, connected, and satisfies items (ii), (iii) of the Theorem (with X_1 playing the role of Y). Moreover, X_1 is invariant under $\mathrm{Aut}(X)$ and the action of $\mathrm{Aut}(X)$ on $V(X_1)$ has *fewer orbits* than it had on $V(X)$. (We deleted the orbits of the vertices in atoms.) This completes one round if $|A| \geq 2$.

On the other hand, if $|A| = 1$, this mean there exist vertices of degree 2 in X. Let P be a maximal path in X whose interior vertices have degree 2 in X. (It is easy to see that all such paths are finite.) Replace P by an edge joining the endpoints of P; do this simultaneously with all maximal paths. If multiple edges arise, replace them by single edges. Call the graph obtained X_1. Now X_1 has the same properties as those listed at the end of the preceding paragraph.

If the connectivity of X is $\kappa = 1$, just delete all atoms to obtain X_1.

If X_1 is not 3-connected, repeat the process with X_1. Termination is guaranteed since the number of orbits of $\mathrm{Aut}(X)$ is strictly reduced in each round. ∎

4.2 Tiling. We can now state a weak extension of Theorem 3.1.

THEOREM 4.2. *Let X be a locally finite 3-connected almost vertex-transitive planar graph with at most one end. Then X has an embedding on a natural geometry such that all automorphisms of X are induced by isometries of the geometry.*

This result together with Theorem 4.1 implies Theorem 1.1 for the case when X has one end. The cases with more than one end were settled in Section 2.1. ∎

In Theorem 4.2, we do not claim that the embedding has polygonal faces. However, the edges may be required to be smooth arcs.

We indicate the steps of the proof of Theorem 4.2.

The assumption of 3-connectedness allows us to construct the manifold M and the cell complex Δ as in Section 3.2. Then Δ has no infinite faces by Lemma 2.2. M is simply connected by Lemma 2.3. So we have a map with finite faces on a simply connected topological manifold. X is embedded on the manifold, and a discrete group G of homeomorphisms of M isomorphically restricts to the automorphism group of X. G is a finitely generated group with one end.

CLAIM 4.1. *There exists a connected vertex-transitive graph R, embedded on M, which is invariant under G.*

Proof. First we construct a fundamental domain for G which is homeomorphic to a disk.

Consider the barycentric subdivision B of Δ. Construct a graph Y as follows: the vertices of Y are the triangles of B; two are adjacent if they share an edge. Clearly, Y is connected. G acts on Y almost vertex-transitively; let T be a connected transversal of the orbits (a connected subgraph having one vertex from each orbit). The union of the triangles corresponding to the vertices of T is a connected fundamental domain F for G. The vertex-stabilizer of G in Y is the identity; therefore the same holds for the (setwise) stabilizer of F under the action of G.

We claim that F is simply connected. Indeed, let us consider the graph R obtained by contracting each G-translate of T to a vertex. Then G acts vertex-transitively on R. Moreover, both Y and R have one end; therefore R is 3-connected. Hence T cannot be a cut vertex in R. Consequently, F is simply connected.

R embeds on M as the dual of the map given by the G-translates of the fundamental domain F. ∎

Since R has one end, Theorem 3.1 can be applied to R, yielding the desired geometry on M. ∎

5 The conjectures of Kleinberg and Tardos

5.1 Statement of the problem. I reproduce here Jon Kleinberg's insightful e-mail message which explains the genesis of the conjecture on the growth rate of infinite vertex-transitive planar graphs (the result proved and generalized in this paper). In the message, Kleinberg also states their stronger conjecture on families of finite planar graphs with a "uniform growth function." That conjecture is equivalent to (the positive answer to) Problem 1.1 and remains open. We prove the equivalence in the next section (Theorem 5.1).

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From: kleinber@theory.lcs.mit.edu
Date: Fri, 08 Mar 96 17:50:26 EST
To: laci@cs.uchicago.edu
Subject: Re:  vertex transitive
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Dear Prof. Babai,

Thank you very much for letting me know [about the trichotomy result for growth rates of vertex-transitive planar graphs]. This is indeed the question that I mentioned to Mario [Szegedy] on Wednesday when I was at AT&T. It's one version of a question that Éva Tardos and I were curious about in connection with our disjoint paths approximation algorithm.

Our algorithm was initially designed to handle the $n \times n$ two-dimensional grid graph; and we subsequently showed that it achieved the same approximation ratio (to within constant factors) on the following generalization of the grid: any planar graph of bounded face size whose growth rate is "uniformly quadratic." By this we mean the following. Let $B(v,r)$ denote the "ball of radius r" about a vertex v, i. e., the set of vertices that can be reached from v by crossing at most r edges. We say that a family of finite graphs $\{G_n\}$ is "uniformly quadratic" if there exist constants a and b such that for all n, G_n is an n-node graph and for all $v \in V(G_n)$,

- $|B(v,r)| \leq br^2$;
- $|B(v,r)| \geq \min(ar^2, n)$.

(Of course, it follows that the maximum degree in G_n is bounded by $|B(v,1)| - 1 \leq b - 1$.)

Now the class of $n \times n$ two-dimensional grid graphs shows that a uniformly quadratic family of planar graphs exists. Thus, we were curious what other functions could be used for the "uniform growth function" in place of $f(r) = r^2$. We did not see candidates for anything else but linear or exponential growth, and so conjectured that these three were the only possiblities.

Thus we were in turn led to what seemed like a cleaner question of the same general flavor – namely, the growth rate of an infinite, locally finite, vertex-transitive planar graph.

I'd be interested to know of references for this "trichotomy" result on growth rates. Do you know whether the same sort of result would hold for this weaker notion of uniform growth that does not require vertex-transitivity?

Best,
Jon Kleinberg

5.2 Proof of equivalence. In this section we prove that the Kleinberg–Tardos problem outlined above (on a family of finite planar graphs) is equivalent to Problem 1.1 (on the uniform growth rate of an infinite planar graph).

Implicit in Kleinberg's letter is the following definition.

Let $\{X_n\}$ be a family of finite graphs; assume X_n has n vertices. For $v \in V(X_n)$, let $B_n(v,r)$ denote the ball of radius r about v in X_n. For a monotone non-decreasing function $f : \mathbf{R}^+ \to \mathbf{R}^+$ we say that the family $\{X_n\}$ has *uniform growth rate* f if there exist positive constants c_1, c_2, C_1, C_2, r_0 such that for all $n, r \geq r_0$ and for every vertex $v \in V(X_n)$,

$$(5.2) \qquad \min\{n, C_1 f(c_1 r)\} \leq |B_n(v,r)| \leq C_2 f(c_2 r).$$

THEOREM 5.1. *For a monotone nondecreasing function $f : \mathbf{R}^+ \to \mathbf{R}^+$, the following are equivalent:*

(a) There exists a family of finite connected planar graphs of uniform growth rate f.

(b) There exists an infinite connected planar graph of uniform growth rate f.

(The "uniform growth rate" concept referred to in item (b) was defined in Section 1.2.)

One direction of the proof requires a "compactness lemma" (below). A *rooted graph* (X,v) is a graph v with a specified vertex $v \in V(X)$ (the root). Isomorphisms of rooted graphs preserve the root by definition. We say that the balls $B_X(v,r)$ and $B_Y(w,r)$ are isomorphic if they are isomorphic as rooted graphs (rooted at their respective centers).

We say that the sequence (X_n, v_n) of connected rooted graphs *converges* to the connected rooted graph (X,v) if there exists a function $n(r)$ such that for all $r \geq 0$ and all $n \geq n(r)$, the ball $B_{X_n}(v_n, r)$ is isomorphic to the ball $B_X(v,r)$.

LEMMA 5.1. *Let $\{(X_n, v_n)\}$ be an infinite family of finite connected rooted graphs of bounded degree. Then the sequence has a convergent subsequence.*

The proof requires a simple limit construction, discussed in detail in [Ba2]. We indicate the proof. Let $B_n(w,r)$ denote the ball $B_{X_n}(w,r)$.

Let b be a bound on the degrees of the vertices of all X_n. Then $|B_n(v_n,r)| \leq 2b^r$. Therefore there are only finitely many possible isomorphism types of balls of radius r that can occur. From this observation a standard "König path" argument[1] ([Ko1, Ko2]) yields that there exists a function $n(r)$ and an infinite subset

[1] Let (Y,y) be a rooted DAG with infinitely many nodes, such that every node has a finite number of children, and every node other than the root has at least one parent. Then, *König's lemma*

N of the positive integers with the property that for all $r \geq 0$ and all $n \in N$, $n \geq n(r)$, the isomorphism type of the balls $B_n(v_n, r)$ does not depend on n. Let $B(v,r)$ denote this isomorphism type; then, with appropriate identification, the rooted graph $(X, v) = \bigcup_{r=1}^{\infty} B(v,r)$ will be the limit of $\{(X_n, v_n) : n \in N\}$. ∎

For a more detailed discussion and applications of this limit construction we refer to [Ba2].

Now we turn to the proof of Theorem 5.1. First we prove (b) ⇒ (a). Assume X is an infinite connected planar graph with uniform growth rate f and let c_1, c_2, C_1, C_2, r_0 be positive constants so that inequality (1.1) is satisfied for all $v \in V(X)$ and $r \geq r_0$.

Let us fix a vertex v of X and let us list the vertices of X as $v_1, v_2, \ldots$ in *breadth-first* order, i. e., $v_1 = v$ is followed by the neighbors of v, then come the vertices at distance 2, and so on. Let X_n be the subgraph of X induced by $V_n = \{v_1, \ldots, v_n\}$.

The X_n are obviously planar. We claim the family $\{X_n\}$ has uniform growth rate f.

Let $w \in V_n$. Then $B_n(w,r) \subseteq B_X(w,r)$, therefore $|B_n(w,r)| \leq C_2 f(c_2 r)$, verifying the upper bound on the growth of the family $\{X_n\}$ with the same constants c_2, C_2, r_0 as for X. For the lower bound, we consider two cases. First, if $V_n \subseteq B_X(v, \lfloor r/2 \rfloor)$ then $V_n = B_n(w,r)$ and therefore $|B_n(w,r)| = n$. Assume now that $V_n \not\subseteq B_X(v, \lfloor r/2 \rfloor)$. Then, by the breadth-first ordering, it follows that $V_n \supseteq B_X(v, \lfloor r/2 \rfloor - 1)$. Now let $u = v$ if $w \in B_X(v, \lfloor r/2 \rfloor)$ and let u be a vertex at distance $\lfloor r/2 \rfloor$ from w on a shortest path from w to v in X. In both cases, $B_n(w,r) \supseteq B_n(u, \lfloor r/2 \rfloor - 1) = B_X(u, \lfloor r/2 \rfloor - 1) \supseteq B_X(u, \lceil r/3 \rceil)$ (assuming $r \geq 8$). Therefore, $|B_n(w,r)| \geq |B_X(u, \lceil r/3 \rceil)| \geq C_1 f(c_1 r/3)$. We conclude that the family $\{X_n\}$ has uniform growth rate f with constants $c_1/3, c_2, C_1, C_2, 3r_0$.

For the implication (a) ⇒ (b), assume $\{X_n\}$ is a family of finite planar graphs with uniform growth rate f and corresponding constants c_1, c_2, C_1, C_2, r_0. Let $V_n = V(X_n)$. First we observe that for $n \geq r_0$, all vertices of X_n have degree $\leq b := C_2 f(c_2 r_0)$.

Let us now fix a vertex $v_n \in V_n$ for each n. Since the $\{X_n\}$ have bounded degree, the rooted graphs $\{(X_n, v_n)\}$ have a convergent subsequence $\{(X_n, v_n) : n \in N\}$ (N is an infinite subset of the positive integers). Let (X, v) be the limit of this subsequence and let $V(X) = V$. The set V is infinite and by the definition of limit, we can identify each V_n with a subset of V such that v_n is identified with v and the following holds.

> There exists a function $n(r)$ such that for all $r \geq 0$ and all $n \in N, n \geq n(r)$, we have $B_n(v_n, r) = B_X(v, r)$.

It is clear that X is planar: otherwise, X would have a finite connected non-planar subgraph, and that subgraph would occur as a subgraph in some X_n.

We claim that X has uniform growth rate f.

Let $w \in V$ and let the distance of w and v in X be s. Then for $n \in N$, $n \geq n(r+s)$, we have $B_X(w,r) = B_n(w,r)$. Therefore $\min\{n, C_1 f(c_1 r)\} \leq |B_X(w,r)| \leq C_2 f(c_2 r)$. Since n can be chosen to be arbitrarily large, we infer $C_1 f(c_1 r) \leq |B_X(w,r)|$. This proves that X has uniform growth f with the same constants c_1, c_2, C_1, C_2, r_0 as the family $\{X_n\}$. ∎

asserts, there exists a (one-way, ascending or descending) infinite directed path in Y. – For our application, let the nodes of level $r \geq 0$ be the isomorphism types of the balls of radius r about the root in the graphs $\{(X_n, v_n)\}$. For $r \geq 1$, the parent of such a ball is the ball of radius $r-1$ about the root. All descending chains being finite, an infinite ascending chain must exist by König's lemma. The family of graphs $\{(X_n, v_n); n \in N\}$ corresponding to such a chain has the required properties.

6 Appedix: Why vertex-transitive?

Regarding the significance of results of this type from the point of view of discrete mathematics and the theory of algorithms, the following might be mentioned.

First of all, the growth rate is a fundamental characteristic of infinite locally finite graphs. It raises particularly intriguing questions for classes of graphs with substantial symmetry (such as almost vertex transitive graphs), whose significance reaches to distant areas of mathematics.

Moreover, local characteristics, such as growth rates, of families of connected finite graphs of bounded degree are related to analogous characteristics of infinite, locally finite, connected graphs via a limit construction (see Section 5.2 and [Ba2]).

Second, the discrete structure used to organize the execution of algorithms in various models tends to be an infinite, locally finite, connected, vertex-transitive graph, and often planar. For example, the plane square grid represents the space-time for a single-tape Turing machine; the square and hexagonal plane grids are favorite domains for cellular automata and for maze problems. Cayley graphs which tile the hyperbolic plane (an important subcase featured in our structural analysis) have also been considered in the context of automata (cf. [Ep+]).

Taking a broader view, it should also be mentioned that vertex-transitive *finite* graphs constitute interesting *testing grounds for theories in combinatorial optimization.*

Compare, for instance, the very well understood structure theory of matchings (cf. [LP]) with our dim ideas on Hamilton cycles. It is an easy consequence of the Gallai-Edmonds structure theorem (cf. [Lo, Problem 7.32]) that all connected finite vertex-transitive graphs with an even number of vertices have a *perfect matching;* and if the number of vertices is odd, they are matching critical, i.e., there will be a perfect matching after the deletion of any vertex ([LGH], cf. [LP, p. Ex. 3.2.5], [Lo, Problem 12.16]). On the other hand, only four finite connected vertex-transitive graphs are known to be non-Hamiltonian. I believe that this state of affairs is explained by our lack of understanding of *Hamiltonicity obstacles.* In variance with others who have considered this question, I conjecture that there exist infinitely many finite connected vertex-transitive graphs without cycles of length $\geq (1-c)|V|$ for some constant $c > 0$. A result in this direction would be an indication of real progress in identifying significant new Hamiltonicity obstacles.

A powerful theory of *connectivity* of vertex-transitive graphs and digraphs was developed by Watkins [Wa], Mader [Ma], and Hamidoune [Ham] (cf. [Ba4, Sec. 3.2]). One of the results is that finite connected vertex-transitive graphs are maximally edge-connected, i.e., their edge-connectivity is equal to the degree. An immediate consequence is that connected finite vertex-transitive graphs are *tough*, i.e., deletion of any subset A of the vertex set of such a graph splits the rest of the graph into $\leq |A|$ connected components. – This result eliminates non-toughness, an appealing Hamiltonicity obstacle, from consideration for vertex-transitive graphs.

A result stating that *all* vertex-transitive graphs possess a certain (modest) *local expansion* property [Ba3] plays a central role in the complexity analysis of randomized algorithms in groups. The result is currently indispensable for the justification of a growing number of "polynomial time" and "nearly linear time" claims in the area [BCFS, BeS, NeP, BBR, BeB, Be, CFL, Ba5].

Vertex-transitivity implies curious connections between important parameters of a graph. As an illustration, let me state an observation of Mario Szegedy [Sz]: *Let X be a finite vertex-transitive graph with n vertices, chromatic number χ, and independence (anti-clique) number α. Then $n/\alpha \leq \chi \leq (n/\alpha)(1+\ln n)$.*

For more information and literature about combinatorial properties of vertex-transitive graphs, we refer to [Ba4].

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