Multiplicative Ergodicity and Large Deviations for an Irreducible Markov Chain*

S. Balaji† S.P. Meyn‡

March 11, 2000

Abstract

The paper examines multiplicative ergodic theorems and the related multiplicative Poisson equation for an irreducible Markov chain on a countable state space. The partial products are considered for a real-valued function on the state space. If the function of interest satisfies a monotone condition, or is dominated by such a function, then

(i) The mean normalized products converge geometrically quickly to a finite limiting value.

(ii) The multiplicative Poisson equation admits a solution.

(iii) Large deviation bounds are obtained for the empirical measures.

Keywords: Markov chain, Ergodic Theory, Harmonic functions, Large Deviations

1991 AMS Subject Classification: 60J10, 60F10, 58F11

*Work supported in part by NSF grant ECS 940372, and JSEP grant N00014-90-J-1270.

†Mathematical Sciences Department, New Jersey Institute of Technology, Newark, New Jersey (balaji@m.njit.edu)

‡Coordinated Sciences Laboratory and Department of Electrical and Computer Engg., University of Illinois at Urbana-Champaign, Urbana, Illinois (s-meyn@uiuc.edu).

1 Introduction and Main Results

Consider a recurrent, aperiodic, and irreducible Markov chain $\mathbf{\Phi} = \{\Phi_0, \Phi_1, \dots\}$ with transition probability P on a countably infinite state space X. We denote by $F\colon \mathsf{X} \to \mathbb{R}_+$ a fixed, positive-valued function on X, and let S_n denote the partial sum,

$$S_n = \sum_{i=0}^{n-1} F(\Phi_i), \qquad n \geq 1. \tag{1}$$

We show in Lemma 3.2 below that the simple multiplicative ergodic theorem always holds:

$$\frac{1}{n} \log \mathsf{E}_x \Big[\exp\Big(S_n \Big) \mathbb{1}_C(\Phi_n) \Big] \to \Lambda, \qquad n \to \infty, x \in \mathsf{X}, \tag{2}$$

where C is an arbitrary finite subset of X, and Λ is the log-Perron Frobenius eigenvalue (*pfe*) for a positive kernel induced by the transition probability P, and the function F [16, 14]. A limit of the form (2) is used in [12, 13] to establish a form of the large deviations principle for the chain. Because of the appearance of the indicator function $\mathbb{1}_C$ in (2) it is necessary in [13] to introduce a similar constraint in the LDP. It is pointed out on page 562 of [12] that the use of the convergence parameter and the consequent use of an indicator function in the statement of the LDP represents a strong distinction between their work and related results in the area.

We are interested in (2) in the situation when the set C is all of X, rather than a finite set, and this requires some additional assumptions on the function F or on the chain $\mathbf{\Phi}$. This is the most interesting instance as it represents a natural generalization of the mean ergodic theorem for Markov chains. The main result of this paper establishes the desired multiplicative ergodic theorem under a simple monotonicity assumption on the function of interest.

There is striking symmetry between linear ergodic theory, as presented in [10], and the multiplicative ergodic theory established in this paper. This is seen most clearly in the following version of the V-Uniform Ergodic Theorem of [10], which establishes an equivalence between a form of geometric ergodicity, and the Foster-Lyapunov drift condition (3). In the results below and throughout the paper we denote by θ some fixed, but arbitrary state in X.

Theorem 1.1 *Suppose that* $\mathbf{\Phi}$ *is an irreducible and aperiodic Markov chain with countable state space* X*, and that the sublevel set* $\{x : F(x) \leq n\}$ *is finite for each* n*. Suppose further that there exists* $V : \mathsf{X} \to [1, \infty)$*, and constants* $b < \infty$*,* $\eta < 1$ *all satisfying*

$$\mathsf{E}_x[V(\Phi_1)] = \sum_{y \in \mathsf{X}} P(x,y)V(y) \leq \eta V(x) - F(x) + b. \tag{3}$$

Then there exists a function $\hat{F} : \mathsf{X} \to \mathbb{R}$ *such that*

$$\mathsf{E}_x\Big[S_n - \gamma n\Big] \to \hat{F}(x),$$

at a geometric rate as $n \to \infty$*, and hence also*

$$\lim_{n\to\infty} \frac{1}{n} \mathsf{E}_x\Big[S_n\Big] = \gamma,$$

where

(i) *the constant* $\gamma \in \mathbb{R}_+$ *is the unique solution to*

$$\mathsf{E}_\theta\Big[\sum_{k=0}^{\tau_\theta - 1} (F(\Phi_k) - \gamma)\Big] = 0,$$

and τ_θ *is the usual return time to the state* θ*.*

(ii) *The function* $\hat{F}$ *solves the Poisson equation*

$$P\hat{F} = \hat{F} - F + \gamma.$$

Proof. The existence of the two limits is an immediate consequence of the Geometric Ergodic Theorem of [10]. That the limit $\hat{F}$ solves the Poisson equation is discussed on page 433 of [10].

The characterization of the limit γ in (i) is simply the characterization of the steady state mean $\pi(F)$ given in Theorem 10.0.1 of [10], where π is an invariant probability measure. ∎

A multiplicative ergodic theorem of the form that we seek is expressed in the following result, which is evidently closely related to Theorem 1.1.

Theorem 1.2 *Suppose that $\mathbf{\Phi}$ is an irreducible and aperiodic Markov chain with countable state space X, and that the sublevel set $\{x : F(x) \leq n\}$ is finite for each n. Suppose further that there exists $V_0 \colon \mathsf{X} \to \mathbb{R}_+$, and constants $B_0 < \infty$, $\alpha_0 > 0$ all satisfying*

$$\mathsf{E}_x[\exp(V_0(\Phi_1))] = \sum_{y \in \mathsf{X}} P(x,y) \exp\bigl(V_0(y)\bigr) \leq \exp\bigl(V_0(x) - \alpha_0 F(x) + B_0\bigr). \tag{4}$$

Then there exists a convex function $\Lambda \colon \mathbb{R} \to (-\infty, \infty]$, finite on a domain $\mathcal{D} \subset \mathbb{R}$ whose interior is of the form $\mathcal{D}^o = (-\infty, \bar{\alpha})$, with $\bar{\alpha} \geq \alpha_0$.

For any $\alpha < \bar{\alpha}$ there is a function $\check{f}_\alpha \colon \mathsf{X} \to \mathbb{R}_+$ such that

$$\mathsf{E}_x\Bigl[\exp\Bigl(\alpha S_n - n\Lambda(\alpha)\Bigr)\Bigr] \to \check{f}_\alpha(x), \tag{5}$$

*geometrically fast as $n \to \infty$, and for **all** α,*

$$\lim_{n \to \infty} \frac{1}{n} \log \mathsf{E}_x\Bigl[\exp\Bigl(\alpha S_n\Bigr)\Bigr] = \Lambda(\alpha).$$

Moreover, for $\alpha < \bar{\alpha}$,

(i) *the constant $\Lambda(\alpha) \in \mathbb{R}$ is the unique solution to*

$$\mathsf{E}_\theta\Bigl[\exp\Bigl(\sum_{k=0}^{\tau_\theta - 1} [\alpha F(\Phi_k) - \Lambda(\alpha)]\Bigr)\Bigr] = 1. \tag{6}$$

(ii) *The function $\check{f}_\alpha$ solves the multiplicative Poisson equation*

$$P\check{f}_\alpha\,(x) = \check{f}_\alpha(x) \exp\bigl(-\alpha F(x) + \Lambda(\alpha)\bigr)$$

Proof. The limit (5) follows from Theorem 5.2. The characterizations given in (i) and (ii) follow from Theorem 5.1, Theorem 4.1, and Theorem 4.2 (i). ∎

Theorem 1.2 is related tangentially to the multiplicative ergodic theorem of Oseledec (see e.g. [1]), which is a sample path limit theorem for products of random variables taking values in some non-abelian group. In the case of scalar F considered here, Oseledec's theorem reduces to Birkhoff's ergodic theorem since the sample path behavior of $\prod \exp(\alpha F(\Phi(i)))$ can be reduced to the strong law of large numbers by taking logarithms. The corresponding p^{th}-moment Lyapunov exponent considered for linear models in [1] also involves the moment generating function $\Lambda(\alpha)$ and, consistent with existing results, we do have $\Lambda'(0) = \pi(F)$ (see Theorem 6.2 below).

From Theorem 1.2 and the Gärtner-Ellis Theorem we immediately obtain a version of the Large Deviations Principle for the empirical measures (see [7] and Section 6 below). An application to risk sensitive optimal control (see [18]) is developed in [2, 5].

The remainder of the paper is organized as follows. In the next section we give the necessary background on geometric ergodicity of Markov chains. Section 3 develops some properties of the convergence parameter, and Section 4 then gives related criteria for the existence of a solution to the multiplicative Poisson equation. The main results are given in Section 5, which includes results analogous to Theorem 1.2 for general functions via domination. Large deviations principles for functionals of a Markov chain and the empirical measures are derived in Section 6.

2 Geometric Ergodicity

Throughout the paper we assume that $\mathbf{\Phi}$ is an irreducible and aperiodic Markov chain on the countable state space X, with transition probability $P\colon \mathsf{X}\times\mathsf{X} \to [0,1]$. When $\Phi_0 = x$ we denote by $\mathsf{E}_x[\,\cdot\,]$ the resulting expectation operator on sample space, and $\{\mathcal{F}_n, n \geq 1\}$ the natural filtration $\mathcal{F}_n = \sigma(\Phi_k :$

$k \leq n$).

The results in this paper concern primarily functions $F\colon \mathsf{X} \to \mathbb{R}$ which are *near-monotone*. This is the property that the sublevel set

$$C_\zeta \triangleq \{x : F(x) \leq \zeta\} \tag{7}$$

is finite for any $\zeta < \|F\|_\infty \triangleq \sup_y |F(y)|$. A near-monotone function is always bounded from below. If it is unbounded ($\|F\|_\infty = \infty$) then F is called *norm-like* [10]. These assumptions have been used in the analysis of optimization problems to ensure that a 'relative value function' is bounded from below [4, 11]. The relative value function is nothing more than a solution to Poisson's equation. A 'multiplicative Poisson equation' is central to the development here, and the near-monotone condition will again be used to obtain lower bounds on solutions to this equation.

The present paper is based upon the V-Uniform Ergodic Theorem of [10]. In this section we give a version of this result and briefly review some related concepts.

For a subset $C \subset \mathsf{X}$ we define the first entrance time and first return time respectively by

$$\sigma_C = \min(k \geq 0 : \Phi_k \in C); \qquad \tau_C = \min(k \geq 1 : \Phi_k \in C),$$

where as usual we set either of these stopping times equal to ∞ if the minimum is taken over an empty set. For a recurrent Markov chain there is an invariant probability measure π which takes the form, for any integrable $F\colon \mathsf{X} \to \mathbb{R}$,

$$\pi(F) = \pi(\theta)\mathsf{E}_\theta\Big[\sum_0^{\tau_\theta - 1} F(\Phi_k)\Big]. \tag{8}$$

The measure π is finite in the positive recurrent case where $\mathsf{E}_\theta[\tau_\theta] < \infty$.

The Markov chain $\mathbf{\Phi}$ is called *geometrically recurrent* if $\mathsf{E}_\theta[R^{\tau_\theta}] < \infty$ for one $\theta \in \mathsf{X}$ and one $R > 1$. Because the chain is assumed irreducible, it then follows that $\mathsf{E}_x[R^{\tau_\theta}] < \infty$ for all x, and the chain is called *geometrically* *regular*. Closely related is the following form of ergodicity. Let $V\colon \mathsf{X} \to \mathbb{R}_+$ with $\inf_{x\in\mathsf{X}} V(x) > 0$, and consider the vector space L_∞^V of real-valued functions $g : \mathsf{X} \to \mathbb{R}$ satisfying

$$\|g\|_V \triangleq \sup_{x\in\mathsf{X}} |g(x)|/V(x) < \infty.$$

Specializing the definition of [10] to this countable state space setting, we call the Markov chain V*-uniformly ergodic* if there exist $B < \infty$, $R > 1$ such that

$$\|P^k g - \pi(g)\|_V = \sup_{x\in\mathsf{X}} \frac{|\mathsf{E}_x[g(\Phi_k)] - \pi(g)|}{V(x)} \le B\|g\|_V R^{-k}.$$

Equivalently, if P and π are viewed as linear operators on L_∞^V, then V-uniform ergodicity is equivalent to convergence in norm:

$$|\!|\!|P^n - \pi|\!|\!|_V \triangleq \sup_{\|g\|_V\le 1} \|P^n g - \pi(g)\|_V \to 0, \qquad n\to\infty.$$

Theorem 2.1 *The following are equivalent for an irreducible and aperiodic Markov chain*

(i) *For some $V\colon \mathsf{X} \to [1,\infty)$; $\eta < 1$; a finite set C; and $b < \infty$,*

$$PV \le \eta V + b\mathbb{1}_C. \tag{9}$$

(ii) *$\boldsymbol{\Phi}$ is geometrically recurrent.*

Moreover, if either (i) or (ii) holds then the chain is V-uniformly ergodic, where V is given in (i).

Proof. Any finite set is necessarily petite, as defined in [10], and hence the result follows from Theorem 15.0.1 of [10]. ∎

If $\boldsymbol{\Phi}$ is V-uniformly ergodic then a version of the Functional Central Limit Theorem holds. We prove a special case below which will be useful when we consider large deviations. Consider any $F \in L_\infty^V$, with $\pi(F) = 0$, define S_n as in (1), and set

$$\gamma^2 = \pi(\theta)\mathsf{E}_\theta[(S_{\tau_\theta})^2].$$

This is known as the *time-average variance constant.* Let F denote the distribution function for a standard normal random variable.

Theorem 2.2 *Suppose that (9) holds for some $V\colon \mathsf{X} \to [1,\infty)$; $\eta < 1$; a finite set C; and $b < \infty$. Then for any $F\colon \mathsf{X} \to \mathbb{R}$ with $F^2 \in L_\infty^V$, and $\pi(F) = 0$, the time average variance constant is finite. For any $-\infty \le c < d \le \infty$, any $g \in L_\infty^V$, and any initial condition $x \in \mathsf{X}$,*

$$\lim_{n\to\infty} \mathsf{E}_x\Big[\mathbb{1}\Big\{\frac{1}{\sqrt{n}}S_n \in (c,d)\Big\}g(\Phi_n)\Big] = \Big(\mathsf{F}(d/\gamma) - \mathsf{F}(c/\gamma)\Big)\pi(g). \qquad (10)$$

Proof. For any $t \ge 0$, $n \in \mathbb{N}$, define

$$W_n(t) = \frac{1}{\sqrt{n}}S_{\lfloor nt\rfloor}, \qquad t \ge 0,$$

so that $W(1) = \frac{1}{\sqrt{n}}S_n$. Theorem 17.4.4 of [10] shows that W_n converges in distribution to γB, where B is a standard Brownian motion. If $\gamma = 0$ then from Theorem 17.5.4 of [10] we can conclude that $W_n(t) \to 0$ a.s. as $n \to \infty$ for each t. This leads to the two equations,

$$\lim_{n\to\infty} \mathsf{E}\Big[\mathbb{1}\Big\{W_n(1) \in (c,d)\Big\}\Big] = \mathsf{F}(d/\gamma) - \mathsf{F}(c/\gamma) \quad \text{and} \quad \lim_{n\to\infty} \mathsf{E}[g(\Phi_n)] = \pi(g).$$

This will prove the theorem provided we can prove asymptotic independence of $W_n(1)$ and $g(\Phi_n)$.

Let $\varepsilon_n = \log(n)/n$, $n \ge 1$. Using V-uniform ergodicity we do have, for any bounded function $h\colon \mathbb{R} \to \mathbb{R}$,

$$\mathsf{E}\Big[h(W_n(1-\varepsilon_n))g(\Phi_n)] = \pi(g)\mathsf{E}[h(W_n(1-\varepsilon_n))\Big] + o(1),$$

and then by the FCLT, for bounded continuous h,

$$\mathsf{E}\Big[h(W_n(1-\varepsilon_n))g(\Phi_n)\Big] \to \pi(g)\mathsf{E}[h(\gamma B(1))], \qquad n \to \infty.$$

The error $|W_n(1) - W_n(1-\varepsilon_n)| \to 0$ a.s., and by uniform integrability of $\{g(\Phi_n)\}$ we conclude that

$$\mathsf{E}[h(W_n(1))g(\Phi_n)] \to \pi(g)\mathsf{E}[h(\gamma B(1))], \qquad n \to \infty.$$

This is the required asymptotic independence. ∎

We will see in Theorem 3.1 (i) below that, under the conditions we impose, the drift condition (4) will always be satisfied for some non-negative V_0. It is useful then that such chains are V-uniformly ergodic.

Theorem 2.3 *Suppose that there exists $V_0 \colon \mathsf{X} \to \mathbb{R}_+$, and constants $B_0 < \infty$, $\alpha_0 > 0$ all satisfying (4), and suppose that the set C_ζ defined in (7) is finite for some $\zeta > B_0/\alpha_0$. Then $\mathbf{\Phi}$ is V-uniformly ergodic with $V = \exp(V_0)$.*

Proof. Under (4) we then have for some b_0,

$$PV \le e^{-\varepsilon}V + b_0 \mathbb{1}_{C_\zeta},$$

where $\varepsilon = \zeta\alpha_0 - B_0 > 0$. This combined with Theorem 2.1 establishes V-uniform ergodicity. ∎

The assumption that the function V in (9) is bounded from below is crucial in general. Take for example the Bernoulli random walk on the positive integers with *positive* drift so that $\lambda \triangleq P(x, x+1) > P(x, x-1) \triangleq \mu$, $x \ge 1$. Let $V(x) = \exp(-\varepsilon x)$, $C = \{0\}$, and choose $\epsilon > 0$ so that $\eta = \lambda e^{-\epsilon} + \mu e^{\epsilon} < 1$. The bound (9) then holds, but the chain is transient. This shows that a lower bound on the function V is indeed necessary to deduce any form of recurrence for the chain. This is unfortunate since frequently we will find that the drift criterion (9) holds for some function V which is not apriori known to be bounded from below. The lemma below resolves this situation.

Theorem 2.4 *Suppose that*

(i) *there exists* $V\colon \mathsf{X} \to \mathbb{R}_+$, $\eta < 1$, *a finite set* C, *and* $b < \infty$, *satisfying (9).*

(ii) $V(x) > 0$ *for* $x \in C$;

(iii) $\mathbf{\Phi}$ *is recurrent.*

Then $\inf_{x\in\mathsf{X}} V(x) > 0$, *and hence* $\mathbf{\Phi}$ *is* V*-uniformly ergodic.*

Proof. Let $M_n = V(\Phi_{n\wedge\tau_C})\eta^{-(n\wedge\tau_C)}$. We then have the supermartingale property,

$$\mathsf{E}[M_n \mid \mathcal{F}_{n-1}] \le M_{n-1},$$

and from recurrence of $\mathbf{\Phi}$ and Fatou's lemma we deduce that for any x,

$$\Bigl(\min_{y\in C} V(y)\Bigr)\mathsf{E}_x[\eta^{-\tau_C}] \le \liminf_{n\to\infty} \mathsf{E}_x[M_n] \le M_0 = V(x).$$

This gives a uniform lower bound on V from which V-uniform ergodicity immediately follows from Theorem 2.1. ∎

3 The Convergence Parameter

Let $\widehat{P}_\alpha$ denote the positive kernel defined for $x, y \in \mathsf{X}$ by

$$\widehat{P}_\alpha(x,y) = \exp(\alpha F(x))P(x,y).$$

If we set $f_\alpha(x) = \exp(\alpha F(x))$, then this definition is equivalently expressed through the formula $\widehat{P}_\alpha = I_{f_\alpha}P$, where for any function g the kernel I_g is the multiplication kernel defined by $I_g(x,A) = g(x)\mathbb{1}_A(x)$.

Let $\theta \in \mathsf{X}$ denote some fixed state. The Perron-Frobenius eigenvalue (or *pfe*) is uniquely defined via

$$\lambda_\alpha \triangleq \inf\Bigl(\lambda \in \mathbb{R}_+ : \sum_{n=0}^{\infty} \lambda^{-n}\widehat{P}_\alpha^n(\theta,\theta) < \infty\Bigr). \tag{11}$$

Equivalently, $\Lambda(\alpha) = \log(\lambda_\alpha)$ can be expressed as

$$\Lambda(\alpha) = \inf\Big(\Lambda \in \mathbb{R} : \mathsf{E}_\theta\Big[\exp\Big(\alpha S_{\tau_\theta} - \Lambda\tau_\theta\Big)\mathbb{1}(\tau_\theta < \infty)\Big] \leq 1\Big). \tag{12}$$

The equivalence of the two definitions (11) and (12) is well known [14, 16].

We set $\Lambda(\alpha) = \infty$ if the infimum in (11) or (12) is over a null set, and we let $\mathcal{D}(\Lambda) = \{\alpha : \Lambda(\alpha) < \infty\}$. Let Λ' denote the right derivative of Λ, and set

$$\bar{\alpha} \stackrel{\Delta}{=} \sup\{\alpha : \Lambda'(\alpha) < \|F\|_\infty\}. \tag{13}$$

If $\|F\|_\infty = \infty$ so that F is unbounded then $\mathcal{D}^o(\Lambda) = (-\infty, \bar{\alpha})$.

It follows from (12) and Fatou's Lemma that

$$\exp(-\xi(\alpha)) \stackrel{\Delta}{=} \mathsf{E}_\theta\Big[\exp\Big(\alpha S_{\tau_\theta} - \Lambda(\alpha)\tau_\theta\Big)\mathbb{1}(\tau_\theta < \infty)\Big] \leq 1. \tag{14}$$

In the definition of ξ here we supress the possible dependency on θ since the starting point θ is assumed fixed throughout.

Result (iii) below may be interpreted as yet another Foster-Lyapunov drift criterion for stability of the process. Refinements of (iii) will be given below.

Lemma 3.1 *We have the following bounds on Λ:*

(i) *If $\mathbf{\Phi}$ is positive recurrent with invariant probability measure π then for all α,*

$$\Lambda(\alpha) \geq \alpha\pi(F),$$

where $\pi(F)$ is the steady state mean of F.

(ii) *For all α,*

$$\Lambda(\alpha) \leq \max(0, \alpha\|F\|_\infty);$$

(iii) *Suppose there exists $\alpha_0 \in \mathbb{R}$, $\overline{\lambda} \in \mathbb{R}$, and $V : \mathsf{X} \to \mathbb{R}_+$ such that V is not identically zero, and*

$$\widehat{P}_{\alpha_0} V \leq \overline{\lambda} V. \tag{15}$$

Then $\alpha_0 \in \mathcal{D}(\Lambda)$ and $\Lambda(\alpha_0) \leq \log(\overline{\lambda})$.

Proof. The bound (i) is a consequence of Jensen's inequality applied to (14), and the formula (8). The bound (ii) is obvious, given the definition of Λ given in (12).

To see (iii), suppose without loss of generality that $V(\theta) = 1$. If the inequality holds then for any $\lambda > \overline{\lambda}$,

$$\sum_{n=0}^{\infty} \lambda^{-n} \widehat{P}_{\alpha}^{n}(\theta,\theta) \le \sum_{n=0}^{\infty} \lambda^{-n} \widehat{P}_{\alpha_0}^{n} V(\theta) \le \frac{1}{1-\overline{\lambda}/\lambda}$$

It follows from (11) that $\alpha \in \mathcal{D}(\Lambda)$, and that $\lambda_\alpha \le \lambda$. We conclude that $\lambda_\alpha \le \overline{\lambda}$ since $\lambda > \overline{\lambda}$ is arbitrary. ∎

Under the aperiodicity assumption imposed here, $\Lambda(\alpha)$ is also the limiting value in a version of the multiplicative ergodic theorem.

Lemma 3.2 *For any non-empty, finite set $C \subset \mathsf{X}$ and any $\alpha \in \mathcal{D}(\Lambda)$,*

$$\frac{1}{n} \log \mathsf{E}_x \Big[\exp\Big(\alpha S_n\Big) \mathbb{1}_C(\Phi_n)\Big] \to \Lambda(\alpha), \qquad n \to \infty, x \in \mathsf{X}. \tag{16}$$

Proof. The proof follows from Kingman's subadditive ergodic theorem [9] for the sequence $\{\log(\widehat{P}_{\alpha}^{n}(\theta,\theta)) : n \ge 0\}$, which gives (16) for $x = \theta$, and $C = \{\theta\}$. The result for general x follows from irreducibility, and for general finite C by additivity: $\mathbb{1}_C = \sum_{\theta \in C} \mathbb{1}_\theta$. ∎

We define for $\alpha \in \mathcal{D}(\Lambda)$,

$$\check{f}_\alpha(x) \triangleq \mathsf{E}_x\Big[\exp\Big(\sum_{k=0}^{\sigma_\theta} [\alpha F(\Phi_k) - \Lambda(\alpha)]\Big) \mathbb{1}(\sigma_\theta < \infty)\Big]. \tag{17}$$

The following relation then follows from the Markov property:

$$\begin{aligned} P\check{f}_\alpha(x) &= \mathsf{E}_x\Big[\exp\Big(\sum_{k=1}^{\tau_\theta} [\alpha F(\Phi_k) - \Lambda(\alpha)]\Big) \mathbb{1}(\tau_\theta < \infty)\Big] \\ &= \begin{cases} \lambda_\alpha \check{f}_\alpha(x) f_\alpha^{-1}(x), & x \ne \theta; \\ \exp(-\xi(\alpha)), & x = \theta, \end{cases} \end{aligned}$$

where $\xi(\alpha)$ is defined in (14). Since $\check{f}_\alpha(\theta) = \lambda_\alpha^{-1} f_\alpha(\theta)$, this establishes the identity

$$P\check{f}_\alpha\,(x) = \lambda_\alpha \exp\bigl(-\xi(\alpha)\mathbb{1}_\theta(x)\bigr) f_\alpha^{-1}(x)\check{f}_\alpha(x). \tag{18}$$

Sufficient conditions ensuring that $\xi(\alpha) = 0$ will be derived in Section 4 below.

Theorem 3.1 (i) provides a converse to Lemma 3.1 (iii).

Theorem 3.1 *Suppose that $\mathbf{\Phi}$ is recurrent, $\Lambda(\alpha_0)$ is finite for some $\alpha_0 > 0$, and suppose that the sublevel set C_ζ is finite for some $\zeta > \Lambda(\alpha_0)/\alpha_0$. Then*

(i) *There exists $V\colon \mathsf{X} \to [1,\infty)$ satisfying (15), and hence also a solution $V_0\colon \mathsf{X} \to \mathbb{R}_+$ satisfying (4);*

(ii) *The function $\check{f}_{\alpha_0}(x)$ defined in (17) satisfies,*

$$\inf_{x\in\mathsf{X}} \check{f}_{\alpha_0}(x) > 0;$$

(iii) *The multiplicative ergodic theorem holds,*

$$\frac{1}{n}\log \mathsf{E}_x\Bigl[\exp\Bigl(\alpha_0 S_n\Bigr)\Bigr] \to \Lambda(\alpha_0), \qquad n\to\infty, x\in\mathsf{X} \tag{19}$$

Proof. We first prove (ii). From Jensen's inequality applied to (17) and recurrence of the chain we have

$$\begin{aligned}\log \check{f}_\alpha(x) &\geq \mathsf{E}_x\Bigl[\sum_{k=0}^{\sigma_\theta}[\alpha F(\Phi_k) - \Lambda(\alpha)]\Bigr]\\ &\geq -\Lambda(\alpha)\mathsf{E}_x\Bigl[\sum_{k=0}^{\sigma_\theta}\mathbb{1}_{C_\zeta}(\Phi_k)\Bigr]\end{aligned}$$

where $C_\zeta = \{x : \alpha F(x) \leq \zeta\}$ is finite. Since C_ζ is finite, it is also *special* [14]. That is, the expectation $\mathsf{E}_x[\sum_{k=0}^{\tau_\theta}\mathbb{1}_{C_\zeta}(\Phi_k)]$ is uniformly bounded in x. Hence the inequality above gives the desired lower bound.

To prove (i), note first that the equivalence of the two inequalities is purely notational, where we must set $V_0 = \log(V)$. To show that the assumptions imply that (i) holds we take $V = c\check{f}_{\alpha_0}$ for some $c > 0$. By (18) the required drift inequality holds, and by (ii) we may choose c so that $V\colon \mathsf{X} \to [1,\infty)$.

To establish (iii), first observe that Lemma 3.2 gives the lower bound,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathsf{E}_x \Big[\exp\Big(\alpha S_n\Big)\Big] \geq \Lambda(\alpha).$$

To obtain an upper bound on the limit supremum, first observe that (18) gives the inequality

$$P\check{f}_\alpha\,(x) \leq \lambda_\alpha f_\alpha^{-1}(x)\check{f}_\alpha(x).$$

On iterating this bound we obtain, by the discrete Feynman-Kac formula,

$$\mathsf{E}_x\Big[\exp\Big(\alpha S_n - n\Lambda(\alpha)\Big)\check{f}_\alpha(\Phi_n)\Big] \leq \check{f}_\alpha(x).$$

Applying (ii) we have that $\check{f}_\alpha(x) > c > 0$ for some c and all x, which combined with the above inequality gives the desired upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathsf{E}_x \Big[\exp\Big(\alpha S_n\Big)\Big] \leq \Lambda(\alpha),$$

and completes the proof. ∎

4 The Multiplicative Poisson Equation

For an arbitrary function $F\colon \mathsf{X} \to \mathbb{R}_+$ and $\alpha \in \mathcal{D}(\Lambda)$ we say that $\check{f}_*$ solves the Multiplicative Poisson Equation (MPE) for f_α provided the following identity holds:

$$P\check{f}_*\,(x) = \lambda_\alpha \check{f}_*(x) f_\alpha^{-1}(x), \qquad x \in \mathsf{X}.$$

Equivalently, $\check{f}_*$ solves the eigenvector equation

$$\hat{P}_\alpha \check{f}_* = \lambda_\alpha \check{f}_*.$$

The function $\check{f}_*$ is known as the *Perron-Frobenius eigenvector* for the kernel $\hat{P}_\alpha$ [16]. In [15] it is called the *ground state*. From (18) it is evident that the function defined in (17) solves the MPE if and only if $\xi(\alpha) = 0$. One of the main goals of this section is to derive conditions under which this is the case.

For $\alpha \in \mathcal{D}(\Lambda)$ define the 'twisted' transition kernel $\check{P}_\alpha$ by

$$\check{P}_\alpha(x, y) = \exp(\xi(\alpha)\mathbb{1}_\theta(x))\frac{f_\alpha(x)}{\lambda_\alpha \check{f}_\alpha(x)} P(x, y)\check{f}_\alpha(y), \qquad x, y \in \mathsf{X}.$$

In operator-theoretic notation this is written,

$$\check{P}_\alpha = \lambda_\alpha^{-1} I_{\exp(\xi(\alpha)\mathbb{1}_\theta)} I_{f_\alpha/\check{f}_\alpha} P I_{\check{f}_\alpha}.$$

We denote by $\check{\boldsymbol{\Phi}}^\alpha = \{\check{\Phi}_0^\alpha, \check{\Phi}_1^\alpha, \dots\}$ the Markov chain with transition probability $\check{P}_\alpha$. When $\check{\Phi}_0^\alpha = x$, the induced expectation operator will be denoted $\check{\mathsf{E}}_x^\alpha[\,\cdot\,]$.

Lemma 4.1 *Suppose that $\boldsymbol{\Phi}$ is recurrent. Then, for any $\alpha \in \mathcal{D}(\Lambda)$, $\check{\boldsymbol{\Phi}}^\alpha$ is also recurrent, and for any set $A \in \mathcal{F}_{\tau_\theta}$*

$$\check{\mathsf{E}}_x^\alpha[\mathbb{1}_A] = \check{P}_\alpha\{A \mid \check{\Phi}_0 = x\} = \frac{\mathsf{E}_x\Big[\exp\Big(\alpha S_{\tau_\theta} - \tau_\theta\Lambda(\alpha)\Big)\mathbb{1}_A\Big]}{\mathsf{E}_x\Big[\exp\Big(\alpha S_{\tau_\theta} - \tau_\theta\Lambda(\alpha)\Big)\Big]} \tag{20}$$

Proof. It is easily seen that for $A \in \mathcal{F}_n$,

$$\check{\mathsf{E}}_x^\alpha[\mathbb{1}_A] = \frac{1}{\check{f}_\alpha(x)}\mathsf{E}_x\Big[\exp\Big(\sum_{k=0}^{n-1}[\alpha F(\Phi_k) - \Lambda(\alpha) + \xi(\alpha)\mathbb{1}_\theta(\Phi_k)]\Big)\check{f}_\alpha(\Phi_n)\mathbb{1}_A\Big]. \tag{21}$$

Since we have $A \cap \{\tau_\theta = n\} \in \mathcal{F}_n$ for every n whenever A is $\mathcal{F}_{\tau_\theta}$-measurable, the above identity implies that for such A,

$$\begin{aligned}
\check{\mathsf{E}}_x^\alpha[\mathbb{1}_A\mathbb{1}_{\{\tau_\theta=n\}}] &= \frac{1}{\check{f}_\alpha(x)}\mathsf{E}_x\Big[\exp\Big(\sum_{k=0}^{n-1}[\alpha F(\Phi_k) - \Lambda(\alpha) + \xi(\alpha)\mathbb{1}_\theta(\Phi_k)]\Big)\check{f}_\alpha(\Phi_n)\mathbb{1}_A\mathbb{1}_{\{\tau_\theta=n\}}\Big] \\
&= \frac{\check{f}_\alpha(\theta)}{\check{f}_\alpha(x)}\exp\big(\xi(\alpha)\mathbb{1}_\theta(x)\big)\mathsf{E}_x\Big[\exp\Big(\alpha S_{\tau_\theta} - \tau_\theta\Lambda(\alpha)\Big)\mathbb{1}_A\mathbb{1}_{\{\tau_\theta=n\}}\Big].
\end{aligned}$$

Summing over $n \geq 1$ and applying Fubini's Theorem then gives

$$\check{\mathsf{E}}_x^\alpha[\mathbb{1}_A \mathbb{1}_{\{\tau_\theta < \infty\}}] = \frac{\check{f}_\alpha(\theta)}{\check{f}_\alpha(x)} \exp\bigl(\xi(\alpha)\mathbb{1}_\theta(x)\bigr)\mathsf{E}_x\Bigl[\exp\Bigl(\alpha S_{\tau_\theta} - \tau_\theta \Lambda(\alpha)\Bigr)\mathbb{1}_A\Bigr],$$

where we have used recurrence of $\mathbf{\Phi}$. This formula holds for any $\mathcal{F}_{\tau_\theta}$-measurable event A: letting A denote the 'full set', $A = \bigcup\{\Phi_k \in \mathsf{X}\}$, then gives $\check{\mathsf{E}}_\theta^\alpha[\mathbb{1}_{\{\tau_\theta < \infty\}}] = 1$, so that $\check{\mathbf{\Phi}}^\alpha$ is recurrent. The representation formula (20) follows immediately for arbitrary $A \in \mathcal{F}_{\tau_\theta}$. ∎

Let $\Lambda^{(\alpha)}(\delta)$ denote the log-pfe for the kernel $I_{f_\delta}\check{P}_\alpha$.

Lemma 4.2 *If $\alpha \in \mathcal{D}(\Lambda)$ then, for any $\delta > 0$,*

$$\Lambda^{(\alpha)}(\delta) \geq \Lambda(\alpha + \delta) - \Lambda(\alpha).$$

Proof. From the representation formula given in Lemma 4.1 we have for any Λ,

$$\begin{aligned}
&\check{\mathsf{E}}_\theta^\alpha\Bigl[\exp\Bigl(\delta S_{\tau_\theta} - \tau_\theta\Lambda\Bigr)\Bigr] \\
&\quad = \exp(\xi(\alpha))\mathsf{E}_\theta\Bigl[\exp\Bigl(\alpha S_{\tau_\theta} - \tau_\theta\Lambda(\alpha)\Bigr)\exp\Bigl(\delta S_{\tau_\theta} - \tau_\theta\Lambda\Bigr)\Bigr] \\
&\quad \geq \mathsf{E}_\theta\Bigl[\exp\Bigl((\alpha+\delta)S_{\tau_\theta} - \tau_\theta(\Lambda(\alpha) + \Lambda)\Bigr)\Bigr]
\end{aligned}$$

The right hand side is > 1 whenever $\Lambda(\alpha) + \Lambda < \Lambda(\alpha + \delta)$, from which the lower bound follows. ∎

The following characterization is also a corollary to Lemma 4.1.

Theorem 4.1 *Suppose that $\mathbf{\Phi}$ is recurrent. Then the following are equivalent for any $\alpha \in \mathcal{D}(\Lambda)$.*

(i) *The chain $\check{\mathbf{\Phi}}^\alpha$ is geometrically recurrent.*

(ii) *there exists $\Lambda < \Lambda(\alpha)$ such that*

$$\mathsf{E}_\theta\Bigl[\exp\Bigl(\alpha S_{\tau_\theta} - \tau_\theta\Lambda\Bigr)\Bigr] < \infty. \tag{22}$$

(iii) *For some* $\lambda < \lambda_\alpha$, $b < \infty$, *a finite set* C, *and a function* $V : \mathsf{X} \to (0, \infty)$,

$$PV \le \lambda f_\alpha^{-1} V + b\mathbb{1}_C.$$

Moreover, if V *is any solution to (iii) then* $\check{f}_\alpha \in L_\infty^V$.

Proof. The equivalence of (i) and (ii) follows from the identity

$$\check{\mathsf{E}}_\theta^\alpha[R^{\tau_\theta}] = \exp(\xi(\alpha))\mathsf{E}_\theta\Big[\exp\Big(\alpha S_{\tau_\theta} - \tau_\theta \Lambda\Big)\Big]$$

where $R = \exp(\Lambda(\alpha) - \Lambda)$ (see Lemma 4.1). By definition, the chain $\check{\mathbf{\Phi}}^\alpha$ is geometrically recurrent if and only if the LHS is finite for some $R > 1$. This establishes the desired equivalence between (i) and (ii) since $\xi(\alpha)$ is always finite.

To see that (i) $\Longrightarrow$ (iii) let $\check{V} \ge 1$, $\check{\lambda} < 1$, and $\check{b} < \infty$ be a solution to the inequality

$$\check{P}_\alpha \check{V} \le \check{\lambda}\check{V} + \check{b}\mathbb{1}_\theta.$$

A function $\check{V}$ satisfying this inequality exists by the geometric recurrence assumption and Theorem 2.1. Letting $V = \check{f}_\alpha \check{V}$, the above inequality becomes, for some $b < \infty$,

$$PV \le \check{\lambda}\lambda_\alpha f_\alpha^{-1} V + b\mathbb{1}_\theta,$$

which is a version of the inequality assumed in (iii).

Conversely, if (iii) holds then we may take $\check{V} = V/\check{f}_\alpha$ to obtain the inequality

$$\begin{aligned}
\check{P}_\alpha \check{V}(x) &\le \frac{f_\alpha(x)}{\lambda_\alpha \check{f}_\alpha(x)} \sum_y P(x,y)\check{f}_\alpha(y)\check{V}(y) \\
&\le \frac{f_\alpha(x)}{\lambda_\alpha \check{f}_\alpha(x)} \Big(\lambda f_\alpha^{-1}(x) V(x) + b\mathbb{1}_C(x)\Big) \\
&= \frac{1}{\lambda_\alpha}\Big(\lambda \check{V}(x) + \frac{f_\alpha(x)}{\check{f}_\alpha(x)} b\mathbb{1}_C(x)\Big)
\end{aligned}$$

This bound shows that the chain $\check{\mathbf{\Phi}}_\alpha$ satisfies all of the conditions of Theorem 2.4, and hence (i) also holds.

Using Theorem 2.4 we also see that $\check{V}$ is bounded from below, or equivalently that $\check{f}_\alpha \in L_\infty^V$. ∎

We can now formulate existence and uniqueness criteria for solutions to the MPE.

Theorem 4.2 *Suppose that $\mathbf{\Phi}$ is recurrent. Then for any $\alpha \in \mathcal{D}(\Lambda)$,*

(i) *If $\check{P}_\alpha$ is geometrically recurrent then $\xi(\alpha) = 0$, and hence the function $\check{f}_\alpha$ given in (17) solves the MPE;*

(ii) *Suppose that $\xi(\alpha) = 0$, and suppose that h is a positive-valued solution to the* ***inequality****,*

$$\widehat{P}_\alpha h\,(x) \le \lambda_\alpha h(x), \qquad x \in \mathsf{X}.$$

Then $h(x)/h(\theta) = \check{f}_\alpha(x)/\check{f}_\alpha(\theta)$, $x \in \mathsf{X}$, where $\check{f}_\alpha$ is given in (17). Hence the inequality above is in fact an equality for all x.

Proof. The proof of (i) is a consequence of the definition (12), Theorem 4.1, and the Dominated Convergence Theorem.

To prove (ii) we first note that the function $\check{h} = h/\check{f}_\alpha$ is superharmonic and positive for the kernel $\check{P}_\alpha$. Since this kernel is recurrent we must have $\check{h}(x) = \check{h}(\theta)$ for all x ([10, Theorem 17.1.5] can be extended to positive harmonic functions). ∎

5 Multiplicative Ergodic Theorems

In this section we present a substantial strengthening of the multiplicative ergodic theorems given in Lemma 3.2 and Theorem 3.1 (iii), and give more readily verifiable criteria for the existence of solutions to the multiplicative Poisson equation. Throughout the remainder of the paper we assume that the chain is recurrent, and in the majority of our results the function F is assumed to be near-monotone. These assumptions are summarized in the following statement:

$$\mathbf{\Phi} \textit{ is recurrent, } F \textit{ is near-monotone, and } \bar{\alpha} > 0. \tag{23}$$

The constant $\bar{\alpha}$ is defined in (13). When $\alpha < \bar{\alpha}$ the twisted kernel defines a geometrically ergodic Markov chain $\check{\mathbf{\Phi}}^{\alpha}$, and specializing to $\alpha = 0$ we see that $\mathbf{\Phi}$ itself is geometrically ergodic:

Theorem 5.1 *Suppose that (23) holds.*

(i) *For each $\alpha < \bar{\alpha}$ the chain $\check{\mathbf{\Phi}}^{\alpha}$ with transition kernel $\check{P}_{\alpha}$ is V_{α}-uniformly ergodic. The function V_{α} can be chosen so that, for some constant $b_0 = b_0(\alpha) > 0$,*

$$V_{\alpha}(x) \geq \frac{b_0}{\check{f}_{\alpha}(x)} \quad \textit{and} \quad V_{\alpha}(x) \geq \exp(b_0 F(x)), \qquad x \in \mathsf{X}. \tag{24}$$

(ii) *If $\alpha \geq \bar{\alpha}$ then $\check{\mathbf{\Phi}}^{\alpha}$ is not geometrically recurrent.*

Proof. Take $V_{\alpha} = \frac{\check{f}_{\beta}}{\check{f}_{\alpha}}$ with $0 < \beta < \bar{\alpha}$ and $\beta > \alpha$. The lower bounds in (24) holds by Theorem 3.1 (ii). Since $\Lambda'(\alpha) < \|F\|_{\infty}$ we have

$$\begin{aligned}
\check{P}_{\alpha} V &\leq \lambda_{\alpha}^{-1} f_{\alpha} \check{f}_{\alpha}^{-1} P \check{f}_{\beta} \\
&= \lambda_{\alpha}^{-1} f_{\alpha} \check{f}_{\alpha}^{-1} \big(\lambda_{\beta} \exp(\xi(\beta) \mathbb{1}_{\theta}) \check{f}_{\beta}^{-1} \check{f}_{\beta}\big) \\
&= \exp\Big(\xi(\beta) \mathbb{1}_{\theta} - \delta\big(F - (\Lambda(\alpha+\delta) - \Lambda(\alpha))/\delta\big)\Big) V,
\end{aligned}$$

where $\delta = \beta - \alpha > 0$. We then have, by the definition of the right derivative,

$$(\Lambda(\alpha+\delta) - \Lambda(\alpha))/\delta \leq \Lambda'(\beta) < \|F\|_{\infty}.$$

From the near-monotone condition it then follows that for some $\eta < 1$, a finite set C, and some $b < \infty$,

$$\check{P}_{\alpha} V \leq \eta V + b \mathbb{1}_{C}.$$

The set C is a sublevel set of F together with the state θ. By Theorem 2.4 we conclude that $\check{\mathbf{\Phi}}^\alpha$ is geometrically recurrent, which proves (i).

Theorem 4.1 implies part (ii). ∎

Theorem 5.2 *Under the assumption (23) the following limits hold:*

(i) *For $\alpha < \bar\alpha$ there exists $R = R(\alpha) > 1$, $0 < c(\alpha) < \infty$ such that for all x,*

$$R^n\Big(\mathsf{E}_x\Big[\exp\Big(\alpha S_n - \Lambda(\alpha)n\Big)\Big] - c(\alpha)\check f_\alpha(x)\Big) \to 0, \qquad n\to\infty.$$

(ii) *For all $\alpha \in \mathbb{R}$,*

$$\frac{1}{n}\log \mathsf{E}_x\Big[\exp\Big(\alpha S_n\Big)\Big] \to \Lambda(\alpha), \qquad n\to\infty, x\in\mathsf{X}.$$

Proof. The proof of (ii) is contained in parts (i) and (iii) of Theorem 3.1. It is given here for completeness.

To see (i) we apply Theorem 5.1, which together with Theorem 2.1 implies that there exists $R > 1$ such that

$$R^n\Big(\check{\mathsf{E}}_x^\alpha[\check f_\alpha^{-1}(\check\Phi_n^\alpha)] - \check\pi_\alpha(\check f_\alpha^{-1})\Big) \to 0, \qquad n\to\infty.$$

From this and (21) we immediately obtain the result with $c(\alpha) = \check\pi_\alpha(\check f_\alpha^{-1})$.

∎

A straightforward approach to general functions on X which are not near-monotone is through domination. Let $F\colon \mathsf{X}\to\mathbb{R}$ be an arbitrary function, and suppose that $G_0\colon \mathsf{X}\to[1,\infty)$ is norm-like. We write $F = o(G_0)$ if the following limit holds,

$$\lim_{n\to\infty}\frac{1}{n}\sup(|F(x)| : G_0(x)\le n) = 0. \tag{25}$$

The proof of the following is exactly as in Theorem 5.2. We can assert as in Theorem 5.1 that $V = \frac{\check g_0}{\check f_\alpha}$ serves as a Lyapunov function, where $\check g_0$ is the solution to the multiplicative Poisson equation using G_0.

Theorem 5.3 *Suppose that $\mathbf{\Phi}$ is recurrent, that $G_0 \colon \mathsf{X} \to [1,\infty)$ is norm-like, $\Lambda(G_0) < \infty$, and $F = o(G_0)$. Then for any $\alpha \in \mathbb{R}$,*

(i) $\Lambda(\alpha) < \infty$;

(ii) *There exists a solution $\check{f}_\alpha$ to the multiplicative Poisson equation*

$$P\check{f}_\alpha\,(x) = \check{f}_\alpha(x)\exp\bigl(-\alpha F(x) + \Lambda(\alpha)\bigr)$$

satisfying,

$$\sup_{x\in\mathsf{X}} \frac{\check{f}_\alpha(x)}{\check{g}_0(x)} < \infty;$$

(iii) *There exists $R = R(\alpha) > 1$, $0 < c(\alpha) < \infty$ such that for all x,*

$$R^n\Bigl(\mathsf{E}_x\Bigl[\exp\Bigl(\alpha S_n - \Lambda(\alpha)n\Bigr)\Bigr] - c(\alpha)\check{f}_\alpha(x)\Bigr) \to 0, \qquad n\to\infty.$$

■

The '$o(\cdot)$ condition' may be overly restrictive in some models. The following result requires only geometric recurrence, but the domain of Λ may be limited.

Theorem 5.4 *Suppose that $\mathbf{\Phi}$ is V-uniformly ergodic, so that (9) holds for some $V \colon \mathsf{X} \to [1,\infty)$, $\eta < 1$, a finite set C, and $b < \infty$. Suppose that the function $F\colon \mathsf{X} \to \mathbb{R}$ is bounded. Then the following hold for all $\alpha \in \mathbb{R}$ satisfying,*

$$|\alpha| < \frac{|\log(\eta)|}{\|F - \pi(F)\|_\infty}.$$

(i) *There exists a solution $\check{f}_\alpha$ to the multiplicative Poisson equation*

$$P\check{f}_\alpha\,(x) = \check{f}_\alpha(x)\exp\bigl(-\alpha F(x) + \Lambda(\alpha)\bigr)$$

satisfying $\check{f}_\alpha \in L_\infty^V$;

(ii) *There exists $R = R(\alpha) > 1$, $0 < c(\alpha) < \infty$ such that for all x,*

$$R^n\Bigl(\mathsf{E}_x\Bigl[\exp\Bigl(\alpha S_n - \Lambda(\alpha)n\Bigr)\Bigr] - c(\alpha)\check{f}_\alpha(x)\Bigr) \to 0, \qquad n\to\infty.$$

Proof. We have, for $x \in C^c$,

$$\check{P}_\alpha V \le \lambda_\alpha \exp(\alpha F - \Lambda(\alpha) - |\log(\eta)|)V.$$

Also, by convexity we know that $\Lambda(\alpha) \ge \alpha\pi(F)$ for all α, so that

$$\check{P}_\alpha V \le \lambda_\alpha \exp(\alpha(F - \pi(F)) - |\log(\eta)|)V.$$

As in the previous results, Theorem 4.1 completes the proof of (i) since $|\Lambda(\alpha)| \le \alpha\|F\|_\infty$. Part (ii) is proved as in Theorem 5.2. ■

6 Differentiability and Large Deviations

The usual proof of Cramer's Theorem for i.i.d. random variables suggests that a multiplicative ergodic theorem will yield a version of the Large Deviations Principle for the chain. While this is true, a useful LDP requires some structure on the log-pfe Λ. We establish smoothness of Λ together with a version of the LDP in this section.

6.1 Regularity and differentiability

A set $C \subset \mathsf{X}$ will be called *F-multiplicatively regular* if for any $A \subset \mathsf{X}$ there exists $\varepsilon = \varepsilon(C, A) > 0$ such that

$$\sup_{x \in C} \mathsf{E}_x \Big[\exp(\varepsilon S_{\tau_A})\Big] < \infty. \tag{26}$$

The chain is called F-multiplicatively regular if every singleton is an F-multiplicatively regular set.

If the function F is bounded from above below, so that for some $\varepsilon > 0$,

$$\varepsilon \le F(x) \le \varepsilon^{-1}, \qquad x \in \mathsf{X},$$

then multiplicative regularity is equivalent to geometric regularity. When F is unbounded this is substantially stronger. From Theorem 2.1 we see that geometric regularity is equivalent to a Foster-Lyapunov drift condition. An exact generalization is given here for norm-like F.

Theorem 6.1 *Suppose that F is norm-like. Then, the chain is F-multiplicatively regular if and only if there exists $\alpha > 0$; a function $V\colon \mathsf{X} \to [1,\infty)$; and a finite constant $\overline{\lambda}$ such that*

$$\widehat{P}_\alpha V\,(x) \le \overline{\lambda} V(x), \qquad x \in \mathsf{X}. \tag{27}$$

Proof. We may assume without loss of generality that $F\colon \mathsf{X} \to \mathbb{R}_+$.

For the "only if" part we set $V(x) = \mathsf{E}_x\Big[\exp(\varepsilon S_{\sigma_C+1})\Big]$ with C an arbitrary finite set and $\varepsilon > 0$ chosen so that $\mathsf{E}_x\Big[\exp(\varepsilon S_{\tau_C})\Big]$ is bounded on C. We then have with $\alpha = \varepsilon$,

$$\widehat{P}_\alpha V\,(x) = \mathsf{E}_x\Big[\exp(\varepsilon S_{\tau_C+1})\Big].$$

The right hand side is equal to V on C^c, and is bounded on C. Note that V is finite valued since the set $S_V = \{x : V(x) < \infty\}$ is absorbing.

To establish the "if" part is more difficult. Suppose that (27) holds. To establish (26) for fixed A we construct a new function $W\colon \mathsf{X} \to [1,\infty)$ such that for some $\beta > 0$,

$$\widehat{P}_\beta W\,(x) \le W(x), \qquad x \in A^c. \tag{28}$$

We may then conclude that the stochastic process

$$M_t = \exp(\beta S_{\tau_A \wedge t}) W(\Phi_{\tau_A \wedge t}), \qquad t \ge 1; \qquad\qquad M_0 = W(x),$$

is a $\mathcal{F}_t$-super martingale whenever $\Phi_0 = x \in A^c$. We then have by the optional stopping theorem, as in the proof of Theorem 2.4,

$$\mathsf{E}_x\Big[\exp(\beta S_{\tau_A})\Big] \le B_A(x)$$

for $x \in A^c$, with $B_A = W$. For $x \in A$ we obtain an identical bound with $B_A = W + f_\beta$ by stopping the process at $t = 1$ and considering separately the cases $\tau_A = 1$ and $\tau_A > 1$.

It remains to establish (28), assuming that (27) holds for some V, and some $\overline{\lambda}$. Fix $0 < \varepsilon_0 < \overline{\lambda}^{-1}$, and for $\beta \le \alpha$ set

$$\widehat{K}_\beta = (1-\varepsilon_0)\sum_{n=0}^{\infty} \varepsilon_0^n \widehat{P}_\beta^{n+1}$$

Using (27) we have $\widehat{K}_\alpha V \le \exp(b)V$ with $\exp(b) = \overline{\lambda}(1-\varepsilon_0)/(1-\varepsilon_0\overline{\lambda}) < \infty$. We thus have

$$\begin{aligned}\widehat{K}_{\alpha/2}V\,(x) &\le \exp(-(\alpha/2)F(x))\widehat{K}_\alpha V\,(x)\\ &\le \exp(b-(\alpha/2)F(x))V(x)\\ &\le \exp(b\mathbb{1}_C(x))V(x)\end{aligned}$$

where C is a finite set.

We may find $\delta > 0$ so that, for $\beta > 0$,

$$\widehat{K}_\beta(x,A) \ge \widehat{K}_0(x,A) \ge \delta, \qquad x \in C. \tag{29}$$

This is possible since C is finite and $\mathbf{\Phi}$ is irreducible and aperiodic.

Let $V_1(x) = V(x)$ for $x \in C$, and set $V_1 \equiv 1$ on C. Then by increasing b if necessary we continue to have $\widehat{K}_{\alpha/2}V_1\,(x) \le \exp(b\mathbb{1}_C(x))V_1(x)$.

We now set $V_2 = V_1^\varepsilon$ where $\varepsilon < 1$ will be determined below. Jensen's inequality gives

$$\widehat{K}_{\varepsilon\alpha/2}V_2\,(x) \le \exp(b\varepsilon\mathbb{1}_C(x))V_2(x) \qquad x \in \mathsf{X}.$$

Letting $\beta = \varepsilon\alpha/2$ we have thus establish a bound of the form

$$\widehat{K}_\beta V_2\,(x) \le \exp(b\beta\mathbb{1}_C(x))V_2(x)$$

where again the constant b must be redefined, but it is still finite, and it is independent of β for $0 < \beta < \alpha/2$.

To remove the indicator function in the last bound set

$$V_3(x) = 2V_2(x) - \mathbb{1}_A(x), \qquad x \in \mathsf{X}.$$

We have for $x \in A^c \cap C^c$,

$$\widehat{K}_\beta V_3\,(x) \le 2\widehat{K}_\beta V_2\,(x) \le 2V_2(x) = V_3(x).$$

For $x \in A^c \cap C$,

$$\widehat{K}_\beta V_3\,(x) \le 2\widehat{K}_\beta V_2\,(x) - \widehat{K}_\beta(x, A) \le 2\exp(\beta b) - \delta$$

where in the last inequality we are using (29) and the definition that $V_2 \equiv 1$ on C. We now define $\beta = \log((\delta + 2)/2)/b$ so that $\widehat{K}_\beta V_3 \le 2 = V_3$ on $x \in A^c \cap C$.

We have thus shown that (28) holds with the kernel $\widehat{K}_\beta$, and with $W = V_3$. The function $W = (1 + \varepsilon_0)V_3 + \varepsilon_0 \widehat{K}_\beta V_3$ must then satisfy (28) for $\widehat{P}_\beta$, which proves the proposition. ■

As an immediate corollary we find that each of the chains $\check{\mathbf{\Phi}}^\alpha$ is F-multiplicatively regular, $\alpha < \bar{\alpha}$, since the Lyapunov function V can be taken as $V = \check{f}_\beta/\check{f}_\alpha$ as in Theorem 5.1 above. Using this fact we may establish differentiability of Λ. Similar results are established in [12] under the *assumption* that the set below is open,

$$\mathcal{W} = \Big\{(\alpha, \Lambda) : \mathsf{E}_\theta\Big[\exp\Big(\alpha S_{\tau_\theta} - \tau_\theta \Lambda\Big)\Big] < \infty\}$$

This assumption fails in general under the assumptions here. However we still have,

Theorem 6.2 *If F is near-monotone then the log-pfe Λ is C^∞ on O where $O = (-\infty, \bar{\alpha})$. For any $\alpha \in O$,*

(i) $\quad \Lambda'(\alpha) = \check{\pi}_\alpha(\theta)\check{\mathsf{E}}_\theta[S_{\tau_\theta}] = \check{\pi}_\alpha(F);$

(ii) $\quad \Lambda''(\alpha) = \check{\pi}_\alpha(\theta)\check{\mathsf{E}}_\theta\Big[\Big(S_{\tau_\theta} - \check{\pi}_\alpha(F)\tau_\theta)\Big)^2\Big] = \check{\gamma}^2(\alpha).$

The quantity $\check{\gamma}^2(\alpha)$ is precisely the time-average variance constant for the centered function $F - \check{\pi}_\alpha(F)$ applied to $\check{\mathbf{\Phi}}^\alpha$.

Proof. The proof is similar to Lemma 3.3 of [12]: one simply differentiates both sides of the identity (6). The justification for differentiating within the expectation follows from F-multiplicative regularity.

That $\check{\gamma}^2(\alpha)$ is the time-average variance constant is discussed above Theorem 2.2. ∎

In the same way we can prove,

Theorem 6.3 *The conclusions of Theorem 6.2 continue to hold, and $\bar{\alpha}$ can be taken infinite, under the assumptions of Theorem 5.3.* ∎

6.2 Large deviations

A version of the large deviations principle is now immediate. For $c \in \mathbb{R}$ and $C \subseteq \mathbb{R}$ we set

$$\Lambda^*(c) \triangleq \sup_{\alpha \in \mathbb{R}} \{c\alpha - \Lambda(\alpha)\}; \qquad \underline{\Lambda}^*(C) \triangleq \inf_{c \in C} \Lambda^*(c). \tag{30}$$

It is well known that Λ^* is a convex function whose range lies in $[0, \infty]$. Its domain is denoted $\mathcal{D}(\Lambda^*) = \{c : \Lambda^*(c) < \infty\}$.

There is much prior work on large deviations for Markov chains, with most results obtained using uniform bounds on the transition kernel (see [17] or [7]). Large deviations bounds are obtained under minimal assumptions in [13]. Specialized to this countable state space setting, the main result can be expressed as follows: For suitable sets $C \subset \mathbb{R}$, and any singleton $i \in \mathsf{X}$,

$$\frac{1}{n} \log\Bigl(\mathsf{P}_x\{\frac{1}{n} S_n \in C \textit{ and } \Phi_n = i\}\Bigr) \sim -\underline{\Lambda}^*(C), \qquad n \to \infty.$$

Following [13], and using similar methodology, the constraint that Φ_n is equal to i is relaxed in [6]. However the imposed assumptions amount to V-uniform ergodicity with $V = 1$. The assumption (23), or the domination condition in Theorem 5.3 is much more readily verified in practice, and the conclusions obtained through these assumptions and the preceding ergodic theorems are very strong.

We define $\mathcal{O}$ to be the range of possible derivatives,

$$\mathcal{O} \triangleq \{\Lambda'(\alpha) : \alpha \in \mathcal{D}^o(\Lambda)\} \subseteq \mathcal{D}(\Lambda^*).$$

When F is near-monotone then $\mathcal{D}^o(\Lambda) = (-\infty, \bar{\alpha})$. For any $a, b \in \mathcal{O}$ we let $\alpha, \beta \in \mathcal{D}^o(\Lambda)$ denote the corresponding values satisfying $\Lambda'(\alpha) = a$ and $\Lambda'(\beta) = b$. From the definitions we then have,

$$\Lambda^*(a) = \alpha a - \Lambda(\alpha) \qquad \Lambda^*(b) = \beta b - \Lambda(\beta).$$

We let $\{\check{f}_\alpha\}$ denote the solutions to the multiplicative Poisson equation, normalized so that $\check{\pi}_\alpha(1/\check{f}_\alpha) = 1$. We define $\check{\gamma}^2(\alpha)$ to be the time-average variance constant,

$$\check{\gamma}^2(\alpha) = \Lambda''(\alpha), \qquad \alpha \in \mathcal{D}^\circ(\Lambda).$$

Recall that we let F denote the distribution function for a standard normal random variable. For any real α, c set

$$\mathsf{B}(\alpha, c) = \mathsf{F}\Big(\frac{c}{\check{\gamma}(\alpha)}\Big) - \frac{1}{2}$$

Theorem 6.4 *Suppose that (23) holds. For any constants $a < \pi(F) < b$ with $a, b \in \mathcal{O}$, and any $0 < c \leq \infty$,*

(i)

$$\limsup_{n\to\infty} \frac{\mathsf{P}_x\{\frac{1}{n}S_n \in (a - c/\sqrt{n}, a)\}}{\exp(-\Lambda^*(a)n)} \quad \leq \quad \mathsf{B}(\alpha, c)\check{f}_\alpha(x),$$

$$\limsup_{n\to\infty} \frac{\mathsf{P}_x\{\frac{1}{n}S_n \in (b, b + c/\sqrt{n})\}}{\exp(-\Lambda^*(b)n)} \quad \leq \quad \mathsf{B}(\beta, c)\check{f}_\beta(x),$$

(ii)

$$\liminf_{n\to\infty} \frac{\mathsf{P}_x\{\frac{1}{n}S_n \in (a, a + c/\sqrt{n})\}}{\exp(-\Lambda^*(a)n)} \quad \geq \quad \mathsf{B}(\alpha, c)\check{f}_\alpha(x),$$

$$\liminf_{n\to\infty} \frac{\mathsf{P}_x\{\frac{1}{n}S_n \in (b - c/\sqrt{n}, b)\}}{\exp(-\Lambda^*(b)n)} \quad \geq \quad \mathsf{B}(\beta, c)\check{f}_\beta(x),$$

(iii) *For any closed set $A \subseteq \mathbb{R}$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log\Bigl(\mathsf{P}_x\{\frac{1}{n}S_n \in A\}\Bigr) \le -\underline{\Lambda}^*(A),$$

(iv) *For any open set $A \subseteq \mathbb{R}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log\Bigl(\mathsf{P}_x\{\frac{1}{n}S_n \in A\}\Bigr) \ge -\underline{\Lambda}^*(A \cap \mathcal{O}).$$

Proof. To prove (i) and (ii) write

$$\check{W}_n(t) = \frac{1}{\sqrt{n}}\Bigl(S_{\lfloor nt\rfloor} - an\Bigr), \qquad t \ge 0.$$

The probability of interest takes the form,

$$\begin{aligned}
&\mathsf{P}_x\left\{\frac{1}{n}S_n \in \Bigl(a + \frac{c_0}{\sqrt{n}}, a + \frac{c_1}{\sqrt{n}}\Bigr)\right\}\exp\Bigl(\Lambda^*(a)n\Bigr)\\
&= \mathsf{P}_x\{\check{W}_n(1) \in (c_0, c_1)\}\\
&= \check{f}_\alpha(x)\check{\mathsf{E}}_x^\alpha\Bigl[\exp\Bigl(-\alpha(S_n - an)\Bigr)\mathbb{1}\Bigl\{\check{W}_n(1) \in (c_0, c_1)\Bigr\}\Bigl(1/\check{f}_\alpha(\check{\Phi}_n)\Bigr)\Bigr]\\
&= \check{f}_\alpha(x)\check{\mathsf{E}}_x^\alpha\Bigl[\exp\Bigl(-\alpha\sqrt{n}\check{W}_n(1)\Bigr)\mathbb{1}\Bigl\{\check{W}_n(1) \in (c_0, c_1)\Bigr\}\Bigl(1/\check{f}_\alpha(\check{\Phi}_n)\Bigr)\Bigr]
\end{aligned}$$

For the first bound in (i) take $c_0 = -c$ and $c_1 = 0$. Since $\alpha < 0$ we obtain,

$$\frac{\mathsf{P}_x\{\frac{1}{n}S_n \in (a - c/\sqrt{n}, a)\}}{\exp(-\Lambda^*(a)n)} \le \check{f}_\alpha(x)\check{\mathsf{E}}_x^\alpha\Bigl[\mathbb{1}\Bigl\{\check{W}_n(1) \in (-c, 0)\Bigr\}\Bigl(1/\check{f}_\alpha(\check{\Phi}_n)\Bigr)\Bigr].$$

Theorem 2.2 gives the first bound in (i), and all of the other bounds are obtained in the same way.

Parts (iii) and (iv) immediately follow. ■

We obtain slightly stronger conclusions under a domination condition.

Theorem 6.5 *Suppose that F satisfies the assumptions of Theorem 5.3. Then parts (i)–(iii) of Theorem 6.4 continue to hold, and part (iv) is strengthened: For any open set $A \subseteq \mathbb{R}$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log\Bigl(\mathsf{P}_x\{\frac{1}{n}S_n \in A\}\Bigr) \ge -\underline{\Lambda}^*(A).$$

Proof. Theorem 5.3 and Theorem 6.3 tell us that $\Lambda\colon \mathbb{R} \to \mathbb{R}$ is C^∞. We can conclude that $\Lambda^*(a) = \infty$ for $a \in \bar{\mathcal{O}}^c$, and it follows that $\underline{\Lambda}^*(A \cap \mathcal{O}) = \underline{\Lambda}^*(A)$ when A is open. ∎

6.3 Empirical measures

These results can be extended to the empirical measures of the chain through domination as in Theorem 5.3. There is again a large literature in this direction, but the results typically hold only for uniformly ergodic Markov chains (see [3, 7, 6]).

Let $\mathcal{M}$ denote the set of all finite signed measures on X, endowed with the weak topology, and define the *empirical measures*,

$$L_n \triangleq \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\Phi_i}, \qquad n \geq 1. \tag{31}$$

L_n is, for each $n \geq 1$, an $\mathcal{M}$-valued random variable.

Assume that $G_0\colon \mathsf{X} \to [1,\infty)$ is given, and that $G\colon \mathsf{X} \to [1,\infty)$ is a norm-like function satisfying $G = o(G_0)$. It follows that G_0 is also norm-like. We consider the vector space L_∞^G of functions $F\colon \mathsf{X} \to \mathbb{R}$ satisfying

$$\|F\|_G \triangleq \sup_{x\in\mathsf{X}} \frac{|F(x)|}{G(x)} < \infty.$$

Its dual, $\mathcal{M}_1^G \subset \mathcal{M}$, is the set of signed measures μ satisfying,

$$\|\mu\|_G \triangleq \sup(\mu(F) : \|F\|_G \leq 1) < \infty.$$

The Banach-Alaoglu Theorem implies that the unit ball in $\mathcal{M}_1^G$ is a compact subset of $\mathcal{M}$ since we have assumed that G is norm-like.

For any $F \in L_\infty^G$ we define $\Lambda(F)$ to be the associated log-gpe, which is finite by Theorem 5.3. We let $\Lambda^*\colon \mathcal{M} \to [0,\infty]$ denote its conjugate dual,

$$\Lambda^*(\mu) = \sup_{F\in L_\infty^G} \big(\langle \mu, F\rangle - \Lambda(F)\big), \qquad \mu \in \mathcal{M}_1^G. \tag{32}$$

Under the assumptions imposed here the function Λ^* is bounded from below:

Proposition 6.1 *Under the assumptions of this section the rate function Λ^* given in (32) satisfies, for some $\varepsilon_0 > 0$,*

$$\Lambda^*(\mu) \geq \varepsilon_0 \|\mu - \pi\|_G^2, \qquad \textit{when } \Lambda^*(\mu) \leq 1.$$

Proof. Define for any $F \in L_\infty^G$ the directional second derivative,

$$\Lambda''(F) \triangleq \frac{d^2}{d\alpha^2}\Lambda(\alpha F)\Big|_{\alpha=1}.$$

Using Theorem 6.2 we can show the second derivative is bounded for bounded F:

$$B_0 \triangleq \sup(\Lambda''(F) : \|F\|_G \leq 1) < \infty.$$

We then have by convexity and a Taylor series expansion, for any $\varepsilon \leq 1$ and any F satisfying $\|F\|_G \leq 1$,

$$\begin{aligned} \langle \mu - \pi, \varepsilon F\rangle &\leq -\varepsilon\pi(F) + \Lambda^*(\mu) + \Lambda(\varepsilon F) \\ &\leq -\varepsilon\pi(F) + \Lambda^*(\mu) + \varepsilon\pi(F) + \varepsilon^2 B_0. \end{aligned}$$

Setting $\varepsilon = \sqrt{\Lambda^*(\mu)}$ then gives,

$$\langle \mu - \pi, F\rangle \leq (1 + B_0)\sqrt{\Lambda^*(\mu)}.$$

This bound holds for arbitrary $\|F\|_G \leq 1$ whenever $\Lambda^*(\mu) \leq 1$, and hence proves the proposition with $\varepsilon_0 = (1 + B_0)^{-2}$. ∎

For any subset $A \subset \mathcal{M}$ write,

$$\underline{\Lambda}^*(A) \triangleq \inf_{\mu \in A} \Lambda^*(\mu)$$

The proof of the following is standard following Proposition 6.1 and Theorem 5.3 (see [7]).

Theorem 6.6 *Under the assumptions of this section the following bounds hold for any open $\mathcal{O} \subseteq \mathcal{M}$, and any closed $\mathcal{K} \subseteq \mathcal{M}$, when $\mathcal{M}$ is endowed with the weak topology:*

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \log\Bigl(\mathsf{P}_x\{L_n \in \mathcal{K}\}\Bigr) &\leq -\underline{\Lambda}^*(\mathcal{K}). \\
\liminf_{n\to\infty} \frac{1}{n} \log\Bigl(\mathsf{P}_x\{L_n \in \mathcal{O}\}\Bigr) &\geq -\underline{\Lambda}^*(\mathcal{O}).
\end{aligned}$$

∎

7 Conclusions

This paper provides a collection of tools for deriving multiplicative ergodic theorems and associated large deviations bounds for Markov chains on a countable state space. For the processes considered it provides a complete story, but it also suggests numerous open problems.

(i) Some generalizations, such as the continuous time case, or models on general state spaces can be formulated easily given the methods introduced here. The general state space case presents new technical difficulties due to the special status of finite sets appealed to in this paper. In some cases this can be resolved by assuming appropriate bounds on the kernels $\{\widehat{P}_\alpha\}$, similar to the bounds used in [17].

(ii) We would like to develop in further detail the structural properties of the pfe λ. We saw in Theorem 6.5 that Λ will be essentially smooth under a domination condition. The case of general near-monotone F is not well understood, and we have seen that even in elementary examples this basic condition fails.

(iii) The large deviation bounds provided by Theorems 6.4 – 6.6 could certainly be strengthened given the very strong form of convergence seen in Theorem 1.2.

We are currently considering all of these extensions, and are developing applications to both control and large deviations.

Acknowledgements

Part of the research for this paper was done while the second author was a Fulbright research scholar and visiting professor at the Indian Institute of Science, and a visiting professor at the Technion. The author gratefully acknowledges support from these institutions.

The authors would like to express their sincere thanks to Ioannis Kontoyiannis, currently at Purdue University, for invaluable comments on an earlier draft of this manuscript. In particular, the strong version of the LDP given in Theorem 6.4 followed from discussions with Prof. Kontoyiannis. The referees also provided numerous useful suggestions for improvements.

References

[1] L. Arnold and W. Kliemann. Large deviations for linear Stochastic differential equations. *Stochastic differential systems*, J.Engelbrecht and W.Schmidt eds., Springer Verlag, 117-151, 1987.

[2] S. Balaji, V.S. Borkar, S.P. Meyn. Risk sensitive dynamic programming with unbounded cost. Presented at the *38th IEEE Conference on Decision and Control*, Pheonix, AZ, Dec. 1999, and published in the proceedings.

[3] E. Bolthausen, J-D. Deuschel, and Y. Tamura. Laplace Approximations for large deviations of nonreversible Markov processes: the nondegenerate case. *Annals of Probability*, 23, 286–297, 1995.

[4] V. S. Borkar. Topics in controlled Markov chains. Pitman Research Notes in Mathematics Series #240, Longman Scientific & Technical, UK, 1991.

[5] V.S. Borkar and S.P. Meyn. Risk Sensitive Optimal Control: Existence and Synthesis for Models with Unbounded Cost submitted, *MOR*, February, 1999.

[6] A. de Acosta. Large deviations for empirical measures of Markov chains. *Journal of Theoret. Probab*, 3, 395-431, 1990.

[7] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications.* Jones and Bartlett Publishers, London, England, 1993.

[8] P.W. Glynn and S.P. Meyn. A Liapunov bound for solutions of Poisson equation. *Annals of Probability*, 24, 916-931, 1996.

[9] J.F.C. Kingman. Subadditive Ergodic Theory. *Annals of Probability*, 1, 883-909, 1973.

[10] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability.* Springer-Verlag, London, 1993.

[11] S.P. Meyn. The policy improvement algorithm for Markov decision processes with general state space. *IEEE Transactions on Automatic Control*, AC-42:191–196, 1997.

[12] P. Ney and E. Nummelin. Markov additive processes I: eigenvalue properties and limit theorems. *Annals of Probability*, 15:561–592, 1987.

[13] P. Ney and E. Nummelin. Markov additive processes II: Large deviations. *Annals of Probability*, 15:593–609, 1987.

[14] E. Nummelin. *General Irreducible Markov Chains and Non-Negative Operators.* Cambridge University Press, Cambridge, MA, 1984.

[15] R. G. Pinsky. *Positive harmonic functions and diffusion.* Cambridge University Press, Cambridge, UK ; New York, 1995.

[16] E. Seneta. *Non-Negative Matrices and Markov Chains.* Springer, New York, NY, 2nd edition, 1981.

[17] S.R.S. Varadhan. Large deviations and Applications. SIAM Publications, Philadelphia, 1984.

[18] P. Whittle. *Risk-Sensitive Optimal Control.* John Wiley and Sons, Chichester, NY, 1990.