A Two-Dimensional Minkowski ?(x) Function

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Abstract

A one-to-one continuous function from a triangle to itself is defined that has both interesting number theoretic and analytic properties. This function is shown to be a natural generalization of the classical Minkowski $?(x)$ function. It is shown there exists a natural class of pairs of cubic irrational numbers in the same cubic number field that are mapped to pairs of rational numbers, in analog to $?(x)$ mapping quadratic irrationals on the unit interval to rational numbers on the unit interval. It is also shown that this new function satisfies an analog to the fact that $?(x)$, while increasing and continuous, has derivative zero almost everywhere.

1 Introduction

Any real number with an eventually periodic continued fraction expansion must be a quadratic irrational. This property linking periodicity of a number's continued fraction expansion with its being quadratic led Minkowski to define his remarkable question-mark function

$$?:[0,1]\to[0,1].$$

(See page 50 of Volume II in [27]; see also p. 754, article 196 in [15], which appears to be essentially a translation of all of Minkowski's number theory papers.) The question-mark function is increasing, continuous, maps each rational number $\frac{p}{q}$ to a pure dyadic number of the form $\frac{k}{2^n}$, maps each quadratic irrational to a rational number, and has the property that the inverse image of the rational numbers is exactly the set of quadratic irrationals. In order to understand the number theoretic properties of quadratic irrationals, it is natural to look at the function theoretic properties of $?(x)$. In particular, the question-mark function is not only continuous and monotonically increasing but has derivative zero almost everywhere. As such, it is a naturally occuring example of a singular function. Moreover, it is, in fact, the diophantine properties of continued fractions that lead to its derivative being zero a.e.. Thus the analytic property of $?(x)$ being both increasing and having derivative zero almost everywhere is actually number theoretic in origin.

In this paper, we will construct a function similar to Minkowski's question-mark function, and will use that function in order to understand the properties of cubic irrationals.

A. Denjoy, in [4], [5] and independently R. Salem, in [36], were the first to realize that $?(x)$ is singular, although earlier, F. Ryde [34] proved in essence that $?(x)$ was singular. However, Ryde showed that $?(x)$ was singular without realizing its connection with Minkowski's function (see also Ryde's [35]). Recent work on $?(x)$ is the work of Kinney [16], Girgensohn [9], Ramharter [33], of Tichy and Uitz [42], and of Viader, Paradis and Bibiloni [43] [31]. (In fact, the idea for the inequality that we prove in section 6.1 and use in 6.2 was inspired by the work of Viader, Paradis and Bibiloni in [43] .)

A natural question to ask is: do cubic irrationals and other higher order algebraic numbers lend themselves to similar analysis? An even more basic question to ask is how to generalize the relation between periodicity for continued fractions and quadratic irrationals to cubics. In 1848, in a letter to Jacobi, Hermite [14] asked for such a generalization. Specifically, the Hermite problem is:

Find methods for expressing real numbers as sequences of positive integers so that the sequence is eventually periodic precisely when the initial number is a cubic irrational.

Over the years there has been much work in trying to solve the Hermite problem. For an overview, see Schweiger's *Multidimensional Continued Fractions* [40]. For work up to 1980, see Brentjes' overview in [3]. Other work is in [1], [2], [7],[6], [8], [11], [12], [17],[19],[20],[21], [22], [24], [25], [37],[38],[39].

On the other hand, there has been little attempt to approach the Hermite problem by generalizing the Minkowski $?(x)$. The only such attempt that we have found is in the thesis of Louis Kollros [18]. Kollros generalizes $?(x)$ to a map from the unit square to itself. However, while he sets up various methods for associating points in the unit square with sequences of integers, he does not concern himself with the function-theoretic properties of this function. It does not appear that Kollros has solved the Hermite problem. In particular, he was not interested in the differentiability properties of his analogue to $?(x)$.

In this paper we develop a different, more natural, analog to $?(x)$. In section two, a review of the Minkowski question-mark function is given. In section three, we construct a map from a two dimensional simplex (a triangle) to itself, as an analog to the map of $?(x)$ from a one dimensional simplex to itself. The map will be determined by partitioning the triangle, first via a "Farey" partition, and then by a barycentric (triadic) partitioning, which we will frequently call the "bary" partitioning. We define a function $\delta(x, y)$ from the Farey triangle to the barycentric triangle. In section four, we see that the Farey iteration can be viewed as a multidimensional continued fraction. We show that periodicity of the Farey iteration corresponds to a class of cubic irrationals. In section five we show, by contrast, that periodicity for the barycentric iterations corresponds to a class of rational points. This results in that our function will map a natural class of cubic points to a natural class of rational points. Finally, in section six, we prove an analog of singularity by showing that, a.e., the area of image triangles in the barycentric partitioning approaches zero far more quickly than the area of the domain triangles in the Farey partitioning.

We note that using Farey partitioning, or Farey nets, to solve the Hermite problem has been considered by both Monkemeyer [30] and more recently by

Grabiner [10]. Both papers are quite interesting; neither use Farey nets to generalize the Minkowski $?(x)$ function. In actuality, this analytic approach would not have been a natural succession in either of these papers, as Monkemeyer's and Grabiner's goals were not function theoretic.

We would like to thank Lori Pedersen for making all of the diagrams. Also, we would like to thank Keith Briggs for pointing out some errors in the bibliography of an earlier version of this paper.

2 A Review of the Minkowski Question-Mark Function

All of the discussion in this section is well-known. We include it here for sake of completeness.

Recall that given two rational numbers $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, each in lowest terms, the *Farey sum*, $\hat{+}$, of the numbers is

$$\frac{p_1}{q_1} \hat{+} \frac{p_2}{q_2} = \frac{p_1 + p_2}{q_1 + q_2}.$$

The ? function is then defined as follows. Suppose we know the value of $?(\frac{p_1}{q_1})$ and $?(\frac{p_2}{q_2})$. We then set

$$?(\frac{p_1 + p_2}{q_1 + q_2}) = \frac{?(\frac{p_1}{q_1}) + ?(\frac{p_2}{q_2})}{2}.$$

Specifying the intial values

$$?(0) = 0 \text{ and } ?(1) = 1,$$

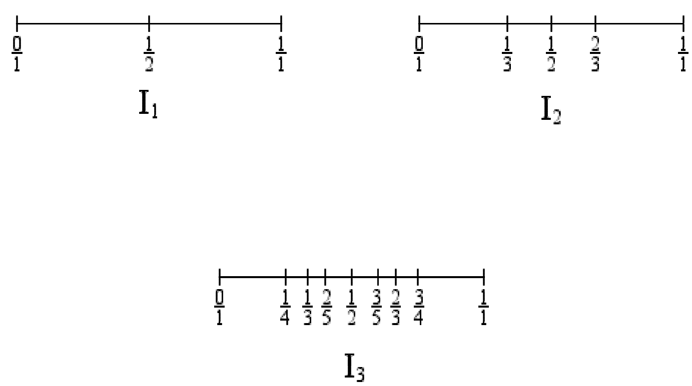
we now know the values of $?(x)$ for any rational number x.

By continuity arguments we can determine the values of $?(x)$ for any real number x in the unit interval. Since we will be generalizing this continuity argument in the next section, we discuss this now in some detail.

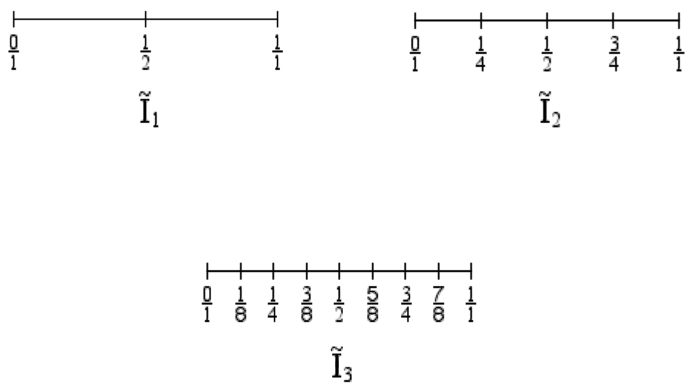
We produce two sequences of partitions, $\mathcal{I}_k$ and $\tilde{\mathcal{I}}_k$, of the unit interval. For each $k \geq 0$, each partition will split the unit interval into 2^k subintervals. Both start with just the unit interval itself:

$$\mathcal{I}_0 = \tilde{\mathcal{I}}_0 = [0, 1].$$

Note that $0 = \frac{0}{1}$ and $1 = \frac{1}{1}$. Now, given the partition $\mathcal{I}_{k-1}$, suppose that the endpoints of each of its 2^{k-1} open subintervals are rational numbers. Form the next partition $\mathcal{I}_k$ by taking the Farey sum of the endpoints of the partition $\mathcal{I}_{k-1}$. Thus the endpoints of $\mathcal{I}_k$ consist of the Farey fractions of order k.



The partition $\tilde{\mathcal{I}}_k$ is even simpler. It is just the partition given by the subintervals $[\frac{l-1}{2^k}, \frac{l}{2^k}]$.



Then the function $?(x)$ can be seen to map the endpoints of each $\mathcal{I}_k$ to the corresponding endpoints of $\tilde{\mathcal{I}}_k$.

Now, as is shown, for example, in [36], $?(x)$ is singular and hence has derivative zero almost everywhere. Using the partitions defined above, we can recast the fact that $?(x)$ is a singular function into the language of lengths of intervals. Fix $\alpha \in [0, 1]$. For each k, let I_k and $\tilde{I}_k$ be the subintervals of the respective partitions $\mathcal{I}_k$ and $\tilde{\mathcal{I}}_k$ that contain the point α. Then, as shown page 437 in [36],

Theorem 1 *For almost all $\alpha \in [0, 1]$,*

$$\liminf_{k\to\infty} \frac{\textit{length of } \tilde{I}_k}{\textit{length of } I_k} = 0.$$

It is this theorem that provides the most natural language for generalizing the failure of differentiability for our analog of the question-mark function.

The proof involves the idea that the Diophantine approximations properties of continued fractions make the above denominator approach zero more slowly than the numerator.

3 The Farey-Bary Map: A generalization of the Minkowski Question-Mark Function.

Our goal is to define a continuous map from a two-dimensional simplex (a triangle) to itself that generalizes the Minkowski question-mark function. This will involve two separate partitionings of the triangle. We would like to have periodicity in the domain correspond to cubic irrationals while periodicity in the range to imply rationality. Both of these goals will only be achieved in part, as we will show that periodicity will imply cubic irrationality in the domain case and rationality for the range. At the same time, we want our generalization to obey some sort of singularity property.

Although in a sense it would be most natural to denote our generalization by the symbol $?(x, y)$, we have found that it is both awkward to say and awkward to read. Thus we will denote our generalization by $\delta(x, y)$.

3.1 The Farey Sum in the Plane

We will often need to refer to a point in the plane of the form

$$v = \begin{pmatrix} p/r \\ q/r \end{pmatrix}.$$

Here, since the coordinates share the same denominator, we can associate to this point a unique vector in space, namely

$$\bar{v} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Conversely, a vector $\bar{v} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ can be associated uniquely to the point

$$v = \begin{pmatrix} p/r \\ q/r \end{pmatrix}$$

in the plane. In what follows, we will usually refer to both the point and its corresponding vector as v.

Consider three points in the plane, each of whose entries are nonnegative integers, each $r_i \neq 0$, and such that each vector's entries share no common factors:

$$v_1 = \begin{pmatrix} p_1/r_1 \\ q_1/r_1 \end{pmatrix}, v_2 = \begin{pmatrix} p_2/r_2 \\ q_2/r_2 \end{pmatrix}, v_3 = \begin{pmatrix} p_3/r_3 \\ q_3/r_3 \end{pmatrix}.$$

These points define a triangle in the plane and, as noted above, can also be represented as the vectors,

$$v_1 = \begin{pmatrix} p_1 \\ q_1 \\ r_1 \end{pmatrix}, v_2 = \begin{pmatrix} p_2 \\ q_2 \\ r_2 \end{pmatrix} \text{ and } v = \begin{pmatrix} p_3 \\ q_3 \\ r_3 \end{pmatrix}.$$

Summing the three vectors, we get

$$v = v_1 + v_2 + v_3 = \begin{pmatrix} p_1 + p_2 + p_3 \\ q_1 + q_2 + q_3 \\ r_1 + r_2 + r_3 \end{pmatrix}.$$

This vector sum can be converted into a point v in the plane, where

$$v = \begin{pmatrix} \frac{p_1+p_2+p_3}{r_1+r_2+r_3} \\ \\ \frac{q_1+q_2+q_3}{r_1+r_2+r_3} \end{pmatrix}.$$

This correspondence between points in the plane and a vector representation allows us to define the Farey sum.

Definition 2 *Let*

$$v_1 = \begin{pmatrix} p_1/r_1 \\ q_1/r_1 \end{pmatrix}, v_2 = \begin{pmatrix} p_2/r_2 \\ q_2/r_2 \end{pmatrix}, v_3 = \begin{pmatrix} p_3/r_3 \\ q_3/r_3 \end{pmatrix},$$

where, for each i, the p_i, q_i and r_i share no common factor. The Farey sum, $\hat{v}$ of the v_i is then

$$\hat{v} = v_1 \hat{+} v_2 \hat{+} v_3 = \begin{pmatrix} \frac{p_1+p_2+p_3}{r_1+r_2+r_3} \\ \\ \frac{q_1+q_2+q_3}{r_1+r_2+r_3} \end{pmatrix}.$$

Note that the point $\hat{v}$ is inside the triangle determined by the vertices v_1, v_2 and v_3 and that $\hat{v}$ corresponds to the vector $\begin{pmatrix} p_1+p_2+p_3 \\ q_1+q_2+q_3 \\ r_1+r_2+r_3 \end{pmatrix}$.

3.2 Farey and Barycentric Partitions

In this section we will define two partitions of the triangle

$$\triangle = \{(x,y) : 1 \geq x \geq y \geq 0\}.$$

The first partition of $\triangle$ will yield the domain of our desired function δ, while the second partition will yield the range.

The Farey Partition

We will define a sequence of partitions $\{\mathcal{P}_n\}$ such that each $\mathcal{P}_n$ will consist of 3^n subtriangles of $\triangle$ and each $\mathcal{P}_n$ will be a refinement of the previous $\mathcal{P}_{n-1}$.

Let $\mathcal{P}_0$ be the initial triangle $\triangle$. The three vertices of $\triangle$ are

$$v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0/1 \\ 0/1 \end{pmatrix},$$

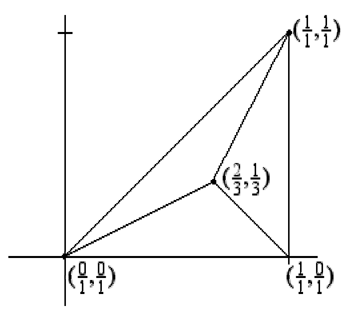
$$v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/1 \\ 0/1 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/1 \\ 1/1 \end{pmatrix}.$$

Taking the Farey sum of these vertices, we have

$$v_1 \hat{+} v_2 \hat{+} v_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

This Farey vector corresponds to the point $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$. In particular, the point $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$ is an interior point of the triangle $\triangle$ and, in a natural way, partitions $\triangle$ into three subtriangles. We will refer to the resulting interior point as the *Farey-center*.



This determines the partition $\mathcal{P}_1$. We now proceed inductively. Suppose we have the partition $\mathcal{P}_n$, that determines 3^n triangles. We now partition each of these triangle into three subtriangles, as follows. Suppose one of the triangles in $\mathcal{P}_n$ has vertices $\begin{pmatrix} p_1/r_1 \\ q_1/r_1 \end{pmatrix}$, $\begin{pmatrix} p_2/r_2 \\ q_2/r_2 \end{pmatrix}$ and $\begin{pmatrix} p_3/r_3 \\ q_3/r_3 \end{pmatrix}$. Computing the Farey sum of the three vertices of the triangle gives a point $\hat{v}$, the Farey-center, in the interior of the subtriangle. The Farey-center, $\hat{v}$, yields a partition of the subtriangle. Computing in this way the partition of each subtriangle of $\triangle$ determined by $\mathcal{P}_n$, gives us the desired next partition $\mathcal{P}_{n+1}$ of $\triangle$.

We denote this full partitioning of $\triangle$ by $\triangle_F$ and call it the *Farey* partitioning.

The Barycentric Partition

Again we will define a sequence of partitions $\tilde{\mathcal{P}}_n$ of $\triangle$ such that each $\tilde{\mathcal{P}}_n$ will consist of 3^n triangles of $\triangle$ and each $\tilde{\mathcal{P}}_n$ will be a refinement of the previous $\tilde{\mathcal{P}}_{n-1}$.

As with the Farey partitioning, the zeroth level partition $\tilde{\mathcal{P}}_0$ is simply the initial triangle $\triangle$.

To compute $\tilde{\mathcal{P}}_1$, we again start with the original three vertices of $\triangle$ and compute the barycenter of $\triangle$, namely $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$. This point, called the *barycenter*, happens, in this case, to be the same point obtained by computing the Farey sum of the coordinates of the vertices of $\triangle$, the Farey-center. This is just a coincidence.

Proceed inductively as follows. Assume we have a partition $\tilde{\mathcal{P}}_n$ of $\triangle$ into 3^n subtriangles. Further, assume at the n^{th} stage that the coordinates of the vertices of any subtriangle can be expressed as rational numbers with 3^n in the denominator. Then, if a given subtriangle in $\tilde{\mathcal{P}}_n$ has vertices $\begin{pmatrix} a_1/3^n \\ b_1/3^n \end{pmatrix}, \begin{pmatrix} a_2/3^n \\ b_2/3^n \end{pmatrix}$, and $\begin{pmatrix} a_3/3^n \\ b_3/3^n \end{pmatrix}$, we compute the Farey sum of the three vertices. This again gives the barycenter of the subtriangle, namely, $\begin{pmatrix} \frac{a_1+a_2+a_3}{3^{n+1}} \\ \\ \frac{b_1+b_2+b_3}{3^{n+1}} \end{pmatrix}$. Computing, in this way, the partition of each subtriangle of $\tilde{\triangle}$ determined by $\tilde{\mathcal{P}}_n$ gives us the desired next partition $\tilde{\mathcal{P}}_{n+1}$ of $\triangle$.

We call this full partitioning of $\triangle$ the *barycentric*, or *Bary*, partitioning and denote it by $\triangle_B$.

3.3 The Farey-Bary Map

We are now ready to define the extension of the Minkowski question-mark function to $\delta : \triangle_F \rightarrow \triangle_B$. We will proceed in stages. At first, we will define a function $\delta_n(x, y)$ from the vertices of the n^{th} partition of $\triangle_F$ to the vertices of the n^{th} partition of $\triangle_B$ and then extend linearly δ_n to all of $\triangle_F$. Then we will show that the functions in the sequence $\{\delta_n\}$ are continuous and uniformly convergent. The limit will be our desired function $\delta(x, y)$ on $\triangle_F$.

We first need to introduce some notation. Each of the partitions $\mathcal{P}_n$ and $\tilde{\mathcal{P}}_n$ determine subtriangles of $\triangle_F$ and $\triangle_B$, respectively. Let $\triangle_{n,F}$ and $\triangle_{n,B}$ denote $\triangle_F$ and $\triangle_B$ after the nth partitioning, respectively. The expression $\langle v_1(n), v_2(n), v_3(n)\rangle$ will denote a general subtriangle of $\triangle_{n,F}$ with vertices $v_1(n), v_2(n)$, and $v_3(n)$. When we need to refer to the 3^n specific subtriangles, we will use $\langle {v_1}^s(n), {v_2}^s(n), {v_3}^s(n)\rangle$, where s $\in \{1, \ldots, 3^n\}$. In a similar fashion, we will refer to the subtriangles of $\tilde{\mathcal{P}}_n$ by $\langle \tilde{v}_1(n), \tilde{v}_2(n), \tilde{v}_3(n)\rangle$, in the general case, and $\langle \tilde{v}_1^s(n), \tilde{v}_2^s(n), \tilde{v}_3^s(n)\rangle$ in the specific case.

Note that it happens to be the case that

$$\mathcal{P}_0 = \tilde{\mathcal{P}}_0,\ \mathcal{P}_1 = \tilde{\mathcal{P}}_1 \text{ and } \triangle_{1,F} = \triangle_{1,B}.$$

Definition 3 *Define $\delta_0, \delta_1 : \triangle_F \to \triangle_B$ to be the identity maps on the vertices of the subtriangles determined by $\mathcal{P}_0$ and $\mathcal{P}_1$. For any n, define δ_n to send any vertex in the n^{th} partition $\mathcal{P}_n$ to the corresponding vertex in the partition $\tilde{\mathcal{P}}_n$. That is, define δ_n on any subtriangle $\langle v_1(n), v_2(n), v_3(n)\rangle$ of the partition $\mathcal{P}_n$ by*

$$\delta(v_i(n)) = \tilde{v}_i(n)$$

for i = 1,2,3. Finally, for any point (x, y) in the subtriangle with vertices $\langle v_1(n), v_2(n), v_3(n)\rangle$, set

$$\delta(x, y) = \alpha\tilde{v}_1(n) + \beta\tilde{v}_2(n) + \gamma\tilde{v}_3(n),$$

where

$$(x, y) = \alpha v_1(n) + \beta v_2(n) + \gamma v_3(n).$$

Note that, since the point (x, y) is in the interior of the triangle $\langle v_1(n), v_2(n), v_3(n)\rangle$, we have that

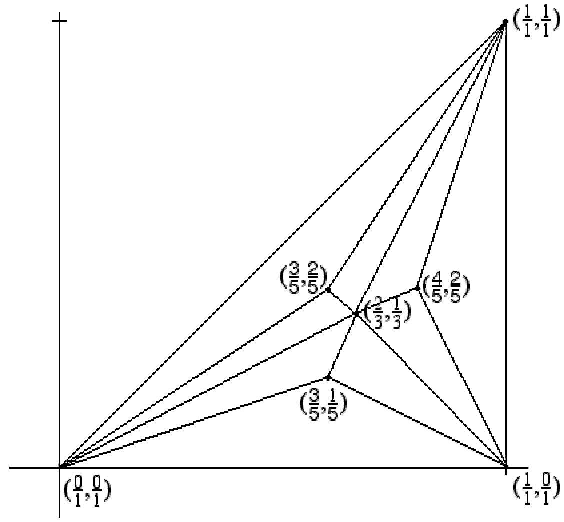
$$\alpha + \beta + \gamma = 1$$

with

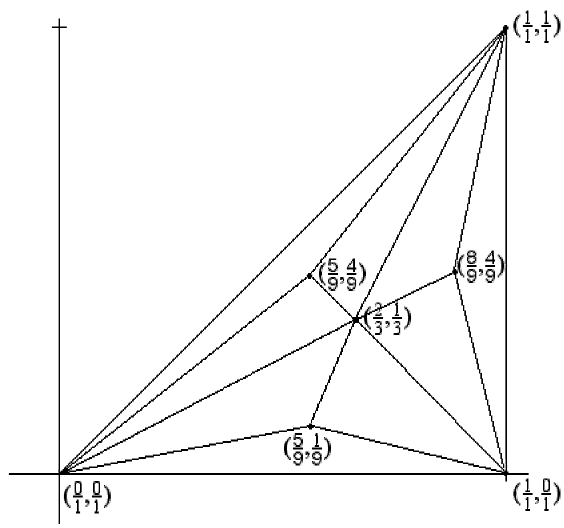
$$0 \le \alpha, \beta, \gamma \le 1.$$

As defined, δ_0 and δ_1 are both the identity map since $\triangle_{0,F} = \triangle_{0,B}$ and $\triangle_{1,F} = \triangle_{1,B}$. However, the mappings start to become more complicated with δ_2.

At this stage we have the Farey partition:



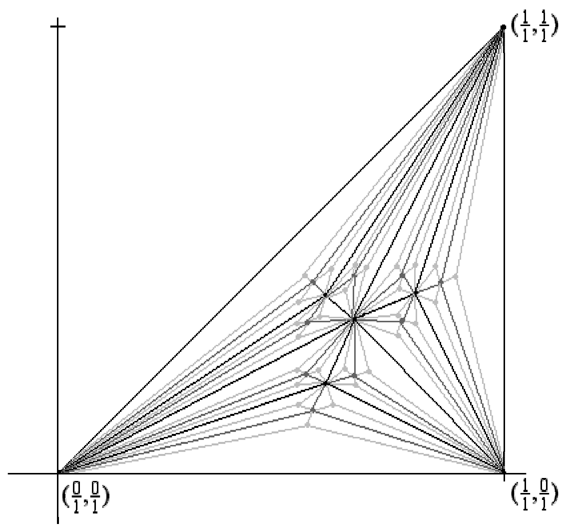
and the Bary partition:



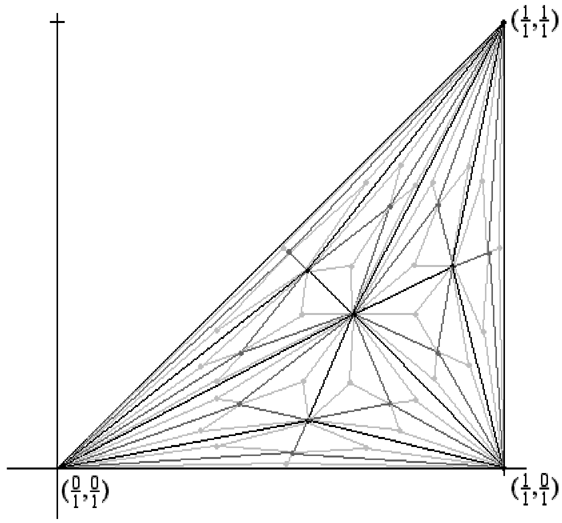
The correspondence between the vertices becomes,

$$
\begin{aligned}
\delta_2 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} &= \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\
\delta_2 \begin{pmatrix} 3/5 \\ 1/5 \end{pmatrix} &= \begin{pmatrix} 5/9 \\ 1/9 \end{pmatrix} \\
\delta_2 \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} &= \begin{pmatrix} 8/9 \\ 4/9 \end{pmatrix} \\
\delta_2 \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} &= \begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix}.
\end{aligned}
$$

Going a few stages further, we get for the Farey partition:



and for the Bary partition:



(We find it interesting that the diagram for the Farey partition is much more aesthetically pleasing than the one for the barycentric partition.)

By definition, $\delta_3(v_i(2)) = \delta_2(v_i(2)$ for any vertex in a subtriangle $\langle v_1(2), v_2(2), v_3(2)\rangle$, of $\triangle_{2,F}$. Thus to describe δ_3, we need only specify what happens on the new vertices obtained in $\triangle_{3,F}$ and $\triangle_{3,B}$.

This new correspondence becomes:

$$\delta_3 \begin{pmatrix} 11/27 \\ 4/27 \end{pmatrix} = \begin{pmatrix} 5/9 \\ 2/9 \end{pmatrix}$$
$$\delta_3 \begin{pmatrix} 11/27 \\ 7/27 \end{pmatrix} = \begin{pmatrix} 5/9 \\ 3/9 \end{pmatrix}$$
$$\delta_3 \begin{pmatrix} 14/27 \\ 1/27 \end{pmatrix} = \begin{pmatrix} 4/9 \\ 1/9 \end{pmatrix}$$
$$\delta_3 \begin{pmatrix} 14/27 \\ 13/27 \end{pmatrix} = \begin{pmatrix} 4/9 \\ 3/9 \end{pmatrix}$$

$$\delta_3 \begin{pmatrix} 20/27 \\ 4/27 \end{pmatrix} = \begin{pmatrix} 6/9 \\ 2/9 \end{pmatrix}$$
$$\delta_3 \begin{pmatrix} 20/27 \\ 16/27 \end{pmatrix} = \begin{pmatrix} 6/9 \\ 4/9 \end{pmatrix}$$
$$\delta_3 \begin{pmatrix} 23/27 \\ 7/27 \end{pmatrix} = \begin{pmatrix} 7/9 \\ 3/9 \end{pmatrix}$$
$$\delta_3 \begin{pmatrix} 23/27 \\ 16/27 \end{pmatrix} = \begin{pmatrix} 7/9 \\ 4/9 \end{pmatrix}$$
$$\delta_3 \begin{pmatrix} 26/27 \\ 13/27 \end{pmatrix} = \begin{pmatrix} 6/9 \\ 3/9 \end{pmatrix}.$$

Theorem 4 *The sequence of functions* $\{\delta_n\}$ *is uniformly convergent.*

Proof: For any point $v \in \triangle_F$, its image $\delta_n(v)$, for any n, must land in one of the 3^n subtriangles in the partition $\tilde{\mathcal{P}}_n$, the n^{th} partition of $\triangle_B$. Label this triangle by $\langle \tilde{v}_1^s(n), \tilde{v}_2^s(n), \tilde{v}_3^s(n) \rangle$, for $s = 1, \ldots, 3^n$. By the definition, we can see that for any $m > n$, the image $\delta_m(v)$, while rarely equal to $\delta_n(v)$, remains in the triangle $\langle \tilde{v}_1^s(n), \tilde{v}_2^s(n), \tilde{v}_3^s(n) \rangle$

Each subtriangle $\langle \tilde{v}_1^s(n), \tilde{v}_2^s(n), \tilde{v}_3^s(n) \rangle$, for $s = 1, \ldots, 3^n$ of the partition $\tilde{\mathcal{P}}_n$, will gain a new barycenter in the next step. We will denote the barycenter of each such partitioned triangle by $\tilde{v}_0^s(n+1)$, where, of course,

$$\tilde{v}_0^s(n+1) = \tilde{v}_1^s(n) \hat{+} \tilde{v}_2^s(n) \hat{+} \tilde{v}_3^s(n).$$

Let $\epsilon > 0$ be given. Clearly, there exist an N such that for the N^{th} partition of $\triangle_B$, the maximum distance between any vertex of the s^{th} subtriangle and its new barycenter is given by

$$\max\{d(\tilde{v}_i^s(N), \tilde{v}_0^s(N+1))\} \leq \frac{\epsilon}{4},$$

for each $s \in \{1, 2, \ldots, 3^N\}$. Also, given any points u and v in the subtriangle $\langle \tilde{v}_1^s(N), \tilde{v}_2^s(N), \tilde{v}_3^s(N) \rangle$, we have

$$\begin{aligned} d(u,w) &\leq d(u, \tilde{v}_0^s(N+1)) + d(\tilde{v}_0^s(N+1), w) \\ &\leq 2 \text{ max d}(\tilde{v}_i^s(N), \tilde{v}_0^s(N+1) \\ &\leq \frac{\epsilon}{2}. \end{aligned}$$

Now, let $v \in \triangle_F$. Then $\delta_N(v) \in \langle \tilde{v}_1^s(N), \tilde{v}_2^s(N), \tilde{v}_3^s(N) \rangle$ for some $s = 1, 2, \ldots, 3^N$. In particular, for all m,n $\geq N$, we have $\delta_m(v)$ and $\delta_n(v)$ also in the subtriangle $\langle \tilde{v}_1^s(N), \tilde{v}_2^s(N), \tilde{v}_3^s(N) \rangle$. But this implies that

$$d(\delta_m(v), \delta_n(v)) \leq \frac{\epsilon}{2}.$$

Thus $\{\delta_n\}$ is uniformly Cauchy and the result follows. $\square$

Definition 5 *Define the Farey-Bary map to be $\delta : \triangle_F \to \triangle_B$ where δ is the limit of the sequence $\{\delta_n\}$.*

Theorem 6 *The Farey-Bary map is continuous.*

Proof: Clearly, since each δ_n is a linear map, the sequence $\{\delta_n\}$ consists of continuous functions. The result follows. $\square$

4 Farey Iteration in the Domain as Multi-dimensional Continued Fraction

Minkowski's $?(x)$ provides a link between algebraic properties of numbers and the failure of differentiabilty, almost everywhere, for $?(x)$. Our goal is to find analogous links for the Farey-Bary map $\delta(x, y)$. The key algebraic property of the Minkowski question mark function is that $?(x)$ maps quadratic irrationals to rational numbers. The goal of this section is to show that $\delta(x, y)$ maps a class of pairs of cubic irrationals to pairs of rationals. Unfortunately, we cannot make the claim that δ maps all pairs of cubics (even in the same number field) to pairs of rationals.

4.1 Preliminary Notation

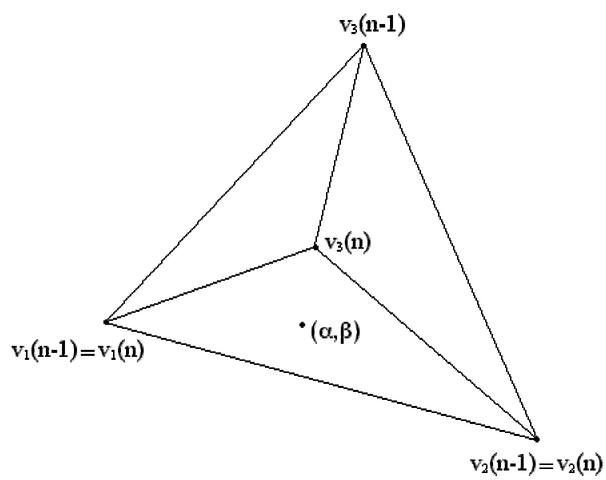
Let $(\alpha, \beta) \in \triangle_F$. The Farey partitions of $\triangle_F$ yield a sequence of triangles converging to the point (α, β). Suppose that at the n^{th} stage of the Farey partitioning, the triangle that contains (α, β) is $\langle v_1(n), v_2(n), v_3(n)\rangle$. We will maintain the notation $v_i(n)$ to mean either the cartesian version of the vertex or the vector in space that corresponds to the vertex. That is, $v_i(n)$ will refer to $\begin{pmatrix} p_i(n)/r_i(n) \\ q_i(n)/r_i(n) \end{pmatrix}$, as well as to $\begin{pmatrix} p_i(n) \\ q_i(n) \\ r_i(n) \end{pmatrix}$. Furthermore, we will order the vertices so that for all n,

$$r_1(n) \leq r_2(n) \leq r_3(n).$$

We want to relate the vertices of the $(n-1)^{st}$ subtriangle that contains(α, β) with the vertices of the subtriangle at the next iteration. For that, suppose that $(\alpha, \beta) \in \langle v_1(n-1), v_2(n-1), v_3(n-1)\rangle \subseteq \triangle_{n-1,F}$. Applying the next partition, $\mathcal{P}_n$, to $\triangle_{n-1,F}$, we decompose $\langle v_1(n-1), v_2(n-1), v_3(n-1)\rangle$ into three new subtriangles. If we let $\langle v_1(n), v_2(n), v_3(n)\rangle$ denote the subtriangle into which (α, β) falls, we see that there are three possibilities for the vertices of $\langle v_1(n), v_2(n), v_3(n)\rangle$.

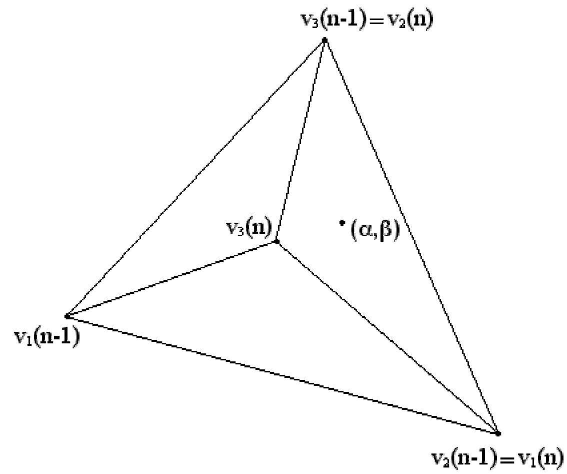
In case I, the vertices of the newly partitioned triangle will be:

$$\begin{aligned}
v_1(n) &= v_1(n-1) \\
v_2(n) &= v_2(n-1) \\
v_3(n) &= v_1(n-1)\hat{+}v_2(n-1)\hat{+}v_3(n-1)
\end{aligned}$$



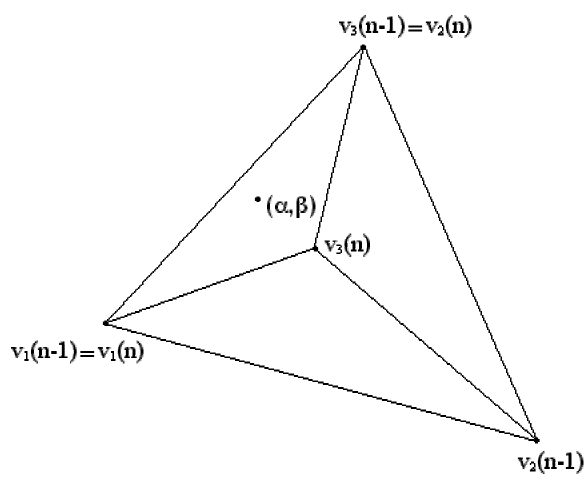
Similarly, the vertices in case II will be:

$$\begin{aligned}
v_1(n) &= v_2(n-1) \\
v_2(n) &= v_3(n-1) \\
v_3(n) &= v_1(n-1)\hat{+}v_2(n-1)\hat{+}v_3(n-1).
\end{aligned}$$



For case III we have:

$$\begin{aligned} v_1(n) &= v_1(n-1) \\ v_2(n) &= v_3(n-1) \\ v_3(n) &= v_1(n-1)\hat{+}v_2(n-1)\hat{+}v_3(n-1). \end{aligned}$$



In the next section, we will streamline this notation.

4.2 Fixing Notation

For each (α, β) in $\triangle_F$ we now associate a sequence of positive integers that will uniquely determine the precise convergence of the Farey subtriangles to (α, β).

To motivate the eventual notation, consider the following three possibilities. Start with a triangle, with vertices v_1,v_2, and v_3, still keeping the convention that $r_1 \leq r_2 \leq r_3$. Suppose we perform k type I operations in a row. The resulting new triangle will have vertices in the following form:

$$v_1, v_2, kv_1 \hat{+} kv_2 \hat{+} v_3.$$

If we perform a type II operation on the triangle, and then $k-1$ type I operations, the new triangle will have vertices:

$$v_2, v_3, v_1 \hat{+} kv_2 \hat{+} kv_3.$$

If we perform a type III operation on the triangle, and then $k-1$ type I operations, the new triangle will have vertices:

$$v_1, v_3, kv_1 \hat{+} v_2 \hat{+} kv_3.$$

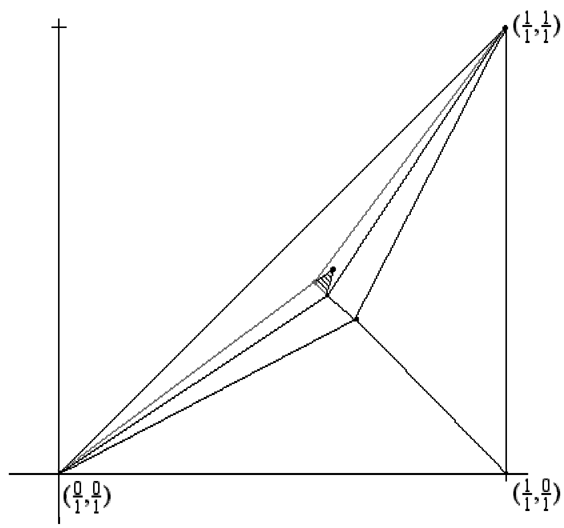
This suggests the following notation.

Define a sequence $\{a_1(i_1), a_2(i_2), \ldots\}$ to be such that each $a_k(i_k)$ is a positive integer and each i_k represents either case I, II or III. The value of $a_k(i_k)$ denotes the operation of first applying a type i_k operation and then $a_k(i_k) - 1$ type I operations. We use the further convention that for $k \geq 2$, i_k can only be of type II or III.

Note that, in the notation of the previous section, by the time we are at step $a_k(i_k)$, we have performed $n = a_1(i_1) + a_2(i_2) + \ldots + a_k(i_k)$ Farey partitions of $\triangle_F$. We associate to each $(\alpha, \beta) \in \triangle_F$ the sequence that yields the corresponding Farey partitions that converge to (α, β). This sequence will be unique. We will also use $\langle v_1(k), v_2(k), v_3(k)\rangle$ to denote the subtriangle of $\triangle_{n,F}$ containing (α, β) after $n = a_1(i_1) + a_2(i_2) + \ldots + a_k(i_k)$ steps partitioning $\triangle_F$. Finally, if we know what case we are in, that is, if we know i_k, we will simply write a_k instead of $a(i_k)$.

Example 7

The shaded region below corresponds to all points $(2(III), 1(II), 1(I))$. Note that in the notation of last section $n = 4$ but that in the notation of this section and for the rest of the paper $k = 3$.



We now have the following recursion formulas for the vertices. For case I at the kth step we get:

$$\begin{aligned} v_1(k) &= v_1(k-1) \\ v_2(k) &= v_2(k-1) \\ v_3(k) &= a_k v_1(k-1) \hat{+} a_k v_2(k-1) \hat{+} v_3(k-1), \end{aligned}$$

For case II we have:

$$\begin{aligned} v_1(k) &= v_2(k-1) \\ v_2(k) &= v_3(k-1) \\ v_3(k) &= v_1(k-1) \hat{+} a_k v_2(k-1) \hat{+} a_k v_3(k-1), \end{aligned}$$

Finally, for case III we get:

$$\begin{aligned} v_1(k) &= v_1(k-1) \\ v_2(k) &= v_3(k-1) \\ v_3(k) &= a_k v_1(k-1) \hat{+} v_2(k-1) \hat{+} a_k v_3(k-1). \end{aligned}$$

We can put these recursion relations naturally into a matrix language. At each step k, define M_k to be the three-by-three matrix

$$M_k = (v_1(k)\ v_2(k)\ v_3(k)) = \begin{pmatrix} p_1(k) & p_2(k) & p_3(k) \\ q_1(k) & q_2(k) & q_3(k) \\ r_1(k) & r_2(k) & r_3(k) \end{pmatrix}$$

If, from the (k-1)st step to the kth step, we are in case I, then

$$M_k = M_{k-1} \begin{pmatrix} 1 & 0 & a_k \\ 0 & 1 & a_k \\ 0 & 0 & 1 \end{pmatrix},$$

for case II we have:

$$M_k = M_{k-1} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & a_k \\ 0 & 1 & a_k \end{pmatrix},$$

and for case III,

$$M_k = M_{k-1} \begin{pmatrix} 1 & 0 & a_k \\ 0 & 0 & 1 \\ 0 & 1 & a_k \end{pmatrix}.$$

Denote in each of these cases the matrix on the right by $A_k(I), A_k(II)$ and $A_k(III)$, respectively. Then we have that each M_k is the product of M_0 with a sequence of various A_m.

Theorem 8 *Each M_k is in the special linear group* **SL(3,Z)**.

Proof: All we need to show is that for all k,

$$\det(M_k) = \pm 1.$$

This follows immediately from observing that $\det(M_0) = 1$ and that the determinants of each of the various $A_k(I), A_k(II)$ and $A_k(III)$ are also plus or minus one. □

4.3 Areas of Farey Subtriangles

Given a finite sequence $\{a_1(i_1), a_2(i_2), \ldots, a_k(i_k)\}$ of positive integers, we define

$$\triangle_k = \{(x, y) : \{a_1(i_1), \ldots, a_k(i_k)\} \text{ are the } 1^{\text{st}}\ k \text{ terms in Farey sequence}\}.$$

A major goal of this paper is showing that the areas of these triangles $\triangle_k$ cannot go to zero too quickly. For these calculations, we will need an easy formula for the areas of the $\triangle_k$.

Theorem 9 *The area of a triangle with vertices* $(p_1/r_1, q_1/r_1)$, $(p_2/r_2, q_2/r_2)$, *and* $(p_3/r_3, q_3/r_3)$ *is*

$$Area\ of\ triangle = \frac{1}{2}\frac{|\det\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}|}{r_1 r_2 r_3}.$$

This is just a calculation involving cross products.

Corollary 10 *Given any finite sequence* $\{a_1(i_1), a_2(i_2), \ldots, a_k(i_k)\}$ *of positive integers,*

$$2Area\ (\triangle_k) = \frac{1}{r_1(k)r_2(k)r_3(k)}.$$

This follows since $\det(M_k) = \pm 1$.

4.4 Farey Periodicity Implies Cubic Irrationals

As we iterate our procedure, the vertices of our triangles converge to a single vector. We want to show:

Theorem 11 *Suppose that* $(\alpha, \beta) \in \triangle_F$ *has an eventually periodic Farey sequence. Then both* α *and* β *are algebraic numbers with* $deg(\alpha) \leq 3$, $deg(\beta) \leq 3$ *and*

$$\dim_{\mathbf{Q}} \mathbf{Q}[\alpha, \beta] \leq 3.$$

This is why the Farey partitioning can be viewed as a multi-dimensional continued fraction algorithm.

Proof: We will be heavily using two facts. First, an eigenvector $(1, a, b)$ of a 3×3 matrix with rational coefficients has the property that

$$\dim_{\mathbf{Q}} \mathbf{Q}[a, b] \leq 3,$$

as seen in a similar argument in [8] in section eight. Second, if we multiply a matrix, which has a largest real eigenvalue, repeatedly by itself, in the limit the columns of the matrix converge to the eigenvector corresponding to the largest eigenvalue.

Suppose that $(\alpha, \beta) \in \triangle$ has an eventually periodic Farey sequence. Even if it is not periodic, the vertices of the corresponding Farey partition triangles converge to the point (α, β). We have seen above that the vertices of the partition triangles correspond to the columns of matrices that are the products of various $A_k(I)$, $A_k(II)$ and $A_k(III)$. With the assumption of periodicity, denote the product of the initial non-periodic matrices be B and the product of the periodic part be A. Then some of the Farey partition triangles about the point (α, β) are given by

$$B, BA, BA^2, BA^3, \ldots.$$

The columns of the matrices $A, A^2, A^3, \ldots$ must converge to a multiple of $B^{-1}(1, \alpha, \beta)^T$. But the columns of the A^k must also converge to an eigenvector and hence $B^{-1}(1, \alpha, \beta)^T$ is an eigenvector of the matrix A. This will give us that α and β must have the desired properties. $\square$

5 Iteration in the Barycentric Range

We have defined Farey partitions, $\mathcal{P}_n$, in $\triangle_F$ and barycentric partitions, $\tilde{\mathcal{P}}_n$ in $\triangle_B$. In $\triangle_F$, the Farey partitions yielded an association between each (α, β) and a sequence obtained from the convergence of the subtriangles resulting from successive applications of the partitions, $\mathcal{P}_n$. This association depended only on the successive partitioning of each subtriangle into three more subtriangles and not on the relative positioning of the new subtriangles. We can follow the same procedure in $\triangle_B$. That is, if we let $(a, b) \in \triangle_B$, we can again associate with (a, b) a sequence of positive integers $\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \ldots\}$ which come from a sequence of barycentric triangles converging to the point (a, b).

Label the triangle corresponding to $\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \ldots, \tilde{a}_k(i_k\}$ by

$$\tilde{\triangle}\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \ldots, \tilde{a}_k(i_k\}.$$

Recall that

$$\tilde{\triangle}_B = \langle \tilde{v}_1(0), \tilde{v}_2(0), \tilde{v}_3(0) \rangle,$$

where

$$\tilde{v}_1(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tilde{v}_2(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \tilde{v}_3(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Associated with the sequence $\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \ldots, \tilde{a}_k(i_n\}$ will be vertices $v_1(k), v_2(k)$ and $v_3(k)$ and corresponding vectors

$$v_1(k) = \begin{pmatrix} * \\ * \\ 3^{a_1+\ldots+a_k} \end{pmatrix}, v_2(k) = \begin{pmatrix} * \\ * \\ 3^{a_1+\ldots+a_k} \end{pmatrix}, v_3(k) = \begin{pmatrix} * \\ * \\ 3^{a_1+\ldots+a_k} \end{pmatrix},$$

where the other entries for the vectors are nonnegative integers. There are, of course, matrices $\tilde{M}_n$ that map the vertices from a given level to the vertices of the next level, in analogue to the matrices M_n. The $\tilde{M}_n$ are products of matrices of the form:

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

depending if we are in case I, II or III, respectively.

Note that at each individual step of the barycentric partitioning, we are cutting the area down by a factor of a 1/3. This leads to the following theorem.

Theorem 12 *Twice the area of* $\tilde{\triangle}\{\tilde{a}_1(i_1), \tilde{a}_2(i_2), \ldots, \tilde{a}_k(i_k)\}$ *is* $\frac{1}{3^{\tilde{a}_1+\tilde{a}_2+\ldots+\tilde{a}_k(i_k)}}$.

5.1 Ternary Periodicity implies rationality

Suppose that we have a point $(a, b) \in \triangle_B$ for which the barycentric partitioning is eventually periodic. We want to show that both a and b are rational numbers. That is, we want the following theorem.

Theorem 13 *If* $(a, b) \in \triangle_B$ *has an eventually perdiodic Barycentric sequence, then both* a *and* b *are rational.*

Proof: This proof is almost exactly the same as the corresponding proof for the Farey case, whose notation we adopt. There is one significant difference, namely that the matrices whose columns yield the vertices of the barycentric partitioning are all multiples of a stochastic matrix. This means that each matrix is a multiple of a matrix whose columns add to one. If the columns add to one, then it can easily be shown that the limit of the products of such a matrix converges to a matrix whose rows are multiples of $(1, 1, 1)$ (see chapter six in [26]). Thus the matrices $A, A^2, A^3, \ldots$ converge to a matrix whose rows are multiples of $(1, 1, 1)$. Since everything in sight is rational, we can show that $B^{-1}(1, \alpha, \beta)^T$ will converge to a triple of rational numbers. Since the entries of B are integers, this yields that α and β are rational numbers. $\square$

6 The Farey-Bary Analog of Singularness

The original Minkowksi ?(x) function is singular, meaning that even though it is increasing and continuous, it has derivative zero almost everywhere. The key to the proof lies in showing that at almost all points

$$\liminf_{k\to\infty} \frac{\text{length of interval in range}}{\text{length of interval in domain}} = 0,$$

for appropriately defined intervals. We will show a direct analog of this, where the lengths of intervals are replaced by areas of triangles. Thus we will show

$$\liminf_{k\to\infty} \frac{\text{area of subtriangle in range}}{\text{area of subtriangle in domain}} = 0,$$

again for appropriately defined subtriangles.

6.1 Almost everywhere $\lim\sup \frac{a_1+\ldots a_n}{n} = \infty$

This is the most technically difficult section of the paper. The goal is to show the following theorem, which will be critical in the next section. Recall that given any point $(\alpha, \beta) \in \triangle$, we have associated a sequence $\{a_1, a_2, \ldots\}$ of positive integers. We want to show that this sequence must increase to infinity, in some sense, almost everywhere. The precise statement is:

Theorem 14 *The set of* $(\alpha, \beta) \in \triangle$ *for which*

$$\limsup_{n\to\infty} \frac{a_1 + \ldots + a_n}{n} < \infty$$

has measure zero.

As it will only be apparent in the next section why to we need this theorem, we recommend on the first reading of this paper to go to the next section first.

Before proving the theorem, we need a preliminary lemma. First, let

$$v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, v_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

and let

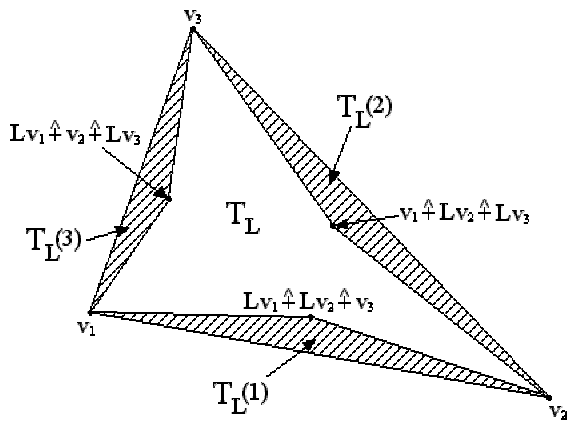
$$T = \langle v_1, v_2, v_3 \rangle$$

be the corresponding triangle in the plane, where the v_i are now viewed as points in the plane.

Suppose that $\det v_1 v_2 v_3 = 1$. Then we know that

$$2 \cdot \text{area of} \langle v_1, v_2, v_3 \rangle = \frac{1}{z_1 z_2 z_3}.$$

Given a positive number $L > 1$, define $T_L(1)$ to be the triangle with vertices $\hat{v}_1, \hat{v}_2$ and $Lv_1 \hat{+} Lv_2 \hat{+} v_3$, $T_L(2)$ the triangle with vertices $\hat{v}_2, v_1 \hat{+} Lv_2 \hat{+} Lv_3$ and $\hat{v}_3$ and $T_L(3)$ the triangle with vertices $\hat{v}_1, Lv_1 \hat{+} v_2 \hat{+} Lv_3$ and $\hat{v}_3$.



Define

$$T_L = T - T_L(1) - T_L(2) - T_L(3).$$

We now state and then prove a lemma that is the technical heart of the proof of the theorem:

Lemma 15 *For all $L \geq 1$,*

$$area(T_L) \leq \frac{L-1}{L} area(T).$$

Proof of Lemma: We know that

$$2 \cdot \text{area}(T) = \frac{1}{z_1 z_2 z_3}.$$

For ease of notation, we set $z_1 = x, z_2 = y, z_3 = z$. Then

$$2 \cdot \text{area}(T_L) = \frac{1}{xyz} - \frac{1}{x(Lx+y+Lz)z} - \frac{1}{xy(Lx+Ly+z)} - \frac{1}{(x+Ly+Lz)yz}$$

$$= \frac{1}{xyz}[1 - \frac{y}{(Lx+y+Lz)} - \frac{z}{(Lx+Ly+z)} - \frac{x}{(x+Ly+Lz)}].$$

Thus we must show that

$$[1 - \frac{y}{(Lx+y+Lz)} - \frac{z}{(Lx+Ly+z)} - \frac{x}{(x+Ly+Lz)}] \leq \frac{L-1}{L}.$$

Setting

$$\alpha = x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2,$$

we have that

$$[1 - \frac{y}{(Lx+y+Lz)} - \frac{z}{(Lx+Ly+z)} - \frac{x}{(x+Ly+Lz)}]$$
$$= \frac{\alpha(L^3-L) + 2xyz(L^3-1)}{(Lx+Ly+z)(Lx+y+Lz)(x+Ly+Lz)}.$$

After a series of calculations, we get that this is equal to:

$$(L-1)[\frac{L(L+1)\alpha + 2xyz(L^2+L+1)}{L(L^2+L+1)\alpha + L^2(x^3+y^3+z^3) + xyz(3L^2+2L^3+1)}].$$

Thus we must show that

$$[\frac{L(L+1)\alpha + 2xyz(L^2+L+1)}{L(L^2+L+1)\alpha + L^2(x^3+y^3+z^3) + xyz(3L^2+2L^3+1)}] \leq \frac{1}{L},$$

which is equivalent to showing that

$$L^2(L+1)\alpha + 2xyzL(L^2+L+1) \leq$$

$$L(L^2+L+1)\alpha + L^2(x^3+y^3+z^3) + xyz(2L^3+3L^2+1),$$

which in turn, reduces to showing that

$$2Lxyz \leq \alpha + L^2(x^3+y^3+z^3) + L^2xyz + xyz.$$

This last inequality follows from the fact that $L \geq 1$. Thus the proof of the lemma is done.

Proof of Theorem: For each positive integer N, set

$$M_N = \{(\alpha, \beta) \in \triangle : \text{for all } n \geq 1, \frac{a_1 + \ldots + a_n}{n} \leq N\}.$$

We will show that

$$\text{measure}(M_N) = 0.$$

Since the union of all of the M_N is the set we want to show has measure zero, we will be done.

Now, $\frac{a_1+\ldots+a_n}{n} \leq N$ if and only if

$$a_1 + \ldots + a_n \leq nN.$$

Since each $a_i \geq 1$, this last inequality implies

$$n - 1 + a_n \leq nN$$

or

$$a_n \leq n(N-1) + 1.$$

Set

$$\tilde{M}_N = \{(\alpha, \beta) \in \triangle : \text{for all } n \geq 1, a_n \leq n(N-1) + 1\}.$$

Since $M_N \subset \tilde{M}_N$, if we can show that measure$(\tilde{M})_N = 0$, we will be done.

Set

$$\tilde{M}_N(1) = \{(\alpha, \beta) \in \triangle : a_1 \leq (N-1) + 1\}$$

and in general

$$\tilde{M}_N(k) = \{(\alpha, \beta) \in \tilde{M}_N(k-1) : a_k \leq k(N-1)+1\}$$

Then we have a decreasing nested sequence of sets with

$$\tilde{M}_N = \bigcap_{k=1}^{\infty} \tilde{M}_N(k).$$

But this puts us into the language of the above lemma. Letting $L = k(N-1)+1$, we can conclude that

$$\text{measure}(\tilde{M}_N(k)) \leq \frac{k(N-1)}{k(N-1)+1} \text{measure}(\tilde{M}_N(k-1))$$

and hence

$$\text{measure}(\tilde{M}_N) \leq \prod_{k=2}^{\infty} \frac{k(N-1)}{k(N-1)+1}.$$

We must show this infinite product is zero, which is equivalent to showing that its reciprical

$$\prod_{k=2}^{\infty} \frac{k(N-1)+1}{k(N-1)} = \prod_{k=2}^{\infty} (1 + \frac{1}{k(N-1)}) = \infty.$$

Taking logarithms, this is the same as showing that the series

$$\sum_{k=2}^{\infty} \log(1 + \frac{1}{k(N-1)}) = \infty.$$

This in turn follows since, for large enough k, we have

$$\log(1 + \frac{1}{k(N-1)}) \geq \frac{1}{2k(N-1)}.$$

We are done.

6.2 Almost everywhere $\liminf(\textbf{area}(\tilde{\Delta}_n)/\textbf{area}(\Delta_n)) = 0$

The goal of this section, and for the entire paper, is:

Theorem 16 *For any point $(\alpha, \beta) \in \triangle$, off of a set of measure zero,*

$$\liminf_{n\to\infty} \frac{area(\tilde{\triangle}\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\})}{area(\triangle\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\})} = 0.$$

This is capturing the intuition that the determinant of the Jacobian of the map $\delta : \triangle_F \to \triangle_B$ is zero almost everywhere, which in turn is a direct generalization that the Minkowski question-mark function is singular. In fact, our proof is in spirit a generalization of Viader, Paradis and Bibiloni's work in [43].

Proof: We know that, letting $s_n = a_1 + \ldots a_n$,

$$\text{area}(\tilde{\triangle}\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\}) = \frac{1}{2 \cdot 3^{s_n}}$$

and that

$$\text{area}(\triangle\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\}) = \frac{1}{2 \cdot r_1(n) r_2(n) r_3(n)}.$$

Thus we want to show that, almost everywhere,

$$\liminf_{n\to\infty} \frac{r_1(n) r_2(n) r_3(n)}{3^{s_n}} = 0.$$

We know that $r_3(n)$ is

$$a_n r_1(n-1) + a_n r_2(n-1) + r_3(n-1),$$

$$a_n r_1(n-1) + r_2(n-1) + a_n r_3(n-1),$$

or

$$r_1(n-1) + a_n r_2(n-1) + a_n r_3(n-1).$$

Thus we have, by the convention of our notation,

$$r_1(n) \leq r_2(n) \leq r_3(n) \leq (2a_n + 1) r_3(n-1).$$

By iterating this inequality, we have

$$r_1(n) \leq r_2(n) \leq r_3(n) \leq \prod_{i=1}^{n} (2a_j + 1).$$

Thus

$$\frac{\text{area}(\tilde{\triangle}\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\})}{\text{area}(\triangle\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\})} \leq \frac{\prod_{i=1}^{n}(2a_j+1)^3}{3^{s_n}}.$$

By the arithmetic-geometric mean,

$$\prod_{i=1}^{n}(1 + b_i) \leq (1 + \frac{b_1 + \ldots b_n}{n})^n.$$

Setting $b_j = 2a_j$, we get

$$\begin{aligned}
\frac{\text{area}(\tilde{\triangle}\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\})}{\text{area}(\triangle\{a_1(i_1), a_2(i_2), \ldots, a_n(i_n)\})} &\leq \frac{(1 + \frac{2s_n}{n})^{3n}}{3^{s_n}} \\
&\leq \frac{(\frac{3s_n}{n})^{3n}}{3^{s_n}} \\
&\leq (\frac{27 \cdot (\frac{s_n}{n})^3}{3^{s_n/n}})^n.
\end{aligned}$$

From the previous section, we know that $s_n/n \to \infty$, almost everywhere. Since the above denominator has a $3^{s_n/n}$ term while the numerator only has a $(s_n/n)^3$ term, the entire ratio must approach zero, giving us our result.

7 Questions

There are a number of natural questions. First, all of this can almost certainly be generalized to higher dimensions.

More importantly, how much does the function theory of δ influence the diophantine properties of points in $\triangle$?

There are many multi-dimensional continued fraction algorithms. For any of these that involve partitioning a given triangle into three new subtriangles, a map analogous to our δ can of course be defined. What are the properties of these new maps?

Underlying most work on multidimensional continued fractions, though frequently hidden behind view, are Lie theoretic properties of the special linear group. Can this be made more explicit?

Finally, the initial Hermite problem remains open.

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