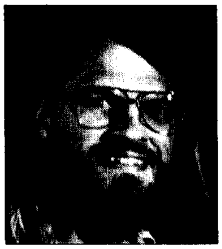

Once you see the vector/matrix generalization, what was a problem before becomes a thing of beauty. So this time I'm going to discuss what's really going on with homogeneous coordinates.

Uppers and Downers

James F. Blinn
California Institute of Technology

Tenser, said the tensor.
Tenser, said the tensor.
Tension, apprehension,
And dissension have begun.
—Alfred Bester, *The Demolished Man*

Mathematical research is largely a process of successive generalization. Generalizing the square root operation to negative values leads to complex number theory. Generalizing Euclidean transformations to perspective transformations leads to projective geometry and homogeneous coordinates. Often stuff we use in day to day calculations is just a special case of some more general theory that we aren't even aware of. In fact, sometimes this can lead to confusion—an example of "too little knowledge is a dangerous thing." This is the case for the standard vector/matrix formulation used to solve geometric problems in homogeneous coordinates. But once you see the generalization, what was a problem before now becomes a thing of beauty. So this time I'm going to discuss what's *really* going on with homogeneous coordinates.

This involves adopting a new notational scheme for vectors and matrices. The whole story is a bit lengthy, so I'm going to split it over two columns (the second column will appear in a later issue of *CG&A*). This column is largely a review of the old notation, an exposé of its deficiencies and an introduction of the new notation. In a later column, I'll show how the new notation solves some complex problems easily, and I'll discuss an interesting representation based on graph theory.

But first, some ground rules. We're dealing with pure projective geometry and homogeneous coordinates here. That means that concepts such as distance, angle measurement, and parallelism are meaningless. We are interested in the things that remain constant after a perspective transformation: intersections, tangency, and so forth. Also, any nonzero scalar multiple of any of our entities still represents the same entity. Sometimes I will discard troublesome scales without much comment.

Naming conventions

Since this is all about notation, let's review some typical conventions for naming things to give you a taste of the things we are trying to unify.

We'll start with the distinction between scalars, vectors, and matrices. For the time being, I will write scalars, vector components, and matrix elements using italic letters: a, p_i, T_{ij}. I'll use Roman letters for vectors: P. Different vectors of the same type will sometimes be distinguished with subscripts: P_1, P_2. I'll use boldface letters for matrices: **T**. We see right away that there is some potential for confusion between the meaning of subscripts; they identify vector elements or name different vectors. This is one of the things we want to avoid when we get to our improved technique.

Often we will need to represent vectors or matrices that are simple modifications of other vectors or matrices. I'll represent these by using the letter of the original with some diacritical mark appended. (The definitions of some of these operations are given below.) A transformed version of a vector P is P′. The adjoint of matrix **M** is written $\mathbf{M}^*$. The transpose of matrix **M** is $\mathbf{M}^t$. The dual of a matrix **M** is $\tilde{\mathbf{M}}$.

2DH review

First, let's go through the standard litany of homogeneous representation in two dimensions. We represent 2D homogeneous entities (2DH) in three-dimensional space.

Points and lines

Points are three-element row vectors:

$$P = (x, y, w)$$

Lines are three-element column vectors:

$$L = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The matrix (or dot) product of the row and column vector is a scalar:

$$P \cdot L = ax + by + cw$$

If the value is zero, the point lies on the line.

Transformations

We transform points by multiplying on the right by a 3 × 3 matrix:

$$P' = P\mathbf{T}$$

Transform lines by multiplying on the left by the adjoint of the matrix we used to transform points:

$$L' = \mathbf{T}^* L$$

The adjoint is the transpose of the matrix of cofactors of the original matrix. An example element of the adjoint of **T** is

$$(T^*)_{23} = T_{21}T_{13} - T_{11}T_{23}$$

The adjoint is the same as the inverse except for a scale factor. But we don't care about scale factors, so the adjoint and inverse are all the same to us.

Intersections

To find the line containing two given points, take their 3D cross product and write the result as a column vector.

$$P_1 \times P_2 = (x_1, y_1, w_1) \times (x_2, y_2, w_2) = \begin{bmatrix} y_1w_2 - y_2w_1 \\ x_2w_1 - x_1w_2 \\ x_1y_2 - x_2y_1 \end{bmatrix}$$

To find the intersection of two lines, take their 3D cross product and write the result as a row vector.

$$L_1 \times L_2 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \times \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

$$= [b_1c_2 - b_2c_1, a_2c_1 - a_1c_2, a_1b_2 - a_2b_1]$$

Quadrics

Now for something a little less well known. A second order algebraic equation such as

$$Ax^2 + 2Bxy + 2Cxw + Dy^2 + 2Eyw + Fw^2 = 0$$

represents an arbitrary conic section (also called a quadric curve). We can write this equation in matrix form as

$$[x, y, w] \begin{bmatrix} A & B & C \\ B & D & E \\ C & D & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = [x, y, w]\mathbf{Q} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

That is, a point P is on the conic if

$$P\mathbf{Q}P^t = 0$$

Note that the matrix **Q** is symmetric.

There is a variant of **Q** that is good for testing line tangency. We'll call this the "dual" of **Q** and write it as $\tilde{\mathbf{Q}}$. It so happens that the dual of **Q** is equal to the adjoint of **Q**. A line L is tangent to the conic if

$$L^t\tilde{\mathbf{Q}}L = 0$$

Note that we had to put in a few transposes in the above two equations to make the row and column conformability rules of matrix multiplication work out.

Starting from an arbitrary point, we can draw two lines tangent to a given quadric. Connecting these two points of tangency gives a line called the "polar line." The vector for this line is just the product of the quadric matrix and the point vector.

$$P\mathbf{Q} = L^t$$

What you get out, according to the rules of matrix multiplication, is a row vector. You have to transpose it to get the line into the column-vector notation.

To transform a conic section, we transform the **Q** matrix by

$$\mathbf{Q}' = \mathbf{T}^*\mathbf{Q}(\mathbf{T}^*)^t$$

that is, by pre- and post-multiplying by the adjoint of the point-transformation matrix **T**. To transform the dual of **Q** we must multiply by the point transformation matrix

$$(\mathbf{Q}^*)' = \mathbf{T}^t(\mathbf{Q}^*)\mathbf{T}$$

Oops

Now wait a minute. There's something fishy here. I've been preaching all along that points are row vectors and lines are column vectors. But in the above equations, points show up sometimes as columns, and lines show up sometimes as rows. It gets even worse when we go to three dimensions.

3DH review

Now, let's generalize this to 3D homogeneous.

The obvious part

Most of the 2DH stuff generalizes pretty easily to three dimensions: just make the vectors four elements long and make the matrices 4×4. You then have to reinterpret the geometric meaning of things a bit. What was a line in 2DH (column vector) is now a plane in 3DH. I typically use the letter E for planes.

$$E = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

What was a conic section in 2DH (symmetric matrix) is a family of 3DH surfaces consisting of ellipsoids, cones, cylinders, saddle points, and the like. Tangency of lines with a conic section becomes tangency of planes against the above surfaces.

The only slightly tricky part in going to 3DH involves the four-dimensional generalization of the cross product. Geometrically this is the problem of finding a plane passing through three points. Now we can write the three-dimensional cross product of P_1 and P_2 as

$$P_1 \times P_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where

$$a = \det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix}$$

with similar expressions for b and c. By analogy, the 4D cross product of P_1, P_2, and P_3 is a column vector with an a component of

$$a = \det\begin{bmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{bmatrix}$$

with similar expressions for b, c, and d.

A symmetric formulation finds a point common to the three planes.

Lines

Again, less well known is a homogeneous formulation for lines in three dimensions. I babbled about this in a Siggraph paper in 1977 (*Computer Graphics* (Proc. Siggraph), Vol. 11, No. 2, 1977, page 237). Here are the highlights.

We represent a 3DH line as an asymmetric 4×4 matrix. Giving the six unique elements of the matrix the names p, q, r, s, t, and u, we can write the matrix as

$$\mathbf{L} = \begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix}$$

Given two points P_1 and P_2, you can calculate tne values of p, q, r, s, t, u for the line connecting them by

$$p = \det\begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix}, q = \det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix}, r = \det\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}$$

$$s = \det\begin{bmatrix} x_1 & w_1 \\ x_2 & w_2 \end{bmatrix}, t = \det\begin{bmatrix} x_1 & z_1 \\ x_2 & z_2 \end{bmatrix}, u = \det\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

Given two planes E_1 and E_2, you can calculate p, q, r, s, t, u for their intersection line by using a similar set of expressions. It turns out that these calculations will always generate a singular matrix for $\mathbf{L}$. Working out the determinant of the matrix gives $(pu - qt + sr)^2$. This means that the components of the line matrix will always satisfy the constraint

$$pu - qt + sr = 0$$

A given point $[x, y, z, w]$ lies on the line $\mathbf{L}$ if their vector/matrix product gives four zeros

$$[x, y, z, w]\mathbf{L} = [0, 0, 0, 0]$$

If the point is not on the line, the two of them together determine a plane in space. The four numbers you get out of the product will be the components of the plane.

$$[x, y, z, w]\mathbf{L} = [a, b, c, d]$$

You just have to transpose the result to get it to be a column vector.

There is also a different form of the line matrix that is good for intersections with planes. We'll call this $\tilde{\mathbf{L}}$. This consists of the same six values as $\mathbf{L}$ but arranged differently

$$\tilde{\mathbf{L}} = \begin{bmatrix} 0 & -u & -t & -s \\ u & 0 & -r & -q \\ t & r & 0 & -p \\ s & q & p & 0 \end{bmatrix}$$

A given plane includes the line $\mathbf{L}$ if the vector/matrix product with the $\tilde{\mathbf{L}}$ form gives four zeros. If it doesn't, the four values give the point of intersection of the line and plane.

$$\tilde{\mathbf{L}}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Again you must rewrite the result as a row vector.

It so happens that $\tilde{\mathbf{L}}$ is *almost* the adjoint of **L**. In fact, if you go through the adjoint calculation machinery, you find that

$$\mathbf{L}^* = (pu - qt + sr)\tilde{\mathbf{L}}$$

If it weren't for the embarrassing fact that $pu - qt + sr = 0$, we'd be in business. As it is, we find the dual by just rearranging the elements.

You transform a line matrix by

$$\mathbf{L}' = \mathbf{T}^*\mathbf{L}(\mathbf{T}^*)^t$$

You transform its dual form by

$$\tilde{\mathbf{L}}' = \mathbf{T}^t\tilde{\mathbf{L}}\mathbf{T}$$

Yikes!

So what's going on here? Our whole concept of row and column vectors' distinguishing between points and planes is crumbling! And well it should. It turns out that the somewhat pictorial matrix representation of all these geometric entities is simply not powerful enough to express all the things that can happen.

The solution

In order to do this right, we have to borrow some notation from the world of tensor analysis. It turns out that physicists have been faced with this sort of thing for some time. They are concerned with somewhat different problems than we are here, but their notation is readily adaptable. Let's build up to this gradually.

We recognize that there are two kinds of things: point-like and plane-like. Rather than cloud our minds with such concepts as rows and columns, we'd like to identify which kind of vector something is by some other sort of notational mechanism. Hmm . . . how about writing points in green and planes in red? Well, that's a bit impractical. What to do? . . .

The physicists' solution, translated into our terms, is to write the point-like things with superscript indices (the uppers of the title) and plane-like things with subscript indices (the downers). The point-like indices are called *contravariant,* and plane-like ones are called *covariant.*

Putting the two different types of indices in two different locations keeps them distinguishable but also creates an ambiguity: superscripts used to mean exponentiation, now they are contravariant coordinate indices. Subscripts used to be available to construct different names, now they are covariant coordinate indices, and we have to use entirely different letters for different names. This ambiguity is another example of a growing problem with mathematical notation:

> There aren't enough squiggles to go around.

Anyway, we can now write our point as

$$P = (P^1, P^2, P^3, P^4)$$

and a 3DH plane as

$$E = (E_1, E_2, E_3, E_4)$$

Matrices have two indices. Each one can be either covariant or contravariant. This makes for three possibilities: pure covariant (m_{ij}), pure contravariant (m^{ij}), and what we call "mixed" (m^i_j). The inability to distinguish between these has been causing all our troubles.

One further note. The different type styles we previously needed to distinguish between scalars, vectors, and matrices are no longer necessary. Since we can now easily determine the species of creature by how many indices it has, I'll just use roman letters from now on. Anything with one index is a vector; anything with two indices is a matrix. In fact, we can now have triply indexed critters (cubical matrices?) or quadruply indexed critters. These actually have practical uses, as we will see below.

The multiplication machine

We will now represent vector and matrix multiplication in a different way. Remember, in the old style, the laws of matrix multiplication are just a shorthand notation for

$$(x, y, z, w)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = ax + by + cz + dw$$

Now, instead of using single letter names, we use index numbers for the components.

$$\mathrm{P} \cdot \mathrm{E} = (\mathrm{P}^1, \mathrm{P}^2, \mathrm{P}^3, \mathrm{P}^4)\begin{bmatrix} \mathrm{E}_1 \\ \mathrm{E}_2 \\ \mathrm{E}_3 \\ \mathrm{E}_4 \end{bmatrix}$$
$$= \mathrm{P}^1\mathrm{E}_1 + \mathrm{P}^2E_2 + \mathrm{P}^3\mathrm{E}_3 + \mathrm{P}^4\mathrm{E}_4 = \sum_i \mathrm{P}^i\mathrm{E}_i$$

In fact, we can save valuable ink by noticing that the summation occurs so often that we can declare that it is implied because the point and plane indices are the same letter. We often emphasize this by using Greek letters for indices that are summed over. Thus the product of a point and a plane is

$$\mathrm{P}^\alpha\mathrm{E}_\alpha$$

This expression is a sort of prototype of the terms that are summed over. This might take a bit of getting used to, but it's worth it. This notation is credited to Albert Einstein, who invented it to shorten his calculations for general relativity; it is therefore called *Einstein Index Notation.*

We can make all our row/column confusion go away by the following rule:

> Each index that is summed over must occur someplace in the prototype term exactly once as a covariant index and once as contravariant index. These indices "annihilate"

each other, and the resultant product has one less of each kind of index.

Indices that are *not* summed over are called *free indices*. They must occur just once in the prototype term, and the same index must occur in the resultant term. For example, the product of two matrices in the old notation is

$$\mathbf{T} = \mathbf{MN}$$

In the new notation this would be

$$T^i_j = M^\alpha_j N^i_\alpha$$

Now for a shocker:

$$M^\alpha_j N^i_\alpha = N^i_\alpha M^\alpha_j$$

But, you say, matrix multiplication is not commutative! And you are right. The above expression is not the matrix product; it is a prototype for each term in the product. Within the term you just have numbers being multiplied; that *is* commutative. So, to mix metaphors somewhat, the two orderings of multiplication are

$$\mathbf{MN} = M^\alpha_j N^i_\alpha = N^i_\alpha M^\alpha_j$$

and

$$\mathbf{NM} = N^\alpha_j M^i_\alpha = M^i_\alpha N^\alpha_j$$

The new order

Let's now reinterpret all of the above confusion in terms of this notation. First let's do the obvious stuff.

Points and Planes

A point has a single contravariant index

$$P^i$$

A 2DH line or 3DH plane has a single covariant index

$$E_i$$

A point times a line/plane is a scalar

$$P^\alpha E_\alpha = s$$

If it's zero, the point lies on the line.

Quadrics

A quadric curve or surface has a pure covariant form for dealing with points. Note that both the points have their accustomed contravariant indices.

$$P^\alpha Q_{\alpha\beta} P^\beta = 0$$

The polar line to a quadric curve and the polar plane to a quadric surface is

$$E_i = P^\alpha Q_{\alpha i}$$

Note again the delightful consistency of indices.

The dual form of the quadric for testing line/plane tangency is pure contravariant:

$$E_\alpha \tilde{Q}^{\alpha\beta} E_\beta = 0$$

Both line/planes have their accustomed covariant indices. The tilde over the Q is kind of redundant since the placement of the indices tells us everything, covariant for the normal form and contravariant for the dual form. We won't bother with tildes any more; yet more cleanups.

This notation is credited to Albert Einstein, who invented it to shorten his calculations for general relativity; it is therefore called Einstein Index Notation.

The contravariant form of Q is the adjoint of the covariant form. This leads to another convenient rule that we will expand upon later. For now I'll just state

> Taking an adjoint flips the type of its indices.

This means that the adjoint of a mixed tensor (which is a transformation matrix) is also a mixed tensor. If the only type of matrix you ever encountered was a transformation matrix, you wouldn't know there was such a thing as covariant or contravariant tensors.

3DH lines

A 3DH line is a pure covariant tensor or a pure contravariant tensor. The plane containing the line and a point P uses the covariant form.

$$P^\alpha L_{\alpha j} = E_j$$

The point of intersection of the line and a plane E uses the contravariant form.

$$L^{\alpha i} E_\alpha = P^i$$

Transformations

A transformation matrix is a mixed tensor. It can transform a point:

$$(P')^i = P^\alpha T^i_\alpha$$

or it can transform a line (2DH) or a plane (3DH):

$$(E')_i = T_i^{\alpha} E_{\alpha}$$

The covariant form of Q transforms like

$$(Q')_{ij} = (T^*)_i^{\alpha} Q_{\alpha\beta} (T^*)_j^{\beta}$$

Notice that we no longer need to use the superscript t to express transpose; the relevant indices are just swapped. Yet more economization. Explicit notation for adjoints *is* still necessary for now.

The contravariant form of Q transforms using T:

$$(Q')^{ij} = T_{\alpha}^{i} Q^{\alpha\beta} T_{\beta}^{j}$$

Likewise, the covariant form of the 3DH line L transforms like

$$(L')_{ij} = (T^*)_i^{\alpha} L_{\alpha\beta} (T^*)_j^{\beta}$$

The contravariant form transforms like T:

$$(L')^{ij} = T_{\alpha}^{i} L^{\alpha\beta} T_{\beta}^{j}$$

This is an example of the general transformation rule:

> To transform something, multiply in a T for each contravariant (point-like) index and a T^* for each covariant (plane-like) index.

In fact, all these things are called tensors precisely because they transform according to this rule.

There is an interesting consequence of this. In order to transform a *transformation matrix,* we must multiply both by T and T^*.

$$(M')_j^{i} = (T^*)_j^{\beta} M_{\beta}^{\alpha} T_{\alpha}^{i}$$

This is a nice way of representing the standard trick of scaling about an arbitrary point by transforming the point to the origin (T^*), scaling about the origin (M), and transforming back (T).

The magic epsilon

You might have noticed that we've encountered a lot of expressions of the form

$$\det \begin{bmatrix} p_1 & p_2 \\ r_1 & r_2 \end{bmatrix}$$

There's another gimmick that the physicists have come up with that's useful to abbreviate this type of thing: the Levi-Civa epsilon. In this column I can give just a hint of the wonders in store for us when we use epsilon. To start out, let's discuss this just in 2DH (3D) terms.

The 3D (2DH) epsilon

The 3D epsilon tensor has three indices, so it looks like

$$\varepsilon_{ijk}$$

Its elements are defined to be

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$
$$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$$
$$\varepsilon_{ijk} = 0 \text{ otherwise}$$

You can visualize ε by thinking of it as a cube of numbers made by stacking up the matrices:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiplying two points P and S by epsilon gives

$$P^{\alpha} S^{\beta} \varepsilon_{\alpha\beta i} = L_i$$

a covariant vector. What is it? To find, say, the first element, let's write down all the terms containing an epsilon that is nonzero:

$$L_1 = P^{\alpha} S^{\beta} \varepsilon_{\alpha\beta 1}$$
$$= P^2 S^3 \varepsilon_{231} + P^3 S^2 \varepsilon_{321} = P^2 S^3 - P^3 S^2$$

The other elements look similar. L is, in fact, the cross product of P and S, that is, the line connecting them.

There is also a contravariant epsilon that has the same numerical values. You can use it to take cross products of lines to find their point of intersection.

$$L_{\alpha} K_{\beta} \varepsilon^{\alpha\beta i} = P^k$$

Notice that I didn't even need to tell you that K was a line. You could tell by seeing that it has a single covariant index.

The epsilon can also be used to calculate adjoints. It turns out that

$$(M^*)^{ij} = \frac{1}{2} \varepsilon^{j\alpha\beta} \varepsilon^{i\gamma\delta} M_{\alpha\gamma} M_{\beta\delta}$$

Where does the 1/2 come from? Well, as an exercise, write down all the nonzero epsilon terms implied by the summation over α, β, γ, and δ. You will find that each product of the M terms will appear twice, requiring the 1/2 to compensate. We can, however, ignore the 1/2 since we are being homogeneous.

The above is for pure covariant matrices; note how the result is contravariant. If you start with a contravariant matrix, you multiply by two covariant epsilons and get a covariant result. For a mixed matrix, you must multiply by one covariant and one contravariant epsilon: you will get a mixed result.

The 4D (3DH) epsilon

There is also a 4D epsilon that has four indices. It's defined by

$$\varepsilon_{ijkl} = 1 \text{ if } ijkl \text{ an even permutation of 1234}$$
$$\varepsilon_{ijkl} = -1 \text{ if } ijkl \text{ an odd permutation of 1234}$$
$$\varepsilon_{ijkl} = 0 \text{ otherwise}$$

Given this, we can compress a lot of the formulas for 3DH stuff.

The plane through three points is

$$E_i = P^\alpha S^\beta R^\gamma \varepsilon_{\alpha\beta\gamma i}$$

The point common to three planes uses the contravariant ε

$$P^i = E_\alpha F_\beta G_\gamma \varepsilon^{\alpha\beta\gamma i}$$

The adjoint of a 4 × 4 matrix is

$$(M^*)^{ij} = \frac{1}{6}\varepsilon^{j\alpha\beta\gamma}\varepsilon^{iabc} M_{\alpha a} M_{\beta b} M_{\gamma c}$$

Again, this is for a covariant matrix. Contravariant and mixed matrices require the same treatment as for 2DH.

The 3DH line through two points P and R is the covariant matrix

$$L_{ij} = P^\alpha R^\beta \varepsilon_{\alpha\beta ij}$$

The plane containing this line and another point S is

$$E_i = S^\alpha L_{\alpha i}$$

You find the contravariant version of a line by intersecting the two planes E and F:

$$L^{ij} = E_\alpha F_\beta \varepsilon^{\alpha\beta ij}$$

The point of intersection of this line with another plane G is

$$P^i = L^{i\alpha} G_\alpha$$

The covariant and contravariant line forms are related by

$$L_{ij} = \varepsilon_{ij\alpha\beta} L^{\alpha\beta}$$

Admittedly, implementing some of the above calculations by explicitly multiplying by epsilon is a bit idiotic. You wind up multiplying by a whole lot of zeroes and ones. But the epsilon notation is good as a bookkeeping convenience. I'll describe even more uses for it in a later column.

So what?

We just reduced all of geometry to tensor multiplication (well, almost all). And there are no embarrassing transposes. Row-ness or column-ness is superseded by the more general concept of covariant contravariant indices. Plus we can feel really cool by sharing notation with General Relativity. ❑

Acknowledgment

Thanks to Al Barr for several of the formulas here.