mine the number of independent elements which go to make up the terms in the final summation. It is not correct that if sampling is done faster than $2W$ the analysis in the paper does not apply, because there is no concept of sampling in the paper at all.

The last point covered by Pridham is, I think, such that no extended reply is necessary. Of course, the signal energy to which I refer is the energy contained in the sample applied to the energy detector. This is the only energy which has any meaning in the paper. Thus the statement that an increase in the time-bandwidth product of the sample requires an increase in the signal component of the energy in the sample should not cause any confusion.

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On Real Eigenvalues of Real Nonsymmetric Matrices

Abstract—**Sufficient conditions for a real nonsymmetric matrix either to have all real eigenvalues or to be totally devoid of any real eigenvalue are asserted. The sufficient conditions stated can be easily and quickly implemented, and do not require adoption of any numerical technique.**

It is well known that all the eigenvalues of a real symmetric matrix are real, and that some real nonsymmetric matrices also have all real eigenvalues. Given a real nonsymmetric matrix, however, it is impossible to tell whether or not all the eigenvalues are real without actually solving for the roots of the characteristic equation—a formidable task when the order of the matrix is high. The impossibility of algebraically resolving equations unrestricted in form, of a degree higher than fourth, leaves one with the numerical methods available to determine the realness of eigenvalues of an arbitrary real nonsymmetric matrix of order greater than four.[1] The numerical methods are necessarily approximate and usually cumbersome to implement. In this letter, the feasibility of attacking such problems primarily from the realizability-theoretic viewpoint is discussed.

First, use is made of an established result in realizability theory, due to Reza,[2] quoted here in the form of a lemma, to arrive at sufficient conditions for the realness of the eigenvalues of a real nonsymmetric matrix.

Reza's Lemma

The "logarithmic derivative" $P'(s)/P(s)$, where $P'(s)=dP(s)/ds$, of any polynomial $P(s)$ with roots restricted to the nonpositive real axis represents the driving-point impedance of an RC network.

Reza's lemma immediately leads to the following assertion.

Assertion 1: A sufficient condition for a real nonsymmetric matrix A to have all nonpositive real eigenvalues is that

$$Z(s) = \frac{\frac{d}{ds}\left[\det(A - sI)\right]}{\det(A - sI)},$$

where I is the identity matrix of the same order as A, be an RC impedance function.

The foregoing sufficiency test can easily be implemented as any two-element (RC, RL, LC) positive real function can be realized by Cauer's continued fraction expansion technique, which does not require factorization of the polynomial. It is noted that as the "logarithmic derivative" of any real polynomial with roots restricted to the axis of real frequencies represents the driving-point impedance of an LC network,[2] a result similar to the above can be arrived at for a real nonsymmetric matrix to have all imaginary eigenvalues. Again, the fact that the "logarithmic derivative" of any Hurwitz polynomial is a positive real function[2] implies that a sufficient condition for a real nonsymmetric matrix B to have all eigenvalues with nonpositive real parts is that the function,

$$Z_1(s) = \frac{\frac{d}{ds}\left[\det(B - sI)\right]}{\det(B - sI)},$$

be positive real.

However, this test is more difficult to apply, in general, as no simple realization technique analogous to the Cauer continued fraction expansion procedure exists for RLC networks. Henceforth, attention will be centered around the real eigenvalues of real nonsymmetric matrices. The first assertion can be extended to lead to the second.

Assertion 2: A sufficient condition for the eigenvalues of a real nonsymmetric matrix to be all real is that there exist a real number $h>0$, such that

$$\frac{\frac{d}{ds}\left[\det(A - (s + h)I)\right]}{\left[\det(A - (s + h)I)\right]}$$

is an RC impedance function.

Assertion 2 follows from Reza's lemma after translation of the zeros of a polynomial $P(s)$ to the nonpositive real axis by a real number h, selected to be larger than the largest positive real root of $P(s)$.

Next, attention is directed toward establishing a sufficient condition for the absence of any real eigenvalue in a real nonsymmetric matrix.[3] For this, the quadratic form representation of a polynomial is taken. Suppose the characteristic equation of a given matrix A is $\det[A - sI]$. Then, $\det[A-sI]$ is either a polynomial $P_e(s)$ of even degree or a polynomial $P_o(s)$ of odd degree, having real coefficients. The general forms for $P_e(s)$ and $P_o(s)$ are

$$P_e(s) = a_{2n}s^{2n} + a_{2n-1}s^{2n-1} + \cdots + a_1 s + a_o$$

$$P_o(s) = a_{2n-1}s^{2n-1} + a_{2n-2}s^{2n-2} + \cdots + a_1 s + a_o.$$

As shown in an earlier paper,[3] if we define the column matrix X as $X=\text{col.}\left[1\ s\ s^2 \cdots s^n\right]$, then polynomials $P_e(s)$ and $P_o(s)$ can be identified as quadratic forms associated with the real symmetric tridiagonal matrices M_e and M_o, respectively:

$$M_e = \begin{bmatrix} a_0 & \frac{a_1}{2} & 0 & \cdots & 0 \\ \frac{a_1}{2} & a_2 & \frac{a_3}{2} & \cdots & 0 \\ 0 & \frac{a_3}{2} & a_4 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \frac{a_{2n-1}}{2} & a_{2n} \end{bmatrix} \qquad M_o = \begin{bmatrix} a_0 & \frac{a_1}{2} & 0 & \cdots & 0 \\ \frac{a_1}{2} & a_2 & \frac{a_3}{2} & \cdots & 0 \\ 0 & \frac{a_3}{2} & a_4 & \cdots & 0 \\ & & & & \\ a & \cdots & \cdots & \frac{a_{2n-1}}{2} & 0 \end{bmatrix},$$

$$P_e(s) = (X, M_e X) = X' M_e X$$

$$P_o(s) = (X, M_o X) = X' M_o X,$$

where the prime denotes "transpose."

The above-mentioned quadratic form representation leads to the third assertion.

Assertion 3: A sufficient condition for the real nonsymmetric matrix A to be totally devoid of real eigenvalues is that the matrix M_e (if A is of even order) or the matrix M_o (if A is of odd order) be positive definite.

The proof is simple and therefore omitted for brevity.

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[2] F. M. Reza, "Some investigations on positive real functions," Syracuse University, Syracuse, N. Y., Rept. EE, 561, 599E, September 1959.

[3] F. M. Reza and N. K. Bose, "Some links between theory of equations and reliability theory," *1967 Proc. 10th Midwest Symp. on Circuit Theory*, pp. V-2-1–V-2-11.