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Coupling time of a (max,plus) matrix

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Abstract: In this note, we give a bound on the coupling time of an irreducible (max,plus) linear system $X(n) = A \otimes X(n-1)$ with its pseudo-periodic regime. This bounds uses a decomposition of the matrix A into its critical part A_c and its non-critical part B, as well as the maximal average weight of a circuit in B, which could be seen as the counterpart of the second largest eigenvalue in the classical linear case.

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Temps de couplage d'une matrice (max,plus)

Résumé : Dans cette note, nous donnons une borne sur le temps de couplage d'un système (max,plus) linéaire irréductible $X(n) = A \otimes X(n-1)$ avec son régime pseudo-périodique. Cette borne utilise une décomposition de la matrice A en sa partie critique A_c et sa partie non-critique B, ainsi que le poids moyen maximal d'un circuit de B, ce qui peut être vu comme l'analogue dans (max,plus) de la seconde plus grande valeur propre dans le cas linéaire classique.

Mots-clés : (max,plus) algebra, coupling time

1 Introduction

In linear algebra, when dealing with positive matrices, the Perron-Frobenius theorem states that if A is a non-negative primitive matrix, there exists a real eigenvalue λ_1 with algebraic as well as geometric multiplicity 1 such that $\lambda_1 > 0$ and $|\lambda_1| > |\lambda_j|$ for any other eigenvalue λ_j. Moreover, the left and right eigenvectors u_1 and v_1 associated with λ_1 can be chosen positive and such that $u_1^T v_1 = 1$. Let $\lambda_2, \cdots \lambda_r$ be the other eigenvalues ordered such that $\lambda_1 > |\lambda_2| \geq \cdots \geq |\lambda_r|$. Then if m_2 is the multiplicity of λ_2,

$$A^n = \lambda_1^n v_1 u_1^T + O(n^{m_2-1}|\lambda_2|^n).$$

If A is a stochastic matrix that is irreducible and aperiodic, then the above equation becomes $A^n = \mathbf{1}\pi^T + O(n^{m_2-1}|\lambda_2|^n)$, where π is the stationary measure of a Markov chain with transition matrix A. The term $n^{m_2-1}|\lambda_2|^n$ can be seen as the rate of convergence to the steady state (see [3] for a detailed development of the computational issues related to that problem).

The (max,plus) semi-ring is the set $\mathbb{R}_{max} \stackrel{\text{def}}{=} \mathbb{R} \cup \{-\infty\}$ equipped with the operations max denoted $\oplus$ and $+$ denoted $\otimes$. Similarly to the classical linear algebra, one can define matrices with coefficients in $\mathbb{R}_{max}$. The "sum" of two matrices of adequate sizes is defined by $(A \oplus B)_{ij} \stackrel{\text{def}}{=} A_{ij} \oplus B_{ij}$ and the product is $(A \otimes B)_{ij} \stackrel{\text{def}}{=} \oplus_k A_{ik} \otimes B_{kj}$. For $n \in \mathbb{N}$, $A^{\otimes n} \stackrel{\text{def}}{=} A \otimes \cdots \otimes A$ (n times). It is natural to study the asymptotic behavior of the sequence $\{A^{\otimes n}\}_{n\in\mathbb{N}}$ when n goes to infinity.

The analogy between (max,plus) matrices and positive matrices is rather deep since one can construct a more general framework [9] unifying the Perron-Frobenius theorem and the (max,plus) theorem (presented in Section 3). However, there exists substantial differences. As stated in the (max,plus) theorem, for (max,plus) matrices,

- coupling occurs in finite time, and
- an irreducible matrix has a single eigenvalue.

In spite of these differences, we will show that the general property that the coupling time has to do with the second eigenvalue has a counterpart for (max,plus) matrices, by considering the simple circuit containing no critical nodes and with the largest possible average weight.

2 Preliminaries

If A is a (max,plus) matrix of size N. We define

$$A^* \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} A^{\otimes n}.$$

We will denote by $G(A)$ (or simply G when no confusion is possible) the graph with N nodes associated with A. The weight of the arc (i, j) in G is A_{ij}. The weight of a path P of G, denoted $w(P)$, is the sum of the weights of its arcs. The length of P denoted by $\ell(P)$ is the number of arcs contained in P. Note that A^*_{ij} is the maximal weight (possibly infinite) of all the paths in G from i to j.

- A is *irreducible* if $G(A)$ is strongly connected.
- A path P in G is *simple* if P does not contain any circuit.
- A circuit c is *critical* if its average weight $(w(c)/\ell(c))$ is maximal among all circuits in G.
- A node is *critical* if it belongs to a critical circuit.
- A path is *critical* if all its arcs are critical.
- The *critical graph* G_c is the union of all critical circuits. It is a fact rather easy to establish that all circuits in the critical graph are critical circuits.

By a simple renumbering of the nodes in the graph, we can write matrix A by block:

$$A = \begin{pmatrix} A_c & T_1 \\ T_2 & B \end{pmatrix}, \tag{1}$$

where sub-matrix A_c corresponds to the critical nodes.

3 The (max,plus) Theorem

Pending to the Perron Frobenius theorem, there exists a fundamental theorem in the (max,plus) framework, which states that after a finite transient behavior, the powers of a (max,plus) matrix become pseudo-periodic.

Theorem 1. *Let A be an irreducible (max,plus) matrix. Then there exist $c \in \mathbb{N}^*$, (cyclicity of A), $\lambda \in \mathbb{R}$ (unique eigenvalue of A) and $n_0 \in \mathbb{N}$ (coupling time of A) such that*

$$\forall n \geq n_0 \quad A^{\otimes n+c} = \lambda c \otimes A^{\otimes n}. \tag{2}$$

λ is the average weight of the critical circuits of A and

$$c = \bigvee_{\Gamma \text{ strongly connected component of } G_c} \; \bigwedge_{\gamma \text{ circuit in } \Gamma} \ell(\gamma).$$

The proof of this theorem can be traced back to the work of Maslov [10], Cuninghame-Green [7] or the work of the Max-Plus group [6]. For a precise historical background on the (max,plus) theorem as well as for a detailed proof, see for example [1].

Several algorithms have been developed to compute λ (see [1]), such as the trace method (with doubly quadratic complexity), or the Karp algorithm [1] (with cubic complexity). Up to now, the fastest method is an adaptation of the Howard algorithm to the (max,plus) case which complexity is linear in practice [5, 4]. The cyclicity c is very close to the period of a Markov chain. The most efficient algorithm to compute it is due to Denardo [8] and has a quadratic complexity.

However, no effort seems to have been dedicated in the literature to compute n_0. The only method which exists up to the authors knowledge is to compute iteratively the powers of matrix A and check if convergence has occurred. However this simple method may have a very high cost, as shown in the following example:

Let

$$A = \begin{pmatrix} 0 & -10^x \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Then $\lambda = 0$, $c = 1$ and the coupling time is obviously $n_0 = 10^x$. It is not acceptable to compute $A^{\otimes 2}, \cdots, A^{\otimes 10^x}$ before concluding that $n_0 = 10^x$.

On the other hand, it is very doubtful that a simple close formula exists for the minimal coupling time (*i.e.* the smallest n_0 such that equation (2) is satisfied). This remark is based on the following lemma.

Lemma 1. *Computing the minimal coupling time of a (max,plus) matrix A is NP-complete (in the number of circuits of $G_c(A)$).*

Proof. The proof of NP-hardness is based on a reduction of a NP-hard problem in number theory known as the Frobenius problem.

Let $a_1 \leq \cdots \leq a_k$ be integer numbers such that $gcd(a_1, \cdots, a_k) = 1$. Then, there exists $m_0 > 0$ which is the smallest integer number such that $\forall m \geq m_0, m = \mu_1 a_1 + \cdots + \mu_k a_k$ with all μ_i non-negative.

The existence of m_0 is rather easy to establish. Bounds on m_0 have been derived (such as the bound due to Schur [2] which states that $m_0 \leq (a_1 - 1)(a_k - 1)$). However, the computation of m_0 has been shown NP-hard in [11].

The reduction of this problem to the coupling problem goes along the following lines. Consider a graph G with a_k vertices and arcs with weight 0 between nodes $(i, i+1)$ for all $i < a_k$ and between nodes $(a_j, 1)$ for all $j \leq k$. In this simple case, the minimal coupling time of the (max,plus) matrix associated to G is the smallest integer n_0 such that there exist paths from any i to any j in G of any length larger than n_0. All circuits in G correspond to integers of the form $\mu_1 a_1 + \cdots + \mu_k a_k$ with all μ_i non-negative. A path from i to j contains a circuit and a simple path from i to j. Therefore, $m_0 \leq n_0 - a_k + 1$. On the other hand, there always exists a path of length $m_0 + a_k - 1$ in the graph. Therefore, $m_0 = n_0 - a_k + 1$ and the problem of computing n_0 is also NP-hard.

As for completeness, given n_0, it is easy to check in polynomial time if $A^{\otimes n_0 - 1} \neq A^{\otimes n_0 - 1 - c} + c\lambda$ and $A^{\otimes n_0} = A^{\otimes n_0 - c} + c\lambda$. □

As mentioned before, this result leaves little hope for finding a close formula for the minimal coupling time. However, it is possible to give a close formula (with rather low complexity) for a coupling time, which will be an upper bound of the minimal coupling time. This is what we are going to do in the following. The remaining question will be how close to the minimal coupling time our result will be. We will address this question experimentally in Section 5.

4 Coupling time

We will assume in the following that the matrix A is irreducible, acyclic with eigenvalue 0.

Regarding periodicity, if the matrix A has a cyclicity equal to c, then $A^{\otimes c}$ is acyclic. The coupling time of A is bounded by the coupling time of $A^{\otimes c}$ multiplied by c.

The last assumption ($\lambda = 0$) can be made with no loss of generality, since the coupling time of A and of $A - \lambda$ are equal.

Under these assumptions, the (max,plus) theorem can be stated as: there exists $n_0 \geq 0$ such that

$$\forall n \geq n_0, A^{\otimes n} = A^{\otimes n-1}. \tag{4}$$

When $\lambda \leq 0$, note that A^* is always finite and

$$A^* = \bigoplus_{n=0}^{N} A^{\otimes n}.$$

4.1 All nodes are critical

This corresponds to the case where $A = A_c$ in Equation 1. This case is very close to the coupling of boolean matrices, as treated in [8] for example. We will provide the proofs of some well known lemmas in the sake of self containment but they are not new.

4.1.1 The critical graph is strongly connected

Lemma 2. *Under the foregoing assumptions, for all i and j there exists a path from i to j in the critical graph whose weight is maximal.*

Proof. If no path from i to j with maximal weight is critical, then by for strong connectedness, there is a circuit with positive weight in A, which is impossible. □

Lemma 3 (Denardo). *Let G be a strongly connected acyclic graph with N nodes. Assume G contains a circuit with l nodes and let i belong to that circuit. Then for all node j in G, there exists a path of length k from i to j provided $k > (N-1)l$.*

The following proof is from [8].

Proof. Let S be the following sequence

- $S(0) = \{i\}$ and
- $\forall k \geq 1$, $S(k) = \{j | \exists$ a path from i to j of length $lk\}$.

Since node i belongs to a circuit of length l, $S(k+1)$ is the union of $S(k)$ and all the circuits made of a path in $S(k)$ and a path of length l. Therefore,

$$\forall k \in \mathbb{N}, \quad S(k+1) \supseteq S(k). \tag{5}$$

S is an increasing sequence in 2^N and converges after a finite number of steps, $K \leq N-1$. Since G is aperiodic, then $S(K)$ is the set of all the nodes in G. Therefore, there exists a path from i to any node j in G of length $(N-1)l$. But there is at least one arc terminates at every node at therefore, there is a path of length $(N-1)l+1$ between i and every node j. Repeating the last argument finishes the proof. □

Lemma 4 (Denardo). *Let G be a strongly connected acyclic graph with N nodes. Let l be the length of its shortest circuit. Then, there exist paths from any node i to any node j of length t for all integer t satisfying $t \geq k_0 \stackrel{\text{def}}{=} N + (N-2)l$.*

Proof. This is simple corollary of Lemma 3 which proof is given in [8]. Let l be the length of the shortest circuit c. The shortest path from i to c has length at most $N-l$ and reaches c at some node denoted k. Then we use Lemma 3 to find paths from k to j of any length larger than $(N-1)l$. Finally, $N - l + (N-1)l = N + (N-2)l$. □

Now, let us apply this result to our case. From Lemma 2, we can focus on the critical graph and forget all the non-critical arcs in A. By assumptions, the critical graph is strongly connected and acyclic. Therefore, from Lemma 4, for any $n \geq k_0$, $A_{ij}^{\otimes n}$ is the weight of a path in the critical graph of length n from i to j which contains the maximal simple path from i to j and some circuits with weight 0. Therefore, $A^{\otimes n} = A^*$ and the system has coupled at k_0.

4.1.2 The critical graph is not strongly connected

In this case, G is made of H critical strongly connected components. They are all acyclic since the global cyclicity is the smallest common multiple of all the cyclicities.

Lemma 5. *A path with maximal weight from any node i to any other node j contains at most $H-1$ non-critical arcs.*

Proof. First, a path of maximal weight does not contains any non-critical arc included in one component. Indeed, one can find a heaviest path by replacing such an arc by a critical path within that component.

Second, if a path P contains more than H non-critical arcs, each of them connecting two critical components, then, this path contains two nodes i and j from the same component and a critical path from i to j in that component out-weights some portion of P. □

For each component h, let us denote by N_h its number of nodes, and by l_h the length of its shortest circuit.

Let us define $k'_0 \stackrel{\text{def}}{=} H - 1 + \sum_{h=1}^{H}((N_h - 2)l_h + N_h)$.

Then, there exist paths from i to j of any length t in G for all $t \geq k'_0$. Once again those paths are made of maximal simple paths from i to j and circuits with null weights. Therefore $A^{\otimes n} = A^*$ for all $n \geq k'_0$ and coupling has occurred at k'_0.

Note that k'_0 depends on the detailed decomposition of A into strongly connected component. To get a simpler bound, which only depends on the number of components and one circuit, we replace l_h by $L \stackrel{\text{def}}{=} \max_{h=1}^{H} l_h$, then $k'_0 \leq k_1 \stackrel{\text{def}}{=} N + H - 1 + (N - 2H)L$.

4.2 Some nodes are not critical

This is the case where block B in Equation (1) is not empty and may play an important role. We let $\mathcal{C}$ denote the set of critical nodes with cardinal C and let $\mathcal{B}$ denote the set of non-critical nodes with cardinal $N - C$. We denote by β the maximal (max,plus) eigenvalue of B. By definition of B, $\beta < 0$. If B does not contain any cycle, we set $\beta = -\infty$.

4.2.1 Paths through critical nodes

It is rather simple to see that there exists a path from any node i to any other node j and containing critical nodes of length smaller than $(N - C) + N - 1$.

Let us define $k_2 \stackrel{\text{def}}{=} (N - C) + N - 1 + k_1$.

Let $M_{ij} \stackrel{\text{def}}{=} \bigoplus_{k \in C} A^*_{ik} \otimes A^*_{kj}$. M_{ij} is the weight of the maximal path from i to j containing critical nodes.

Lemma 6. *For all i and j, There exits a path from i to j of weight M_{ij} and of length t for all $t \geq k_2$.*

Proof. One can build a path of length t and weight M_{ij} using the following construction. Choose a simple path from i to k, a simple path from k to j (for an adequate critical node k) which form paths of length $h_1 \leq N - C$ for the first part and $h_2 \leq N - 1$ for the second part and of total weight M_{ij}. This is possible since all circuits have non-positive weight in G. Now, to build a path all length n, we just need to build a critical circuit from k to k of length $n - h_1 - h_2 > k_1$. This is possible using Lemma 4. This circuit is critical and has a weight equal to 0. $\square$

4.2.2 Paths through non-critical nodes only

If u and v are non-critical nodes, B^*_{uv} is the weight of the maximal path from u to v which does not contain any critical nodes.

Lemma 7. $B^{\otimes N-C+k}_{uv} \leq B^*_{uv} + \beta k$ *for all $k > 0$.*

Proof. If there is no path in B from u to v of length $N-C+k$, then $B_{uv}^{\otimes N-C+k} = -\infty \leq B_{uv}^* + \beta k$.

If there exist paths in B from u to v of length $N-C+k$, such a path P is not simple and contains a simple path, of length $h < N-C$ and weight less than B_{uv}^* and circuits of average weight less than β. Therefore, the weight of P is less than $B_{uv}^* + \beta(N-C+k-h) \leq B_{uv}^* + \beta k$. □

4.2.3 Non-critical nodes versus critical nodes

Lemma 8. *Let u and v be non-critical nodes. If $k > \frac{B_{uv}^* - M_{uv}}{-\beta}$, then the paths from u to v and length $k+N-C$ have a weight smaller than M_{uv}.*

Proof. This is a direct consequence of Lemma 7. Indeed, If P is a path from u to v with length $k+N-C$, then

$$w(P) \leq B_{uv}^{\otimes k+N-C} \leq B_{uv}^* + \beta k \leq M_{uv}.$$

□

Lemma 9. *Let i and j be two arbitrary nodes. If*

$$n \geq k_3 \stackrel{\text{def}}{=} \max_{u \in \mathcal{B}} \max_{v \in \mathcal{B}} \left(\max \left(\frac{B_{uv}^* - M_{uv}}{-\beta} + N - C, k_2 \right) \right),$$

then
1)paths of length n from i to j exist.
2)The paths of maximal weight and length n have weight M_{ij}.

Proof. 1) This is a direct consequence of Lemma 6.

2) Assume P is a path from i to j of maximal weight and length n. By Lemma 8, $w(P) \leq M_{ij}$. On the other hand, Lemma 6 says that there exists paths of weight M_{ij} and length n. □

As a direct consequence of Lemma 9, for all $n \geq k_3$, $A_{ij}^{\otimes n} = M_{ij}$, so that coupling has occurred before k_3. Also note that this result not only provides a coupling time for $A^{\otimes n}$ but also gives the value of $A^{\otimes n}$ once the periodic regime is reached: $A^{\otimes n} = M$.

We rewrite the formula of k_3, under a more compact form using the sup-norm $|.|$ and by restricting matrix M to its non-critical part: M'.

$$\begin{aligned} k_3 &= \max \left(\frac{|B^* - M'|}{-\beta} + N - C, k_2 \right), \\ &= N + \max \left(\frac{|B^* - M'|}{-\beta} - C, N - 2 + H + (C - 2H)L \right). \end{aligned}$$

We summarize the result in the following theorem.

Theorem 2. *Let A be an irreducible acyclic (max,plus) matrix. Then the sequence $\{A^{\otimes n}\}_{n\in\mathbb{N}}$ couples with its periodic regime: $A^{\otimes n} = M + \lambda n$ when $n \geq n_0$, with*

$$n_0 = N + \max\left(\frac{|B^* - M'|}{-\beta} - C, N - 2 + H + (C - 2H)L\right),$$

where

- *λ is the eigenvalue of A,*
- *M is the matrix defined by $M_{ij} = \oplus_{k\in\mathcal{C}}(A-\lambda)^*_{ik} \otimes (A-\lambda)^*_{kj}$,*
- *N is the size of A,*
- *C is the number of critical nodes in $G(A)$,*
- *H is the number of strongly connected components in $G_c(A)$,*
- *L is the largest length of the shortest circuit in each strongly connected components of $G_c(A)$,*
- *B is the sub-matrix of the non-critical nodes of $A - \lambda$,*
- *M' is the sub-matrix of the non-critical nodes of M,*
- *β is the largest eigenvalue of B.*

Note that β is not the second largest average weight of simple circuits in A, since some circuits containing critical and non-critical nodes may have an average weight larger than β.

Complexity The computation of n_0 involves the computation of C which can be done in $O(N^3)$, H in $O(N^2)$, L in $O(N^3)$, B^* in $O((N-C)^3)$, M' in $O(C(N-C)^2)$ and β in $O((N-C)^3)$. Which yields a final complexity which is cubic in the size of the matrix. This is acceptable since a cubic complexity is needed to compute λ using the Karp algorithm.

Note however, that computing C, H and L is not essential since they can be bounded by 0 or N.

5 Experiments

To make computations easier, we have only considered irreducible matrices A with a null eigenvalue. Everything remains the same with arbitrary eigenvalue λ using an initial transformation (adding $-\lambda$ to all the coefficients in A).

5.1 Cases where β is not involved.

First, we consider a matrix

$$A = \begin{pmatrix} -\infty & 0 & 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -1 & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty & 0 & 0 \\ -\infty & -\infty & -\infty & -\infty & -\infty & 0 \\ -\infty & -\infty & -1 & 0 & -\infty & -\infty \end{pmatrix},$$

A is acyclic and all its nodes are critical (N = C). Therefore, β is not defined, and our bound is reduced to the second part that is $N + N - 2 + H + (N - 2H)L$ which adds up to $n_0 = 16$ using the fact that $N = 6, H = 2$ and $L = 2$. By computing iteratively the powers of A, the minimal coupling time is 7.

Actually this case is the worse case (among all the experiments we have done) for our bound, this is due to the fact that k_2 is not a tight coupling time for a critical matrix with several components. When all the nodes are critical, then k_1 is a better coupling time (here, $k_1 = 11$).

Now, we consider a matrix

$$A = \begin{pmatrix} -\infty & -2 & -\infty & -\infty \\ 2 & -\infty & 1 & 1 \\ 1 & -\infty & -\infty & -\infty \\ 1-x & -\infty & -\infty & -\infty \end{pmatrix},$$

Where x is a paramater with values in $\mathbb{R}_+$. Note that A is an acyclic irreducible matrix with eigenvalue 0. Since $x > 0$, the critical nodes are $\{1, 2, 3\}$ and $\{4\}$ is the only non-critical node.

We have: $N = 4, C = 3, H = 1, L = 2, \beta = -\infty$. Since $\beta = -\infty$, it is useless to compute B^* or M'. The coupling time given by Theorem 2 is: $n_0 = 9$.

This is a case where the circuits with the second largest average weight contain critical nodes. Therefore, they are not taken into account in the computation of our bound even if their average weight $(-x/2)$ can be arbitrarily close to 0.

By computing the minimal coupling time using simulation, we get a minimal coupling time equal to 6 for all positive x, which does not depend on x either and which is rather close to our upper bound.

5.2 Cases where β is involved in the bound

We consider the matrix

$$A = \begin{pmatrix} 0 & 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & 10 & x & -\infty \\ -10 & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty & 0 \\ -\infty & -\infty & 0 & -2y & -\infty \end{pmatrix},$$

where x and y are real parameters such that $y > 0$ and $x < 10$. Once again, A is an acyclic irreducible matrix with eigenvalue 0.

We have: $N = 5, C = 3, H = 1, L = 1, \beta = -y$. Matrices B and M' are of size 2.

$$B = \begin{pmatrix} -\infty & 0 \\ -2y & -\infty \end{pmatrix}.$$

Therefore,

$$B^* = \begin{pmatrix} 0 & 0 \\ -2y & 0 \end{pmatrix} \text{ and } M' = \begin{pmatrix} x-10 & x-10 \\ x-10 & x-10 \end{pmatrix}.$$

Finally, $|B^* - M'| = 10 - x$ and from Theorem 2, we get $n_0 = 2 + \max(\frac{10-x}{y}, 8)$.

This case shows the importance of block B and of blocks T_1 and T_2 in the coupling time of A.

Computations of the powers of matrix A for several values of x and y have been performed to obtain the minimal coupling times which is given in the following table together with the bound n_0.

x	y	n_0	minimal coupling time
9	100	10	7
9	2	10	7
0	1	12	11
-100	1	112	111
0	0.1	102	101
-10	0.1	202	201

In the cases where both x is very large (close to 10) (and y is very small (smaller than 1)) then the minimal coupling time is 7. Our bound gives $n_0 = 10$ and does not depend on x and y. However, as soon as x is small enough and/or y is close to 0, our bound depends on x and y: $n_0 = \frac{10-x}{y} + 2$. All the experiments reported in the table seem to say that the minimal coupling time is $\frac{10-x}{y} + 1$, which is remarkably close to our upper bound.

Finally, we have studied a case where matrix A contains several non-critical circuits which compete with the largest of them.

$$A = \begin{pmatrix} 0 & 10 & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 5 & -\infty & -10 & -\infty & -\infty \\ -15 & -\infty & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -10 & -\infty & -1 & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 & -\infty & -10 & -\infty \\ -\infty & -\infty & -\infty & -\infty & 5 & -\infty & 0 \\ -\infty & -\infty & -\infty & -\infty & -\infty & -2y & -\infty \end{pmatrix}.$$

Here $0 < y < 1$ so that A is acyclic, $\lambda = 0$ and $\beta = -y$. We have $N = 7, C = 3, L = 1, H = 1$. After some easy computations, one gets $|B^* - M'| = 30 + 2y$. Therefore, $n_0 = 6 + \max(\frac{30}{y}, 9)$. The following table gives some comparisons between n_0 and the minimal coupling time.

y	n_0	minimal coupling time
0.9	40	36
0.8	44	40
0.5	66	62
0.1	306	302
0.01	3006	3002

It shows that our bound behaves very well compared with the minimal coupling time (up to a constant additive factor).

It is rather easy to see that whenever the coupling time does not only depend on the boolean structure of the matrix (number of strongly connected components in the critical graph, length of critical cycles) but depends on certain important values in A, then our bound is close to the minimal coupling time since it may only differ from the minimal coupling time by an additive constant.

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