Hard Constraints and the Bethe Lattice: Adventures at the Interface of Combinatorics and Statistical Physics

Graham R. Brightwell* and Peter Winkler†

Abstract

Statistical physics models with hard constraints, such as the discrete hard-core gas model (random independent sets in a graph), are inherently combinatorial and present the discrete mathematician with a relatively comfortable setting for the study of phase transition.

In this paper we survey recent work (concentrating on joint work of the authors) in which hard-constraint systems are modeled by the space $\text{Hom}(G, H)$ of homomorphisms from an infinite graph G to a fixed finite constraint graph H. These spaces become sufficiently tractable when G is a regular tree (often called a Cayley tree or Bethe lattice) to permit characterization of the constraint graphs H which admit multiple invariant Gibbs measures.

Applications to a physics problem (multiple critical points for symmetry-breaking) and a combinatorics problem (random coloring), as well as some new combinatorial notions, will be presented.

2000 Mathematics Subject Classification: 82B20, 68R10.
Keywords and Phrases: Hard constraints, Bethe lattice, graph homomorphisms, combinatorial phase transition.

1 Introduction

Recent years have seen an explosion of activity at the interface of graph theory and statistical physics, with probabilistic combinatorics and the theory of computing as major catalysts. The concept of "phase transition", which a short time

---

[*] Department of Mathematics, London School of Economics, Houghton St., London WC2A 2AE England. E-mail: g.r.brightwell@lse.ac.uk

[†] Bell Labs 2C-365, 700 Mountain Ave., Murray Hill NJ 07974-0636 USA. E-mail: pw@lucent.com

ago most graph theorists would barely recognize, has now appeared and reappeared in journals as far from physics as the *Journal of Combinatorial Theory* (Series B).

Traffic between graph theory and statistical physics is already heavy enough to make a complete survey a book-length proposition, even if one were to assume a readership with knowledge of both fields.

This article is intended for a general mathematical audience, *not* necessarily acquainted with statistical physics, but it is not to serve as an introduction to the field. Readers are referred to texts such as [1, 12, 17, 19] for more background. We will present only a small part (but we hope an interesting one) of the interface between combinatorics and statistical physics, with just enough background in each to make sense of the text. We will focus on the most combinatorial of physical models—those with hard constraints—and inevitably on the authors' own research and related work.

We hope it will be clear from our development that there is an enormous amount of fascinating mathematics to be uncovered by studying statistical physics, quite a lot of which has been or will be connected to graph theory. What follows is only a sample.

2 Random Independent Sets

In what follows a *graph* $G = \langle V, E\rangle$ consists of a set V (finite or countably infinite) of *nodes* together with a set E of *edges*, each of which is an unordered pair of nodes. We will sometimes permit loops (edges of the form $\{v, v\}$) but multiple edges will not be considered or needed. We write $u \sim v$, and say that u is "adjacent" to v, if $\{u, v\} \in E$; a set $U \subset V$ is said to be *independent* if it contains no edges.

The *degree* of a node u of G is the number of nodes adjacent to u; all graphs considered here will be *locally finite*, meaning that all nodes have finite degree. A *path* in G (of length k) is a sequence $u_0, u_1, \ldots, u_k$ of distinct nodes with $u_i \sim u_{i+1}$; if in addition $u_k \sim u_0$ we have a *cycle* of length $k+1$. If every two nodes of G are connected by a path, G is said to be *connected.*

The plane grid $\mathbb{Z}^2$ is given a graph structure by putting $(i, j) \sim (i', j')$ iff $|i-i'| + |j-j'| = 1$. Let us carve out a big piece of $\mathbb{Z}^2$, say the box $B_n^2 := \{(i, j) \mid -n \leq i, j \leq n\}$. Let I be a uniformly random independent set in the graph B_n^2; in other words, of all sets of nodes (including the empty set) not containing an edge, choose one uniformly at random. What does it look like?

Plate 1 shows such an I (in a rectangular region). Here the nodes are represented by squares, two being adjacent if they have a common vertical or horizontal border segment. The sites belonging to I are colored, the others omitted. It is by no means obvious how to obtain such a random independent set in practice; one cannot simply choose points one at a time subject to the independence constraint. In fact the set in Plate 1 was generated by *Markov chain mixing*, an important and fascinating method in the theory of computing, which has by itself motivated much recent work at the physics-combinatorics interface.

It is common in statistical physics to call the nodes of B_n^2 *sites* (and the edges, *bonds*). Sites in I are said to be *occupied*; one may imagine that each occupied site contains a molecule of some gas, any two of which must be at distance greater than 1.

In the figure, *even* occupied sites (nodes $(i,j) \in I$ for which $i+j \equiv 0 \mod 2$) are indicated by one color, odd sites by another. A certain tendency for colors to clump may be observed; understandably, since occupied sites of the same parity may be as close as $\sqrt{2}$ (in the Euclidean norm) but opposite-parity particles must be at least $\sqrt{5}$ apart.

It stands to reason that if more "particles" were forced into I, then we might see more clumping. Let us weight the independent sets according to size, as follows: a positive real λ, called the *activity* (or sometimes *fugacity*) is fixed, and then each independent set I is chosen with probability proportional to $\lambda^{|I|}$. We call this the "λ-measure". Of course if $\lambda = 1$ we are back to the uniform measure, but if $\lambda > 1$ then larger independent sets are favored.

The λ-measure for $\lambda \neq 1$ is, to a physicist, no less natural than the uniform. In a combinatorial setting, such a measure might arise e.g. if the particles happen to be of two different types, with all "typed" independent sets equiprobable; then the probability that a particular set of sites is occupied is given by the λ-measure with $\lambda = 2$.

Plate 2 shows a random I chosen when $\lambda = 3.787$. The clusters have grown hugely as more particles were packed in. Push λ up just a bit more, to 3.792, and something like Plate 3 is the result: one color (parity) has taken over, leaving only occasional islands of the other.

Something qualitative has changed here, but what exactly? The random independent sets we have been looking at constitute what the physicists call the *hard-core lattice gas model*, or "hard-core model" for short. Readers are referred to the exceptionally readable article [2] in which many nice results are obtained for this model[1]. On the plane grid, the hard-core model has a "critical point" at activity about 3.79, above which the model is said to have experienced a *phase transition*.

3 What is Phase Transition?

There is no uniformity even among statistical physicists regarding the definition of phase transition; in fact, there is even disagreement about whether the "phases" above are the even-dominated versus odd-dominated configurations at high λ, or the high-λ regime versus the low. Technical definitions involving points of non-analyticity of some function miss the point for us.

The point really is that a slight change in a parameter governing the local behavior of some statistical system, like the hard-core model, can produce a

---

[1] Readers, however, are cautioned regarding conducting a web search with key-words "hard-core" and "model".

global change in the system, which may be evidenced in many ways. For example, suppose we sampled many independent sets in B_n^2 at some fixed λ, and for each computed the ratio of the number of even occupied sites to the number of odd. For low λ these numbers would cluster around $1/2$, but for high λ they would follow a bimodal distribution; and the larger the box size n, the sharper the transition.

Here's another, more general, consideration. Suppose we look only at independent sets which contain all the even sites on the boundary of B_n^2. For these I the origin would be more likely to be occupied than, say, one of its odd neighbors. As n grows, this "boundary influence" will fade—provided λ is low. But when λ is above the critical point, the boundary values tend to make I an even-dominated set, giving any even site, no matter how far away from the boundary, a non-disappearing advantage over any odd one.

Computationally-minded readers might be interested in a third approach. Suppose we start with a fixed independent set I_0, namely the set of all even sites in B_n^2, and change it one site at a time as follows: at each tick of a clock we choose a site u at random. If any of u's neighbors is occupied, we do nothing. Otherwise we flip a biased coin and with probability $\lambda/(1+\lambda)$ we put u in I (where it may already have been), and with probability $1/(1+\lambda)$ we remove it (or leave it out). The result is a Markov chain whose states are independent sets and whose stationary distribution, one can easily verify, is exactly our λ-measure. Thus if we do this for many steps, we will have a nearly perfect sample from this distribution—but how many steps will that take? We *believe* that when λ is below its critical value, only polynomially (in n) steps are required—the Markov chain is said to be *rapidly mixing*; even polylogarithmic, if we count the number of steps per site. But for high λ it appears to take time exponential in n (or perhaps in $\sqrt{n}$) before we can expect to see an odd-dominated independent set. The exact relationship between phase transition and Markov chain mixing is complex and the subject of much study.

All these measures rely on taking limits as the finite box B_n^2 grows; the very nice discovery of Dobrushin, Lanford and Ruelle [9, 15] is that there is a way to understand the phenomenon of phase transition as a property of the infinite plane grid. The idea is to extend the λ-measure to a probability distribution on independent sets on the whole grid, then ask whether the extension is unique.

We cannot extend the definition of the λ-measure directly since $\lambda^{|I|}$ is generally infinite, but we can ask that it behave locally like the finite measure. We say that a probability distribution μ on independent sets in the plane grid is a *Gibbs measure* if for any site u the probability that u is in I, given the sites in $I \cap (\mathbb{Z}^2 \setminus \{u\})$, is $\lambda/(1+\lambda)$ if the neighborhood of u is unoccupied and, of course, 0 otherwise.

It turns out that Gibbs measures always exist (here, and in far greater generality) but may or may not be unique. When there is more than one Gibbs measure we will say that there is a phase transition. For the hard-core model on $\mathbb{Z}^2$, there is a unique Gibbs measure for low λ; but above the critical value, there is a Gibbs measure in which the even occupied sites are dominant and another in which the odd sites are dominant (all other Gibbs measures are convex combinations of these two). How can you construct these measures? Well, for example, the even measure can be obtained as a limit of λ-measures on boxes whose even boundary sites are forced to be in I. The fact that the boundary influence does not fade (in the high λ case) implies that the even and odd Gibbs measures are different.

We have noted that the critical value of λ for the hard-core model on $\mathbb{Z}^2$ is around 3.79. This is an empirical result and all we mathematicians can prove is that there is at least one critical point, and all such are between 1.1 and some high number. It is believed that, for each d, there is just one critical value λ_d on $\mathbb{Z}^d$. It is also to be expected that λ_d is decreasing in d, but only recently has it been shown that the largest critical value on $\mathbb{Z}^d$ tends to 0 as $d \to \infty$. This result was obtained by David Galvin and Jeff Kahn [11], two combinatorialists, using graph theory, geometry, topology, and lots of probabilistic combinatorics. A consequence of their work is that $\lambda = 1$ is above the critical value(s) for sufficiently large d; this can be stated in a purely combinatorial way: for sufficiently high d and large n, *most* independent sets in B_n^d are dominated by vertices of one parity.

In the next section we explain how we can use graphs to understand models with hard constraints; then, in the section following that, we will switch from $\mathbb{Z}^2$ to a much easier setting, in which we can get our hands on nice Gibbs measures.

4 Hard Constraints and Graph Homomorphisms

We are interested in what are sometimes called "nearest neighbor" hard constraint models, where the constraints apply only to adjacent sites. Each site is to be assigned a "spin" from some finite set, and only certain pairs of spins are permitted on adjacent sites. We can code up the constraints as a finite graph H whose nodes are the spins, and whose edges correspond to spins allowed to appear at neighboring sites. This *constraint graph* H may have some loops; a loop at node $v \in H$ would mean that neighboring sites may both be assigned spin v. We adopt the statistical physics tradition of reserving the letter "q" for the number of spins, that is, the number of nodes in H.

The graph G (e.g. $\mathbb{Z}^2$, above) of sites, usually infinite but always countable and locally finite, is called (by us) the *board.* A legal assignment of spins to the sites of G is nothing more or less than a graph homomorphism from G to H, i.e. a map from the sites of G to the nodes of H which preserves edges. We denote the set of homomorphisms from G to H by $\mathrm{Hom}(G, H)$, and give it a graph structure by putting $\varphi \sim \psi$ if φ and ψ differ at exactly one site of G.

We will often confuse a graph with its set of nodes (or sites). In particular, if U is a subset of the nodes of G then U together with the edges of G contained in U constitute the "subgraph of G induced by U", which we also denote by U.

In the hard-core model, the constraint graph H consists of two adjacent nodes, one of which is looped: a function from a board G to this H is a homomorphism iff the set of sites mapped to the unlooped node is an independent set. Plate 4 shows some constraint graphs found in the literature.

When H is complete and every node is looped as well, there is no constraint and nothing interesting happens.

When H is the complete graph K_q (without loops), homomorphisms to H are just ordinary, "proper" q-colorings of the board. (A proper q-coloring of a graph G is a mapping from the nodes of G to a q-element set in which adjacent nodes are never mapped to the same element.) This corresponds to something called the "anti-ferromagnetic Potts model at zero temperature". In the anti-ferromagnetic Potts model at positive temperature, adjacent sites are merely discouraged (by an energy penalty), not forbidden, from having the same spin; thus this is not a hard constraint model in our terminology. The $q = 2$ case of the Potts model is the famous Ising model.

For a general constraint graph H, we need to elevate the notion of activity to vector status. To each node i of H we assign a positive real activity λ_i, so that H now gets an activity vector $\lambda := (\lambda_1, \ldots, \lambda_q)$. When the board G is finite, each homomorphism $\varphi \in \mathrm{Hom}(G,H)$ is assigned probability proportional to

$$\prod_{v \in G} \lambda_{\varphi(v)} \ .$$

We can think of λ_i as the degree to which we try to use spin i, when it is available. For example, if we know the spins of the neighbors of site v and consequently, say, spins i, j and k are allowed for v, then the λ-measure forces $\Pr(\varphi(v) = i) = \lambda_i/(\lambda_i + \lambda_j + \lambda_k)$.

When G is infinite, things get a little more complicated. A finite subset (and its induced subgraph) $U \subset G$ will be called a "patch" and its boundary ∂U is the set of sites not in U but adjacent to some site of U. We define $U^+ := U \cup \partial U$. If φ is a function on G, then $\varphi\restriction U$ denotes its restriction to the subset U.

We say that μ is a Gibbs measure for λ if: for any patch $U \subset G$, and almost every $\psi \in \mathrm{Hom}(G,H)$,

$$\Pr_{\mu}\big(\varphi\restriction U = \psi\restriction U \,\big|\, \varphi\restriction(G-U) = \psi\restriction(G-U)\big) = \Pr_{U^+}\big(\varphi\restriction U = \psi\restriction U \,\big|\, \varphi\restriction\partial U = \psi\restriction\partial U\big)$$

where "$\Pr_{U^+}$" refers to the finite λ-measure on U^+.

This definition looks messy but it just means that the probability distribution of a random φ inside a patch U depends only on its value on the boundary of U, and is the same as if U and its boundary comprised all of the board. We will see later that when H has a certain nice property, as it does in the case of the hard-core model, it suffices to check the Gibbs condition only on patches consisting of a single site—we call this the one-site condition.

It is a special case of a theorem of Dobrushin [9] that there is always at least one Gibbs measure for any λ on $\mathrm{Hom}(G,H)$; we are concerned with questions about when there is a unique Gibbs measure, and when there is a phase transition (i.e. more than one Gibbs measure).

Let us again look briefly at possible implications for phase transition in the setting of finite boards. Given a finite board G, a constraint graph H and activities λ, we define the *point process* $\mathcal{P}(G,H,\lambda)$ as follows: starting from any element of $\mathrm{Hom}(G,H)$, choose a site u of G uniformly at random, and give it a fresh spin according to the Gibbs condition, so that each 'legal' spin j is chosen with probability proportional to λ_j. The point process is a Markov chain on $\mathrm{Hom}(G,H)$, and it is easy to check that the λ-measure is a stationary distribution (which will be unique provided $\mathrm{Hom}(G,H)$ is connected, a point we will return to later).

Running the point process for sufficiently long will thus generate a random homomorphism according to the λ-measure. However, suppose that the finite board G is a large piece of an infinite board G' exhibiting a phase transition for our λ. Then, if we start with a homomorphism arising from one Gibbs measure on $\mathrm{Hom}(G',H)$ (restricted to G), it is reasonable to expect that the point process will take a long time to reach a configuration resembling that from any other Gibbs measure on $\mathrm{Hom}(G',H)$. Thus it is generally believed that, in some necessarily loose sense, phase transition on an infinite graph corresponds to slow convergence for the point process on finite subgraphs.

5 Cayley Trees and Branching Random Walks

Gibbs measures can be elusive and indeed it is generally a difficult task to prove that phase transitions occur on a typical board of interest, like $\mathbb{Z}^d$. In order to get results and intuition physicists sometimes turn to a more tractable board, called by them the Bethe lattice (after Hans Bethe) and by combinatorialists, usually, the Cayley tree.

We denote by $\mathbb{T}^r$ the r-branching Cayley tree, equivalently the unique connected (infinite) graph which is cycle-free and in which every site has degree $r+1$. $\mathbb{T}^r$ is a vastly different animal from $\mathbb{Z}^d$. It is barely connected, falling apart with the removal of any site; its patches have huge boundaries, comparable in size with the patch itself; its automorphism group is enormous. It's surprising that we can learn anything at all about $\mathrm{Hom}(\mathbb{Z}^d,H)$ from $\mathrm{Hom}(\mathbb{T}^r,H)$, and indeed we must be careful about drawing even tentative conclusions in either direction. Basic physical parameters like entropy become dodgy on non-amenable (big-boundary) boards like $\mathbb{T}^r$ and a number of familiar statistical physics techniques become useless. More than making up for these losses, though, are the combinatorial techniques we can use to study $\mathrm{Hom}(\mathbb{T}^r,H)$. There are even situations (e.g. in the study of information dissemination) where $\mathbb{T}^r$ is the natural setting.

We are particularly interested in Gibbs measures on $\mathrm{Hom}(\mathbb{T}^r,H)$ which have the additional properties of being *simple* and *invariant*.

For any site u in a tree T, let $d(u)$ be the number of edges incident with u and let $C_1(u), C_2(u), \ldots, C_{d(u)}(u)$ be the connected components of $T \setminus \{u\}$.

Definition 5.1 A Gibbs measure μ on $\mathrm{Hom}(T,H)$ is *simple* if, for any site $u \in T$ and any node $i \in H$, the μ-distributions of

$$\varphi \restriction C_1(u), \ldots, \varphi \restriction C_{d(u)}(u)$$

are mutually independent given $\varphi(u) = i$.

This condition, which is trivially satisfied by the λ-measure for finite T, would follow from the Gibbs condition itself if fewer than two of the $C_i(u)$'s were infinite.

Definition 5.2 Let $\mathcal{A}(G)$ be the automorphism group of the board G, and for any subset $S \subset \mathrm{Hom}(G,H)$ and $\kappa \in \mathcal{A}(G)$ let $S \circ \kappa := \{\varphi \circ \kappa : \ \varphi \in S\}$. We say that a measure μ on $\mathrm{Hom}(G,H)$ is *invariant* if, for any μ-measurable $S \subset \mathrm{Hom}(G,H)$ and any $\kappa \in \mathcal{A}(G)$, we have $\mu(S \circ \kappa) = \mu(S)$.

Again, this condition is trivially satisfied for finite G; but for an infinite board with as many automorphisms as $\mathbb{T}^r$, it is quite strong. Later we consider relaxing it slightly. For now, we might well ask, how can we get our hands on any Gibbs measure for $\mathrm{Hom}(\mathbb{T}^r,H)$, let alone a simple, invariant one?

The absence of cycles in $\mathbb{T}^r$ makes it plausible that we can get ourselves a Gibbs measure by building random configurations in $\mathrm{Hom}(\mathbb{T}^r,H)$ one site at a time. We could choose a root $x \in \mathbb{T}^r$, assign it a random spin $i \in H$, then assign the neighbors of i randomly to the $r+1$ children of x; thereafter, each time a site u gets spin j we give its r children random spins from among the neighbors of j.

The process we have described can be thought of as a branching random walk on H. Imagine amoebas staggering from node to adjacent node of H; each time an amoeba steps it divides into r baby amoebas which then move independently at the next time step. Of course, the (usually tiny) constraint graph H is shortly piled high with exponentially many amoebas, but being transparent they happily ignore one another and go on stepping and dividing.

To get started we have to throw the first amoeba onto H where we imagine that its impact will cause it to divide $r+1$ ways instead of the usual r.

Note that the $r = 1$ case is just ordinary random walk, started somewhere on the doubly-infinite path $\mathbb{T}^1$ and run both forward and backward.

To determine what probabilities are used in stepping from one node of H to an adjacent node, we assign a positive real *weight* w_i to each node. For convenience we denote by z_i the sum of the weights of the neighbors of i (including i itself, if there is a loop at i). An amoeba-child born on node i then steps to node j with probability w_j/z_i. If there is a loop at i, the amoeba stays at i with the appropriate probability, w_i/z_i.

Assuming H is connected and not bipartite[2], the random walk (branching or not) will have a stationary distribution π; it is easily verified that π_i is proportional to $w_i z_i$ for each i, and somewhat less easily verified that the mapping $w \hookrightarrow \pi$ is one-to-one provided $\sum w_i$ has been normalized to 1. We use the stationary distribution to pick the starting point for the first amoeba, i.e. to assign a spin to the root of $\mathbb{T}^r$.

Finally, the payoff: not only does this node-weighted branching random walk give us a simple invariant Gibbs measure; it's the *only* way to get one.

---

[2] A graph is bipartite if its nodes can be partitioned into two sets neither of which contains an edge. Thus, for example, the existence of a looped node already prevents H from being bipartite.

The following theorem appears in [4] but it is not fundamentally different from characterizations which can be found in Georgii [12] and elsewhere.

Theorem 5.3 *Let H be a fixed connected constraint graph with node-weights w and let r be a positive integer. Then the measure μ induced on* $\mathrm{Hom}(\mathbb{T}^r, H)$ *by the r-branching w-random walk on H is a simple, invariant Gibbs measure, for some activity λ on H. Conversely, if H, r and λ are given, then every simple, invariant Gibbs measure on* $\mathrm{Hom}(\mathbb{T}^r, H)$ *is given by the r-branching random walk on H with nodes weighted by some w.*

The proof is actually quite straightforward, and worth including here. Invariance of μ with respect to root-preserving automorphisms of $\mathbb{T}^r$ is trivial, since the random walk treats all children equally; the only issue is whether the selection of root makes a difference. For this we need only check that for two neighboring sites u and v of $\mathbb{T}^r$, μ is the same whether u is chosen as root or v is. But, either way we may choose $\varphi(u)$ and $\varphi(v)$ as the first two spins and the rest of the procedure is the same; so it suffices to check that for any (adjacent) nodes i and j of H, the probability that $\varphi(u) = i$ and $\varphi(v) = j$ is the same with either root choice. But these two probabilities are

$$\pi_i p_{ij} = z_i w_i \frac{w_j}{z_i} = w_i w_j = z_j w_j \frac{w_i}{z_j} = \pi_j p_{ji}$$

as desired.

To show that μ is simple is, indeed, simple: if we condition on $\varphi(u) = i$ then, using invariance to put the root at u, the independence of φ on the $r+1$ components of $\mathbb{T}^r \setminus \{u\}$ is evident from the definition of the branching random walk.

The activity vector λ for which μ is a Gibbs measure turns out to be given by

$$\lambda_i = \frac{w_i}{z_i^r} \ .$$

Let U be any finite set of sites in $\mathbb{T}^r$, with exterior boundary ∂U. On account of invariance of labeling, we may assume that the root x does not lie in $U^+ = U \cup \partial U$.

Let $g \in \mathrm{Hom}(U^+, H)$; we want to show that the probability that a branching random walk φ matches g on U, given that it matches on ∂U, is the same as the corresponding conditional probability for the λ-measure.

Let T be the subtree of $\mathbb{T}^r$ induced by U^+ and the root x; for any $f \in \mathrm{Hom}(T, H)$,

$$\begin{aligned}\Pr(\varphi \restriction T = f) &= \pi_{f(x)} \cdot \prod_{u \to v} p_{f(u),f(v)} \\ &= z_{f(x)} w_{f(x)} \prod_{u \to v} \frac{w_{f(v)}}{z_{f(u)}}\end{aligned}$$

where $u \to v$ means that v is a child of u in the tree. The factors $z_{f(u)}$ corresponding to sites u in U each occur as denominator r times in the above expression, since each site in U has all of its r successors in T; and of course each $w_{f(u)}$ occurs once as a numerator as well. It follows that if we compare $\Pr(\varphi\restriction T = f)$ with $\Pr(\varphi\restriction T = f')$, where f' differs from f only on U, then the value of the first is proportional to

$$\prod_{u\in U}\frac{w_{f(u)}}{z^r_{f(u)}} = \prod_{u\in U}\lambda_{f(u)}$$

which means that μ coincides with the finite measure, as desired.

Now let us assume that μ is a simple, invariant Gibbs measure on $\mathrm{Hom}(\mathbb{T}^r, H)$ with activity vector λ, with the intent of showing that μ arises from a node-weighted branching random walk on H.

We start by constructing a μ-random φ, site by site. Choose a root x of $\mathbb{T}^r$ and pick $\varphi(x)$ from the *a priori* distribution σ of spins of x (and therefore, by invariance, of any other site). We next choose a spin for the child y of x according to the conditional distribution matrix $P = \{p_{ij}\}$ given by

$$p_{ij} := \Pr\left(\varphi(y) = j \,|\, \varphi(x) = i\right) \;\; ;$$

again, by invariance of μ, P is the same for any pair of neighboring sites. It follows that $\sigma = \sigma \cdot P$, and moreover that P is the transition matrix of a reversible Markov chain, since the roles of x and y can be interchanged.

Next we proceed to the rest of the children of x, then to the grandchildren, etc., choosing each spin conditionally according to all sites so far decided.

We claim, however, that the distribution of possible spins of the non-root v depends only on the spin of its parent u; this is so because μ is simple and all sites so far "spun" are in components of $\mathbb{T}^r \setminus \{u\}$ other than the component containing v. Thus the value of $\varphi(v)$ is given by P for *every* site $v \neq x$, and it follows that μ arises from an r-branching Markov chain with state-space H, starting at distribution σ.

Evidently for any (not necessarily distinct) nodes i, j of H, there will be pairs (u, v) of adjacent sites with $\varphi(u) = i$ and $\varphi(v) = j$ if and only if $i \sim j$ in H. Hence P allows transitions only along edges of H, and there is a unique distribution π satisfying $\pi \cdot P = \pi$; thus $\sigma = \pi$.

It remains only to show that P is a node-weighted random walk, and it turns out that a special case of the Gibbs condition for one-site patches suffices. Let j and j' be nodes of H which have a common neighbor i, and suppose that all of the neighbors of the root x have spin i. Such a configuration will occur with positive probability and according to the Gibbs condition for $U = \{x\}$,

$$\frac{\Pr(\varphi(x) = j')}{\Pr(\varphi(x) = j)} = \frac{\lambda_{j'}}{\lambda_j}$$

but

$$\Pr(\varphi(x) = j) = \frac{\pi_j p_{ji}^{r+1}}{\sum_{k\sim i}\pi_k p_{ki}^{r+1}}$$

and similarly for j', so

$$\frac{\Pr(\varphi(x) = j')}{\Pr(\varphi(x) = j)} = \frac{\pi_{j'} p_{j'i}^{r+1}}{\pi_j p_{ji}^{r+1}} \ .$$

Thus the ratio

$$\frac{p_{j'i}}{p_{ji}} = \left(\frac{\lambda_{j'} \pi_j}{\lambda_j \pi_{j'}} \right)^{\frac{1}{r+1}}$$

is independent of i.

Since P is reversible we have $p_{ij} = \pi_j p_{ji} / \pi_i$, hence

$$\frac{p_{ij'}}{p_{ij}} = \frac{\pi_{j'} p_{j'i}}{\pi_j p_{ji}}$$

is also independent of i, and it follows that P is a node-weighted random walk on H. This concludes the proof of Theorem 5.3.

In view of Theorem 5.3, if we can understand the behavior of the map $w \hookrightarrow \lambda$, we will know, given λ, whether there is a nice Gibbs measure and if so whether there is more than one. The first issue is settled nicely in the following theorem, a proof of which can be found in [4] and requires some topology. A similar result was proved by Zachary [23].

Theorem 5.4 *For every $r \geq 2$, every constraint graph H and every set λ of activities for H, there is a node-weighted branching random walk on $\mathbb{T}^r$ which induces a simple, invariant Gibbs measure on* $\mathrm{Hom}(\mathbb{T}^r, H)$.

It's nice to know that we haven't required so much of our measures that they can fail to exist.

A statistical physics dictum (true in great, but not unlimited, generality) says that there's never a phase transition in dimension 1; that holds here:

Theorem 5.5 *For any connected constraint graph H and any activity vector λ, there is a unique simple invariant Gibbs measure on* $\mathrm{Hom}(\mathbb{T}^1, H)$.

Furthermore, in any dimension, there's always *some* region where the map $w \hookrightarrow \lambda$ is one-to-one:

Theorem 5.6 *For any r and H there is an activity vector λ for which there is only one simple invariant Gibbs measure on* $\mathrm{Hom}(\mathbb{T}^r, H)$.

6 Fertile and Sterile Graphs

The fascination begins when we hit an H and a λ which boast multiple simple, invariant Gibbs measures. Let us examine a particular case, involving a constraint graph we call the "hinge".

The hinge has three nodes, which we associate with the colors green, yellow and red; all three nodes are looped and edges connect green with yellow, and yellow with red. Thus the only missing edge is green-red, and a $\varphi \in \mathbb{T}^r$ may be thought of as a green-yellow-red coloring of the tree in which no green site is adjacent to a red one.

The hinge constraint in fact corresponds to a discrete version of the Widom-Rowlinson model, in which two gases (whose particles are represented by red and green) compete for space and are not permitted to occupy adjacent sites; see e.g. [3, 21, 22]. When λ_{red} and λ_{green} are equal and large relative to λ_{yellow}, the Widom-Rowlinson model tends to undergo a phase transition as one gas spontaneously dominates the other. Plate 5 shows a red-dominated sample from the Widom-Rowlinson model on $\mathbb{Z}^2$, with the unoccupied sites left uncolored instead of being colored yellow.

We can see the phase transition operate on the Cayley tree $\mathbb{T}^2$. If the green, yellow and red nodes are weighted 4, 2 and 1 respectively, λ (normalized to integers) turns out to be $(49, 18, 49)$—equal activity for green and red. How can a random walk which is biased so strongly toward green end up coloring a tree according to a Gibbs measure with symmetric specification? As a clue, let us examine a site u of $\mathbb{T}^2$ which happens to be surrounded by yellow neighbors. To be colored green requires that a certain amoeba stepped from yellow to green, then both of its children returned to yellow. Thus the conditional probability that u is green is proportional to

$$\frac{4}{4+2+1} \cdot \left(\frac{2}{4+2}\right)^2$$

as opposed to

$$\frac{1}{4+2+1} \cdot \left(\frac{2}{2+1}\right)^2$$

for red, but these values are equal.

Clearly the reversed weights 1, 2 and 4 would yield the same activity vector, and in fact a third, symmetric weighting, approximately 6, 7 and 6, does as well. Plate 6 shows pieces of $\mathbb{T}^2$ colored according to these three weightings. Of course the colorings have different proportions and are easily identifiable; checking the stationary distributions for the three random walks, we see that *a priori* a site is colored green with probability about 59% in the first weighting, 30% with the symmetric weighting and only 7% in the reversed weighting. Yet, from a conditional point of view, the three colorings are identical.

It turns out that the hinge is one of seven minimal graphs each of which can produce a phase transition on $\mathbb{T}^r$ for any $r \geq 2$. The graphs are pictured in Plate 7. We say that a graph H is *fertile* if $\mathrm{Hom}(\mathbb{T}^r, H)$ has more than one simple, invariant Gibbs measure for some r and λ; otherwise it is *sterile*. The fertile graphs are exactly those which contain one or more of the seven baby graphs in Plate 7 as an induced subgraph. It turns out that the value of r does not come into play: if the constraint graph is rich enough to produce a phase transition on any $\mathbb{T}^r$, then it does so for all $r \geq 2$. One way to state the result is as follows:

Theorem 6.1 [4] *Fix $r > 1$ and let H be any constraint graph. Suppose that H satisfies the following two conditions:*

(a) *Every looped node of H is adjacent to all other nodes of H;*

(b) *With its loops deleted, H is a complete multipartite graph.*

Then for every activity vector on H, there is a unique invariant Gibbs measure on the space $\mathrm{Hom}(\mathbb{T}^r, H)$.

If H fails either condition (a) *or condition* (b) *then there is a set of activities λ on H for which* $\mathrm{Hom}(\mathbb{T}^r, H)$ *has at least two simple, invariant Gibbs measures, and therefore λ can be obtained by more than one branching random walk.*

The proof of Theorem 6.1 is far too complex to reproduce here, but reasonably straightforward in structure. First, a distinct pair of weightings yielding the same activity vector must be produced for each of the seven baby fertile graphs, and for each $r \geq 2$. Second, it must be demonstrated that if H contains one of the seven as an induced subgraph, then there are weightings (whose restrictions are close to those previously found) which induce phase transitions on $\mathrm{Hom}(\mathbb{T}^r, H)$. Third, a monotonicity argument is employed to show that if H satisfies conditions (a) and (b) of the theorem, then the map from w to λ is injective. Finally, an easy graph-theoretical argument shows that H satisfies (a) and (b) precisely if it does not contain any of the seven baby fertile graphs as an induced subgraph.

7 An Application to Statistical Physics

Theorem 6.1 has many shortcomings, applying as it does only to hard constraint models on the Bethe lattice, and we must also not forget that it considers only the very nicest Gibbs measures. The constraint graph of the hard-core model is sterile, yet it can have multiple Gibbs measures on $\mathbb{T}^r$ (or, as we saw, on $\mathbb{Z}^2$) if we relax the invariance condition.

For $H =$ the hinge, however, and for any r and λ, there are multiple Gibbs measures on $\mathrm{Hom}(\mathbb{T}^r, H)$ if and only if there are multiple simple, invariant Gibbs measures. Like the Ising model, the Widom-Rowlinson model exhibits spontaneous symmetry-breaking; indeed its relationship to the Ising model parallels the relation between nodes and edges of a graph.

One of the nice properties known for the Ising model is that it can exhibit at most one critical point; but the proof of this fact does not work for the Widom-Rowlinson model. Indeed, in [3] the methods above are used to construct a board G for which $\mathrm{Hom}(G, H)$ has three (or more) calculable critical points, with H the hinge. Set $\lambda = \lambda_{\text{green}} = \lambda_{\text{red}}$, fixing $\lambda_{\text{yellow}} = 1$, so that the single parameter λ controls the Widom-Rowlinson model. Then:

Theorem 7.1 *There exist $0 < \lambda_1 < \lambda_2 < \lambda_3$ and an infinite graph G, such that the Widom–Rowlinson model on G with activity λ has a unique Gibbs measure for $\lambda \in (0, \lambda_1] \cup [\lambda_2, \lambda_3]$, and multiple Gibbs measures for $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \infty)$.*

The board G constructed in [3] is a tree, but not quite a regular one; it is made by dangling seven new pendant sites from each site of $\mathbb{T}^{40}$. Readers are referred to that paper for the calculations, but the intuition is something like this.

For low λ the random coloring of G is mostly yellow, but as λ rises, either green or red tends to take over the interior vertices as in $\mathbb{T}^{40}$. Then comes the third interval, where the septuplets of leaves, wanting to use both green and red, force more yellow on the interior vertices, relieving the pressure and restoring green-red symmetry. Finally the activity becomes so large that the random coloring is willing to give up red-green variety among the septuplets in order to avoid yellow interior vertices, and symmetry-breaking appears once again.

It turns out that multiple critical points can be obtained for the hard-core model in a similar way.

8 Dismantlable Graphs

In addition to the fertile and sterile graphs, a second graph dichotomy appears repeatedly in our studies: dismantlable and non-dismantlable graphs. Coincidentally, the term "dismantlable" as applied to graphs was coined by Richard Nowakowski and the second author [16] almost twenty years ago in another context entirely: a pursuit game on graphs.

Two players, a cop $\mathcal{C}$ and a robber $\mathcal{R}$, compete on a fixed, finite, undirected graph H. We will assume that H is connected and has at least one edge, although the concepts make sense even without these assumptions. The cop begins by placing herself at a node of her choice; the robber then does the same. Then the players alternate beginning with $\mathcal{C}$, each moving to an adjacent node. The cop wins if she can "capture" the robber, that is, move onto the node occupied by the robber; $\mathcal{R}$ wins by avoiding capture indefinitely. In doing so $\mathcal{R}$ is free to move (or even place himself initially) onto the same node as the cop, although that would be unwise if the node were looped since then $\mathcal{C}$ could capture him at her next move.

Evidently the robber can win on any loopless graph by placing himself at the same node as the cop and then shadowing her every move; among graphs in which every node is looped, $\mathcal{C}$ clearly wins on paths and loses on cycles of length 4 or more. (In the game as defined in [16, 18], there is in effect a loop at every node of H.)

The graph on which the game is played is said to be *cop-win* if $\mathcal{C}$ has a winning strategy, *robber-win* otherwise. The following structural characterization of cop-win graphs is proved in [16] for the all-loops case, but in fact the proof (which is not difficult, and left here as an exercise) works fine in our more general context.

Let $N(i)$ be the neighborhood of node i in H and suppose there are nodes i and j in H such that $N(i) \subseteq N(j)$. Then the map taking i to j, and every other node of H to itself, is a homomorphism from H to $H \setminus \{i\}$. We call this a *fold* of the graph H. A finite graph H is *dismantlable* if there is a sequence of folds reducing H to a graph with one node (which will necessarily be looped).

Note that dismantlable graphs are easily recognized in polynomial time. Plate 8 shows some dismantlable and non-dismantlable graphs.

The following theorem, from [5], collects a boatload of equivalent conditions.

Theorem 8.1 *The following are equivalent, for finite connected graphs H with at least one edge.*

1. *H is dismantlable.*
2. *H is cop-win.*
3. *For every finite board G, $\mathrm{Hom}(G,H)$ is connected.*
4. *For every board G, and every pair $\varphi, \psi \in \mathrm{Hom}(G,H)$ agreeing on all but finitely many sites, there is a path in $\mathrm{Hom}(G,H)$ between φ and ψ.*
5. *There is some positive integer m such that, for every board G, every pair of sets U and V in G at distance at least m, and every pair of maps $\varphi, \psi \in \mathrm{Hom}(G,H)$, there is a map $\theta \in \mathrm{Hom}(G,H)$ such that θ agrees with φ on U and with ψ on V.*
6. *For every positive integer r, and every pair of maps $\varphi, \psi \in \mathrm{Hom}(\mathbb{T}^r, H)$, there is a site u in $\mathbb{T}^r$ with $\varphi(u) \neq \psi(u)$, a patch U containing u, and a map $\theta \in \mathrm{Hom}(\mathbb{T}^r, H)$ which agrees with ψ on $\mathbb{T}^r \setminus U$ and with φ on u.*
7. *For every board G and activity vector λ, if μ is a measure on $\mathrm{Hom}(G,H)$ satisfying the one-site condition, then μ is a Gibbs measure.*
8. *For every finite board G and activity vector λ, every stationary distribution for the point process $\mathcal{P}(G,H,\lambda)$ is a Gibbs measure.*
9. *For every board G of bounded degree such that $\mathrm{Hom}(G,H)$ is non-empty, there is an activity vector λ such that there is a unique Gibbs measure on $\mathrm{Hom}(G,H)$.*
10. *For every r, there is an activity vector λ such that there is a unique Gibbs measure on $\mathrm{Hom}(\mathbb{T}^r, H)$.*

We will prove here what we think, to a graph theorist, is the most interesting of these equivalences—(i) and (iii). Recall that two maps in $\mathrm{Hom}(G,H)$ are adjacent if they differ on one site of G.

Let us first assume H is dismantlable. If it has only one node the connectivity of $\mathrm{Hom}(G,H)$ is trivial, since it has at most one element. Otherwise there are nodes $i \neq j$ in H with $N(i) \subseteq N(j)$ and we may assume by induction that $\mathrm{Hom}(G,H')$ is connected for $H' := H \setminus \{i\}$.

Define, for φ in $\mathrm{Hom}(G,H)$, the map φ' in $\mathrm{Hom}(G,H')$ (and also in $\mathrm{Hom}(G,H)$) by changing all i's to j's in the image. If α and β are two maps in $\mathrm{Hom}(G,H)$

then there are paths from α to α', α' to β' and β' to β; so $\mathrm{Hom}(G,H)$ is connected as claimed.

For the converse, let H be non-dismantlable, and suppose that nonetheless $\mathrm{Hom}(G,H)$ is connected for all finite boards G; let $q = |H|$ be minimal with respect to these properties.

If there are nodes i and j of H with $N(i) \subseteq N(j)$, then $H := H \setminus \{i\}$ is also non-dismantlable. In this case, we claim that the connectivity of $\mathrm{Hom}(G,H)$ implies connectivity of $\mathrm{Hom}(G,H')$. To see this, define, for $\varphi \in \mathrm{Hom}(G,H)$, the map $\varphi' \in \mathrm{Hom}(G,H')$ by changing all i's to j's in the image as before. If α and β are two maps in $\mathrm{Hom}(G,H')$, then we may connect them by a path $\varphi_1, \ldots, \varphi_t$ in $\mathrm{Hom}(G,H)$; now we observe that the not-necessarily distinct sequence of maps $\varphi'_1, \ldots, \varphi'_t$ connects α and β in $\mathrm{Hom}(G,H')$. This contradicts the minimality of H, so we may assume from now on that there is no pair of nodes $i \neq j$ in H with $N(i) \subseteq N(j)$.

Now let G be the 'weak' square of H, that is, the graph whose nodes are ordered pairs (i_1,i_2) of nodes of H with $(i_1,i_2) \sim (j_1,j_2)$ just when $i_1 \sim j_1$ and $i_2 \sim j_2$. There are two natural homomorphisms from G to H, the projections π_1 and π_2, where $\pi_1(i_1,i_2) = i_1$ and $\pi_2(i_1,i_2) = i_2$; we claim that π_1 is an isolated point of the graph $\mathrm{Hom}(G,H)$, which certainly implies that $\mathrm{Hom}(G,H)$ is disconnected.

If not, there is a map π' taking (say) (i_1,i_2) to $k \neq i_1$ and otherwise agreeing with π_1. Let j_2 be a fixed neighbor of i_2 and j_1 any neighbor of i_1. Then $(i_1,i_2) \sim (j_1,j_2)$, and hence $k \sim j_1$ in H. We have shown that every neighbor of i_1 is also a neighbor of k, contradicting the assumption that no such pair of nodes exists in H. This completes the proof.

For the last part of the proof, there is also a simpler (and smaller) construction that works provided H has at least one loop: see [5] or Cooper, Dyer and Frieze [8].

We have seen that, for a dismantlable constraint graph H, and any board G of bounded degree, there is some λ (which can be taken to depend only on H and the maximum degree of G) such that there is a unique Gibbs measure on $\mathrm{Hom}(G,H)$. Dyer, Jerrum and Vigoda [10] have proved a "rapid mixing" counterpart to this result: given a dismantlable H, and a degree bound Δ, there is some λ such that the point process $\mathcal{P}(G,H,\lambda)$ is rapidly mixing for all finite graphs G with maximum degree at most Δ. Of course, if H is not dismantlable, then no such result can be true as $\mathrm{Hom}(G,H)$ need not be connected.

9 Random Colorings of the Cayley Tree

We have observed that ordinary "proper" q-colorings of a graph G are maps in $\mathrm{Hom}(G,K_q)$; since K_q is sterile, there is never more than one Gibbs measure for $G = \mathbb{T}^r$. However, we see even in the case $q = 2$ that multiple Gibbs measures exist, because each of the two 2-colorings of $\mathbb{T}^r$ determines *by itself* a trivial Gibbs measure, as does any convex combination. However, only the $\frac{1}{2}$, $\frac{1}{2}$ combination is invariant under parity-changing automorphisms of $\mathbb{T}^r$.

All of the Gibbs measures in the $q = 2$ case are, however, simple and invariant under all the parity-preserving automorphisms of the board. Such Gibbs measures are neededed to realize the phase transition for the hard-core model as well, so it is not surprising that it is useful to relax our requirements slightly and to consider these *semi-invariant* simple Gibbs measures.

Fortunately we don't have to throw away all our work on invariant Gibbs measures in moving to semi-invariant ones. Given a constraint graph H on nodes $1, 2, \ldots, q$ which is connected and not bipartite, we form its bipartite "double", denoted $2H$, as follows: the nodes of $2H$ are $\{1, 2 \ldots, q\} \cup \{-1, -2, \ldots, -q\}$ with an edge between i and j just when $i \sim -j$ or $-i \sim j$ in H. Note that $2H$ is loopless; a loop at node i in H becomes the edge $\{-i, i\}$ in $2H$.

A homomorphism ψ from $\mathbb{T}^r$ to $2H$ induces a homomorphism $|\psi|$ to H via $|\psi|(v) = |\psi(v)|$. In the reverse direction, a map φ in $\mathrm{Hom}(\mathbb{T}^r, H)$ may be transformed to a map $\bar{\varphi}$ in $\mathrm{Hom}(\mathbb{T}^r, 2H)$, by putting $\bar{\varphi}(v) = \varphi(v)$ for even sites $v \in \mathbb{T}^r$ and $\bar{\varphi}(v) = -\varphi(v)$ for odd v.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an activity vector for H and suppose that μ is a simple invariant Gibbs measure on $\mathrm{Hom}(\mathbb{T}^r, H)$ corresponding to λ. From μ we can obtain a simple invariant Gibbs measure $\bar{\mu}$ on $\mathrm{Hom}(\mathbb{T}^r, 2H)$ by selecting φ from μ, and flipping a fair coin to decide between $\bar{\varphi}$ (as defined above) and $-\bar{\varphi}$. Obviously $\bar{\mu}$ yields the activity vector $\bar{\lambda}$ on $2H$ given by $\bar{\lambda}_i = \lambda_{|i|}$. Furthermore, the weights on H which produce μ extend to $2H$ by $w_{-i} = w_i$.

Conversely, suppose ν is a simple invariant Gibbs measure on $\mathrm{Hom}(\mathbb{T}^r, 2H)$ whose activity vector satisfies $\lambda_{-i} = \lambda_i$ for each i. Then the measure $|\nu|$, obtained by choosing ψ from ν and taking its absolute value, is certainly an invariant Gibbs measure on $\mathrm{Hom}(\mathbb{T}^r, H)$ for $\lambda \restriction \{1, \ldots, q\}$, but is it simple?

In fact, if the weights on $2H$ which produce ν do not satisfy $w_{-i} = cw_i$, then $|\nu|$ will fail to be simple. To see this, observe that if the weights are not proportional then there are nodes $i \sim j$ of H such that $p_{-i,-j} \neq p_{i,j}$ in the random walk on $2H$. Suppose that $|\psi|$ is conditioned on the color of the root w of $\mathbb{T}^r$ being fixed at i, and let x and y be distinct neighbors of w. Set $\alpha = \Pr(\psi(w) = i \mid |\psi(w)| = i)$. Then

$$\Pr(|\psi|(x) = j) = (1 - \alpha)p_{-i,-j} + \alpha p_{i,j}$$

but

$$\Pr(|\psi|(x) = j \wedge |\psi|(y) = j) = (1 - \alpha)p_{-i,-j}^2 + \alpha p_{i,j}^2 > \Pr(|\psi|(x) = j)^2$$

so the colors of x and y are not independent given $|\psi|(w)$.

However, we can recover simplicity at the expense of one bit worth of symmetry. Let ν^+ be ν conditioned on $\psi(u) > 0$, and define ν^- similarly. Then $|\nu^+|$ and $|\nu^-|$ are essentially the same as ν^+ and ν^-, respectively, and all are simple; but these measures are only semi-invariant.

On the other hand, suppose μ is a simple semi-invariant Gibbs measure on $\mathrm{Hom}(\mathbb{T}^r, H)$. Let θ be a parity-reversing automorphism of $\mathbb{T}^r$ and define $\mu' := \mu \circ \theta$, so that $\frac{1}{2}\mu + \frac{1}{2}\mu'$ is fully invariant (but generally no longer simple). However, $\nu := \frac{1}{2}\bar{\mu} + \frac{1}{2}(-\overline{\mu'})$ is a simple *and* invariant Gibbs measure on $\mathrm{Hom}(\mathbb{T}^r, 2H)$, thus given by a node-weighted random walk on $2H$. We can recover μ as ν_+, hence:

Theorem 9.1 *Every simple semi-invariant Gibbs measure on* $\mathrm{Hom}(\mathbb{T}^r, H)$ *is obtainable from a node-weighted branching random walk on* $2H$*, with its initial state drawn from the stationary distribution on positive nodes of* $2H$*.*

Suppose, instead of beginning with a measure, we start by weighting the nodes of $2H$ and creating a Gibbs measure as in Theorem 9.1. Suppose the activities of the measure are $\{\lambda_i : i = \pm 1, \ldots, \pm q\}$. By identifying color $-i$ with i for each $i > 0$, we create a measure on H-colorings, but this will not be a Gibbs measure unless it happens that $(\lambda_{-1}, \ldots, \lambda_{-q})$ is proportional to $(\lambda_1, \ldots, \lambda_q)$.

We could assure this easily enough by making the weights proportional as well, e.g. by $w_{-i} = w_i$; then the resulting measure on $\mathrm{Hom}(G, H)$ could have been obtained directly by applying these weights to H, and is thus a fully invariant simple Gibbs measure. To get new, semi-invariant Gibbs measures on $\mathrm{Hom}(G, H)$, we must somehow devise weights for $2H$ such that $w_{-i} \not\propto w_i$ yet $\lambda_{-i} \propto \lambda_i$.

Restated with slightly different notation, simple semi-invariant Gibbs measures are in 1–1 correspondence with solutions to the "fundamental equations"

$$\lambda_i = \frac{u_i}{\left(\sum_{j\sim i} v_j\right)^r} = \frac{v_i}{\left(\sum_{j\sim i} u_j\right)^r}$$

for $i = 1, \ldots, q$. Such a solution will be invariant if $u_i = v_i$ for each i.

Plate 9 illustrates a semi-invariant, but not invariant, simple Gibbs measure for uniform 3-colorings of $\mathbb{T}^3$. Approximate weights of the nodes of $2H = 2K_3$ are given along with part of a sample coloring drawn from this measure. Additional measures may be obtained by permuting the colors or by making all the weights equal (invariant case).

Results for q-colorings of $\mathbb{T}^r$, with $q > 2$ and $r > 1$, are as follows:

When $q < r+1$, all choices of activity vector including the uniform case yield multiple simple semi-invariant Gibbs measures.

When $q > r+1$, there is only one simple semi-invariant Gibbs measure for the uniform activity vector, but multiple simple semi-invariant Gibbs measures for some other choices of activity vector.

The critical case is at $q = r+1$, that is, when the number of colors is equal to the degree of the Cayley tree. Here it turns out that there are multiple simple semi-invariant Gibbs measures for all activity vectors *except* the uniform case, where there is just one.

When $q > r+1$ and the activities are equal, the unique simple semi-invariant Gibbs measure is in fact the only Gibbs measure of any kind. This was conjectured in [6] but proved only for $q > cr$, with fixed $c > 1$; Jonasson [13] has recently, and very nicely, finished the job. Jonasson's result is in a sense a special case of the conjecture that the Markov chain of q-colorings of a finite graph of maximum degree less than q, which progresses by choosing and recoloring sites randomly one at a time, mixes rapidly. So far the best result is Vigoda's [20] which proves this if the maximum degree is at most $6q/11$.

When $q \leq r+1$ there are lots of other Gibbs measures, including ones we call *frozen*. These come about because it is possible for a measure to satisfy the Gibbs condition in a trivial and somewhat unsatisfactory way. For example, suppose we are q-coloring $\mathbb{T}^r$ (with root w) for some $q \leq r+1$, and let ψ be any fixed coloring in which the children of every node exhibit all colors other than the color of the parent. Let μ be the measure which assigns probability 1 to ψ. Then for any finite patch U, which we can assume to be a subtree including the root, the colors on ∂U force the colors on the leaves of U, and we can continue inwards to show that the original coloring $\psi \restriction U$ is the only one consistent with the colors on ∂U. Thus μ satisfies the Gibbs condition trivially, and is also vacuously simple—but not invariant or semi-invariant. We call a Gibbs measure of this type "frozen". A frozen state of $\mathrm{Hom}(\mathbb{T}^2, K_3)$ is illustrated in Plate 10. For more about frozen Gibbs measures the reader is referred to [5].

In a soft constraint model such as the Potts model, frozen Gibbs measures can only occur at zero temperature. Since most of the time statistical physicists are interested only in phases which exist at some positive temperature (and have positive entropy), frozen measures are generally absent from the statistical physics literature. However, they are interesting combinatorially and motivate some definitions in the next section.

10 From Statistical Physics back to Graph Theory

We conclude these notes with a theorem and a conjecture in "pure" graph theory, stripped of probability and physics, but suggested by the many ideas which have appeared in earlier sections.

Suppose H is bipartite and we are given some $\varphi \in \mathrm{Hom}(\mathbb{T}^1, H)$, where the sites of $\mathbb{T}^1$ are labeled by the integers $\mathbb{Z}$. Then knowing $\varphi(n)$ even for a very large n tells us something about $\varphi(0)$, namely which "part" of H it is in. We call this phenomenon *long range action*, and define it on Cayley trees as follows: If there is a $\varphi \in \mathrm{Hom}(\mathbb{T}^r, H)$ and a node $i \in H$ such that for any n, no $\psi \in \mathrm{Hom}(\mathbb{T}^r, H)$ agreeing with φ on the sites at distance n from the root can have spin i at the root, we say $\mathrm{Hom}(\mathbb{T}^r, H)$ has long range action.

Theorem 10.1 *If H is k-colorable then* $\mathrm{Hom}(\mathbb{T}^{k-1}, H)$ *has long range action.*

For example, the coloring described at the end of the previous section, which gives rise to a frozen Gibbs measure, shows that $\mathrm{Hom}(\mathbb{T}^2, K_3)$ (more generally, $\mathrm{Hom}(\mathbb{T}^r, K_{r+1})$) has long range action. We also see from Theorem 8.1 that $\mathrm{Hom}(\mathbb{T}^r, H)$ has long range action for *no* r if and only if H is dismantlable; of course then H has at least one loop and therefore has infinite chromatic number.

Note that the theorem connects a statement about homomorphisms *from* H to a statement about homomorphisms *to* H. However, it is difficult to see how to turn a k-coloring of H into a suitable map in $\mathrm{Hom}(\mathbb{T}^{k-1}, H)$. Suppose, for instance, that H is the 5-cycle C_5, with nodes represented by the integers modulo 5. We can get a completely frozen map in $\mathrm{Hom}(\mathbb{T}^2, C_5)$ by making sure we use both $i+1$ and $i-1$ on the children of any site of spin i. But what has this map got to do with any 3-coloring of C_5?

The proof of Theorem 10.1, found in [7], uses a vector-valued generalization of coloring to construct the required map in $\mathrm{Hom}(\mathbb{T}^{k-1}, H)$.

We now move from long range action to the familiar notion of connectivity. Theorem 8.1—in fact, the part whose proof is given above—tells us that $\mathrm{Hom}(G, H)$ is connected for any finite G just when H is dismantlable. Suppose we restrict ourselves to boards of bounded degree? If, for example, H is bipartite, $\mathrm{Hom}(K_2, H)$ is already disconnected. If $H = K_q$ then $\mathrm{Hom}(K_q, H)$ is extremely disconnected, consisting of $n!$ isolated maps. By analogy with Theorem 10.1, we should perhaps be able to prove:

Conjecture 10.2 *If* H *is* k*-colorable then* $\mathrm{Hom}(G, H)$ *is disconnected for some finite* G *of maximum degree less than* k*.*

A proof for $k = 3$ appears in [7] and Lovász [14] has shown that the conjecture holds for $k = 4$ as well. We think that a proof of Conjecture 10.2 would have to capture some basic truths about graphs and combinatorial topology, and fervently hope that some reader of these notes will take up the challenge.

References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London (1982).

[2] J. van den Berg and J.E. Steif, Percolation and the hard-core lattice gas model, *Stochastic Proc. and their Appls.* **49** (1994), 179–197.

[3] G.R Brightwell, O. Häggström and P. Winkler, Nonmonotonic behavior in hard-core and Widom–Rowlinson models, CDAM Research Report LSE-CDAM-98-13 (June 1998); shorter version to appear in *J. Stat. Physics.*

[4] G.R. Brightwell and P. Winkler, Graph homomorphisms and phase transitions, *J. Comb. Theory (Series B)* **77** (1999), 221–262.

[5] G.R. Brightwell and P. Winkler, Gibbs measures and dismantlable graphs, *J. Comb. Theory (Series B)* **78** (2000), 141–166.

[6] G.R. Brightwell and P. Winkler, Random colorings of a Cayley tree, *Contemporary Combinatorics* (B. Bollobás ed.), Bolyai Society Mathematical Studies series (2002).

[7] G.R. Brightwell and P. Winkler, Graph homomorphisms and long range action, CDAM Research Report LSE-CDAM-20010-7 (2001), London School of Economics. To appear in *Graphs, Morphisms and Statistical Physics*, a DIMACS publication.

[8] C. Cooper, M. Dyer and A. Frieze, On Markov chains for randomly H-colouring a graph, *Journal of Algorithms* **39** (2001) 117–134.

[9] R.L. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, *Thy. of Prob. and its Appls.* **13** #2 (1968), 197–224.

[10] M. Dyer, M. Jerrum and E. Vigoda, Rapidly mixing Markov chains for dismantleable constraint graphs. To appear in *Graphs, Morphisms and Statistical Physics*, a DIMACS publication.

[11] D. Galvin and J. Kahn, On phase transition in the hard-core model on $\mathbb{Z}^d$, preprint (2001).

[12] H.-O. Georgii, *Gibbs Measures and Phase Transitions*, de Gruyter, Berlin (1988).

[13] J. Jonasson, Uniqueness of uniform random colorings of regular trees, preprint (2001), to appear in *Stat. Prob. Letters.*

[14] L. Lovász (2001), private communication.

[15] O.E. Lanford and D. Ruelle, Observables at infinity and states with short range correlations in statistical mechanics, *CMP* **9** (1969), 327–338.

[16] R. Nowakowski and P. Winkler, Vertex-to-vertex pursuit in a graph, *Discrete Math.* **43** (1983), 235–239.

[17] B. Prum and J.C. Fort, *Stochastic Processes on a Lattice and Gibbs Measures*, Kluwer, Dordrecht (1991).

[18] A. Quilliot, Homomorphismes, points fixes, rétractions et jeux de pousuite dans les graphes, les ensembles ordonnés et les espaces métriques, Thése d'Etat, Université de Paris VI, Paris, France (1983).

[19] D. Ruelle, *Statistical Mechanics*, Mathematical Physics Monograph Series, W.A. Benjamin Inc., Reading MA (1969).

[20] E. Vigoda, Improved bounds for sampling colorings, *Proc. 40th Symp. on Foundations of Comp. Sci.*, I.E.E.E. Computer Society, Los Alamitos CA (1999), 51–59.

[21] J.C. Wheeler and B. Widom, Phase equilibrium and critical behavior in a two-component Bethe-lattice gas or three-component Bethe-lattice solution, *J. Chem. Phys.* **52** (1970), 5334–5343.

[22] B. Widom and J.S. Rowlinson, New model for the study of liquid-vapor phase transition, *J. Chem. Phys.* **52** (1970), 1670–1684.

[23] S. Zachary, Countable state space Markov random fields and Markov chains on trees, *Ann. Probab.* **11** (1983), 894–903.

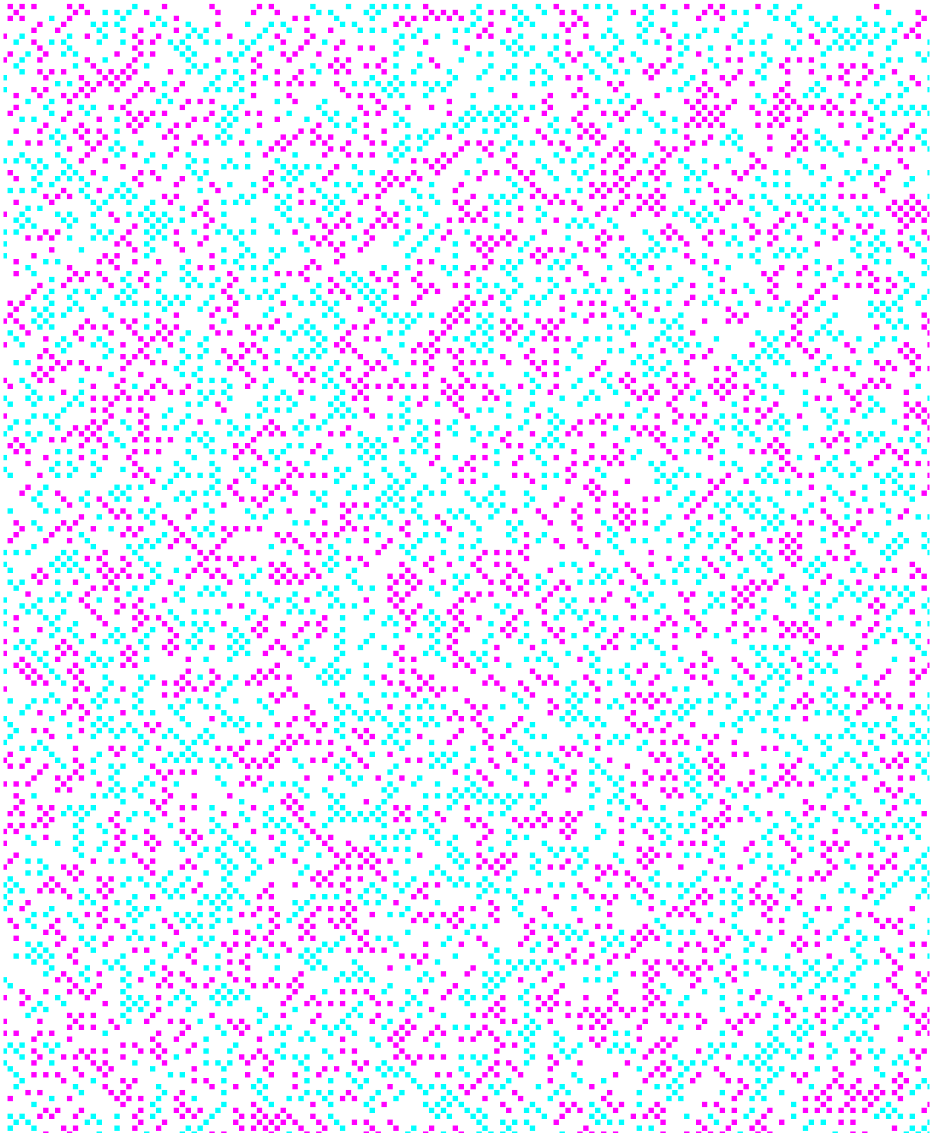


Plate 1

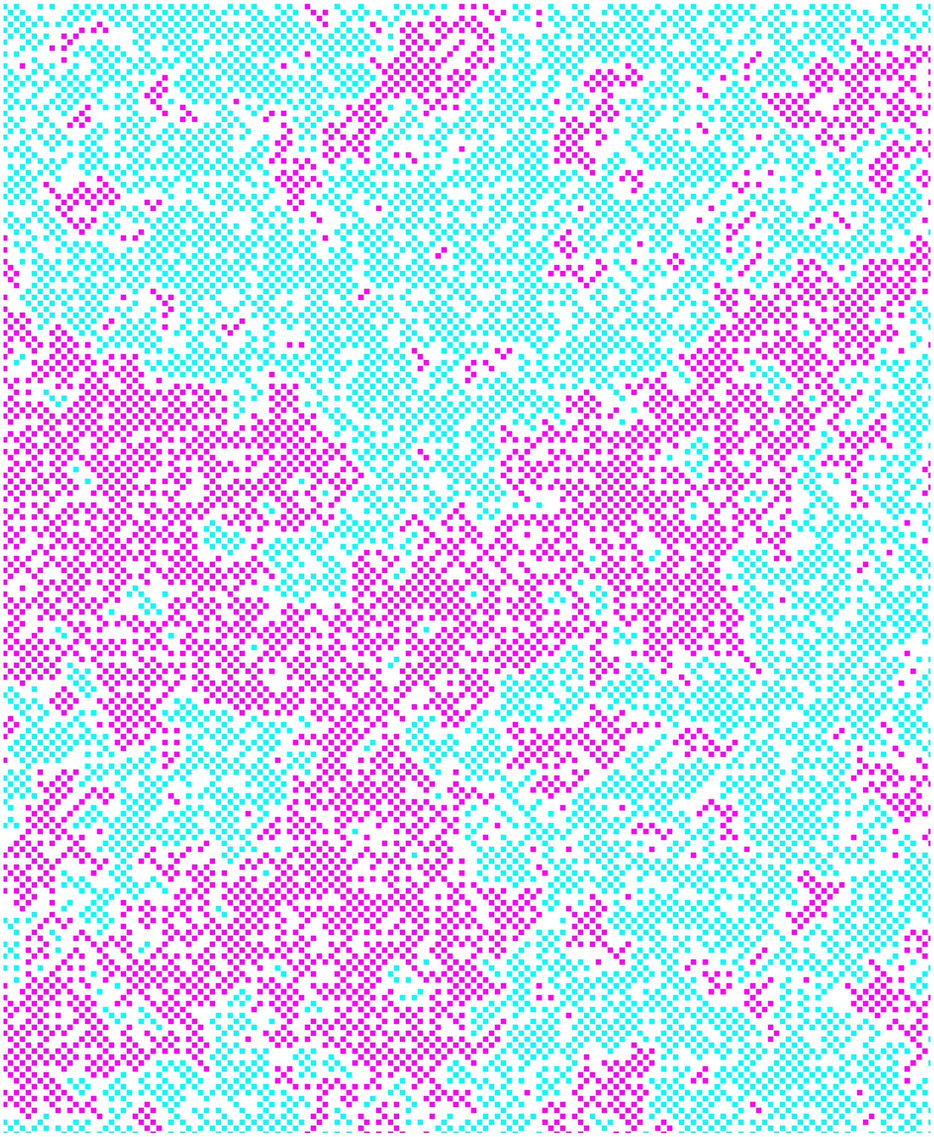


Plate 2

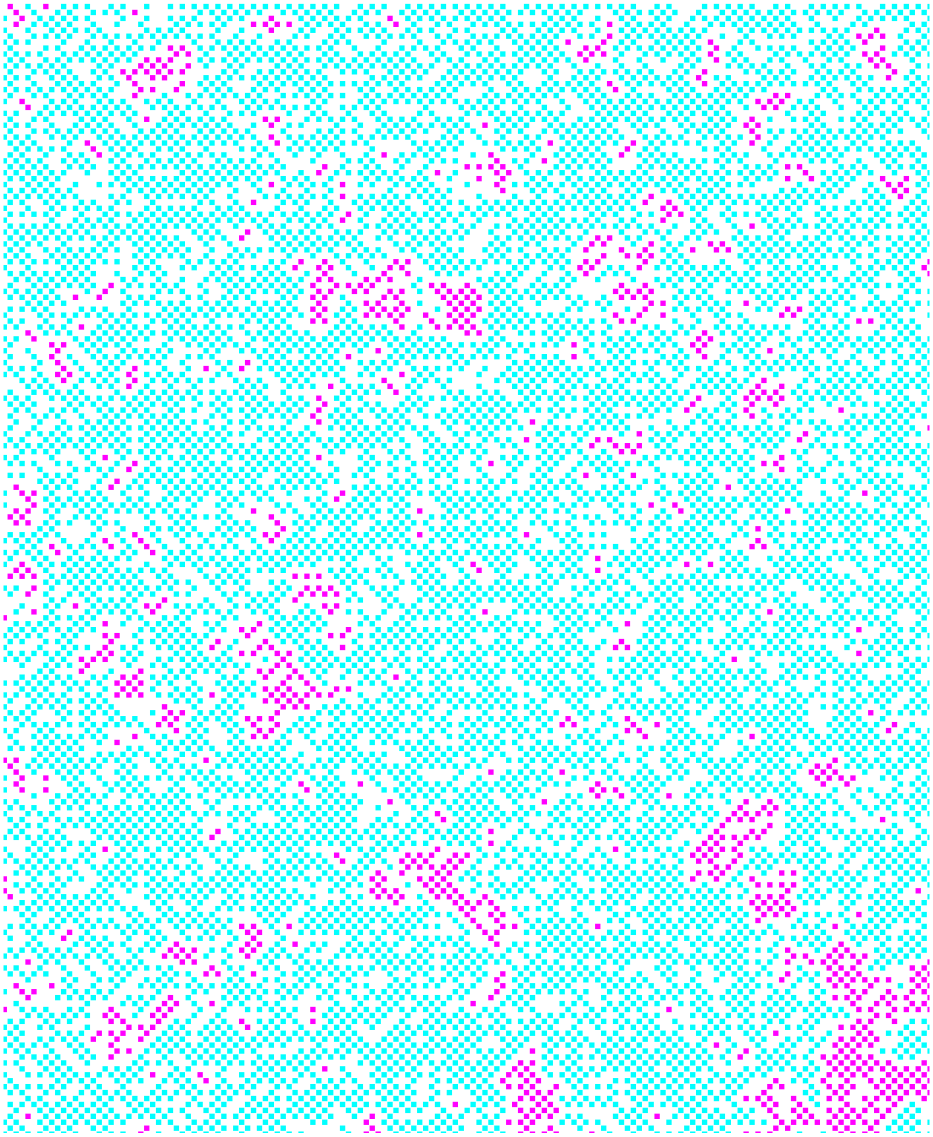
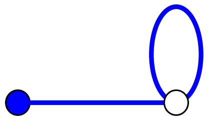
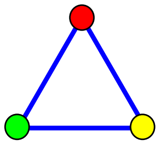


Plate 3

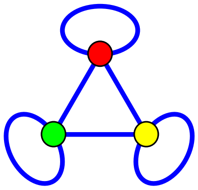
constraint graphs H



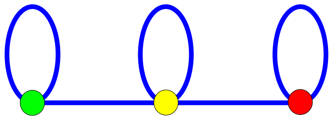
independent
sets
=hard-core model



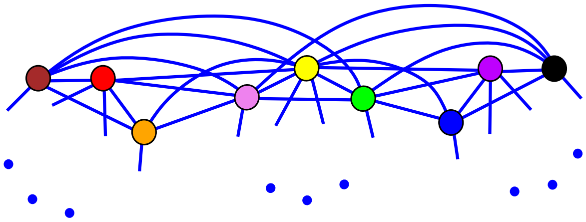
proper
3-colorings



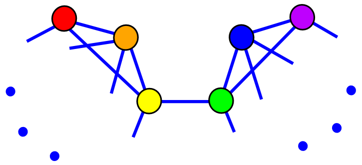
unconstrained
3-colorings



Widom-Rowlinson model



iceberg model

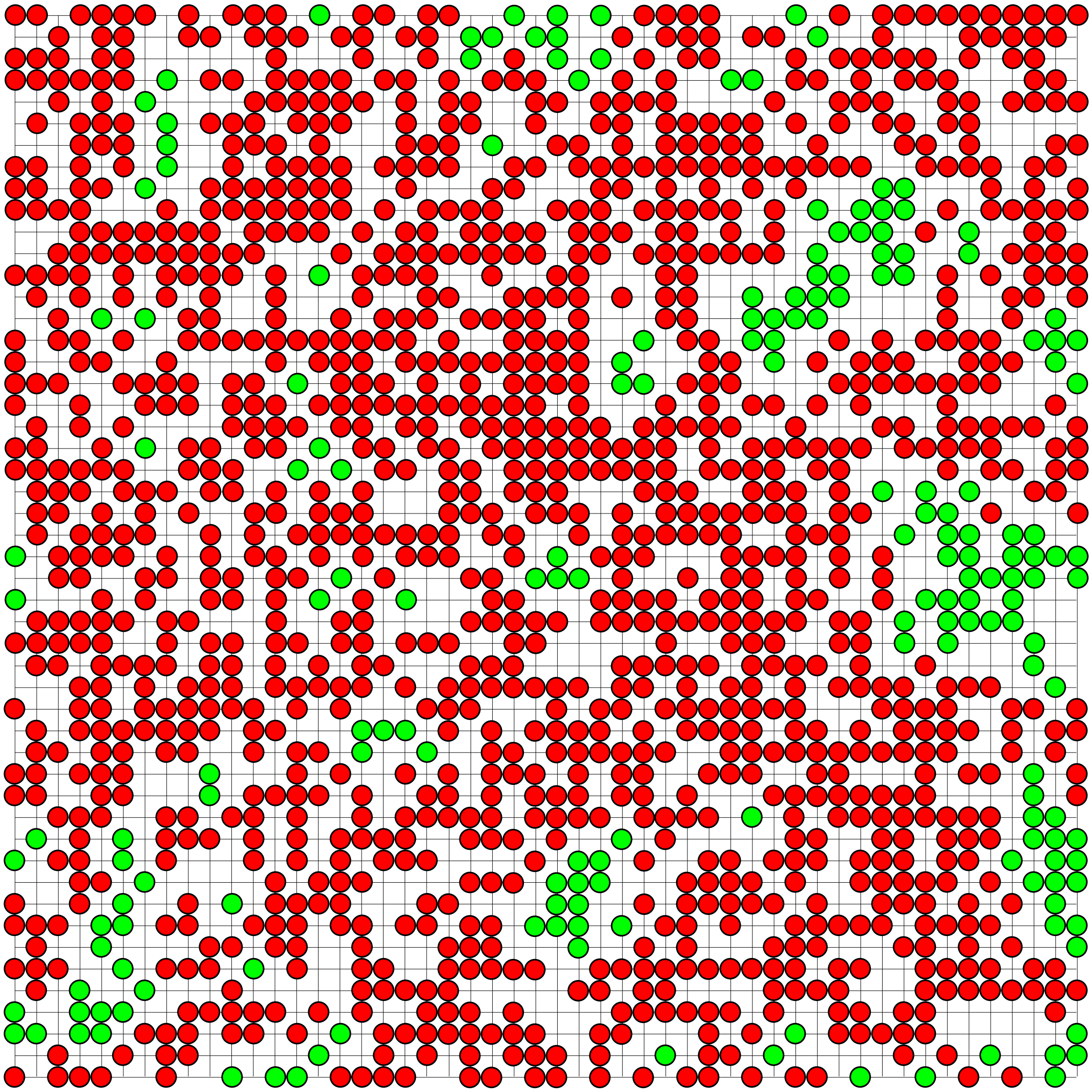


beach model

probability on $\mathrm{Hom}(G,H)$:

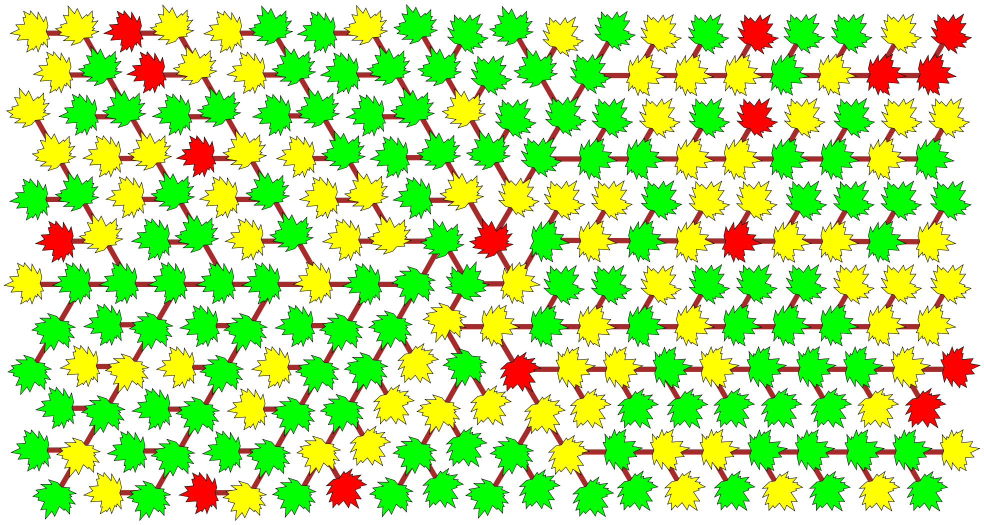
$$\boxed{Pr(\varphi) \sim \prod_{x\in G} \lambda_{\varphi(x)}}$$

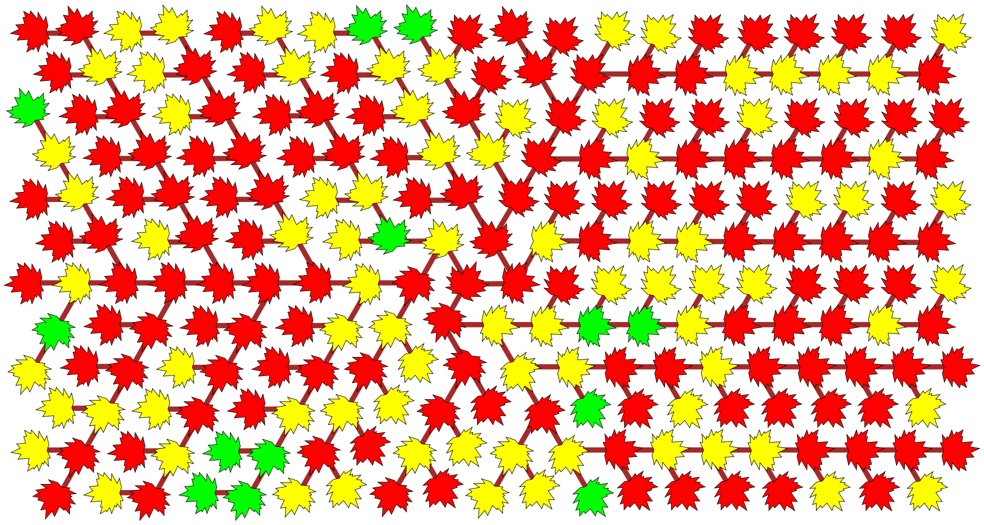
Plate 4



Widom-Rowlinson model on $\mathbb{Z}^2$
(red phase, $\lambda = 1.9$)

Plate 5





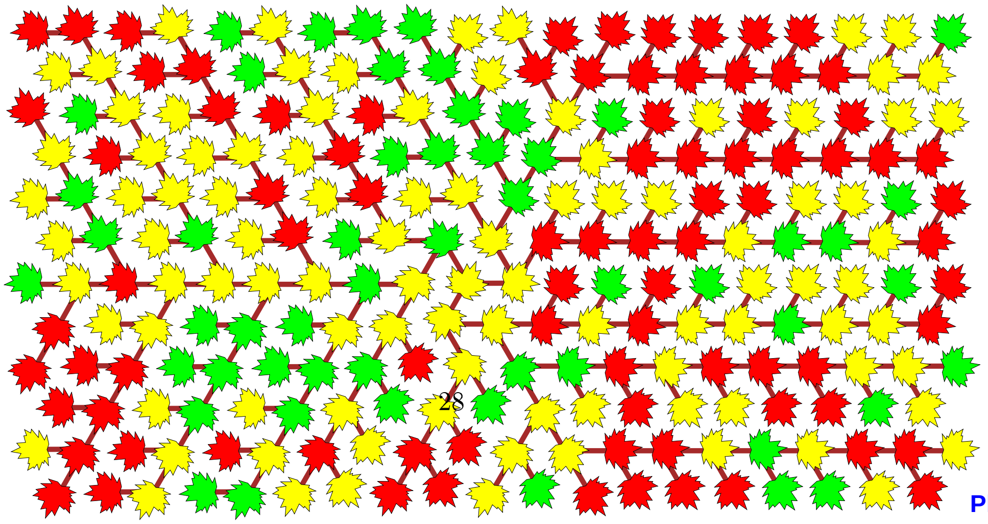
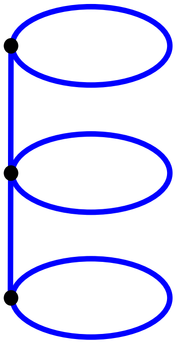
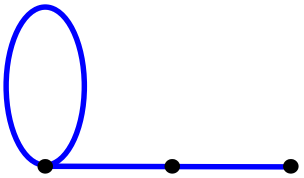


Plate 6

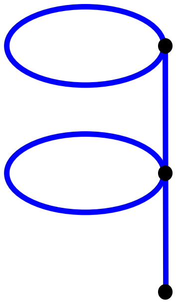
seven “fertile” graphs



the **hinge**

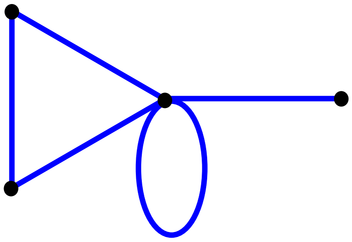


the **pipe**

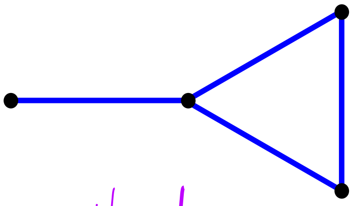


the **wrench**

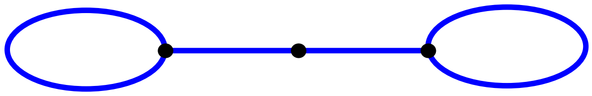
the **stick**



the **gun**



the **key**



the **wand**

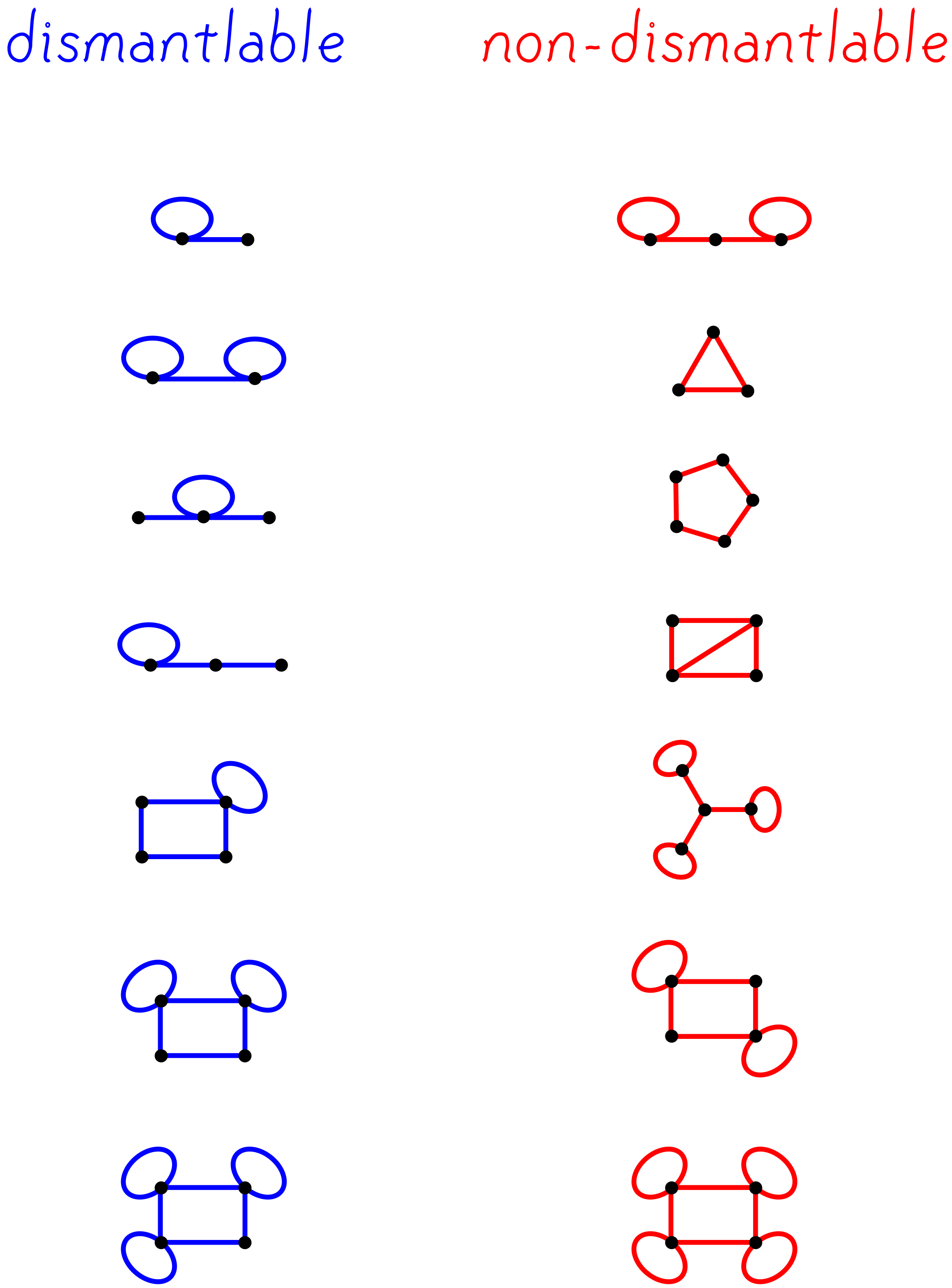
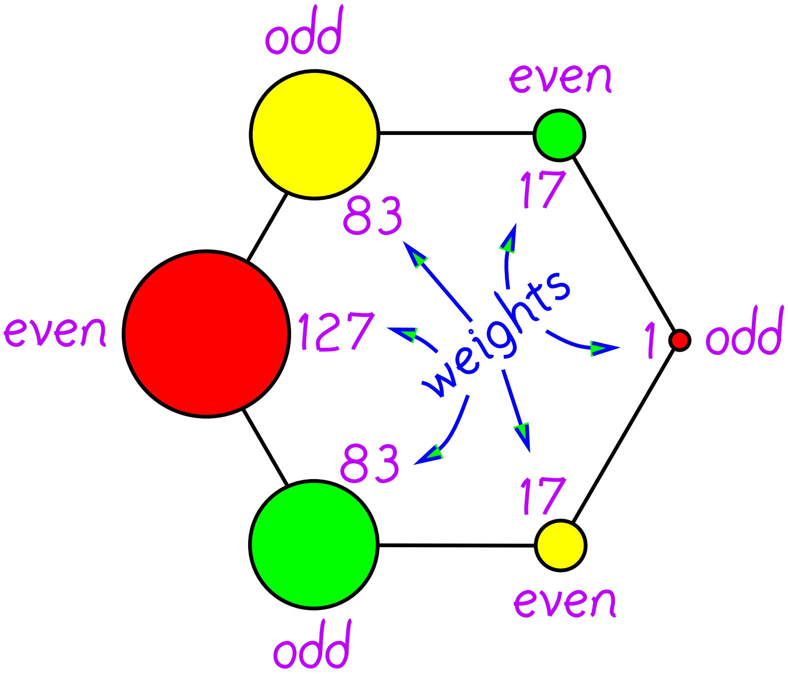
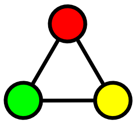


Plate 8



unbalanced phase on $\mathrm{Hom}(T^3,$  $)$

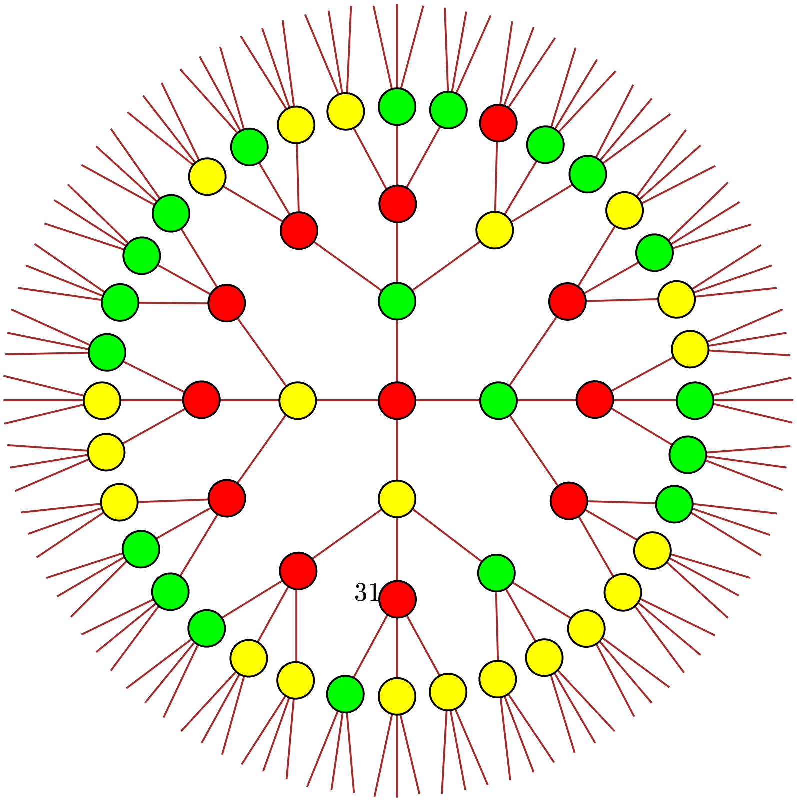
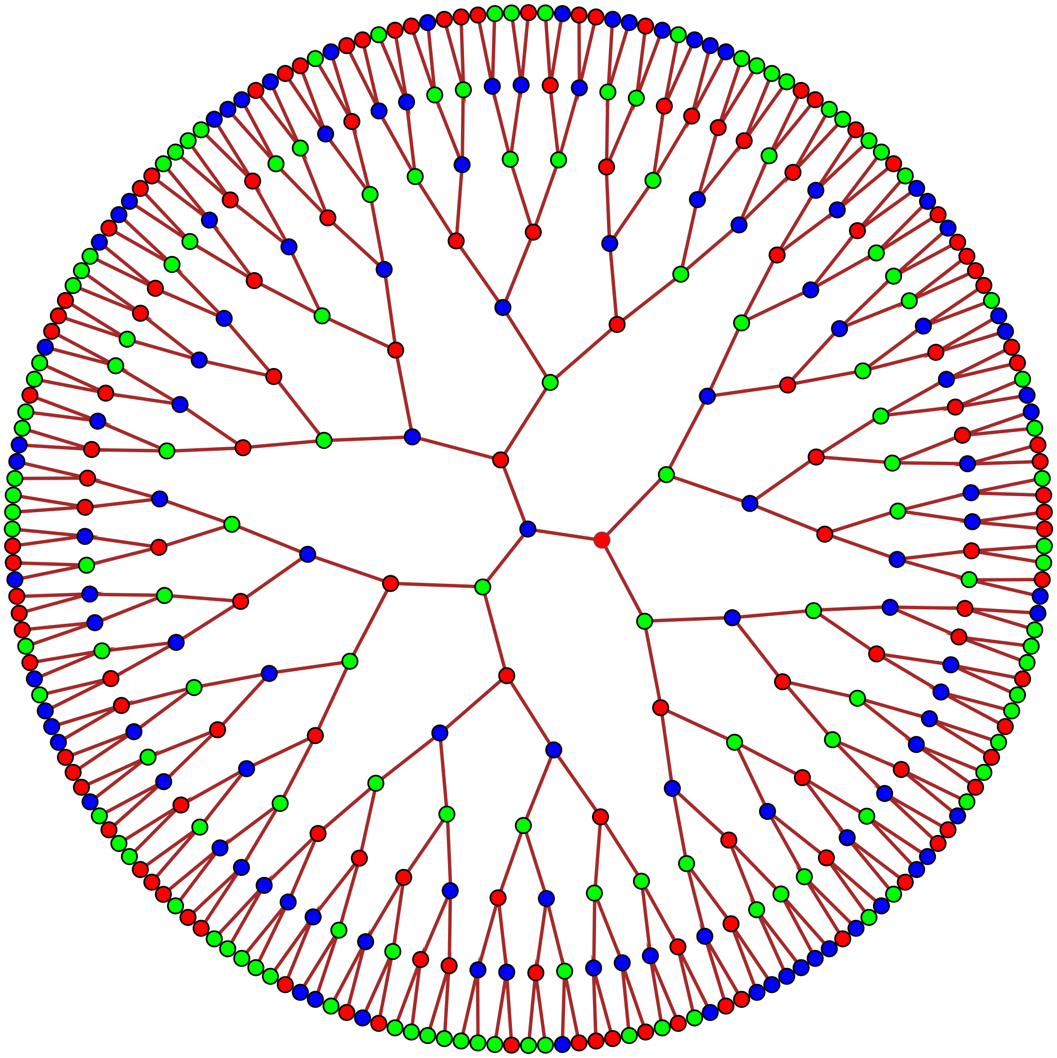


Plate 9



part of a frozen 3-coloring of T^2

Plate 10