Magnetic susceptibility of the 2D Ising model on the finite lattice

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Abstract

The generalization of the form factor representation of the 2D Ising model correlation function to the case of the general disposition of correlating spins on a cylinder is given. The magnetic susceptibility on the lattice where one of the dimensions N is finite is calculated in both the para- and ferromagnetic regions of the Ising coupling parameter. The singularity structure in the complex temperature plane and the thermodynamic limit $N \to \infty$ are discussed.

1 Introduction

The Ising model has long been a subject of great interest. The computation of the free energy [1] and spontaneous magnetization [2], the series [3, 4] and nonlinear differential equations [5, 6] for the correlation functions are the most important advances of the modern mathematical physics. The partition function of the 2D Ising model in zero field was evaluated exactly [7] not only in the thermodynamic limit but also for the finite lattice with different boundary conditions. The simplicity of the corresponding expressions enables to get an idea about the mechanism of the appearance of critical singularities in thermodynamical quantities from both mathematical and physical points of view.

Analytical expressions for the thermodynamical quantities, which contain the dependence on the size of the lattice, have numerous applications. For example, in a computer simulation of thermodynamical systems or field models one often needs such expressions to estimate the number of the degrees of freedom for which discrete numerical model is adequate to initial continuous and infinite system. It is worth mentioning that modern experiments and technologies often deal with finite-size systems. The theoretical analysis of these problems experiences the lack of exactly solvable examples.

In this paper we present exact expressions for the 2-point correlation function and the susceptibility of the 2D Ising model on the lattice with one finite ($N = \text{const}$) and other infinite ($M \to \infty$) dimension. These expressions are very similar to well-known form factor expansions [8], [9]. We investigate the singularity structure of the susceptibility for the finite N and discuss the thermodynamic limit $N \to \infty$.

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2 Correlation function $\langle\sigma(0,0)\sigma(x,0)\rangle$

The Ising model on the $M \times N$ square lattice (Fig. 1) is defined by the hamiltonian $H[\sigma]$

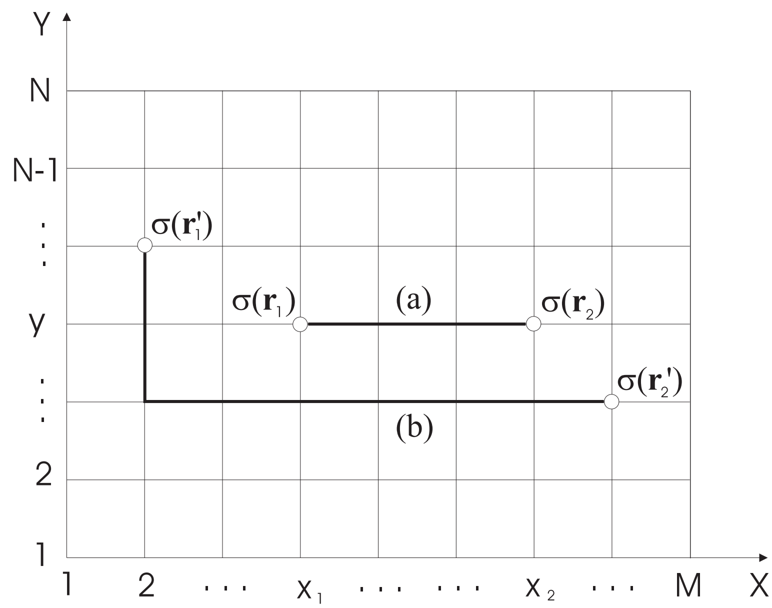

Figure 1: The numeration of the sites of the lattice and the variants of the disposition of correlating spins: a) along the cylinder axis, b) arbitrary disposition of correlating spins on the lattice.

$$H[\sigma] = -J \sum_{\mathbf{r}} \sigma(\mathbf{r})(\nabla_x + \nabla_y)\sigma(\mathbf{r}),$$

where two-dimensional vector $\mathbf{r} = (x, y)$ labels sites of the lattice: $x = 1, 2, \ldots, M$, $y = 1, 2, \ldots, N$; the Ising spin $\sigma(\mathbf{r})$ in each site takes on the values ± 1; $J > 0$ is the coupling constant. Shift operators ∇_x, ∇_y act as follows

$$\nabla_x \sigma(x, y) = \sigma(x+1, y), \quad \nabla_y \sigma(x, y) = \sigma(x, y+1).$$

The partition function and 2-point correlation function at the temperature β^{-1} are defined by

$$Z = \sum_{[\sigma]} \mathrm{e}^{-\beta H[\sigma]}, \tag{2.1}$$

$$\langle\sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)\rangle = Z^{-1} \sum_{[\sigma]} \mathrm{e}^{-\beta H[\sigma]} \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2). \tag{2.2}$$

The summation in these formulae has to be taken over all spin configurations. We will use the following dimensionless parameters

$$K = \beta J, \quad t = \tanh K, \quad s = \sinh 2K. \tag{2.3}$$

We will consider the lattice with periodic boundary conditions for both X and Y directions. This gives two equations for ∇_x, ∇_y

$$(\nabla_x)^M = 1, \quad (\nabla_y)^N = 1.$$

For such boundary conditions the partition function (2.1) can be expressed in terms of four summands [7]

$$Z = (2\cosh^2 K)^{MN} \cdot \frac{1}{2}\Big(Q^{(f,f)} + Q^{(f,b)} + Q^{(b,f)} - Q^{(b,b)}\Big), \tag{2.4}$$

where each of them is the pfaffian of the operator $\widehat{D}$ (the lattice analogue of the Dirac operator)

$$Q = \mathrm{Pf}\,\widehat{D}, \tag{2.5}$$

where

$$\widehat{D} = \begin{pmatrix} 0 & 1 + t\nabla_x & 1 & 1 \\ -1 - t\nabla_{-x} & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 + t\nabla_y \\ -1 & -1 & -1 - t\nabla_{-y} & 0 \end{pmatrix}. \tag{2.6}$$

The upper indices (f, b) of the quantities Q in (2.4) correspond to different types (antiperiodic or periodic) of boundary conditions for the operators ∇_x, ∇_y in (2.6):

$$(\nabla_x^{(b)})^M = (\nabla_y^{(b)})^N = 1, \quad (\nabla_x^{(f)})^M = (\nabla_y^{(f)})^N = -1. \tag{2.7}$$

When, for example, $M \gg N$ (i.e. torus is reduced into cylinder), then in the right hand side of (2.4) only "antiperiodic" term survives:

$$Z = (2\cosh^2 K)^{MN} Q^{(f,f)}. \tag{2.8}$$

Since the operator $\widehat{D}$ is translationally invariant, the pfaffian (2.5) can be easily evaluated. After performing the Fourier transformation, one has the following factorized representation for the partition function (2.8)

$$Z = 2^{MN} \prod_{\mathbf{q}}^{(f,f)} (s^2 + 1 - s \cdot \cos q_x - s \cdot \cos q_y)^{1/2}. \tag{2.9}$$

The upper index (f) in products (or sums hereinafter) implies that the components of quasimomentum q_x and q_y in Brillouin zone run over halfinteger values in the units $2\pi/M$ and $2\pi/N$ respectively; integer values correspond to the index (b). For example,

$$\prod_{q_y}^{(f)} \mathcal{F}(q_y) = \prod_{l=1}^{N} \mathcal{F}\Big(\frac{2\pi}{N}(l + \tfrac{1}{2})\Big), \qquad \prod_{q_y}^{(b)} \mathcal{F}(q_y) = \prod_{l=1}^{N} \mathcal{F}\Big(\frac{2\pi}{N} l\Big).$$

The product over one of the quasimomentum components in the right hand side of (2.9) can be evaluated to the explicit form, so for the partition function one has

$$Z = (2s)^{MN/2} \prod_q^{(f)} \mathrm{e}^{-M\gamma(q)/2} \big(1 + \mathrm{e}^{-M\gamma(q)}\big), \tag{2.10}$$

where the function $\gamma(q)$ is the positive root of the following equation

$$\sinh^2 \frac{\gamma(q)}{2} = \sinh^2 \frac{\mu}{2} + \sin^2 \frac{q}{2}, \tag{2.11}$$

where the parameter μ is the function of s

$$\sinh \frac{\mu}{2} = \frac{1}{\sqrt{2}} \big(\sqrt{s} - 1/\sqrt{s}\big). \tag{2.12}$$

For $q \neq 0$ $\gamma(q)$ remains positive in the whole range of variable $0 < s < \infty$, but $\gamma(0)$ changes its sign after crossing the critical point $s = 1$. Since the product in (2.10) is taken over fermionic spectrum, which does not contain the value $q = 0$, this does not cause any problem here. However, the ambiguity in the definition of $\gamma(0) = \pm\mu$ leads to two different representations for the correlation function.

The sum over spin configurations in the right hand side of (2.2) for the correlation function can also be written in terms of pfaffians [10]. Corresponding matrices, however, are not translationally invariant. This fact crucially complicates the calculations. Nevertheless, the evaluation of the correlation function can be reduced to the evaluation of the determinant of a matrix

$$\langle \sigma(0)\sigma(\mathbf{r}) \rangle = \det A^{(\dim)}, \tag{2.13}$$

with considerably smaller dimension $\dim = \dim(\mathbf{r})$, defined by the distance between correlating spins. Further work is needed to transform the representation (2.13) into the representation with analytical dependence on distance.

Form factor representation for the correlation function of the Ising model is the most acceptable from physical point of view. First it was obtained in [8] for the infinite lattice in the ferromagnetic region ($K > K_c$, $s > 1$). Later it was extended [9] for the paramagnetic case ($K < K_c$, $s < 1$). We note that somewhat earlier similar representation for the 2-point Green function was deduced in [11] via S-matrix approach [12] for a quantum field model with factorized S-matrix ($S_2 = -1$), which is usually associated with the scaling limit of the Ising model. The discovery of the form factor representation for the correlation function has led to the whole trend [13] in the integrable quantum field theory.

For the finite lattice the problem seems to be more difficult, but the result [14] is even simpler in a sense. If correlating spins are located along one of the axes of the lattice, the matrix in the right hand side of (2.13) is of Toeplitz form. For example, when correlating spins are located along the horizontal axis (Fig. 1a), then

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2) \rangle = \det A^{(|x|)}, \qquad \mathbf{r}_2 - \mathbf{r}_1 = (x, 0), \tag{2.14}$$

$|x| \times |x|$ matrix $A^{(|x|)}_{k,k'}$ has the elements [14]

$$A^{(|x|)}_{k,k'} = \frac{1}{MN} \sum_{\mathbf{p}}^{(f,f)} \frac{\mathrm{e}^{ip_x(k-k')}[2t(1+t^2) - (1-t^2)(\mathrm{e}^{ip_x} + t^2\mathrm{e}^{-ip_x})]}{(1+t^2)^2 - 2t(1-t^2)(\cos p_x + \cos p_y)}, \qquad (2.15)$$
$$k, k' = 0,\, 1,\, \ldots\,,\, |x|-1.$$

As it was shown in [14] via Wiener-Hopf integral equations technique [7] adjusted to the finite-sized lattice, the determinant (2.14) can be evaluated analytically and for the correlation function one has

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)\rangle = (\xi \cdot \xi_T)\mathrm{e}^{-|x|/\Lambda} \sum_{l=0}^{[N/2]} g_{2l}(x), \qquad \text{for } \gamma(0) = \mu, \qquad (2.16)$$

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)\rangle = (\xi \cdot \xi_T)\mathrm{e}^{-|x|/\Lambda} \sum_{l=0}^{[(N-1)/2]} g_{2l+1}(x), \qquad \text{for } \gamma(0) = -\mu, \qquad (2.17)$$

$$g_n(x) = \frac{\mathrm{e}^{-n/\Lambda}}{n! N^n} \sum_{[q]}^{(b)} \left(\prod_{i=1}^{n} \frac{\mathrm{e}^{-|x|\gamma_i - \eta_i}}{\sinh \gamma_i} \right) F_n^2[q], \quad g_0 = 1, \qquad (2.18)$$

$$F_n[q] = \prod_{i<j}^{n} \frac{\sin((q_i - q_j)/2)}{\sinh((\gamma_i + \gamma_j)/2)}, \quad F_1 = 1, \qquad (2.19)$$

where $\gamma_i = \gamma(q_i)$, $\eta_i = \eta(q_i)$. The expressions (2.16), (2.17) are the finite sums. However, upper limits of summation can be set infinite, since it follows from (2.19) that form factors $F_n[q]$ vanishes for $n > N$. Note an important detail – summation over the phase volume in (2.18) is taken over bosonic spectrum of quasimomenta, in contrast with initial fermionic spectrum, which defines the matrix (2.15). The other quantities in (2.16)–(2.19) are given by

$$\xi = |1 - s^{-4}|^{1/4}, \qquad (2.20)$$

$$\ln \xi_T = \frac{N^2}{2\pi^2} \int_0^{\pi} \frac{dp\, dq\, \gamma'(p)\gamma'(q)}{\sinh(N\gamma(p)) \sinh(N\gamma(q))} \ln \left| \frac{\sin((p+q)/2)}{\sin((p-q)/2)} \right|, \qquad (2.21)$$

$$\Lambda^{-1} = \frac{1}{\pi} \int_0^{\pi} dp\, \ln \coth(N\gamma(p)/2), \qquad (2.22)$$

$$\eta(q) = \frac{1}{\pi} \int_0^{\pi} \frac{dp\, (\cos p - \mathrm{e}^{-\gamma(q)})}{\cosh \gamma(q) - \cos p} \ln \coth(N\gamma(p)/2). \qquad (2.23)$$

"Cylindrical parameters" ξ_T , Λ^{-1} , $\eta(q)$ explicitly depend on the number of sites N

on the base of the cylinder. Their asymptotic behaviour for $N|\mu| \gg 1$ is following

$$\begin{aligned} \ln \xi_T &\simeq \frac{1}{\pi} \mathrm{e}^{-2N|\mu|}, &(2.24)\\ \Lambda^{-1} &\simeq e^{-N|\mu|} \sqrt{\frac{2\sinh|\mu|}{\pi N}} &(2.25)\\ \eta(q) &\simeq \frac{4\mathrm{e}^{-N|\mu|}}{(\mathrm{e}^{\gamma(q)}-1)} \sqrt{\frac{\sinh|\mu|}{2\pi N}}. &(2.26) \end{aligned}$$

Outside the critical point cylindrical parameters Λ^{-1}, $\ln \xi_T$ and $\eta(q)$ for large N exponentially decrease and turn into zero for the infinite lattice. Finite sums (2.16), (2.17) transform into series, summation over phase volume in (2.18) is substituted by integration and in the issue the form factor representations on the cylinder turn into form factor representations on the infinite lattice [8], [9]. For any finite N both expansions – over even n (2.16) and over odd n (2.17) – are valid in both ferromagnetic $(s > 1)$ and paramagnetic $(s < 1)$ regions. We remind that we started from the determinant (2.14) of a $|x| \times |x|$ matrix. The number of terms in its formal definition rapidly increases when x grows. However, the form factor representations (2.16)–(2.19) are the finite sums for any fixed N, and the number of terms does not depend on $|x|$. This gives a unique opportunity to verify (2.16)–(2.19) by means of comparing with the results of transfer matrix calculations for $N-$ rows Ising chains. For fixed N the dimension of corresponding transfer matrix is equal to $2^N \times 2^N$. One can find analytically all eigenvectors and eigenvalues if N is not too large. We have successfully performed such check analytically for $N = 2, 3, 4$ and numerically – for $N = 5, 6$.

3 Correlation function $\langle\sigma(0,0)\sigma(x,y)\rangle$

The rigorous derivation of the form factor representation on the cylinder was performed in [14] only for the spins displaced along the cylinder axis. We have not yet succeeded in generalization of the method for arbitrary disposition of correlating spins (Fig. 1b). Meanwhile, the evaluation of the momentum representation of correlation function

$$\widetilde{G}(\mathbf{p}) = \sum_{\mathbf{r}} \mathrm{e}^{i\mathbf{pr}} \langle\sigma(0)\sigma(\mathbf{r})\rangle, \tag{3.1}$$

or the susceptibility (which is connected with $\widetilde{G}(\mathbf{p} = 0)$) requires explicit dependence on both components of the vector $\mathbf{r}$. Form factor representations (2.16)–(2.19) have a transparent physical content. This allows to make reasonable assumptions for corresponding generalizations. The above mentioned possibility of independent check allows to eliminate wrong hypotheses and to make correct choice. In principle, when y-component of the vector $\mathbf{r}$ is not zero, all quantities in (2.16)–(2.19) could change their form. Corresponding expressions for free bosons and fermions on the lattice prompt one of the simplest generalizations – just the substitution

$$\mathrm{e}^{-|x|\gamma(q)} \longrightarrow \mathrm{e}^{-|x|\gamma(q)-iyq}.$$

Really amazing that it is enough. If instead of $g_n(x)$ (2.18) one uses the expression

$$g_n(\mathbf{r}) = \frac{\mathrm{e}^{-n/\Lambda}}{n!N^n} \sum_{[q]}^{(b)} \prod_{j=1}^{n} \left(\frac{\mathrm{e}^{-|x|\gamma_j - iyq_j - \eta_j}}{\sinh \gamma_j} \right) F_n^2[q], \quad g_0 = 1, \tag{3.2}$$

then correlation functions (2.16) and (2.17) exactly coincide with transfer matrix results for $N = 2, 3, 4$ in the whole range of variables x, y, K. Numerical calculations confirm this for $N = 5, 6$ also. The validity of (3.2) is out of doubts and we hope that the known answer will simplify the problem of its rigorous derivation.

Let us illustrate the matter by the example of $N = 3$. The expansion (2.16)–(2.17) are very similar to the representation of the correlation function in terms of eigenvalues of the transfer matrix

$$\langle \sigma(0)\sigma(\mathbf{r}) \rangle = a_1(y)(\lambda_1/\lambda_0)^{|x|} + a_2(y)(\lambda_2/\lambda_0)^{|x|} + \cdots, \tag{3.3}$$

where λ_0 is the largest eigenvalue, coefficients $a_j(y)$ are given by some bilinear combinations of eigenvectors. To reduce, for example, (2.17) to (3.3), we use the following expressions for cylindrical parameters ξ_T, Λ^{-1}, $\eta(q)$

$$\Lambda^{-1} = \frac{1}{2} \left(\sum_q^{(f)} \gamma(q) - \sum_q^{(b)} \gamma(q) \right), \tag{3.4}$$

$$\mathrm{e}^{-\Lambda^{-1} - \eta(q_i)} = \frac{\prod_q^{(b)} \sinh\left(\frac{\gamma(q)+\gamma(q_i)}{2}\right)}{\prod_q^{(f)} \sinh\left(\frac{\gamma(q)+\gamma(q_i)}{2}\right)}, \tag{3.5}$$

$$\xi_T^4 = \frac{\prod_q^{(b)} \prod_p^{(f)} \sinh^2\left(\frac{\gamma(q)+\gamma(p)}{2}\right)}{\prod_q^{(b)} \prod_p^{(b)} \sinh\left(\frac{\gamma(q)+\gamma(p)}{2}\right) \prod_q^{(f)} \prod_p^{(f)} \sinh\left(\frac{\gamma(q)+\gamma(p)}{2}\right)}. \tag{3.6}$$

One can derive these expressions from (2.21)–(2.23) by the substitution of the integration variable $z = \mathrm{e}^{iq}$ and computing the residues after proper squeezing the integration contours.

For $N = 3$ we have from (3.4)–(3.6) and (2.20)

$$\Lambda^{-1} = \frac{1}{2} \big[\gamma(\pi) + 2\gamma(\pi/3) - \gamma(0) - 2\gamma(2\pi/3) \big], \tag{3.7}$$

$$\xi\xi_T = \frac{\sinh\frac{\gamma(0)+\gamma(\pi/3)}{2} \sinh\frac{\gamma(\pi)+\gamma(2\pi/3)}{2} \sinh^2\frac{\gamma(2\pi/3)+\gamma(\pi/3)}{2}}{\sinh\frac{\gamma(0)+\gamma(2\pi/3)}{2} \sinh\frac{\gamma(\pi)+\gamma(\pi/3)}{2} \sinh\gamma(\pi/3) \sinh\gamma(2\pi/3)}, \tag{3.8}$$

$$e^{-\Lambda^{-1} - \eta(q)} = \frac{\sinh\frac{\gamma(0)+\gamma(q)}{2} \sinh^2\frac{\gamma(2\pi/3)+\gamma(q)}{2}}{\sinh\frac{\gamma(\pi)+\gamma(q)}{2} \sinh^2\frac{\gamma(\pi/3)+\gamma(q)}{2}}. \tag{3.9}$$

Finally,

$$\ln(\lambda_0/\lambda_1) = \Lambda^{-1} + \gamma(0), \tag{3.10}$$

$$\ln(\lambda_0/\lambda_2) = \Lambda^{-1} + \gamma(2\pi/3), \tag{3.11}$$

$$\ln(\lambda_0/\lambda_3) = \Lambda^{-1} + \gamma(0) + 2\gamma(2\pi/3), \tag{3.12}$$

$$a_1(y) = \frac{1}{3}\frac{\sinh\frac{\gamma(0)+\gamma(2\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(2\pi/3)}{2}\sinh^2\frac{\gamma(2\pi/3)+\gamma(\pi/3)}{2}}{\sinh\frac{\gamma(0)+\gamma(\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(\pi/3)}{2}\sinh\gamma(\pi/3)\sinh\gamma(2\pi/3)}, \tag{3.13}$$

$$a_2(y) = \frac{2}{3}\frac{\sinh\frac{\gamma(0)+\gamma(\pi/3)}{2}\sinh\frac{\gamma(0)+\gamma(\pi)}{2}}{\sinh\gamma(\pi/3)\sinh\frac{\gamma(\pi/3)+\gamma(\pi)}{2}}\cos(2\pi y/3), \tag{3.14}$$

$$a_3(y) = \frac{1}{64}\frac{1}{\sinh\frac{\gamma(0)+\gamma(\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(\pi/3)}{2}\sinh\frac{\gamma(0)+\gamma(2\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(2\pi/3)}{2}}\times \tag{3.15}$$

$$\times\frac{1}{\sinh\gamma(\pi/3)\sinh\gamma(2\pi/3)\sinh^2\frac{\gamma(\pi/3)+\gamma(2\pi/3)}{2}}.$$

The transfer matrix $2^3\times 2^3$ has 8 eigenvalues, some of them are equal. Besides that, some eigenvectors have zero components. As result, the expression for the correlation function (3.3) contains only three (not seven) independent terms. If we take into account the definition (2.11), (2.12) of the function $\gamma(q)$ for particular values of quasimomentum $q = 0,\ \pi/3,\ 2\pi/3,\ \pi$, we get exact correspondence between this three terms and (3.10)–(3.15).

4 Momentum representation of the correlation function

Since we have the expression (3.2) for $g_n(\mathbf{r})$, which depends on both components of $\mathbf{r}$, we can make the Fourier transform. Let us write the momentum representation of (3.1) in the form similar to (2.16)–(2.17)

$$\widetilde{G}(\mathbf{p}) = \xi\xi_T\sum_n \widetilde{g}_n(\mathbf{p}), \tag{4.1}$$

$$\widetilde{g}_n(\mathbf{p}) = \sum_{\mathbf{r}} \mathrm{e}^{-|x|/\Lambda} g_n(\mathbf{r})\mathrm{e}^{i\mathbf{pr}}, \tag{4.2}$$

where

$$\sum_{\mathbf{r}} = \sum_{x=-\infty}^{\infty}\sum_{y=1}^{N}. \tag{4.3}$$

After performing the summation in (4.2) we have

$$\widetilde{g}_n(\mathbf{p}) = \frac{\mathrm{e}^{n/\Lambda}}{n!N^{n-1}}\sum_{[q]}^{(b)}\left(\prod_{j=1}^{n}\frac{\mathrm{e}^{-\eta_j}}{\sinh\gamma_j}\right)\frac{\sinh\left(\Lambda^{-1}+\sum\limits_{j=1}^{n}\gamma_j\right)F_n^2[q]}{\cosh\left(\Lambda^{-1}+\sum\limits_{j=1}^{n}\gamma_j\right)-\cos p_x}\delta\left(p_y-\sum_{j=1}^{n}q_j\right). \tag{4.4}$$

The component p_x of the quasimomentum has a continuous spectrum in the range $[-\pi, \pi]$, but the p_y is discrete

$$p_y = \frac{2\pi l}{N}, \qquad l = 1,\, 2\ \ldots\ N.$$

Corresponding δ-function in the right hand side of (4.4) is understood as the Kronecker symbol

$$\delta\Big(p_y - \sum_{j=1}^{n} q_j\Big) = \delta\Big(l - \sum_{j=1}^{n} l_j\Big)\ \Big|_{\mathrm{mod}\, N}\ .$$

The function $\widetilde{g}_n(\mathbf{p})$ is periodic in p_x, p_y with the period 2π. After inserting the "unity"

$$1 = \int\limits_{\Lambda^{-1}+n\gamma(0)}^{\Lambda^{-1}+n\gamma(\pi)} d\omega\, \delta\Big(\Lambda^{-1} + \sum_{j=1}^{n} \gamma_j - \omega\Big),$$

in the sum (4.4) (here δ denotes Dirac δ-function) and changing the order of integration we obtain

$$\widetilde{g}_n(\mathbf{p}) = \int\limits_{\Lambda^{-1}+n\gamma(0)}^{\Lambda^{-1}+n\gamma(\pi)} d\omega \frac{\sinh\omega}{\cosh\omega - \cos p_x} \rho_n(\omega, p_y), \tag{4.5}$$

$$\rho_n(\omega, p_y) = \frac{\mathrm{e}^{-n/\Lambda}}{n! N^{n-1}} \sum_{[q]}{}^{(b)} \Big(\prod_{j=1}^{n} \frac{\mathrm{e}^{-\eta_j}}{\sinh\gamma_j}\Big) F_n^2[q]\, \delta\Big(\Lambda^{-1} + \sum_{j=1}^{n}\gamma_j - \omega\Big)\, \delta\Big(p_y - \sum_{j=1}^{n} q_j\Big). \tag{4.6}$$

On the infinite lattice in the scaling limit the rotational symmetry is restored and (4.5), (4.6) turn into classical Lehmann representation in the quantum field theory.

5 Magnetic susceptibility

On the $M \times N$ square lattice with equal horizontal and vertical coupling parameters the partition function Z depends on four variables

$$Z = Z(K, h, N, M) = \sum_{[\sigma]} \mathrm{e}^{-\beta H[\sigma] + h\sum\limits_{\mathbf{r}} \sigma(\mathbf{r})}, \tag{5.1}$$

where dimensionless parameter $h = \beta\mathcal{H}$, $\mathcal{H}$ – magnetic field. The specific magnetization $\mathfrak{M}$ and magnetic susceptibility χ can be expressed through field derivatives of the partition function

$$\mathfrak{M}(K, h, N, M) = \frac{1}{MN} \frac{\partial \ln Z}{\partial h} = \langle\sigma\rangle, \tag{5.2}$$

$$\beta^{-1}\chi(K,h,N,M) = \frac{\partial\mathfrak{M}}{\partial h} = \sum_{\mathbf{r}}\Big(\langle\sigma(0)\sigma(\mathbf{r})\rangle - \langle\sigma\rangle^2\Big). \tag{5.3}$$

The magnetization at $h=0$ and finite M, N turns into zero due to Z_2-symmetry of the Ising model. This holds even when one of the dimensions is set infinite. In the last case, when, for example, $M\to\infty$, $N=\mathrm{const}$, 2D Ising model transforms into 1D chain with N rows, for which a spontaneous symmetry breaking is impossible. The susceptibility can be easily computed from (4.1)–(4.4)

$$\chi = \chi_0 + \sum_{l=1}^{[N/2]}\chi_{2l} \qquad \text{for } \gamma(0)=\mu, \tag{5.4}$$

$$\beta^{-1}\chi_0 = \xi\xi_T N\coth(1/2\Lambda), \tag{5.5}$$

$$\chi = \sum_{l=0}^{[(N-1)/2]}\chi_{2l+1} \qquad \text{for } \gamma(0)=-\mu, \tag{5.6}$$

$$\beta^{-1}\chi_n = \frac{\mathrm{e}^{-n/\Lambda}}{n!N^{n-1}}\sum_{[q]}^{(b)}\left(\prod_{i=1}^{n}\frac{\mathrm{e}^{-\eta_i}}{\sinh\gamma_i}\right)F_n^2[q]\coth\left[\frac{1}{2}\left(\Lambda^{-1}+\sum_{i=1}^{n}\gamma_i\right)\right]\delta\left(\sum_{i=1}^{n}q_i\right). \tag{5.7}$$

In paramagnetic region $(s<1)$ the expression (5.6) admits the limit $N\to\infty$ and turns into the susceptibility on the infinite lattice. However, in the ferromagnetic region $(s>1)$ one can make the limit $N\to\infty$ only for the quantity χ_F

$$\chi_F = \chi - \chi_0 = \sum_{l=1}^{\infty}\chi_{2l}, \tag{5.8}$$

which reproduces well-known zero-field ferromagnetic susceptibility of the Ising model in thermodynamic limit. For large but finite N the main contribution to the susceptibility is given by the term χ_0

$$\beta^{-1}\chi_0 \simeq 2\xi N\Lambda \simeq \frac{\sqrt{\pi}\xi N^{3/2}}{\sqrt{\sinh|\mu|}}\mathrm{e}^{N|\mu|}, \tag{5.9}$$

which exponentially increases with the growth of the size of the cylinder base. It follows from (5.9) that the larger N – the smaller field $\delta h\sim \mathrm{e}^{-N|\mu|}$ is needed to order all spins on the lattice.

Unfortunately, the exact solution for the partition function of the Ising model in external field is not known. However, the very fact of the appearance of spontaneous magnetization can be deduced from the analysis of high- and low-temperature expansions. The rigorous definition of spontaneous magnetization is given by the following order of limits according to the Bogolyubov concept of quasiaverages

$$\mathfrak{M}_0(K) = \lim_{h\to 0}\Big[\lim_{M,N\to\infty}\mathfrak{M}(K,h,N,M)\Big]. \tag{5.10}$$

However, if we conjecture the decreasing of correlations at large distances and the possibility of interchanging of corresponding limits, we can find the exact solution for the squared spontaneous magnetization. It is equal to spin-spin correlation function (2.20) with infinite distance between correlating spins

$$\mathfrak{M}_0^2(K) = \lim_{|\mathbf{r}|\to\infty} \langle \sigma(0)\sigma(\mathbf{r})\rangle = \langle \sigma(0)\rangle\langle\sigma(\infty)\rangle = \langle\sigma\rangle^2 = \xi. \tag{5.11}$$

Meanwhile, the sums over lattice of each summand in the right hand side of (5.3) do not converge in the thermodynamic limit. Therefore, the substitution of $\mathfrak{M}^2(K,0,\infty,\infty)$ by the limiting value of correlation function (which equals ξ) under the (infinite) sum in the last step of the limits $h\to 0$, $M,N\to\infty$ requires not only (5.11), but also the existence of the limit

$$\lim_{h\to 0}\Big\{\lim_{M,N\to\infty} MN\big[\mathfrak{M}^2(K,h,M,N)-\xi\big]\Big\} = f(K), \tag{5.12}$$

and, moreover,

$$f(K) = 0. \tag{5.13}$$

The explicit dependence of the correlation function on the size N, namely, the exponential tending of cylindrical parameters to their limiting values (2.24)–(2.26), can be viewed as an argument in favor of the equalities (5.12), (5.13).

The behaviour of correlation function at large distances in the ferromagnetic region is mainly defined by the first term in the expansion (2.16). Note that it does not depend on y-projection of $\mathbf{r}$

$$G_0(|\mathbf{r}|) = \xi\xi_T \mathrm{e}^{-|x|/\Lambda}. \tag{5.14}$$

Therefore, the distance $\sim\Lambda$, for which spins are strongly correlated, rapidly increases (cf (2.25)) with the growth of N. Physically it means that for "ferromagnetic" temperatures the cylinder is grained into "domains" of size $\sim\Lambda$ with nonzero magnetization, the magnetization of the whole infinite cylinder is being equal to zero. It is clear that the squared spontaneous magnetization would be more naturally defined by the value of the correlation function at large distances $|\mathbf{r}| = R(N)$, which do not exceed the size of the domain

$$N \ll R(N) \ll \Lambda.$$

It follows from (2.25) that for sufficiently large N these inequalities can be satisfied. In accordance with this, the sum over x with infinite limits in the definition of the thermodynamic limit of susceptibility (5.3) has to be substituted by the sum with the limits that do not exceed the size of the domain. In this case the condition

$$\sum_{x=-R}^{R}\sum_{y=1}^{N}[G_0(|\mathbf{r}|) - G_0(R)] \simeq \xi N R^2/\Lambda \underset{N\to\infty}{\longrightarrow} 0,$$

can be treated as a formal substantiation of the definition (5.8) of susceptibility in the ferromagnetic phase. We can now estimate the "parameter of thermodynamic cutting" $R(N)$

$$R(N) \ll \sqrt{\Lambda/N\xi} \simeq \mathrm{e}^{N|\mu|/2}[\pi/(2N\sinh|\mu|)]^{1/4}.$$

We suppose that these estimations slightly clarify the physical content of the formal thermodynamic limit procedure.

6 Singularity structure

The initial expression (2.1) for the partition function of the Ising model is a polynomial in s, and the solution (2.9) is the factorized form of this polynomial. It provides an example of the mechanism of Lee-Yang "zeros" [15], which stipulates the appearance of critical singularities in the thermodynamic limit. The roots of the polynomial (2.9) are located on the unit circle $|s| = 1$ in the complex s plane. For finite M and N the zero on the real axis $s = 1$ does not appear, since the fermionic spectrum does not contain the value of quasimomentum $q_x = q_y = 0$. When one of the dimensions increases then zeros are concentrated on the circle $|s| = 1$, forming a dense set. In the limit $M \to \infty$, $N = \mathrm{const}$ they are transformed into finite number (which equals N) of the root type branchpoints, located on the circle $|s| = 1$. To make sure of this, one has to use the representation (2.10) and definition (2.11), (2.12) of the function $\gamma(q)$. These branchpoints, in turn, form a dense set with the growth of N, but in the limit $N \to \infty$ they are transformed into four isolated logarithmic branchpoints $s = \pm 1, \pm i$. As result, the specific heat in the thermodynamic limit acquires the logarithmic divergence $\sim \ln|1-s|$. It is worth noticing that the specific heat is expressed through the same function in both ferromagnetic and paramagnetic regions of s contrary to the susceptibility.

One would think that the similar picture holds for susceptibility. Indeed, the initial expression (2.2) for the correlation function for finite M and N is a ratio of polynomials in s. The formation of the singularities of the partition function, which stands in the denominator, we have just briefly described. Unfortunately, the polynomial in the numerator cannot be written in such simple factorized form. Nevertheless, our form factor representation for $M \to \infty$ and finite N shows that correlation function has a finite number of root branchpoints on the circle $|s| = 1$. Their number is doubled in comparison with the case of partition function, since the expressions (2.16)–(2.19), (3.2) contain functions $\gamma(q)$ (2.11), corresponding to both bosonic and fermionic values of quasimomentum. The susceptibility on the cylinder is given by the infinite sum of correlation functions and this can lead to the appearance of additional singularities. One can show, however, that these singularities do not appear on the first sheet of the Riemann surface.

As an example, let us write down the susceptibility χ (5.4) for $N = 3$, using the expressions (3.13)–(3.15) and representations (3.7)–(3.9) for cylindrical parameters

$$\begin{aligned}\beta^{-1}\chi \;=\; & \frac{\sinh\frac{\gamma(0)+\gamma(2\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(2\pi/3)}{2}\sinh^2\frac{\gamma(2\pi/3)+\gamma(\pi/3)}{2}}{\sinh\frac{\gamma(0)+\gamma(\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(\pi/3)}{2}\sinh\gamma(\pi/3)\sinh\gamma(2\pi/3)}\coth\left(\frac{\Lambda^{-1}+\gamma(0)}{2}\right)+\\ & +\frac{1}{64}\frac{1}{\sinh\frac{\gamma(0)+\gamma(\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(\pi/3)}{2}\sinh\frac{\gamma(0)+\gamma(2\pi/3)}{2}\sinh\frac{\gamma(\pi)+\gamma(2\pi/3)}{2}}\times\\ & \times\frac{1}{\sinh\gamma(\pi/3)\sinh\gamma(2\pi/3)\sinh^2\frac{\gamma(\pi/3)+\gamma(2\pi/3)}{2}}\coth\left(\frac{\Lambda^{-1}+\gamma(0)+2\gamma(2\pi/3)}{2}\right).\end{aligned} \tag{6.1}$$

The singularities in s could appear due to zero denominator in (6.1). It is easily seen, however, that corresponding factors

$$\sinh\frac{\gamma(q)+\gamma(q')}{2} = (\cos q' - \cos q)/\sinh\frac{\gamma(q)-\gamma(q')}{2}$$

for $q \neq q'$ do not equal zero. It can be also shown that on the first sheet of the Riemann surface (which is defined by the condition of positivity of $\gamma(q)$, treated as functions of s, for real $s > 0$) the arguments of cotangents in (6.1) also do not equal zero: these factors appear as the result of summation over coordinate x. Therefore, the complete set of the singularities of susceptibility is exhausted by the branchpoints contained in functions

$$\mathrm{e}^{\gamma(q)} = \left[\sqrt{\frac{1}{2}(s+s^{-1}) + \sin^2\frac{q}{2}} + \sqrt{\frac{1}{2}(s+s^{-1}) - \cos^2\frac{q}{2}}\right]^2 \tag{6.2}$$

For each value of quasimomentum $q \neq 0, \pi$ the function (6.2) has four branchpoints. If we denote them by $s_c = |s_c|\mathrm{e}^{\pm i\varphi_c}$, then

$$|s_c| = 1, \quad \cos\varphi_c = \begin{cases} \cos^2 q/2 \\ -\sin^2 q/2 \end{cases}. \tag{6.3}$$

It is seen from (6.2), that for $q = 0, \pi$ there exist only two branchpoints $s_c = \pm i$. One can now calculate that for any fixed N the whole number of singularities is equal to $4N - 2$, and all singularities are located on the unit circle $|s| = 1$. We represent the corresponding picture for $N = 3$ in the Fig. 2. We do not discuss the limit $N \to \infty$, when the

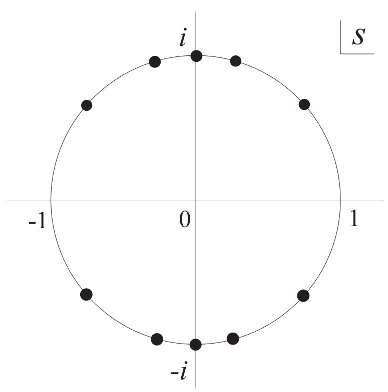

Figure 2: The location of the singularities of susceptibility χ in the complex plane $s = \sinh 2\beta J$ for $N = 3$.

singularities on the circle $|s| = 1$ form a dense set. This problem was seriously analyzed in [16]–[17] and the authors assume (not prove) that the singularities form a natural boundary $|s| = 1$ for the susceptibility.

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