

**Universal Properties of Self-Avoiding
Walks from Two-Dimensional
Field Theory**

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ABSTRACT: We use the recently conjectured exact S -matrix of the massive $O(n)$ model to derive its form factors and ground state energy. This information is then used in the limit $n \rightarrow 0$ to obtain quantitative results for various universal properties of self-avoiding chains and loops. In particular, we give the first theoretical prediction of the amplitude ratio C/D which relates the mean square end-to-end distance of chains to the mean square radius of gyration of closed loops. This agrees with the results from lattice enumeration studies to within their errors, and gives strong support for the various assumptions which enter into the field theoretic derivation. In addition, we obtain results for the scaling function of the structure factor of long loops, and for various amplitude ratios measuring the shape of self-avoiding chains. These quantities are all related to moments of correlation functions which are evaluated as a sum over m -particle intermediate states in the corresponding field theory. We show that in almost all cases, the restriction to $m \leq 2$ gives results which are accurate to at least one part in 10^3 . This remarkable fact is traced to a softening of the $m > 2$ branch cuts relative to their behaviour based on phase space arguments alone, a result which follows from the threshold behaviour of the two-body S -matrix, $S(0) = -1$. Since this is a general property of interacting 2d field theories, it suggests that similar approximations may well hold for other models. However, we also study the moments of the area of self-avoiding loops, and show that, in this case, the 2-particle approximation is not valid.

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1. INTRODUCTION

In the past few years, there has been remarkable progress in obtaining exact results for two-dimensional field theories^{1,2}. One application of these is to the scaling limit of classical statistical mechanics systems, close to a second order phase transition where the correlation length ξ is much larger than the microscopic cut-off. Yet, besides checking the values of critical exponents and other universal finite-size scaling amplitudes which arise from studies of massless, or conformal, field theories, there has been little in the way of direct confrontation with statistical mechanics³. This is partly because the simplest predictions of *massive* quantum field theories are in terms of their particle content and S -matrix elements, as quantum field theories in $1 + 1$ dimensions. In order to make contact with easily observable quantities relevant to statistical mechanics, it is necessary to go further and derive off-shell behaviour. More recently, these problems have become tractable, at least for a class of integrable theories, with the development of methods for calculating off-shell form factors^{4,5} and the technique of the thermodynamic Bethe ansatz^{6,7}.

However, another obstacle is the lack of precise information with which to compare such predictions. Aside from the Ising model^{8,9} little is known about off-critical correlation functions in statistical models. Ideally, of course, one should compare with real experiments, but unfortunately it is notoriously difficult to obtain reliable data on two-dimensional systems, uncontaminated by the effects of impurities and finite-size. While, in principle, it is possible to carry out high-resolution scattering experiments with X-rays, such studies appear to have gone out of fashion with experimentalists.

We are therefore left with comparison with the results of numerical studies. In contrast with real experiments, these are actually easier to carry out for large systems in low dimensions. While Monte Carlo studies can study quite large systems, their results are subject to statistical errors which make comparison with detailed predictions of the correlation functions difficult. Since we wish to study such functions unaffected by finite-size effects, exact diagonalisation of finite transfer matrices is also not useful for our purpose. This leaves series expansions. The problem for which the longest such expansions have been carried out, and therefore the most accurate information is available, is that of self-avoiding walks (see Ref. 10 for a review of the current state of knowledge, and for further references). Although this is strictly not a problem in critical behaviour, its relation to the critical behaviour of the $O(n)$ model in the limit $n \rightarrow 0$ was shown long ago by de Gennes¹¹, and it is

possible to express all the quantities relevant to the asymptotic behaviour of N -step walks in the limit $N \rightarrow \infty$ in terms of correlation functions and amplitudes. The correspondence actually relates the fixed fugacity ensemble, where each step is weighted by a factor x , to the $n \rightarrow 0$ limit of the $O(n)$ model, at a temperature simply related to x . One result of this is that the actual correlation length of the $O(n)$ model is not directly measurable through self-avoiding walks.

The field theory of the two-dimensional $O(n)$ model has been studied extensively. While for values of $n > 2$ this model is always in its disordered phase at non-zero temperature, its analytic continuation to $-2 \leq n \leq 2$ may be carried out, and it does then have a finite T_c . In fact, the lattice version of this model at criticality may be mapped *via* standard techniques onto the Coulomb gas model^{12,13}, with the result that the critical exponents previously conjectured^{14,15} and the value of the central charge c in the corresponding conformal field theory may be easily read off. The standard methods of conformal field theory^{16,17} then show that the energy operator corresponds to a degenerate representation of the Virasoro algebra, labelled by its position (1,3) in the Kac table. It was shown by A. Zamolodchikov¹⁸ that a conformal field theory perturbed by a relevant operator of this type is integrable, in the sense that an infinite number of the conserved charges in the conformal field theory survive the perturbation. If the perturbed theory is massive, which we expect to be the case for $T > T_c$ in the $O(n)$ model, the existence of these conserved charges implies that scattering in the theory is purely elastic, and that the many-body S -matrix factorises into a product of two-body scatterings¹⁹. For the $O(n)$ model one expects to find a simple particle spectrum: an n -plet of particles states transforming according to the vector representation of $O(n)$, and, since the interaction is repulsive, no bound states.

On this basis, A. Zamolodchikov²⁰ conjectured an exact S -matrix for this theory, based on the principles of analyticity, unitarity, crossing, and the Yang-Baxter equations. His result represents the simplest solution to these conditions, but it is not unique, because of the CDD ambiguity.

Once the S -matrix is assumed, however, there is a well-defined program for investigating the off-shell behaviour of the theory, through the form factors^{4,5}. For a local field $\phi(r)$, these are defined as matrix elements $\langle 0 | \phi(0) | \beta_1, a_1; \dots; \beta_m, a_m \rangle$, where the ket represents an asymptotic m -particle state labelled by the rapidities β_j and $O(n)$ colour indices a_j of the particles. The form factors satisfy a set of equations relating them to the S -matrix, and other requirements of analyticity and crossing. Once again, it is generally assumed that the simplest, or ‘minimal’, solution of these equations should be chosen. This program

has been carried through in detail only for theories with non-degenerate single-particle states^{21,22,23,24}, or with only simple symmetries^{4,5,25,26}. The spectrum of the $O(n)$ model is, however, more complicated, and the two-body S -matrix acts on the direct product of two n -dimensional representations. While, for specific integer values of n , it is clearly possible to decompose this into irreducible amplitudes for which the S -matrix is diagonal, this is not useful for continuation to $n = 0$. We are therefore forced into additional complications in order to include the $O(n)$ structure.

Fortunately, in most cases this may be avoided for all intents and purposes. The purpose of computing the form factors is to reconstitute the two-point correlation functions through the unitarity sum

$$\langle \phi(r)\phi(0) \rangle = \sum_{m=0}^{\infty} \sum_{\text{colours}} \int_{\text{phase space}} |\langle 0|\phi|\beta_1, a_1; \dots; \beta_m, a_m \rangle|^2 e^{-Mr \sum_{j=1}^m \cosh \beta_j} \quad . \quad (1)$$

Clearly, for the infrared behaviour (large r) only the lowest values of m are important, while, in the ultraviolet region of small r , in principle all the intermediate states contribute. However, for our purposes, we shall be interested in moments of the correlation functions of the form $\int r^p \langle \phi(r)\phi(0) \rangle d^2r$. For large p , the large r end of the integration region will be emphasised, and the truncation to, say, $m \leq 2$ should be valid. Remarkably, however, we find that, this can be true for $p = 2$ and even $p = 0$. In the case when ϕ is the energy operator (which couples only to states with m even), the value of the $p = 2$ moment is known exactly from a sum rule derived from the c -theorem^{27,28}. The result obtained from the truncation to two-particle states agrees with this up to about one part in 10^3 . This approximation is less good for the $p = 0$ moment (having an error of about 10%), but, as will be discussed below, this moment may be found exactly from the TBA. When ϕ is the $O(n)$ vector field itself (corresponding to the spin of the lattice $O(n)$ model), it couples only to states with odd m . Then the truncation to $m = 1$ appears to be very accurate even for the zeroth order moment.

The reason for this remarkable numerical simplification appears to be that the form factors to the states with higher numbers of particles have a much softer behaviour close to the m -particle threshold than expected on the basis of phase space alone. In fact, each form factor contains an explicit factor of $\prod_{i < j} (\beta_i - \beta_j)$. This may be traced to the threshold behaviour of the two-body S -matrix, $S(0) = -1$. In general, unitarity implies that $S(\beta)S(-\beta) = 1$, so that, in principle, $S(0)$ could be ± 1 . However, the upper sign appears to be realised only in a free boson theory: otherwise, in all interacting theories,

the lower sign holds. Thus, this suppression of the higher particle states would appear to be a very general feature of two-dimensional theories.

In the case of the $O(n)$ model, the restriction to $m \leq 2$ makes the otherwise cumbersome problem of the group theory factors straightforward. As a result, we are able to obtain accurate values for the moments of the spin-spin and energy-energy correlation functions in the $n \rightarrow 0$ limit, based, of course, on all the assumptions of minimality for the S -matrix and the form factors referred to above. As will be shown in detail in the subsequent sections, these moments are simply related, respectively, to moments of the end-to-end distance of self-avoiding walks, and of the mean square distance between points of self-avoiding loops. For these lattice problems, extensive numerical work has been carried out on the mean square end-to-end distance of N -step walks, $\langle R_e^2 \rangle_N \sim CN^{2\nu^{29}}$, and the mean square radius of gyration of loops, $\langle R_g^2 \rangle_N \sim DN^{2\nu^{30}}$, where, from Coulomb gas arguments, $\nu = \frac{3}{4}^{12}$, and C and D are amplitudes. By themselves, they are not universal³⁰ (as will also follow from our analysis below), but the ratio C/D should be free of all metric factors and therefore universal.

The only quantity entering the calculation of this ratio which is not determined to high accuracy by the 2-particle truncation of the form factor approach is the zeroth moment U_0 of the energy-energy correlations. But this is just proportional to the specific heat, and therefore may be found in an alternative way by differentiating the extensive part of the free energy. This may be derived from the thermodynamic Bethe ansatz. The TBA has been applied to the unitary minimal models with $c = 1 - 6/(k+1)(k+2)$, with k integer, perturbed by the $(1,3)$ operator, by Al. Zamolodchikov⁷. Although the $O(n)$ model is not identical in its operator content to the minimal models, the results for the free energy, when perturbed by the same type of operator, must be the same. Thus we may take over Zamolodchikov's result, the only generalisation necessary being the continuation to non-integer k , so as to allow the $n \rightarrow 0$ limit to be taken. This turns out to be straightforward, and leads to an exact result for the universal part of the free energy per correlation volume, which is related to U_0 .

Piecing together all this information, we find $C/D \approx 13.70$. This is to be compared with the estimates¹⁰ of 13.69 for the square lattice, and 13.72 for the triangular lattice. Thus the errors from the 2-particle truncation are less than the systematic errors of the current data from enumerations. This agreement is a non-trivial test of the $O(n)$ model S -matrix and form factors. (By contrast, the ratio C/D for ordinary random walks is 12.)

The mean square radius of gyration of loops is related to the ratio of the $p = 2$ and

$p = 0$ moments of the energy-energy correlation function. The generating function for all the moments is proportional to the mean structure factor of N -step loops, $\mathcal{S}(q) = N^{-1} \sum_{ij} \langle e^{iq \cdot (r_i - r_j)} \rangle$, which could be measured, for example, by light scattering. Since the 2-particle approximation gives the zeroth and second moments to such remarkable accuracy, and its accuracy is supposed to improve for the higher moments, this gives a very precise way of computing $\mathcal{S}(q)$, at least for moderate values of its scaling argument $q^2 \langle R_g^2 \rangle$. Thus we are able to make the first accurate calculation of this function. The higher moments are dominated by the behaviour of the 2-particle cut near threshold. Since, as we remarked above, this is softened due to the fact that $S(0) = -1$, these moments behave as a function of p in a very different way from those of a free theory. Such a free theory describes random self-intersecting loops. Thus we are able to picture in a rather direct way how the condition of self-avoidance influences the statistics of the loops. On the other hand, the behaviour of $S(q)$ at large values of its scaling variable qR is proportional to $(qR)^{-1/\nu}$, as expected for an object of fractal dimension $1/\nu$. The coefficient of this behaviour may be determined by comparing conformal perturbation theory with the results of the thermodynamic Bethe ansatz.

Although the truncation to states with $m \leq 2$ works remarkably well for moments of the spin-spin and energy-energy correlations, this is not true for all correlation functions. For example, the $O(n)$ model possesses conserved currents whose integrals generate the $O(n)$ symmetry. It turns out that moments of the current-current correlation function are related to moments of the *area* $\langle a^p \rangle_N$ of N -step self-avoiding loops, in the limit $n \rightarrow 0$ ^{10,31}. These are expected to scale as $E^{(p)} N^{p\nu}$ ^{10,32}. Thus we might expect to be able to compute the amplitudes $E^{(p)}$ in the 2-particle approximation. The amplitudes of both the first and second moments have been estimated from enumerations. However, it turns out that for these values of p , the 2-particle approximation is hopeless. This may be traced to two causes: first, current conservation forces an additional softening of the 2-particle contribution to the current-current correlation function near threshold; and second, the fact that the current has unit scaling dimension means that the ultraviolet behaviour is more singular, and gives a large contribution to the lower moments which is not adequately represented by the 2-particle contribution.

While the form factor approach has, so far, been applied to the computation of two-point functions, it is not restricted to this case. Several amplitudes governing the statistics of open self-avoiding walks, for example their mean square radius of gyration, are related to moments of higher-point correlation functions. We show how in principle this calculation

may be carried out, and that, in the simplest approximation, we get results which agree with those of numerical lattice calculations to within a few percent.

The lay-out of this paper is as follows. In the next section, we make precise the connection between the self-avoiding walk problem and the $O(n)$ model, and in particular relate the various amplitudes and scaling functions whose values we are trying to compute to moments of appropriate correlation functions in the field theory. In Sec. (3) we recall A. Zamolodchikov's arguments²⁰ for the S -matrix of this theory, and derive the 2-particle form factors. We show how this leads to an accurate estimate for the second moment of the energy-energy correlation function. In the next section we derive the zeroth moment of this correlation function by way of the thermodynamic Bethe ansatz, generalizing the arguments of Al. Zamolodchikov⁷ so that the limit $n \rightarrow 0$ may be taken. This result, combined with those of the previous section, then leads to our estimate for C/D .

In Sec. (5) we present the calculation of the structure factor and comment on its features, and in Sec. (6) we discuss the current-current correlation function and show why the 2-particle approximation fails for its lower moments. In Sec. (7) we consider the radius of gyration of open chains, which involves the calculation of a three-point correlation function, and, finally, we summarise our conclusions and discuss possible further extensions of our results.

2. THE $O(N)$ MODEL AND SELF-AVOIDING WALKS

In this section we summarize the well-known relation¹¹ between these two problems in the limit $n \rightarrow 0$, in order to make precise the quantities we shall need later.

Consider first the set of N -step self-avoiding walks on a given lattice. It is convenient to consider lattices of co-ordination number three (*e.g.* a honeycomb lattice in two dimensions), but, later on, an appeal to universality will imply that in the critical region the results are universal. In the same spirit let us consider a specific lattice model with $O(n)$ symmetry, in which the degrees of freedom are spins $s_a(r)$ at each site r , labelled by a 'colour' index a which takes values from 1 to n . The spins are normalised so that $\text{Tr } s_a s_b = \delta_{ab}$, and the partition function is

$$Z = \text{Tr} \prod_{\text{nn}} \left(1 + x \sum_a s_a(r_i) s_a(r_j) \right) \quad , \quad (2)$$

where the product is over all nearest neighbour pairs of sites (r_i, r_j) . Once again, we expect, on the grounds of universality, that the particular form of the hamiltonian $H_1 = -\sum_{\text{nn}} \ln(1 + x \sum_a s_a(r_i) s_a(r_j))$ implied by (2) should not be important in the critical region. In particular, we could consider the more standard hamiltonian $H_2 = -x \sum_{\text{nn}} \sum_a s_a(r_i) s_a(r_j)$.

The continuum limit of either of these models close their respective critical points is assumed to be described by an $O(n)$ -invariant field theory, and, at the critical point, by a conformal field theory whose central charge $c(n) = 1 - 6/(k+1)(k+2)$, where $n = 2 \cos(\pi/(k+1))$ with $k \geq 0$ ^{16,17}. We note, in particular, that c vanishes when $n = 0$, as it should, since it measures the finite-size correction to the free energy which would vanish in this case. Its derivative dc/dn at $n = 0$ will be important: this takes the value $5/3\pi$. The scaling dimension of the energy operator is $x_e = 2k/(k+2)$ and equals $2/3$ for $n = 0$. It is related to the exponent ν characterising the divergence of the correlation length by $\nu = 1/(2 - x_e) = 3/4$. This also fixes the critical index $\alpha = 2 - 2\nu$ of the singular part of the free energy. The other exponents which will enter are $\gamma = 43/32$ which gives the divergent behaviour $(x_c - x)^{-\gamma}$ of the susceptibility in the limit $n = 0$, and the magnetic scaling index $\eta = 5/24$. They are related by the scaling equation $\gamma = \nu(2 - \eta)$.

On expanding the product in (2), it may be written as a sum over self-avoiding loops:

$$Z = \sum_{\substack{\text{loop} \\ \text{configurations}}} x^{\text{number of links}} n^{\text{number of loops}} \quad . \quad (3)$$

As $n \rightarrow 0$, only configurations with a single loop survive in the $O(n)$ term. Thus, if p_N is defined as the number of N -step self-avoiding loops per lattice site, then

$$\mathcal{N}_s \sum_N p_N x^N = \lim_{n \rightarrow 0} n^{-1} \ln Z \quad , \quad (4)$$

where \mathcal{N}_s is the total number of lattice sites. Note that the right hand side is proportional to the free energy, so is also proportional to the total area \mathcal{A} . The free energy of the $O(n)$ model is believed to be an analytic function of its temperature-like variable x for sufficiently small x , and to have a singularity of the form $(x_c - x)^{2-\alpha}$ at some finite, lattice-dependent x_c . Since the coefficients of the series on the left hand side are all non-negative, the singularities of the free energy on the circle $|x| = x_c$ will determine the behaviour of p_N at large N . Hyperscaling implies that, in two dimensions, $\mathcal{A}^{-1} \ln Z \sim nU\xi^{-2}$, where ξ is the correlation length and U (which we shall calculate exactly in Sec. (4)) is universal. The correlation length itself has the critical behaviour

$$\xi \sim \xi_0 (1 - x/x_c)^{-\nu} \quad , \quad (5)$$

where $\nu = 3/4$ but the metric factor ξ_0 is non-universal. Putting all these factors together we see that

$$\sum_N p_N x^N \sim a_0 \xi_0^{-2} U (1 - x/x_c)^{2\nu} \quad , \quad (6)$$

as $x \rightarrow x_c$, where $a_0 = \mathcal{A}/\mathcal{N}_s$ is the area per site. This implies that, as $N \rightarrow \infty$, $p_N \sim BN^{-2\nu-1}\mu^N$, where $\mu = x_c^{-1}$ and the amplitude B is given by

$$B = \sigma a_0 \xi_0^{-2} \frac{U}{\Gamma(-2\nu)} \quad . \quad (7)$$

For certain lattices p_N actually vanishes by symmetry except when N is divisible by an integer σ . This implies that the left hand side of (6) is actually a series in x^σ , and therefore has σ equivalent singularities on its circle of convergence, thus accounting for the lattice-dependent factor of σ on the right hand side of (7). For example, for the square lattice $\sigma = 2$.

Next, we consider the energy-energy correlation function. Close to the critical point, we write $H_2 = H_2^c + (x_c - x) \sum_{\text{bonds } r} \mathcal{E}^{\text{lat}}(r)$, where $\mathcal{E}^{\text{lat}}(r) = \sum_a s_a(r_<)s_a(r_>)$ is the lattice energy located on the link r which connects the sites $r_<$ and $r_>$. The correlation function $\langle \mathcal{E}^{\text{lat}}(r_1) \mathcal{E}^{\text{lat}}(r_2) \rangle$ then receives contributions from all self-avoiding loops which contain the links r_1 and r_2 . If the number of such loops with N steps is $f_N(r_1, r_2)$, then

$$n \sum_N f_N(r_1, r_2) x^{N-2} = \langle \mathcal{E}^{\text{lat}}(r_1) \mathcal{E}^{\text{lat}}(r_2) \rangle \quad . \quad (8)$$

In particular, if we sum over all $r_1 \neq r_2$ on the left hand side, we sum over all such N -step loops, and obtain

$$n\mathcal{N}_s \sum_N N(N-1)p_N x^{N-2} = \sum_{r_1 \neq r_2} \langle \mathcal{E}^{\text{lat}}(r_1) \mathcal{E}^{\text{lat}}(r_2) \rangle \quad , \quad (9)$$

which is just the second derivative of (4) with respect to x . The right hand side is just the specific heat of the $O(n)$ model.

In the same way, if we multiply by $(r_1 - r_2)^2$ before summing, we get the generating function for p_N weighted by

$$\sum_{r_1, r_2} (r_1 - r_2)^2 = \sum_{r_1, r_2} \left((r_1 - \bar{r}) - (r_2 - \bar{r}) \right)^2 = 2N^2 R_g^2 \quad , \quad (10)$$

so that

$$2n\mathcal{N}_s \sum_N N^2 p_N \langle R_g^2 \rangle_N x^{N-2} = \sum_{r_1, r_2} (r_1 - r_2)^2 \langle \mathcal{E}^{\text{lat}}(r_1) \mathcal{E}^{\text{lat}}(r_2) \rangle \quad . \quad (11)$$

Similarly, the generating function for the structure factor $\mathcal{S}_N(q) = N^{-1} \sum_{r_1, r_2} \langle e^{iq \cdot (r_1 - r_2)} \rangle$ is given by

$$n\mathcal{N}_s \sum_N N p_N \mathcal{S}_N(q) x^{N-2} = \sum_{r_1, r_2} e^{iq \cdot (r_1 - r_2)} \langle \mathcal{E}^{\text{lat}}(r_1) \mathcal{E}^{\text{lat}}(r_2) \rangle \quad . \quad (12)$$

In the scaling region, the dominant contribution to the sum on the right hand side comes from $|r_1 - r_2|$ of the order of the correlation length, and we may therefore use a continuum field theory description. The lattice energy \mathcal{E}^{lat} is replaced by the continuum energy density $\mathcal{E}(r)$ in such a way that they enter into the hamiltonian (action) in the same way:

$$\sum_r \mathcal{E}^{\text{lat}}(r) \rightarrow \int \mathcal{E}(r) d^2r \quad . \quad (13)$$

However, in the field theory, the energy operator is special in that it represents the perturbation $(x_c - x) \int \mathcal{E}(r) d^2r$ of the critical theory. It is therefore proportional to the trace $\Theta(r) = T_\mu^\mu$ of the stress tensor. Explicitly

$$\Theta(r) = -2\pi\nu^{-1}(x_c - x) \mathcal{E}(r) \quad , \quad (14)$$

where the conventional factor of 2π has been included in the definition of the stress tensor. Since the normalisation of the stress tensor is fixed, its correlation functions are completely universal. In particular

$$\langle \Theta(r) \Theta(0) \rangle = n\xi^{-4} \Phi_1(r/\xi) \quad , \quad (15)$$

where Φ_1 is a universal scaling function.

From (11) we then see that, in terms of this scaling function,

$$2n \sum_N N^2 p_N \langle R^2 \rangle_N x^{N-2} = a_0 (\nu/2\pi)^2 (x_c - x)^{-2} \int r^2 \langle \Theta(r) \Theta(0) \rangle d^2r \quad (16)$$

$$= n a_0 (\nu/2\pi)^2 U_2 (x_c - x)^{-2} \quad (17)$$

where $U_2 = \int \rho^2 \Phi_1(\rho) d^2\rho$ is universal, so that

$$2N p_N \langle R_g^2 \rangle_N \sim \sigma a_0 (\nu/2\pi)^2 U_2 \mu^N \quad . \quad (18)$$

The right hand side of (16) is in fact given by a sum rule which is a consequence of Zamolodchikov's c -theorem²⁷:

$$c(n) = (3/4\pi) \int r^2 \langle \Theta(r) \Theta(0) \rangle d^2r \quad , \quad (19)$$

where $c(n)$ is the central charge of the $O(n)$ model. In terms of the amplitude D in the relation $\langle R_g^2 \rangle \sim DN^{2\nu}$, this implies the relation

$$BD = \frac{5}{32\pi^2} \sigma a_0 \quad , \quad (20)$$

which was derived in ²⁸, and generalised in ¹⁰. Alternatively we see that the c -theorem sum rule gives an exact value for U_2 :

$$U_2 = \int \rho^2 \Phi_1(\rho) d^2 \rho = \frac{20}{9} \quad . \quad (21)$$

From (9), the amplitude B may itself be written in terms of the zeroth moment of Φ_1 , since

$$\sum_N N^2 p_N x^{N-2} = a_0 (\nu/2\pi)^2 (x_c - x)^{-2} \int \langle \Theta(r) \Theta(0) \rangle d^2 r = a_0 (\nu/2\pi)^2 (x_c - x)^{-2} \xi^{-2} U_0$$

where $U_0 = \int \Phi_1(\rho) d^2 \rho$. Hence we obtain the alternate result for the amplitude B :

$$B = \sigma a_0 \xi_0^{-2} (\nu/2\pi)^2 \frac{U_0}{\Gamma(2 - 2\nu)} \quad . \quad (22)$$

Comparing with (7), this gives the further sum rule

$$U_0 = \int \Phi_1(\rho) d^2 \rho = (2\pi/\nu)^2 (-2\nu)(1 - 2\nu) U \quad . \quad (23)$$

We stress that this is merely a consequence of the fluctuation sum in (9) being proportional to the specific heat. As we shall see in Sec. (4), the amplitude U may be obtained exactly. The above two sum rules for U_0 and U_2 then give an estimate of the errors of the two-particle approximation to $\langle \Theta(r) \Theta(0) \rangle$.

The amplitude D defined above for the radius of gyration may also be written, from (7, 18, 23) in the more intuitive form

$$D = \xi_0^2 \frac{\Gamma(\alpha)}{2\Gamma(\alpha + 2\nu)} \frac{U_2}{U_0} \quad , \quad (24)$$

As expected, the mean square radius of gyration is proportional to the ratio of the second to the zeroth moments of the energy-energy correlation function.

Finally, the structure factor is related to the Fourier transform of Φ_1 by

$$\sum_N N p_N \mathcal{S}_N(q) x^{N-2} = a_0 (\nu/2\pi)^2 (x_c - x)^2 \xi^{-2} \int e^{iq \cdot \rho \xi} \Phi_1(\rho) d^2 \rho \quad . \quad (25)$$

Disentangling the large N dependence from this equation is more complicated, and will be discussed in Sec. (5).

Next, we consider the case of linear self-avoiding walks. Let $c_N(r)$ be the number of such walks with N steps from the origin to the site r . The generating function for this is just the spin-spin correlation function of the $O(n)$ model as $n \rightarrow 0$:

$$\sum_N c_N(r) x^N = \lim_{n \rightarrow 0} \langle s_1(r) s_1(0) \rangle \quad . \quad (26)$$

In the scaling region, we expect this correlation function to scale as

$$\langle s_1(r) s_1(0) \rangle \sim b \xi^{-\eta} \Phi_2(r/\xi) \quad , \quad (27)$$

where Φ_2 is universal, but the metric factor b is not. Note that the magnetisation is different from the energy operator in that it has no absolute normalisation. From (26) we may easily compute the generating function for the number c_N of all N -step walks:

$$\sum_N c_N x^N = b \xi_0^{2-\eta} (1 - x/x_c)^{-\gamma} V_0 \quad , \quad (28)$$

and for this number weighted by their squared end-to-end distance R_e^2 :

$$\sum_N c_N \langle R_e^2 \rangle_N x^N = b \xi_0^{4-\eta} (1 - x/x_c)^{-\gamma-2\nu} V_2 \quad , \quad (29)$$

where V_0 and V_2 are the zeroth and second moments, respectively, of Φ_2 . We then find, for the amplitude C in the asymptotic behaviour $\langle R_e^2 \rangle \sim CN^{2\nu}$, the result

$$C = \xi_0^2 \frac{\Gamma(\gamma)}{\Gamma(\gamma + 2\nu)} \frac{V_2}{V_0} \quad . \quad (30)$$

Note the close similarity with (24). (The extra factor of 2 in the denominator in (24) is due to the factor of 2 appearing in (10).)

Neither of the amplitudes C or D is universal, because of the metric factor ξ_0^2 . However, this cancels in their ratio, and we find

$$C/D = \frac{2\Gamma(\gamma)\Gamma(\alpha + 2\nu)}{\Gamma(\gamma + 2\nu)\Gamma(\alpha)} \frac{V_2 U_0}{V_0 U_2} \quad . \quad (31)$$

3. S-MATRIX AND FORM FACTORS OF THE THERMAL PERTURBATION OF THE O(N) MODEL

3.1 Scattering theory

Since in the scaling limit the energy operator of the O(n) model on the lattice corresponds to the primary operator $\varphi_{1,3}$ of the conformal model, the thermal perturbation of the critical point action is described by

$$A = A_c + (x_c - x) \int \varphi_{1,3}(x) d^2x \quad . \quad (32)$$

As with any $\varphi_{1,3}$ deformation of the minimal conformal models, the QFT defined by the action (32) presents a number of higher integrals of motion which guarantee its integrability¹⁸. For $x < x_c$ (the only case that we will investigate) the model develops a finite correlation length and the associate scaling QFT is mainly characterised by the factorised S -matrix of its massive excitations. The scattering theory has been proposed by Zamolodchikov²⁰ and its main features may be summarised as follows¹.

First of all, on the basis of the form of the partition function (4), Zamolodchikov argued that it is possible to interpret the loops as trajectories of a set of n particles $A_i(\beta)$ (with mass M) that belong to the vector representation of O(n). Hence the scattering matrix for the process $|A_{i_1}(\beta_1)A_{i_2}(\beta_2)\rangle \rightarrow |A_{j_1}(\beta_1)A_{j_2}(\beta_2)\rangle$ may be written as

$$S_{i_1 i_2}^{j_1 j_2}(\beta_{12}) = S_0(\beta_{12}) \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} + S_1(\beta_{12}) \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} + S_2(\beta_{12}) \delta_{i_1 i_2} \delta^{j_1 j_2} \quad , \quad (33)$$

where $\beta_{12} = \beta_1 - \beta_2$. S_0 is the amplitude for the transmission process whereas S_1 and S_2 are respectively the amplitudes for the reflection and annihilation processes. Equivalently, we may decompose the scattering matrix into channels of definite isospin: for the symmetric traceless, antisymmetric and isosinglet channels we have respectively

$$\begin{aligned} S_S(\beta) &= S_0(\beta) + S_1(\beta) \\ S_A(\beta) &= S_0(\beta) - S_1(\beta) \\ S_I(\beta) &= S_0(\beta) + S_1(\beta) + n S_2(\beta) \quad . \end{aligned} \quad (34)$$

¹ It is worth mentioning that the same S -matrix, although with a different interpretation, has been also derived by Smirnov³³ using a quantum group reduction of the scattering theory of the Sine-Gordon model.

The amplitudes S_1 and S_2 are linked to each other by the crossing symmetry relation

$$S_1(\beta) = S_2(i\pi - \beta) \quad , \quad (35)$$

whereas S_0 is a crossing symmetric function.

In order to take into account the property that the loops entering (4) are non-intersecting paths, Zamolodchikov suggested imposing the condition

$$S_0(\beta) = 0 \quad . \quad (36)$$

Using the Yang-Baxter equations and the unitarity condition, the final form of the minimal S -matrix is then given by

$$\begin{aligned} S_1(\beta) &= -\sinh\left(\frac{i\pi - \beta}{k+1}\right) R(\beta) \\ S_2(\beta) &= -\sinh\left(\frac{\beta}{k+1}\right) R(\beta) \quad , \end{aligned} \quad (37)$$

where

$$R(\beta) = \frac{1}{\sinh\left(\frac{i\pi - \beta}{k+1}\right)} \exp\left[i \int_0^\infty \frac{dx}{x} \frac{\sinh\frac{\pi kx}{2} \sin x\beta}{\sinh\frac{\pi(k+1)x}{2} \cosh\frac{\pi x}{2}}\right] \quad . \quad (38)$$

Its analytic structure may be read off from its infinite product representation

$$\begin{aligned} R(\beta) &= \frac{1}{\sin\pi\left(\frac{1}{k+1} - \frac{\beta}{i\pi(k+1)}\right)} \frac{\Gamma\left(1 - \frac{\beta}{i\pi(k+1)}\right)}{\Gamma\left(1 + \frac{\beta}{i\pi(k+1)}\right)} \prod_{l=1}^{\infty} \frac{\Gamma\left(\frac{2l}{k+1} - \frac{\beta}{i\pi(k+1)}\right)}{\Gamma\left(\frac{2l}{k+1} + \frac{\beta}{i\pi(k+1)}\right)} \\ &\times \frac{\Gamma\left(1 + \frac{2l}{k+1} - \frac{\beta}{i\pi(k+1)}\right) \Gamma\left(\frac{2l-1}{k+1} + \frac{\beta}{i\pi(k+1)}\right) \Gamma\left(1 + \frac{2l-1}{k+1} + \frac{\beta}{i\pi(k+1)}\right)}{\Gamma\left(1 + \frac{2l}{k+1} + \frac{\beta}{i\pi(k+1)}\right) \Gamma\left(\frac{2l-1}{k+1} - \frac{\beta}{i\pi(k+1)}\right) \Gamma\left(1 + \frac{2l-1}{k+1} - \frac{\beta}{i\pi(k+1)}\right)} \quad . \end{aligned} \quad (39)$$

Since there are no poles in the physical sheet $0 \leq \text{Im}\beta \leq \pi$, no bound states appear. Hence the whole particle content of the model only consists in the n degenerate states of the vector representation of $O(n)$. For $n = 1$ the S -matrix of the isosinglet channel correctly coincides with the S -matrix of the thermal perturbation of the Ising model

$$S_1(\beta) + S_2(\beta) = -1 \quad ,$$

whereas in the limit $n \rightarrow 0$, the final form of the scattering amplitudes is given by

$$\begin{aligned} S_1(\beta) &= -G(\beta) \\ S_2(\beta) &= -i \tanh\left(\frac{\beta}{2}\right) G(\beta) \quad , \end{aligned} \quad (40)$$

where

$$G(\beta) = \exp \left[i \int_0^\infty \frac{dx}{x} \frac{\sinh \frac{\pi x}{2} \sin x \beta}{\sinh \pi x \cosh \frac{\pi x}{2}} \right] . \quad (41)$$

Notice that the S -matrix of the $O(n)$ model possesses nontrivial asymptotic behaviour for $\beta \rightarrow \pm\infty$

$$\begin{aligned} S_1(\beta) &\rightarrow e^{\pm i\pi\Delta} \\ S_2(\beta) &\rightarrow \mp e^{\pm i\pi(\Delta + \frac{1}{k+1})} , \end{aligned} \quad (42)$$

where

$$\Delta = \frac{3k+2}{2(k+1)} .$$

According to²⁶, this implies that the particle operators of the $O(n)$ model satisfy a generalised statistics.

3.2 Form Factors

The previous discussion of the scattering theory of the thermal deformation of the $O(n)$ model gives rise to a series of questions related to the minimality assumptions made in its derivation or to the non-integer values assumed by the variable n . In addition the final scattering amplitudes are not directly related to the underlying microscopic formulation of the model, making it apparently problematic to judge their validity. To answer such questions, it is useful to recall that for integrable models the knowledge of the S -matrix is a powerful starting point to compute their correlation functions through the form factor bootstrap approach^{4,5}. Hence we may take the point of view of considering the $O(n)$ scattering theory discussed above as simply the basic tool to implement the program of reconstruction of correlation function for this model, the whole justification of the approach being in the final comparison with quantities extracted by other methods.

In the form factor bootstrap approach the correlation functions are computed by exploiting their spectral representations, that is, their expression as an infinite series over multi-particle intermediate states. For instance, the two-point function of an isoscalar operator $\Phi(x)$ in real Euclidean space is given by

$$\begin{aligned} \langle \Phi(x)\Phi(0) \rangle &= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \dots d\beta_n}{n!(2\pi)^n} \langle 0 | \Phi(x) | \beta_1, a_1; \dots \beta_n, a_n \rangle \langle \beta_1, a_1 \dots \beta_n, a_n | \Phi(0) | 0 \rangle \\ &= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \dots d\beta_n}{n!(2\pi)^n} | F_{a_1, \dots, a_n}^\Phi(\beta_1 \dots \beta_n) |^2 \exp \left(-Mr \sum_{i=1}^n \cosh \beta_i \right) , \end{aligned} \quad (43)$$

where a sum on the colour indices is implied. In (43) r denotes the radial distance, i.e. $r = \sqrt{x_0^2 + x_1^2}$ and

$$F^\Phi(\beta_1, \dots, \beta_n)_{a_1, a_2, \dots, a_n} \equiv \langle 0 | \Phi(0) | \beta_1, a_1; \dots; \beta_n, a_n \rangle \quad (44)$$

are the so-called *form factors* (FF). The normalization of the asymptotic states is fixed as

$$\langle \beta_1, a_1 | \beta_2, a_2 \rangle = 2\pi \delta_{a_1, a_2} \delta(\beta_1 - \beta_2) \quad . \quad (45)$$

Since the spectral representations are based only on the completeness of the asymptotic states, they are general expressions for any QFT. However, for integrable models, they become quite effective because the exact computation of the form factors reduces to finding a solution of a finite set of functional equations. In fact they satisfy the Watson equations 4,5

$$\begin{aligned} F^\Phi(\beta_1 \dots \beta_{i+1}, \beta_i, \dots, \beta_n)_{a_1 \dots a'_{i+1}, a'_i \dots a_n} &= F^\Phi(\beta_1 \dots \beta_i, \beta_{i+1} \dots \beta_n)_{a_1 \dots a_i, a_{i+1} \dots a_n} S_{a'_i, a'_{i+1}}^{a_i, a_{i+1}}(\beta_i - \beta_{i+1}) \\ F^\Phi(\beta_1, \dots, \beta_n + 2\pi i)_{a_1, \dots, a_n} &= F^\Phi(\beta_n, \dots, \beta_{n-1})_{a_n, a_1, \dots, a_{n-1}} \quad . \end{aligned} \quad (46)$$

The first of (46) show that the monodromy properties of the FF are determined by the two-body S -matrix of the model. On the other hand the second equation states that an analytic continuation in the variables β_i simply induces a reordering in the set of the asymptotic particles. Notice that this system of equations does not uniquely fixed the solution since the multiplication of $F^\Phi(\beta_1, \dots, \beta_n)_{a_1, \dots, a_n}$ by a symmetric, $2\pi i$ periodic function leaves (46) untouched. For a specific operator this ambiguity may be generally solved by the knowledge of the asymptotic behaviour of its form factors and their analytic structure. Let us briefly discuss the two aspects separately.

The asymptotic behaviour of a FF under a simultaneous shift of all rapidity variables is simply dictated by relativistic invariance

$$F^\Phi(\beta_1 + \Lambda, \beta_2 + \Lambda, \dots, \beta_n + \Lambda)_{a_1, \dots, a_n} = e^{s\Lambda} F^\Phi(\beta_1, \beta_2, \dots, \beta_n)_{a_1, \dots, a_n} \quad , \quad (47)$$

where s is the spin of the operator Φ . Notice that for scalar operators the FF depend only on the differences $\beta_{ij} = \beta_i - \beta_j$. Secondly, restricting our attention only to those operators which have two-point functions with a power-law ultraviolet behaviour, we have to require that their FF behave asymptotically no worse than $\exp(k\beta_i)$ for $\beta_i \rightarrow \infty$, where k is a constant independent of i . If a perturbative formulation of the theory is available, the constant k may be fixed by matching the asymptotic behaviour of the FF with the asymptotic behaviour of the relevant Feynman diagrams which contribute to the amplitude.

Concerning the analytic nature of the form factors, their pole structure gives rise to a set of recursive equations which relate the l -particle FF ($l > 2$). Since in the $O(n)$ model there are no bound states, the only singularities which appear in the FF are those associated with the annihilation poles $\beta_a = \beta_b + i\pi$ with residues⁵

$$2\pi i \operatorname{res} F^\Phi(\beta + i\pi, \beta, \beta_1, \dots, \beta_l)_{a_0, \bar{a}_0, a_1, \dots, a_l} =$$

$$\left(\delta_{a_1}^{a'_1} \delta_{a_2}^{a'_2} \dots \delta_{a_l}^{a'_l} - S_{a_1, \dots, a_l}^{a'_1, \dots, a'_l}(\beta_1, \dots, \beta_l | \beta) \right) F^\Phi(\beta_1, \dots, \beta_l)_{a'_1, \dots, a'_l} \quad (48)$$

where

$$S_{a_1, \dots, a_l}^{a'_1, \dots, a'_l}(\beta_1, \dots, \beta_l | \beta) = S_{a_1, a_0}^{a'_1, \tau_1}(\beta_1 - \beta) S_{\tau_1, a_2}^{a'_2, \tau_2}(\beta_2 - \beta) \dots S_{a_l, \tau_{l-1}}^{a'_l, a_0}(\beta_l - \beta) \quad . \quad (49)$$

These equations induce a recursive structure in the space of the FF relating matrix elements with $l+2$ and l particles.

After this short discussion on the general properties of the FF, let us turn our attention to the $O(n)$ models. In order to analyse the correlation functions in the high-temperature phase of the model, it becomes convenient to define initially a two-particle form factor $F_{\min}(\beta)$ that does not depend on colour indices, does not contain poles on the physical sheet and has the mildest behaviour for large values of β . The first requirement implies that the monodromy property of such FF is induced by the S -matrix of the isosinglet channel $S_I(\beta)$ i.e.

$$F_{\min}(\beta) = [S_1(\beta) + n S_2(\beta)] F_{\min}(-\beta) \quad (50)$$

$$F_{\min}(i\pi - \beta) = F_{\min}(i\pi + \beta) \quad .$$

The solution of these equations which satisfies the above requirements is given by

$$F_{\min}(\beta) = \sinh \frac{\beta}{2} f_k(\beta) \exp \left[- \int_0^\infty \frac{dx}{x} \frac{\sinh \frac{\pi k x}{2} \sin^2 \frac{\hat{\beta} x}{2}}{\sinh \pi x \sinh \frac{\pi(k+1)x}{2} \cosh \frac{\pi x}{2}} \right] \quad , \quad (51)$$

where $\hat{\beta} = i\pi - \beta$ and

$$f_k(\beta) = \frac{\sin(\hat{\beta}/2)}{\sin(\hat{\beta}/(k+1))} \quad . \quad (52)$$

Note that $f_1(\beta) = 1$. In the self-avoiding case ($n \rightarrow 0$) the monodromy property of $F_{\min}(\beta)$ is dictated only by $S_1(\beta)$ of (40) and its expression reduces to

$$F_{\min}(\beta) = \sinh \frac{\beta}{2} \exp \left[- \int_0^\infty \frac{dx}{x} \frac{\sinh \frac{\pi x}{2} \sin^2 \frac{1}{2}(i\pi - \beta)x}{\sinh^2 \pi x \cosh \frac{\pi x}{2}} \right] \quad . \quad (53)$$

Useful properties of this function are discussed in the Appendix.

Notice that at the threshold ($\beta \rightarrow 0$) $F_{\min}(\beta)$ goes linearly to zero since $S_I(0) = -1$. The vanishing of the $F_{\min}(\beta)$ at the threshold has far-reaching consequences because it induces a suppression of the higher particle contributions in the spectral representation of the correlation functions. In order to see this, let us consider the general parametrisation of a FF of an operator $\Phi^{a,\dots,m}$ with colour indices a, \dots, m

$$\langle 0 | \Phi^{a,\dots,m}(0) | \beta_1, a_1; \dots; \beta_l, a_l \rangle = \mathcal{F}_{a_1,\dots,a_n}^{a,\dots,m}(\beta_1, \dots, \beta_l) \prod_{i < j} \frac{F_{\min}(\beta_{ij})}{\cosh \frac{\beta_{ij}}{2}} . \quad (54)$$

The denominator in (54) takes into account the pole structure of these matrix elements whereas the function $\mathcal{F}_{a_1,\dots,a_n}^{l,m,\dots}(\beta_1, \dots, \beta_n)$ carries the colour structure of this matrix element. The important point is that this function has neither poles or zeros in the physical sheet and satisfies a simplified system of monodromy equations which arise by substituting (54) into (46).

Let us consider now the contribution of the four-particle intermediate states to the 2-point functions. If the corresponding form factor was constant, this contribution would be of the form

$$\int \prod_1^4 d\beta_i e^{-Mr \sum_1^4 \cosh \beta_i} \sim \frac{e^{-4Mr}}{r^2} ,$$

which corresponds in momentum space to a branch point at $q^2 = -16M^2$ of the form $\ln(q^2 + 16M^2)$. But since F_{\min} vanishes at the threshold, neglecting other terms, we have an additional factor $\prod_{i < j} \beta_{ij}^2$ in the integral. This leads to a behaviour $\frac{e^{-4Mr}}{r^8}$, so that the branch cut gets softened to a function of the form $(q^2 + 16M^2)^3 \ln(q^2 + 16M^2)$. This effect actually gets stronger as we consider higher numbers of particles: the m -particle branch cut would be, on the grounds of phase space alone, of the form $(q^2 + m^2 M^2)^{m/4-1}$, but it actually gets softened to $(q^2 + m^2 M^2)^{m^2/4-1}$.

As it is evident from the above discussion, this suppression of the higher particle states is not peculiar to the $O(n)$ model but on the contrary we expect it to be a completely general feature for all interacting theories which have $S(0) = -1$. From a pragmatic point of view, this observation is crucial to the success of the form factor bootstrap approach to the computation of correlation functions. In fact, although for integrable models the exact computation of the FF may be achieved with little effort on the basis of our previous considerations, one may wonder about their final usefulness. After all, to compute the correlation functions we have to sum over the infinite multiparticle FF, and these series are in general hard to analyse. Of course, for large values of Mr only the lowest term

will dominate the sum, the higher particle contributions being exponentially small. For small values of Mr however the correlators possess singularities and, on these scales, all numbers of particles contribute in principle to the sum. But if there is reason to believe that the higher particle terms are anyway suppressed, then the lowest contribution of the spectral representation becomes quite effective in approximating the correlation functions even in these ultraviolet regions. The validity of such approximation is of course linked to a sufficiently mild divergence of the correlation functions in the short distance scales.

A first check of this phenomenon of suppression of higher FF contributions is given by looking at the second moment of the scaling function $\Phi_1(\rho)$

$$U_2 = \int \rho^2 \Phi_1(\rho) d^2\rho \quad . \quad (55)$$

The extra power in ρ^2 helps to smooth the ultraviolet behaviour of the correlation function and we may expect this integral to be almost saturated by the two-particle contribution. For the trace of the stress-energy tensor the two-particle form factor we take¹

$$\langle 0 | \Theta(0) | a, \beta_1; b, \beta_2 \rangle = -2\pi i \delta_{ab} M^2 F_{\min}(\beta_1 - \beta_2) \quad , \quad (56)$$

with $F_{\min}(\beta)$ given by (53). Truncating the spectral representation of the scaling function $\Phi_1(\rho)$ to this term we have for the RHS of (55)

$$2\pi \int_0^\infty \frac{d\beta}{\cosh^4 \beta} |F(2\beta)|^2 = 2.2241 \quad , \quad (57)$$

to be compared with the exact result of U_2 , from (21), extracted by the c -theorem sum rule (in the limit $n \rightarrow 0$)

$$U_2 = \frac{20}{9} = 2.2222 \quad . \quad (58)$$

Therefore we see that by keeping only the two-particle approximation of the correlator, we get an approximation to U_2 with a precision of about one part in 10^3 . However, the careful reader will notice that in this case the two-particle approximation results in an estimate of this quantity slightly larger than the exact value. Since the spectral representation of the scaling function $\Phi_1(\rho)$ involves an infinite series of terms which are integrals over the

¹ We assume the mildest behaviour at infinity of this matrix element to fix it uniquely. Its normalization is easily determined by using the definition of the hamiltonian operator $H = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx^1 T^{00}(x^0, x^1)$ and computing the matrix elements of both terms of this equation on $\langle \beta_1, a_1 |$ and $| \beta_2, a_2 \rangle$. Notice however that our expression differs from that proposed in ²⁰ which diverges at infinity much faster than ours.

modulus squared of its form factors, one may expect an underestimate of the exact value of U_2 from the 2-particle approximation. Although this is what usually happens in ordinary QFT, the situation for the $O(n)$ (in the limit $n \rightarrow 0$) may be different. To see this, let us consider for instance the four-particle FF of the trace of the energy-momentum tensor

$$\langle 0|\Theta|\beta_1, a; \beta_2, b; \beta_3, c; \beta_4; d\rangle \quad . \quad (59)$$

From group theory considerations this has the form

$$A \delta_{ab}\delta_{cd} + B \delta_{ad}\delta_{bc} + C \delta_{ac}\delta_{bd} \quad , \quad (60)$$

where the invariant amplitudes A, B, C (functions of β_1, \dots, β_4 and n) satisfy the functional and recursive equations of the form factors. For our considerations, their explicit form is not needed. In order to consider the four-particle contribution to the scaling function $\Phi_1(\rho)$, we have to take the modulus squared of this form factor and sum on the colour indices. The final expression is given by

$$(|A|^2 + |B|^2 + |C|^2)n^2 + (A\bar{B} + B\bar{A} + B\bar{C} + C\bar{B} + C\bar{A} + A\bar{C})n \quad , \quad (61)$$

which may be rewritten as

$$(|A|^2 + |B|^2 + |C|^2)(n^2 - n) + |A + B + C|^2 n \quad . \quad (62)$$

This is clearly positive for $n \geq 1$, and in this case we have the usual positive contribution of this form factor to the two-point function. But, for $n \rightarrow 0$, we need to pick up the linear term in n , i.e. the cross term in (61), which has no reason to be positive. As a matter of fact, for $n = 1$, which is the Ising model, we know it is negative, because in that case there is no 4-particle form factor so that $A+B+C = 0$. Hence, even though we have not pursued explicitly the calculation of the above amplitudes A, B, C for the $O(n)$ model, one should not be surprised to find that higher particle contributions will contribute negatively to the correlation functions. This effect becomes more evident if we consider the zero moment of $\Phi_1(\rho)$

$$U_0 = \int \Phi_1(\rho) d^2\rho \quad . \quad (63)$$

This time there are no extra powers in ρ which suppress the ultraviolet behaviour of the correlation function, and higher form factors may be expected to be important. As shown in the next section, U_0 may be computed exactly through the TBA

$$U_0 = \frac{4\pi^2}{3} = 13.1595 \quad . \quad (64)$$

On the other hand, the two-particle approximation to this quantity is given by

$$U_0^{(2)} = 2\pi \int_0^\infty dt \frac{|F_{\min}(2t)|^2}{\cosh^2 t} = 14.3651 \quad , \quad (65)$$

which is close to the exact result, but slightly larger.

4. TBA AND THE RATIO C/D .

4.1 The bulk free energy and the amplitude U .

In this section, we discuss the calculation of the universal amplitude U in the extensive part of the free energy, defined by $\ln Z \sim \mathcal{A} n U M^2$, where $M = \xi^{-1}$ is the mass of the $O(n)$ model, in the limit $n \rightarrow 0$. This is performed using the thermodynamic Bethe ansatz (TBA). This method was first applied to integrable perturbed conformal field theories in a pioneering paper by Al. Zamolodchikov⁶, who also later⁷ applied it to the minimal models perturbed by the $(1, 3)$ operator. Since our calculation is a rather simple generalization of his, we shall only sketch the argument, and refer the reader to Zamolodchikov's papers for the rather heavy details. Thus, in this section, we make no attempt to be self-contained.

In the TBA approach, one considers the massive theory as a quantum field theory in $1 + 1$ dimensions, at finite temperature R^{-1} . Denoting the reduced free energy per unit length of this system by $E(R)$, dimensional analysis implies that this has the form $E(R) = (2\pi/R)F(r)$, where $r = MR$. As $r \rightarrow 0$, the scaling function has the form $F(0) + O(r^2) + \text{singular terms}$. The terms proportional to $F(0)$ dominates in the high-temperature limit $R \rightarrow 0$ and gives the central charge of the conformal field theory. The $O(r^2)$ piece gives a term extensive in both the temperature and the spatial size of the system, and is therefore the extensive part of the free energy when we view the system as a classical system in two dimensions, which is what we wish to compute.

The TBA proceeds by enumerating the allowed states in a large spatial box, labelled by the rapidities of the particles, consistent with the fact that the wave functions must change by a factor of the S -matrix whenever two particles are exchanged. The fraction of these states which are actually occupied is then determined by minimizing the free energy. For a simple theory with only one type of excitation of mass M , the result is that

$$E(R) = -M \int L(\beta) \cosh \beta \frac{d\beta}{2\pi} \quad , \quad (66)$$

where

$$L(\beta) = \ln(1 + e^{-\epsilon(\beta)}) \quad , \quad (67)$$

is the solution of the integral equation

$$-MR \cosh \beta + \epsilon(\beta) + (\phi * L)(\beta) = 0 \quad , \quad (68)$$

where $*$ denotes a convolution, and $\phi(\beta) = -i(d/d\beta) \ln S(\beta)$.

Zamolodchikov showed⁶ that, in the limit $R \rightarrow 0$, it is sufficient to consider the ‘kink’ solution of the simpler integral equation

$$-e^\beta + \epsilon^{\text{kink}}(\beta) + (\phi * L^{\text{kink}})(\beta) = 0 \quad , \quad (69)$$

and that

$$F(r) \sim -\frac{1}{2\pi^2} \int L^{\text{kink}}(\beta) e^\beta d\beta - \frac{r^2}{8\pi^2} \int \frac{dL^{\text{kink}}(\beta)}{d\beta} e^{-\beta} d\beta + \text{singular terms} \quad . \quad (70)$$

Furthermore, the integral in the second term may be evaluated easily by examining the behaviour of (69) as $\beta \rightarrow -\infty$, and requiring that the $O(e^\beta)$ terms cancel between the first and third terms. Thus if $\phi(\beta) \sim Ae^\beta$ as $\beta \rightarrow -\infty$, then the value of the integral in the second term of (70) is $2\pi/A$.

In Ref. 7, Zamolodchikov considered the case of the unitary minimal models, labelled by an integer k , which correspond to the critical continuum limit of the lattice RSOS models, perturbed by the operator $(1, 3)$. Although the $O(n)$ model and the minimal model with $n = 2 \cos(\pi/(k+1))$ have the same value of the central charge, their operator content is certainly different. However, within the sector of operators of the type $(1, 2s+1)$, generated by repeated operator product expansions of $(1, 3)$ with itself, they are the same. Thus, to any order in perturbation theory, the correlation functions of the two theories should agree, and therefore so should such quantities as the mass gap and ground state energy. We assume that this will also hold non-perturbatively. In principle, these theories could differ at the non-perturbative level, but we see no physical reason for this to happen. Zamolodchikov argued that, although there is only one physical massive excitation in the RSOS model, in order to count correctly the states it is necessary to introduce pseudo-excitations which carry no energy but do enter into the TBA equations. For the k th minimal model these excitations correspond to the vertices of the A_{k-1} Dynkin diagram, labelled by $a = 1, \dots, k-1$. The physical particle corresponds to $a = 1$. The kink integral equation (69) becomes

$$-\delta_{a1} + \epsilon_a^{\text{kink}}(\beta) + \sum_b l_{ab} (\phi * L_b^{\text{kink}})(\beta) = 0 \quad , \quad (71)$$

where l_{ab} is the incidence matrix of the diagram. Thus, by looking at the $\beta \rightarrow -\infty$ and us-

ing $A = 2$ we find the following system of equations for the integrals $I_a = \int (dL_a^{\text{kink}}/d\beta) e^{-\beta} d\beta$:

$$\begin{aligned}
I_2 &= \pi \\
I_1 + I_3 &= 0 \\
I_2 + I_4 &= 0 \\
&\vdots \\
I_{k-2} &= 0 \quad .
\end{aligned} \tag{72}$$

Strictly speaking, these equations make sense only when k is an integer. However, we may formally find the solution for general k , which is

$$I_j = \pi \frac{\sin((j-k)\pi/2)}{\sin(k\pi/2)} \quad . \tag{73}$$

Since only $a = 1$ has mass, the term we want is proportional to $I_1 = \pi \cot(k\pi/2)$. Putting all the factors together, we find that

$$F(r) = F(0) - \frac{r^2}{8\pi} \cot(k\pi/2) + \text{power series in } r^{4/(k+2)} \quad . \tag{74}$$

Note that when k is an odd integer, the extensive part of the free energy vanishes, as found by Zamolodchikov: this is interpreted as being a consequence of (fractional) supersymmetry. For $k = 2p$ the coefficient diverges, but, since the overall result is finite, there must be a cancellation of this leading term against one of the singular terms, resulting in logarithmic behaviour. This comes from

$$\lim_{k \rightarrow 2p} -\frac{1}{8\pi} \frac{2/\pi}{k-2p} \left(r^2 - r^{4(p+1)/(k+2)} \right) = -\frac{2}{k+2} \left(\frac{r}{2\pi} \right)^2 \ln r \quad , \tag{75}$$

in agreement with (2.22) of Ref. 7.

However, we are interested in the limit $n \rightarrow 0$, corresponding to $k \rightarrow 1$. In that limit we then find for the free energy per unit area $\mathcal{A}^{-1} \ln Z \sim (\pi/8)(k-1)M^2$. Since $dk/dn|_{n=0} = 2/\pi$, we then have the final result that

$$U = \frac{1}{4} \quad . \tag{76}$$

4.2 The ratio C/D .

We are now in a position to calculate the amplitude D defined in Sec. (2). From (21, 23, 76) we find that $U_2/U_0 = 5/3\pi^2$ and so, from (24),

$$D = \frac{5}{6\pi^{\frac{3}{2}}}\xi_0^2 \quad . \quad (77)$$

Now consider the amplitude C , which, by (30), is proportional to the ratio of the second and zeroth moments of the scaling function Φ_2 . At the same level of approximation, we should neglect all m -particle intermediate states with $m > 2$. But, since the spin operator couples only to odd m , this implies that we need include only the single particle intermediate state, which is equivalent to assuming that

$$\Phi_2(\rho) = \int \frac{e^{iq\cdot\rho}}{q^2 + 1} \frac{d^2q}{(2\pi)^2} \quad . \quad (78)$$

Note that this is *not* the same as assuming the Ornstein-Zernicke (free-particle) form for the correlation function, which would imply that $\eta = 0$ and $\gamma = 2\nu$. In writing (78) for the scaling function, we have already extracted in (30) the dependence coming from the exact value γ . From (78) we find the moment ratio $V_2/V_0 \approx 4$, so that finally, from (31), we obtain one of our main results

$$C/D \approx \frac{24\pi^{\frac{3}{2}}}{5} \frac{\Gamma(\frac{43}{32})}{\Gamma(\frac{91}{32})} \approx 13.70 \quad . \quad (79)$$

It is difficult to estimate the errors in this calculation, which arise from the neglect of the 3- and higher particle intermediate states in the spin-spin correlation function. We would expect the error to be larger for the zeroth moment V_0 than for V_2 . On the grounds of phase space alone, one would expect the m -particle state to give a relative contribution of the order of $\int e^{-mMr} d^2r \sim m^{-2}$ to the zeroth moment. This will be modified by a factor of $m^{-m(m-1)/4}$ due to the softening of the multi-particle branch points discussed in Sec. (3), leading to an expected error of a few percent in V_0 . This estimate, however, does not take into account the further suppression of the small r region due to the small value of the magnetic scaling index η .

4.3 Connection with the conformal limit.

There is more information to be gained from the TBA. So far the energy operator $\mathcal{E}(r)$ has appeared only in the combination $(2\pi/\nu)(x_c - x)\mathcal{E}(r)$, as the trace $\Theta(r)$ of the stress tensor. But if we fix on a definite normalization of $\mathcal{E}(r)$, for example, that given naturally by its conformal limit, then there is a definite relation between the temperature-like variable $(x_c - x)$ and the mass M . As shown by Al. Zamolodchikov^{6,7}, this may be determined by comparing the corrections to the TBA with the results of a perturbative analysis. This analysis has been carried through already³⁴ for the $O(n)$ model for a slightly different situation than that considered above, namely, when the non-contractible loops which wind around the imaginary time direction carry a factor 2 rather than n . However, this change of boundary condition should not affect the relationship we wish to determine.

With this boundary condition on the cylinder, the effective UV central charge is $\tilde{c} = 1$, rather than vanishing in the limit $n \rightarrow 0$. This may be traced to the existence of a negative dimension operator with scaling dimensions $(\Delta_0, \bar{\Delta}_0)$, such that

$$\tilde{c} = c - 12(\Delta_0 + \bar{\Delta}_0) \quad (80)$$

so we see that $\Delta_0 = \bar{\Delta}_0 = -\frac{1}{24}$, when $n = 0$. Noticing that these are the scaling dimensions of the operator $\varphi_{2,3}$ in the Kac table, we may interpret this choice of boundary conditions as arising from the insertion of this operator at each end of the cylinder. Thus the perturbative expansion of the ground state energy begins as³⁵

$$E(R) = -\frac{\pi}{6R} + R(x_c - x) \left(\frac{2\pi}{R}\right)^{2/3} C_{(2,3),(2,3)}^{\mathcal{E}} + \dots \quad , \quad (81)$$

where $C_{(2,3),(2,3)}^{\mathcal{E}}$ is the structure constant of the three point function $\langle \mathcal{E}\varphi_{2,3}\varphi_{2,3} \rangle$.

In our problem it is natural to normalize the energy operator so that as $r \rightarrow 0$,

$$\langle \mathcal{E}(r)\mathcal{E}(0) \rangle \sim n/r^{4/3} \quad , \quad (82)$$

so that, with periodic boundary conditions, the free energy will be order n as required. This means that we should take $\mathcal{E} = \sqrt{n}\varphi_{1,3}$, where $\varphi_{1,3}$ is normalized in the conventional fashion. As a result the structure constant is

$$C_{(2,3),(2,3)}^{\mathcal{E}} = \lim_{k \rightarrow 1} \sqrt{n(k)} C_{(2,3),(2,3)}^{(1,3)} = \frac{3^{9/4} \Gamma^8(2/3)}{16\pi^3 \Gamma(4/3)} = 0.302248 \quad . \quad (83)$$

Note that this has a finite limit as $n \rightarrow 0$.

Defining the scaling function $\tilde{c}(MR) = -(6R/\pi)E(R)$ we therefore have

$$\tilde{c}(MR) = 1 - 12(2\pi)^{-1/3} (0.302248) (x_c - x) R^{4/3} + \dots \quad . \quad (84)$$

On the other hand, in Ref. 34 $\tilde{c}(MR)$ has been determined as the numerical solution of the TBA equations, and the first terms of its expansion are given by

$$\tilde{c}(MR) = 1 - 0.4454536 (MR)^{4/3} + \dots \quad . \quad (85)$$

Comparing these, we obtain the number which links the physical mass M to the scale of temperature, given our choice of normalization of \mathcal{E}

$$(x_c - x) = \kappa M^{4/3} \quad , \quad (86)$$

where $\kappa \approx 0.226630$. Moreover, using our parametrization $M = \xi_0^{-1}(1 - x/x_c)^{3/4}$, we also see that

$$x_c = \kappa \xi_0^{-4/3} \quad . \quad (87)$$

This relation will be used in section 5 to fix the asymptotic behaviour of the structure factor.

5. THE STRUCTURE FACTOR OF SELF-AVOIDING LOOPS

An important measure of self-avoiding walks, which is, in principle, accessible by light scattering experiments, is the structure factor of N -step loops

$$\mathcal{S}_N(q) = N^{-1} \left| \sum_i \langle e^{iq \cdot r_i} \rangle \right|^2 = N^{-1} \sum_{i,j} \langle e^{iq \cdot (r_i - r_j)} \rangle \quad . \quad (88)$$

$\mathcal{S}_N(q)$ is a positive definite quantity that plays the role of generating function for all moments of the energy-energy correlator. In the continuum limit we have

$$n \sum_N N p_N \mathcal{S}_N(q) x^{N-2} = a_0 \int d^2x e^{iq \cdot x} \langle \mathcal{E}(x) \mathcal{E}(0) \rangle \quad , \quad (89)$$

and $\mathcal{S}_N(q)$ may be expressed as

$$\mathcal{S}_N(q) \underset{N \rightarrow \infty}{\simeq} NF(y) = N \left(1 - \frac{y^2}{2} + \dots \right) \quad , \quad (90)$$

where y is the scaling variable given in terms of the mean square radius of gyration of loops

$$y^2 = q^2 \langle R_g^2 \rangle_N . \quad (91)$$

As we are going to show, the scaling function $F(y)$ may be analyzed in great detail by means of two different methods, the form factor approach and the conformal field theory. By the former we are able to arrive at quite an accurate estimate of the scaling function $F(y)$ (at least for moderate values of the scaling variable y), whereas, by the latter we may extract its asymptotic behaviour.

5.1 Structure Factor in the Two-Particle Approximation

A precise determination of $F(y)$ may be obtained in terms of the two-particle form factor as follows. Expand initially both terms in (89) in power series of q

$$n \sum_{N,p} N p_N \mathcal{S}_{N,2p} q^{2p} x^{N-2} = a_0 \sum_{p=0}^{\infty} \frac{1}{(2p)!} \int d^2x (iq \cdot x)^{2p} \langle \mathcal{E}(x) \mathcal{E}(0) \rangle . \quad (92)$$

(the odd terms in the series vanish by parity). The key point is that we may compute *exactly* the first two coefficients of the series by exploiting the TBA and the c -theorem sum rule. As the other terms are related to the higher moments of the energy-energy correlation function

$$\int d^2x |x|^{2p} \langle \mathcal{E}(x) \mathcal{E}(0) \rangle , \quad (93)$$

they will be approximated with quite high accuracy by retaining only the two-particle contribution. The reason is that the higher powers $|x|^{2p}$ emphasize more the large distance scales of the correlation function, and on the contrary suppress its short distance singularity. Hence they improve substantially the ultraviolet convergence of the integral and the spectral representation of the correlator $\langle \mathcal{E}(x) \mathcal{E}(0) \rangle$ is effectively truncated to the two-particle term¹.

Let us then write the right hand side of (92) as

$$a_0 \left(I_1 - \frac{1}{2} I_2 + I_3 \right) , \quad (94)$$

¹ As a matter of fact, we have already seen in sect. 3 that the two-particle contribution provides an accurate estimate for the second moment. Hence, we expect the precision obtained by the two-particle contribution to increase for the higher moments.

where

$$\begin{aligned}
I_1 &= \int d^2x \langle \mathcal{E}(x) \mathcal{E}(0) \rangle \\
I_2 &= \int d^2x (q \cdot x)^2 \langle \mathcal{E}(x) \mathcal{E}(0) \rangle \\
I_3 &= \sum_{p=2}^{\infty} (-1)^p \int d^2x \frac{1}{(2p)!} (q \cdot x)^{2p} \langle \mathcal{E}(x) \mathcal{E}(0) \rangle \quad .
\end{aligned} \tag{95}$$

As discussed in Sec. (2), I_1 and I_2 are exactly obtained in terms of the extensive part of the free energy and by the c -theorem:

$$\begin{aligned}
I_1 &= n \frac{2\nu(2\nu-1)}{(x_c-x)^2} M^2 U \\
I_2 &= n \frac{q^2}{2} \frac{\nu^2}{3\pi(x_c-x)^2} c'(0) \quad .
\end{aligned} \tag{96}$$

I_3 is related to the higher moments of $\langle \mathcal{E}(x) \mathcal{E}(0) \rangle$, and, computed in the two-particle approximation, it is given by

$$I_3 = \frac{nM^2\nu^2}{2\pi(x_c-x)^2} \sum_{p=2}^{\infty} (-1)^p \frac{q^{2p}}{(4M^2)^p} \mathcal{I}_{2p} \quad , \tag{97}$$

where

$$\mathcal{I}_{2p} = \int_0^{\infty} dt \frac{|F_{\min}(2t)|^2}{(\cosh t)^{2+2p}} \quad . \tag{98}$$

The numerical evaluation of the first \mathcal{I}_{2p} is reported in Table I. They vanish asymptotically as

$$\mathcal{I}_{2p} \underset{p \rightarrow \infty}{\simeq} \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(p - \frac{3}{4}\right)}{\Gamma\left(p + \frac{3}{4}\right)} \Xi^2(0) \quad , \tag{99}$$

where $\Xi(0)$ is a constant defined in the appendix.

To obtain $\mathcal{S}_N(q)$ from the RHS of (92) we need to express M as $\xi_0^{-1}(1-x/x_c)^{3/4}$ and to take the inverse Laplace transform. The result is

$$\begin{aligned}
\frac{\mathcal{S}_{N,2}}{\mathcal{S}_{N,0}} &= -\frac{5}{12\pi\sqrt{\pi}} \frac{\Gamma(N)}{\Gamma\left(N - \frac{3}{2}\right)} \xi_0^2 \\
\frac{\mathcal{S}_{N,2p}}{\mathcal{S}_{N,0}} &= (-)^p \frac{3}{2\sqrt{\pi}} \frac{\mathcal{I}_{2p}}{2^{2p}} \frac{\Gamma\left(N + \frac{3(p-1)}{2}\right)}{\Gamma\left(N - \frac{3}{2}\right) \Gamma\left(\frac{3p+1}{2}\right)} \xi_0^{2p} \quad .
\end{aligned} \tag{100}$$

The non-universal factor ξ_0 as in (100) may be eliminated by using both the scaling form of $\mathcal{S}_N(q)$ ((90)) and the corresponding definition of $\langle R_g^2 \rangle_N$ obtained by $\mathcal{S}_{N,2}/\mathcal{S}_{N,0}$

$$\langle R_g^2 \rangle_N \underset{N \rightarrow \infty}{\simeq} \frac{5}{6\pi\sqrt{\pi}} N^{2\nu} \xi_0^2 \quad , \quad (101)$$

in agreement with (77). With this substitution, the scaling function $F(y)$ is given by

$$F(y) = \left(1 - \frac{y^2}{2} + \sum_{p=2}^{\infty} B_{2p} y^{2p} \right) \quad , \quad (102)$$

where

$$B_{2p} = (-1)^p \frac{3}{2\sqrt{\pi}} \frac{\mathcal{I}_{2p}}{\Gamma\left(\frac{3p+1}{2}\right)} \left(\frac{3\pi\sqrt{\pi}}{10} \right)^p \quad . \quad (103)$$

Their values for $p \leq 20$ are reported in Table I. The resulting series is convergent for all complex y , and in particular yields a highly precise determination of the scaling function $F(y)$ for small values of y . Although the higher order coefficients become more and more precisely determined by the 2-particle approximation, this does not mean that this approximation captures the correct large y behaviour of the scaling function. Indeed, it is straightforward to see that, within the 2-particle approximation, this is determined by the large β behaviour of $F_{\min}(\beta)$, which leads to a power law $y^{-1/2}$. A more accurate way of finding the large y behaviour is through conformal field theory.

5.2 Asymptotic Behaviour of the Structure Factor from CFT

In order to analyze the asymptotic behaviour of $F(y)$ in the limit $y \rightarrow \infty$, let us first write the RHS of (92) as

$$A(q) = \frac{2\pi a_0}{q^2} \int_0^\infty dt t J_0(t) \langle \mathcal{E}(t/q) \mathcal{E}(0) \rangle \quad . \quad (104)$$

($J_0(t)$ is the Bessel function). In the limit $q \rightarrow \infty$ only the short distance scales of the correlation function are important and therefore $\langle \mathcal{E}(t/q) \mathcal{E}(0) \rangle$ is determined by the operator product expansion:

$$\langle \mathcal{E}(r) \mathcal{E}(0) \rangle \underset{r \rightarrow 0}{\simeq} r^{-4/3} + \mathcal{C} \langle \mathcal{E}(0) \rangle r^{-2/3} + \dots \quad . \quad (105)$$

$\langle \mathcal{E}(0) \rangle$ is the vacuum expectation value of the energy operator determined by TBA

$$\langle \mathcal{E}(0) \rangle = -\frac{\nu}{2\pi(x_c - x)} \langle \Theta(0) \rangle = -\frac{3}{32\pi} \frac{M^2}{(x_c - x)} \quad , \quad (106)$$

whereas \mathcal{C} is the structure constant of the CFT algebra. To compute it, we have to use the formulas of Ref. 36 and take into account the fact that we have chosen to normalize the 2-point function of the energy operator $\mathcal{E}(r)$ to n rather than unity. Therefore \mathcal{C} is given by the limit

$$\mathcal{C} = \lim_{k \rightarrow 1} \sqrt{n(k)} \mathcal{C}_{(1,3),(1,3)}^{(1,3)}(k) = 6\sqrt{6\pi} \left(\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \right)^{\frac{9}{2}} . \quad (107)$$

Inserting (105) into (104), the large q behaviour of $A(q)$ is given by

$$\begin{aligned} A(q) &\underset{q \rightarrow \infty}{\simeq} A^{(2/3)} + A^{(4/3)} + \dots \\ &= \frac{2\pi a_0 J_1}{q^{2/3}} - \frac{3a_0 M^2 \mathcal{C} J_2}{32\pi(x_c - x)} \frac{1}{q^{4/3}} + \dots , \end{aligned} \quad (108)$$

where

$$\begin{aligned} J_1 &= \int_0^\infty dt t^{-1/3} J_0(t) = 2^{1/3} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \\ J_2 &= \int_0^\infty dt t^{1/3} J_0(t) = 2^{-1/3} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \end{aligned} \quad (109)$$

To obtain the scaling function $F(y)$ we still have to perform an inverse Laplace transform on $A(q)$. However the leading order $A^{(2/3)}$ of $A(q)$ does not contain any dependence on $(x_c - x)$ and therefore in the asymptotic expansion of $F(y)$,

$$F(y) \underset{y \rightarrow \infty}{\simeq} \frac{a}{y^{2/3}} + \frac{b}{y^{4/3}} + \dots , \quad (110)$$

the coefficient a vanishes identically! Hence the scaling function $F(y)$ decreases at infinity faster than expected by a power-counting argument. As we will discuss at the end of this section, the validity of this result is not restricted to two-dimensional self-avoiding loops but, on the contrary, is quite general. In two dimensions, however, we may use the exact value of the structure constant \mathcal{C} to extract the universal coefficient b in (110). In fact, expressing M as $\xi_0^{-1}(1 - x/x_c)^{3/4}$ and making an inverse Laplace transform on $A^{(4/3)}$ we have

$$N p_N \mathcal{S}_N(q) \underset{q, N \rightarrow \infty}{\sim} - \frac{3\sigma a_0}{32\pi} \mathcal{C} J_2 \xi_0^{-2} x_c \frac{N^{-3/2} \mu^N}{\Gamma(-\frac{1}{2})} q^{-4/3} . \quad (111)$$

Normalizing this quantity to the corresponding expression for $q \rightarrow 0$, i.e.

$$N p_N \mathcal{S}_N(0) \underset{N \rightarrow \infty}{\sim} \frac{3\sigma a_0}{16} \xi_0^{-2} \frac{N^{-1/2} \mu^N}{\Gamma(\frac{1}{2})} , \quad (112)$$

(in agreement with (7, 76),) and using the relation $\langle R^2 \rangle_N \sim 5\xi_0^2 N^{3/2}/6\pi^{3/2}$ from (77), we see that the coefficient b in (110) depends on the combination $\kappa = x_c \xi_0^{4/3}$, which was computed in Sec. (4.3). The appearance of this quantity in this context is not surprising, since we are considering the ultraviolet behaviour of $\mathcal{S}(q)$, which is normalized at $q = 0$. Putting all this together, we obtain the final result

$$F(y) \underset{y \rightarrow \infty}{\simeq} \frac{b}{y^{4/3}} + \dots \quad , \quad (113)$$

where b is the universal constant

$$\begin{aligned} b &= \frac{\kappa}{4\pi} J_2 \left(\frac{5}{6\pi\sqrt{\pi}} \right)^{\frac{2}{3}} \mathcal{C} = \\ &= \kappa \frac{3^{\frac{5}{6}} 5^{\frac{2}{3}}}{(2\pi)^{\frac{3}{2}}} \left(\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \right)^{\frac{7}{2}} = 9.65065 \times 10^{-3} \quad . \end{aligned} \quad (114)$$

Returning to the question of the leading coefficient a of $F(y)$ which vanishes, it is easy to show that this result is quite general and may be simply justified by a physical argument. In fact, in d dimensions the leading behaviour of the generating function of $\mathcal{S}_N(q)$ decreases as $q^{d-2/\nu}$, but with a corresponding coefficient which is independent of $(x_c - x)$. The sub-leading term goes in any dimension as $q^{-1/\nu}$ but the coefficient in front depends on the contrary on $(x_c - x)$. Therefore, the scaling function $F(y)$ always decreases at infinity as $y^{-1/\nu}$. This is what one expects for an object whose fractal dimension is $1/\nu$. It is easy to see that the same power law also rules the large q behaviour of the structure factor of a linear polymer whose generating function is given by

$$\int e^{iq(r-r')} \langle s(0)\mathcal{E}(r)\mathcal{E}(r')s(r_1) \rangle_c d^d r d^d r' d^d r_1 \quad . \quad (115)$$

Since this is a connected correlation function, in the limit $q \rightarrow \infty$ the leading term comes from the \mathcal{E} term in the OPE of $\mathcal{E}(r)\mathcal{E}(r')$ rather than from the unit operator. Hence the decrease of $F(y)$ as $y^{-1/\nu}$ is consistent with the idea that linear and loop polymers have the same fractal dimension on small scales.

6. AREA OF LOOPS AND THE CURRENT-CURRENT CORRELATION FUNCTION.

In this section we show how the amplitude governing the mean square area of self-avoiding closed loops is related to the second moment of a current-current correlation

function in the $O(n)$ model. A summary of this calculation was given in Ref. 10. Consider a loop of N steps. It has been argued that the mean area $\langle a \rangle_N$ of such loops should behave asymptotically as $EN^{2\nu}$, and, more generally, that the moments $\langle a^p \rangle_N$ of the area should behave as $E^{(p)}N^{2p\nu}$, where the amplitude combinations $E^{(p)}/D^p$ are expected to be universal. In this paper we focus on the case $p = 2$.

Any given loop may be assigned two possible orientations. Consider a given orientation, and let $J_\mu^{\text{lat}}(r)$ be a vector of unit magnitude on the link r in the direction of the orientation. Then the *signed* area of a given loop is

$$a = \frac{1}{2} \sum_r \epsilon_{\mu\nu} r_\mu J_\nu^{\text{lat}}(r) \quad . \quad (116)$$

Of course, this quantity averages to zero when summed over orientations, but its mean square is non-zero. For a given loop

$$a^2 = \frac{1}{4} \sum_{r,r'} \left(r_\mu r'_\mu J_\nu^{\text{lat}}(r) J_\nu^{\text{lat}}(r') - r_\mu r'_\nu J_\nu^{\text{lat}}(r) J_\mu^{\text{lat}}(r') \right) \quad . \quad (117)$$

Using the results $\sum_r J_\nu^{\text{lat}}(r) = 0$ and $\sum_r r_\nu J_\nu^{\text{lat}}(r) = 0$, which follow from the fact that J_ν is conserved, (117) may be rewritten as

$$a^2 = \frac{1}{4} \sum_{r,r'} \left(-\frac{1}{2}(r-r')^2 J_\nu^{\text{lat}}(r) J_\nu^{\text{lat}}(r') + (r_\mu - r'_\mu)(r_\nu - r'_\nu) J_\mu^{\text{lat}}(r) J_\nu^{\text{lat}}(r') \right) \quad . \quad (118)$$

This is for one loop. If we now average over all possible oriented loops, weighted by a factor x^N for each loop, we find that

$$\begin{aligned} 2n\mathcal{N}_s \sum_N p_N \langle a^2 \rangle_N x^N \\ = \frac{1}{4} \sum_{r,r'} \left(-\frac{1}{2}(r-r')^2 \langle J_\nu^{\text{lat}}(r) J_\nu^{\text{lat}}(r') \rangle + (r_\mu - r'_\mu)(r_\nu - r'_\nu) \langle J_\mu^{\text{lat}}(r) J_\nu^{\text{lat}}(r') \rangle \right) \quad , \end{aligned} \quad (119)$$

where the correlation functions on the right hand side are evaluated in a *complex* $O(n)$ lattice model, with hamiltonian $H_2 = -x \sum_{\text{nn}} s_a^*(r) s_a(r')$, in the limit $n \rightarrow 0$. The current is the one which generates the $U(1)$ transformations $s_a \rightarrow e^{i\alpha} s_a$. An explicit expression for $J_\mu^{\text{lat}}(r)$ was given in this model in Ref. 10.

(119) is now ready for taking the continuum limit. (Doing this earlier on (117) leads to erroneous results.) To do this, we simply let $\sum_r J_\mu^{\text{lat}}(r) \rightarrow \int J_\mu(r) d^2r$, so that the right hand side of (119) becomes

$$\frac{1}{4} \mathcal{A} \int \left(-\frac{1}{2} r^2 \delta_{\mu\nu} + r_\mu r_\nu \right) \langle J_\mu(r) J_\nu(0) \rangle d^2r \quad . \quad (120)$$

Since J_μ is conserved, by rotational symmetry the correlation function on the right hand side has the form

$$\langle J_\mu(r)J_\nu(0) \rangle = (\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)G(r) \quad , \quad (121)$$

where G is a scalar. Substituting this into (120) and integrating by parts, the right hand side is simply $\mathcal{A} \int G d^2r$. On the other hand, using the same method, one may show that $\int r^2 \langle J_\mu(r)J_\mu(0) \rangle d^2r = -4 \int G d^2r$. Thus

$$2n \sum_N p_N \langle a^2 \rangle_N x^N = -\frac{1}{4} a_0 \int r^2 \langle J_\mu(r)J_\mu(0) \rangle d^2r \quad . \quad (122)$$

Now, since the current has unit scaling dimension, its two-point function has the scaling form

$$\langle J_\mu(r)J_\mu(0) \rangle = -n\xi^{-2}\Phi_J(r/\xi) \quad . \quad (123)$$

Moreover, the normalization of this current is fixed by the requirement that the ends of a self-avoiding walk, which in the complex $O(n)$ model are represented by insertions of s_1^* and s_1 , are respectively unit sources and sinks for J_μ . Therefore the scaling function Φ_J , like that of the correlation function of the trace of the stress tensor, is completely universal with no metric factors. It follows that the right hand side of (122) has the form $\frac{1}{4}na_0U_J\xi_0^2(1-x/x_c)^{-2\nu}$, where $U_J = \int \rho^2 \Phi_J(\rho) d^2\rho$. Thus

$$p_N \langle a^2 \rangle_N \sim \frac{1}{8}\sigma a_0 \xi_0^2 U_J \frac{N^{2\nu-1}}{\Gamma(2\nu)} \mu^N \quad . \quad (124)$$

Using $p_N \sim BN^{-2\nu-1}\mu^N$ and the relation (20) for BD , we then find that

$$\langle a^2 \rangle_N \sim \frac{4\pi^2}{5} \frac{DU_J\xi_0^2}{\Gamma(2\nu)} N^{4\nu} \quad . \quad (125)$$

Finally, using the exact relation (77) for D , we obtain

$$\frac{\langle a^2 \rangle_N}{\langle R^2 \rangle_N^2} = \frac{E^{(2)}}{D^2} = \frac{48\pi^3}{25} U_J \quad . \quad (126)$$

Now let us see how we might compute the current-current correlation function, and hence U_J , through the form factor approach. Since the current is an $O(n)$ singlet, it couples only to the even particle sector. The simplest non-trivial form factor is therefore

$$\langle 0 | J_\mu(x^1) | \beta_1, a_1, +; \beta_2, a_2, - \rangle = e^{iM(\sinh \beta_1 + \sinh \beta_2)x^1} \langle 0 | J_\mu(0) | \beta_1, a_1, +; \beta_2, a_2, - \rangle \quad . \quad (127)$$

In the above \pm denotes the U(1) charge of the particle, and a_j its O(n) colour index. It is simpler to work with the light-cone combinations $J^\pm = J_0 \pm J_1$. Then, by O(n) symmetry, Lorentz covariance and current conservation,

$$\langle 0 | J^\pm(0) | \beta_1, a_1, +; \beta_2, a_2, - \rangle = \delta_{a_1 a_2} (e^{\pm\beta_1} - e^{\pm\beta_2}) F(\beta_{12}) \quad . \quad (128)$$

It is straightforward to check that this satisfies the conservation condition $q^+ J^- + q^- J^+ = 0$, with $q^\pm = M(e^{\pm\beta_1} + e^{\pm\beta_2})$. In fact the Watson equations for this form factor then show that the simplest solution is to take $F(\beta) = d F_{\min}(\beta)$, where d is a normalization constant, and F_{\min} was defined in Sec. (3). The normalization is fixed by crossing (128) to find

$$\langle \beta_1, a_1, - | J^\pm(x^1) | \beta_2, a_2, - \rangle = -d \delta_{a_1 a_2} (e^{\pm\beta_1} + e^{\pm\beta_2}) F_{\min}(i\pi - \beta_{12}) e^{-iM(\sinh \beta_1 - \sinh \beta_2)x^1} \quad . \quad (129)$$

Thus the matrix element of the total U(1) charge is

$$\begin{aligned} \langle \beta_1, a_1, - | \int J^0(x^1) dx^1 | \beta_2, a_2, - \rangle \\ = -2\pi d \delta_{a_1 a_2} (\cosh \beta_1 + \cosh \beta_2) F_{\min}(i\pi - \beta_{12}) \delta(M(\sinh \beta_1 - \sinh \beta_2)) \quad . \end{aligned} \quad (130)$$

The left hand side is simply $-2\pi \delta_{a_1 a_2} \delta(\beta_1 - \beta_2)$. Therefore, using the fact that $F_{\min}(i\pi) = 1$, we find

$$d = \frac{1}{2} M \quad . \quad (131)$$

In the 2-particle approximation, then, the current-current correlation function is

$$\langle J_\mu(r) J_\mu(0) \rangle_2 = \frac{nM^2}{4} \int \frac{d\beta_1}{2\pi} \frac{d\beta_2}{2\pi} (e^{\beta_1} - e^{\beta_2}) (e^{-\beta_1} - e^{-\beta_2}) |F_{\min}(\beta_{12})|^2 e^{-Mr(\cosh \beta_1 + \cosh \beta_2)} \quad . \quad (132)$$

The factor of n comes from the sum over colour indices in the intermediate state. Note that there is no factor of 1/2! since the particles have opposite U(1) charge and so are not identical.

This leads to the estimate for U_J in this approximation

$$U_{J,2} = -\frac{1}{2\pi} \int_0^\infty \frac{(1 - \cosh 2\beta) |F_{\min}(2\beta)|^2}{\cosh^4 \beta} d\beta \quad . \quad (133)$$

The integral may be evaluated numerically using the forms for F_{\min} discussed in Sec. (3), to give $U_{J,2} = 0.61506701$. The resulting estimate for the ratio $E^{(2)}/D^2$ is 36.62. This figure is unreasonably high. Estimates of the mean area³² lead to $(E^{(1)}/D)^2 \approx 6.39$, and we would not expect this to be significantly less than $E^{(2)}/D^2$. Moreover, since the maximal

area of πR^2 is reached for a circular loop, we would not expect this amplitude ratio to exceed π^2 . In fact, recently Guttmann³⁷ has estimated a value for $E^{(2)}$ for the square lattice which gives $E^{(2)}/D^2 \approx 6.72$ a much more reasonable value.

With hindsight it is not hard to find the reason for the failure of the 2-particle approximation in this case. If one examines the region of integration which is contributing the dominant part to the integral in (133) one finds that it comes from $|\beta| \sim 2 - 5$, which corresponds to a center of mass energy between $7.5M$ and $150M$, which is far above the 4-particle threshold at $4M$. Thus, under these circumstances, there is no justification for ignoring the 4-particle (and perhaps higher) intermediate states. The fact that the 2-particle contribution overestimates the result is not surprising in the light of our result for U_0 , where we showed that higher intermediate states can give negative contributions in the $n \rightarrow 0$ limit. There appear to be two reasons for this failure. First, the contribution near the 2-particle threshold is suppressed by a factor $(1 - \cosh 2\beta)$, whose origin may be traced to current conservation. Second, the $\langle J_\mu(r) J_\nu(0) \rangle$ correlation function behaves like r^{-2} in the ultraviolet limit, and is therefore much more singular than the spin-spin or energy-energy correlation functions. Thus it is much harder to approximate this behaviour keeping only the two-particle state, and the ultraviolet region will give a much more important contribution to the second moment.

7. RADIUS OF GYRATION OF SELF-AVOIDING WALKS.

In the previous sections, we have shown how various amplitudes of self-avoiding walks and loops are related to moments of two-point correlation functions in the $O(n)$ model, which may be estimated using the form factor approach. In this section we consider another such quantity, which is, however, related to a higher correlation function. The evaluation of this therefore provides a further test of the form factor method.

Consider an N -step self-avoiding walk from the origin to the point r . If r_i labels a site visited by the walk, then the squared radius of gyration is given by

$$2N^2 R_g^2 = \sum_{i,j} (r_i - r_j)^2 \quad . \quad (134)$$

Following the same line of argument as in Sec. (2), if c_N the total number of N -step walks,

the generating function $\sum_N c_N N^2 \langle R_g^2 \rangle_N x^N$ is proportional to the moment

$$\int (r_1 - r_2)^2 \langle s(0) \mathcal{E}(r_1) \mathcal{E}(r_2) s(r) \rangle d^2 r_1 d^2 r_2 d^2 r \quad . \quad (135)$$

The calculation of this four-point function using the form factors would be very cumbersome: depending on the time-ordering of the points, different kinds of intermediate states would arise, and, in addition, it would be necessary to know at least the three-particle form factors of the spin operator s , which we have avoided so far in this paper.

However, the radius of gyration is in fact related to other measures by the Cardy-Saleur formula³⁸ (as corrected by Caracciolo *et al.*³⁹), which is a consequence of the c -theorem:

$$\frac{246}{91} \langle R_g^2 \rangle_N - 2 \langle R_m^2 \rangle_N + \frac{1}{2} \langle R_e^2 \rangle_N = 0 \quad , \quad (136)$$

where $\langle R_m^2 \rangle_N$ is the mean square distance of a monomer (a site visited by the walk) from one end of the walk, and, as in Sec. (2), $\langle R_e^2 \rangle_N$ is the mean square end-to-end distance. Thus, if we can compute one universal ratio of these measures, we may find the other. In fact, $\langle R_m^2 \rangle_N$ is easier, because it involves calculating only a three-point amplitude.

In fact, let us consider the integral

$$J = \int (r_1 + r_2)^2 \langle \mathcal{E}(0) s(r_1) s(r_2) \rangle d^2 r_1 d^2 r_2 \quad . \quad (137)$$

It will become clear below why we choose this particular moment. Since $(r_1 + r_2)^2 = 2r_1^2 + 2r_2^2 - (r_1 - r_2)^2$, this is related to generating functions in the following way:

$$J = 4 \sum_N c_N N \langle R_m^2 \rangle_N x^{N-1} - \sum_N c_N N \langle R_e^2 \rangle_N x^{N-1} \quad . \quad (138)$$

Let us now see how to evaluate J using the form factor approach. In general we have to sum over all possible time orderings of the points 0, r_1 and r_2 , with a different expression in terms of the form factors in each region. In addition, the regions where t_1 and t_2 are on the same side of the origin will involve, even in the simplest approximation, knowledge of the form factors $\langle \beta_1 \beta_2 | s | \beta_3 \rangle$. We may avoid this by the following trick. Let $r_1 = R + \rho$ and $r_2 = R - \rho$. If we imagine doing the ρ integral first, the result will depend only on $|R|$, so we may choose the time component of R to be zero. This means that the time ordering will always be $s\mathcal{E}s$ and not $\mathcal{E}ss$ or $ss\mathcal{E}$. Let us therefore write $\rho = (t, x)$ and assume that the intermediate states are saturated by their lowest contribution, which, in

this time ordering, is only the 1-particle state. Then

$$\begin{aligned}
J &\approx 4 \int 4R^2 \langle \mathcal{E}(0) s(R-\rho) s(R+\rho) \rangle d^2 R d^2 \rho \\
&= 64\pi \int_0^\infty R^3 dR \int_{-\infty}^\infty dx \int_0^\infty dt \int \frac{d\beta_1}{2\pi} \int \frac{d\beta_2}{2\pi} \langle 0|s|\beta_1 \rangle e^{-iM(R-x) \sinh \beta_1 - Mt \cosh \beta_1} \\
&\quad \cdot \langle \beta_1 | \mathcal{E} | \beta_2 \rangle e^{iM(R+x) \sinh \beta_2 - Mt \cosh \beta_2} \langle \beta_2 | s | 0 \rangle .
\end{aligned} \tag{139}$$

In writing this we have suppressed the colour indices, which are all set equal to a fixed value by the insertion of the spin operator, rather than being summed over. The integrals over x and t are straightforward (it is for this reason that we choose the particular moment defined by J .) The form factors of the energy operator \mathcal{E} are simply proportional to those of the trace of the stress tensor, Θ , given by (56), so that

$$\langle \beta | \mathcal{E} | -\beta \rangle = i M^2 \nu (x_c - x)^{-1} F_{\min}(i\pi + 2\beta) . \tag{140}$$

This gives

$$J = 32\pi \nu (x_c - x)^{-1} |\langle 0|s|1 \rangle|^2 \int_0^\infty R^3 dR \int \frac{d\beta e^{-2iMR \sinh \beta}}{2\pi \cosh^2 \beta} i F_{\min}(i\pi + 2\beta) . \tag{141}$$

Before performing the integral over R it is advantageous to deform the contour so that $\beta \rightarrow \beta - i\pi/2$, with the new β contour lying just above the real axis. This is possible because of the analyticity properties of F_{\min} . The R integral is then straightforward, and we find

$$J \approx 6\nu (x_c - x)^{-1} M^{-4} |\langle 0|s|1 \rangle|^2 I , \tag{142}$$

where

$$I = \int_C \frac{i F_{\min}(2\beta)}{\cosh^4 \beta \sinh^2 \beta} d\beta . \tag{143}$$

The contour runs just above the pole at $\beta = 0$. The contribution from this small semicircle gives $\pi \Xi(0)$, where the number $\Xi(0)$ is evaluated exactly in the Appendix. The remaining principal value integral may be evaluated numerically using the form for F_{\min} given there. The result is that $I \approx \pi \Xi(0) - 0.456 \approx 3.432$.

Now, in the same approximation,

$$\begin{aligned}
\sum_N c_N \langle R_e^2 \rangle_N x^N &= \int r^2 \langle s(r) s(0) \rangle d^2 r \\
&\approx |\langle 0|s|1 \rangle|^2 3! \int \frac{d\beta}{(M \cosh \beta)^4} = 8M^{-4} |\langle 0|s|1 \rangle|^2 ,
\end{aligned} \tag{144}$$

so, for consistency with (27), we should take (up to a non-universal normalization which will cancel)

$$|\langle 0|s|1\rangle|^2 = M^\eta \quad . \quad (145)$$

Thus, in the one-particle approximation,

$$\sum_N c_N \langle R_e^2 \rangle_N x^N = 8\xi_0^{-4+\eta} (1 - x/x_c)^{-\gamma-2\nu} \quad , \quad (146)$$

so that

$$c_N \langle R_e^2 \rangle_N \sim 8\xi_0^{-4+\eta} \frac{N^{\gamma+2\nu-1}}{\Gamma(\gamma+2\nu)} \mu^N \quad . \quad (147)$$

Now, using the same normalization, the desired combination of generating functions (138) is

$$4 \sum_N c_N N \langle R_m^2 \rangle_N x^N - \sum_N c_N N \langle R_e^2 \rangle_N x^N = 6\nu I \xi_0^{-4+\eta} (1 - x/x_c)^{-\gamma-2\nu-1} \quad , \quad (148)$$

so that

$$c_N N (4 \langle R_m^2 \rangle_N - \langle R_e^2 \rangle_N) \sim 6\nu I \xi_0^{-4+\eta} \frac{N^{\gamma+2\nu+1}}{\Gamma(\gamma+2\nu+1)} \mu^N \quad , \quad (149)$$

and finally

$$4 \frac{\langle R_m^2 \rangle_N}{\langle R_e^2 \rangle_N} - 1 = \frac{6\nu I}{8} \frac{\Gamma(\gamma+2\nu)}{\Gamma(\gamma+2\nu+1)} = \frac{6\nu I}{8(\gamma+2\nu)} \quad . \quad (150)$$

Substituting our estimate for I , we then find $\langle R_m^2 \rangle_N / \langle R_e^2 \rangle_N \approx 0.420$, which corresponds to $\langle R_g^2 \rangle_N \langle R_e^2 \rangle_N \approx 0.126$. The best numerical estimates for these ratios, from Monte Carlo simulations, are 0.43962 ± 0.00033 and 0.14029 ± 0.00012 ³⁹.

The approximately 5% error in the approximation of the first ratio is what might be expected from the earlier results of this paper. The particular moment chosen in (137) does not damp the short-distance behaviour of the correlation function as r_1 or r_2 become small. On the other hand, the singularities in this correlation function are less severe than those in the energy-energy correlation function, and in that case, an estimate of U_0 led to an error of less than 10%. Unfortunately, the error in our estimate gets magnified when we use the Cardy-Saleur formula (136) to estimate the other ratio.

Clearly it is possible to improve this calculation at the expense of considerably more analytic work. However, our result does show that the utility of the form factor approach is not restricted to the computation of two-point functions.

8. DISCUSSION.

In this paper we have attempted to show that the study of two-dimensional field theory applied to critical behaviour has advanced to a point where it is now capable of yielding not only exact values for critical exponents but also other universal amplitude ratios and scaling functions which describe the approach to the critical point. We have chosen the self-avoiding walk problem as a non-trivial example where precise numerical comparisons may be made with the results of series expansions.

Our main results are the estimate for the ratio C/D , given in (79); the calculation of the scaling function of the structure factor for closed loops, whose coefficients of whose power series expansion are tabulated in Table I, and whose asymptotic behaviour is given in (113, 114); and the amplitude ratio for the radius of gyration of open self-avoiding walks, given in Sec. (7).

The S -matrix and form factor approach to correlation functions, while not exact in the sense that the sum over intermediate states must be truncated, turns out to be remarkably useful in practice. In most cases, only a few-particle intermediate states need be retained to give a precision of one part in 10^3 . This is related to the softening of the branch cuts by the interactions. This effect is peculiar to two dimensions and is a consequence of two-particle unitarity. Consider, for example, a scalar field theory with a $g\phi^4$ interaction. Two-particle phase space is the imaginary part of the simple bubble diagram

$$\int \frac{d^2k}{(k^2 + M^2)((q - k)^2 + M^2)} \sim \frac{1}{M\sqrt{q^2 + 4M^2}} \quad , \quad (151)$$

However, in the interacting theory this is modified by the iteration of the bubble diagram so that it becomes

$$\frac{1}{M\sqrt{q^2 + 4M^2}} \frac{1}{1 + g/(M\sqrt{q^2 + 4M^2})} \sim 1/g \quad (152)$$

as $q^2 \rightarrow -4M^2$. Thus, no matter how small g , the two-particle branch cut is always softened. A similar effect is responsible for the behaviour $S \rightarrow -1$ of the two-body S -matrix at threshold. The Watson equations also show that these two results are manifestations of the same phenomenon. For higher particle states, the action of this effect in each two-particle channel separately further suppresses their contribution.

For the self-avoiding walk problem, this effect has a simple physical interpretation: the repulsive interaction between different parts of the walk suppresses those configurations

where the walk repeatedly winds back on itself. It is these which correspond to the multi-particle intermediate states in the form factor approach.

While this method works well in most of the examples considered in this paper, it fails in others. The degree of success appears to be directly related to the softness of the short-distance singularities in the correlation function, which is given by the conformal dimensions of the operators involved. For example, for the spin-spin correlation function, which has a singularity of the form $r^{-5/24}$, even the simple one-particle approximation does very well, even for the zeroth moment. For the energy-energy correlation function, which behaves like $r^{-4/3}$, the 2-particle approximation works to a level of 10% for the zeroth moment, and to 1 part in 10^3 for the second. The TBA and the c -theorem give these moments exactly, in any case. In the three-point function $\langle s\mathcal{E}s \rangle$, which has a singularity $r^{-37/48}$, the level of accuracy is about 5%. However, for the current-current correlation function, which behaves like r^{-2} , the two-particle approximation fails completely for the low moments. It would therefore be useful to have another method of interpolation between the infrared and ultraviolet behaviours of such correlation functions.

The possibility of computing higher-point correlation functions also suggests that these methods may give other interesting geometric information on the sizes and shapes of self-avoiding walks and loops, for example, their asphericity.

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APPENDIX

In this appendix we discuss some useful formulas for the $F_{\min}(\beta)$ in the limit $n \rightarrow 0$. For large values of β , $F_{\min}(\beta)$ behaves as

$$F_{\min}(\beta) \sim e^{\frac{3}{8}\beta} . \tag{153}$$

Its analytic structure may be read off from its infinite product representation

$$F_{\min}(\beta) = \sinh \frac{\beta}{2} \Xi(\beta) \quad (154)$$

where

$$\Xi(\beta) = \prod_{l=0}^{\infty} \left| \frac{\Gamma(l+1 + \frac{i\hat{\beta}}{2\pi}) \Gamma(l+2 + \frac{i\hat{\beta}}{2\pi})}{\Gamma^2\left(l + \frac{3}{2} + \frac{i\hat{\beta}}{2\pi}\right)} \frac{\Gamma^2\left(l + \frac{3}{2}\right)}{\Gamma(l+1) \Gamma(l+2)} \right|^{2(l+1)} \quad (155)$$

and $\hat{\beta} = i\pi - \beta$. An interesting quantity is the value of $\Xi(\beta)$ at the origin

$$\Xi(0) = \exp\left(\frac{7}{4\pi^2}\zeta(3)\right) = 1.23756 \quad (156)$$

(where $\zeta(3)$ is the Riemann function).

For numerical calculations, a useful formula is given by a mixed representation of $\Xi(\beta)$

$$\begin{aligned} \Xi(\beta) = & \prod_{l=0}^{N-1} \left[\frac{\left(1 + \left(\frac{\hat{\beta}/2\pi}{l+\frac{3}{2}}\right)^2\right)^2}{\left(1 + \left(\frac{\hat{\beta}/2\pi}{l+1}\right)^2\right) \left(1 + \left(\frac{\hat{\beta}/2\pi}{l+2}\right)^2\right)} \right]^{\frac{(l+1)(l+2)}{2}} \\ & \times \exp \left[- \int_0^{\infty} \frac{dx \sinh \frac{\pi x}{2} q(N, x) e^{-2N\pi x}}{x \sinh^2 \pi x \cosh \frac{\pi x}{2}} \sin^2 \frac{\hat{\beta} x}{2} \right], \end{aligned} \quad (157)$$

where

$$q(N, x) = \frac{(N+1)(N+2)}{2} - N(N+2)e^{-2\pi x} + \frac{N(N+1)}{2}e^{-4\pi x} \quad (158)$$

In fact, the rate of convergence of the integral may be improved substantially by increasing the value of N .

Table I

p	\mathcal{I}_{2p}	B_{2p}
0	2.28627	1
1	3.53980×10^{-1}	-0.5
2	1.65158×10^{-1}	1.17363×10^{-1}
3	1.00385×10^{-1}	-1.65011×10^{-2}
4	6.92166×10^{-2}	1.58450×10^{-3}
5	5.14233×10^{-2}	-1.12325×10^{-4}
6	4.01424×10^{-2}	6.18847×10^{-6}
7	3.24617×10^{-2}	-2.74819×10^{-7}
8	2.69541×10^{-2}	1.01085×10^{-8}
9	2.28462×10^{-2}	-3.14532×10^{-10}
10	1.96857×10^{-2}	8.41962×10^{-12}
11	1.71925×10^{-2}	-1.96581×10^{-13}
12	1.51847×10^{-2}	4.04936×10^{-15}
13	1.35396×10^{-2}	-7.43064×10^{-17}
14	1.21715×10^{-2}	1.22477×10^{-18}
15	1.10193×10^{-2}	-1.82639×10^{-20}
16	1.00382×10^{-2}	2.47954×10^{-22}
17	9.19444×10^{-3}	-3.08162×10^{-24}
18	8.46266×10^{-3}	3.52340×10^{-26}
19	7.82306×10^{-3}	-3.72244×10^{-28}
20	7.26018×10^{-3}	3.64834×10^{-30}

Table I. The values of the first twenty integrals \mathcal{I}_{2p} defined in (98), and the coefficients B_{2p} of the structure factor scaling function (102), expanded in powers of its argument $y = qR$.