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ELSEVIER Linear Algebra and its Applications 389 (2004) 227–234
www.elsevier.com/locate/laa

Birkhoff's contraction coefficient

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Received 18 December 2003; accepted 18 February 2004

Submitted by R.A. Brualdi

Abstract

Presented here is a new proof of the theorem of Garrett Birkhoff which states that multiplication by any positive square matrix induces a contraction mapping on positive projective space with respect to the Hilbert projective metric and also evaluates the contraction coefficient.

AMS classification: 15A48

Keywords: Contraction mapping; Hilbert projective pseudometric; Perron–Frobenius theorem; Linear programming

1. Introduction

In [1] and [2, pp. 383–386], Birkhoff proves that multiplication by any positive square matrix induces a contraction mapping on positive projective space with respect to the Hilbert projective metric, and then he quickly derives the Perron–Frobenius Theorem. Hilbert had originally introduced this metric for a different purpose in a 1903 paper [3] on Bolyai–Lobachevsky geometry. Birkhoff reduces the problem to the case of a 2×2 matrix and then computes the contraction coefficient, appealing to ideas from projective geometry. A second proof presented in [4, pp. 100–110], which refers to [5–10], relies on no established theory, but is quite long and complicated. Caswell [11, p. 372] and Hartfiel [12, p. 22] both refer to this proof in their recently published books. Cavazos-Cadena [13] has just given a third proof

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doi:10.1016/j.laa.2004.02.039

which is shorter and much easier than the second, using only elementary differential calculus. Here we provide yet another proof, which requires some simple algebra and calculus and the most basic linear programming theory. Its disadvantage is that, unlike the earlier proofs, it verifies, rather than derives, the value of the contraction coefficient, but it demonstrates its minimality constructively and tightens the contraction inequality.

It should be mentioned that a fair amount of work has been directed at generalizing Birkhoff's result. Kohlberg and coworkers [14,15] (and unpublished work with Pratt) have further investigated the Hilbert projective metric. In addition, he [16] and other investigators, e.g., [17], have established that some of those functions on the positive cone which share certain properties with positive matrices, including nonnegativity and homogeneity of degree one, are contraction mappings in the same sense and, therefore, have associated Perron theorems.

To state the theorem, we need first to introduce some notions and notation. Let $n \geqslant 2$ and let $\mathbf{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbf{R}^n - \{\mathbf{0}\} \mid x_i \geqslant 0 \text{ for } i = 1, \dots, n\}$, the nonnegative cone in $\mathbf{R}^n$, to which we shall return later. Consider its subset $\mathbf{R}^n_{++} = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_i > 0 \text{ for } i = 1, \dots, n\}$, the positive cone in $\mathbf{R}^n$. We first note that $\mathbf{R}^n_{++}$ is an abelian group under coordinatewise multiplication, with identity $\mathbf{1} = (1, \dots, 1)$. We then define a norm $\|\cdot\| : \mathbf{R}^n_{++} \to [1, \infty)$, which will be manipulated multiplicatively rather than additively, by $\|\mathbf{x}\| = \frac{\max_{1\leqslant i\leqslant n} x_i}{\min_{1\leqslant i\leqslant n} x_i} = \max_{1\leqslant i\leqslant n} \frac{x_i}{x_j}$. One can easily verify the following four observations, the first three of which are analogous to additive properties of real vector space norms. For all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$, (i) $\|\mathbf{x}\| = 1$ if and only $\mathbf{x}$ is constant or scalar, i.e., $\mathbf{x} = (x, \dots, x)$ for some $x > 0$; (ii) $\|\mathbf{x}\| = \|\mathbf{x}^{-1}\|$; (iii) $\|\mathbf{xy}\| \leqslant \|\mathbf{x}\|\|\mathbf{y}\|$; (iv) for any $c > 0$, $\|c\mathbf{x}\| = \|\mathbf{x}\|$, i.e., $\|\cdot\|$ is homogeneous of degree zero. We next define a distance d on $\mathbf{R}^n_{++}$, again in analogy to the vector space case, by $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{xy}^{-1}\|$. By the properties of the norm we have for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n_{++}$, (i) $d(\mathbf{x}, \mathbf{y}) \geqslant 1$ and $d(\mathbf{x}, \mathbf{y}) = 1$ if and only if $\mathbf{x} = c\mathbf{y}$ for some $c > 0$; (ii) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$; (iii) $d(\mathbf{x}, \mathbf{z}) \leqslant d(\mathbf{x}, \mathbf{y})d(\mathbf{y}, \mathbf{z})$; (iv) for $a, b > 0$, $d(a\mathbf{x}, b\mathbf{y}) = d(\mathbf{x}, \mathbf{y})$. The Hilbert projective pseudometric δ on $\mathbf{R}^n_{++}$ is defined by $\delta(\mathbf{x}, \mathbf{y}) = \log d(\mathbf{x}, \mathbf{y})$. By properties (i), (ii), and (iii) of d, it satisfies the definition of a metric except that $\delta(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = c\mathbf{y}$ for some positive c. Considering this and also property (iv), it may be viewed as a metric on $\mathbf{P}^{n-1}_{++}$, the positive part of real $n - 1$ dimensional projective space.

Using the group structure of $\mathbf{R}^n_{++}$, we see that if $\mathbf{a} \in \mathbf{R}^n_{++}$ is fixed, then multiplicative translation by $\mathbf{a}$, $\mathbf{x} \to \mathbf{ax}$, is a bijection on $\mathbf{R}^n_{++}$. Furthermore, it follows from the definition of d that for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$, $d(\mathbf{ax}, \mathbf{ay}) = d(\mathbf{x}, \mathbf{y})$, i.e., such a translation is an isometry. Finally, we define $\tau : [1, \infty) \to [0, 1)$ by $\tau(r) = \frac{\sqrt{r}-1}{\sqrt{r}+1}$. It is easily verified that τ is strictly increasing and onto.

Let $m, n \geqslant 2$ and let $\mathbf{A}$ be an $m \times n$ matrix with positive terms, so that $\mathbf{A} : \mathbf{R}^n_{++} \to \mathbf{R}^m_{++}$. Because $\mathbf{A}$ is homogenous of degree 1, i.e., $\mathbf{A}(a\mathbf{x}) = a\mathbf{Ax}$ for all $\mathbf{x} \in \mathbf{R}^n_{++}$, and $a > 0$, it also induces a map, again called $\mathbf{A} : \mathbf{P}^{n-1}_{++} \to \mathbf{P}^{m-1}_{++}$. For $i = 1, \dots m$, let $\mathbf{a}_i$ be the ith row vector of $\mathbf{A}$ and let $d(\mathbf{A}) = \max_{1\leqslant i,j\leqslant m} d(\mathbf{a}_i, \mathbf{a}_j)$. Let

$\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$ be arbitrary and first consider the case $d(\mathbf{x}, \mathbf{y})^{\tau(d(\mathbf{A}))} = 1$. This occurs if and only if either $d(\mathbf{x}, \mathbf{y}) = 1$ or if $\tau(d(\mathbf{A})) = 0$, the latter of which is equivalent to $d(\mathbf{A}) = 1$. In the first instance, by property (i) of d and degree 1 homogeneity of $\mathbf{A}$, it is clear that $d(\mathbf{Ax}, \mathbf{Ay}) = 1$. The second instance is equivalent to $\mathbf{A}$ having rank 1, which also implies that $d(\mathbf{Ax}, \mathbf{Ay}) = 1$. Here and later we use d to represent distances in both $\mathbf{R}^n_{++}$and $\mathbf{R}^m_{++}$ simultaneously.

We state Birkhoff's theorem in terms of d rather than δ, but the translation is easy. We do not include the trivial case discussed immediately above so that we may claim a strict inequality and, as in [13], we state the theorem for rectangular, not just square, matrices.

Theorem 1 (Birkhoff). *Let $m, n \geqslant 2$, let $\mathbf{A}$ be an $m \times n$ matrix with positive terms, and assume that $d(\mathbf{A}) > 1$. Then for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$ with $d(\mathbf{x}, \mathbf{y}) > 1$, we have $d(\mathbf{Ax}, \mathbf{Ay}) < d(\mathbf{x}, \mathbf{y})^{\tau(d(\mathbf{A}))}$. Furthermore, if $\eta < \tau(d(\mathbf{A}))$, then there is a pair $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$, such that $d(\mathbf{Ax}, \mathbf{Ay}) > d(\mathbf{x}, \mathbf{y})^{\eta}$.*

Remark 1. "Taking the logarithm" of the theorem, we obtain $\delta(\mathbf{Ax}, \mathbf{Ay}) < \tau(d(\mathbf{A}))\delta(\mathbf{x}, \mathbf{y})$ and, therefore, find that when $m = n$, $\mathbf{A}$ is a contraction mapping on $\mathbf{P}^{n-1}_{++}$ with respect to the Hilbert projective metric, with minimal contraction coefficient $\tau(d(\mathbf{A}))$. From this, much of the Perron–Frobenius Theorem can be obtained, along with a measure of the dominance of the maximal eigenvalue of $\mathbf{A}$ [2, Corollaries 1 and 2, p. 385], which is not a consequence of the more common algebraic proof of that theorem.

2. Proof of the theorem

Let $\mathbf{A}, \mathbf{x}, \mathbf{y}$ be as in the statement of the theorem. First note that if $d(\mathbf{Ax}, \mathbf{Ay}) = 1$, then there is nothing to prove, so we may restrict attention to the case $d(\mathbf{Ax}, \mathbf{Ay}) > 1$. This condition will be rephrased below along with the problem. The remainder of the proof is divided into sections for readability. In Sections 2.1–2.3 we prove that $\tau(d(\mathbf{A}))$ is a contraction coefficient, and in Section 2.4 that there is no smaller one.

2.1. Reduction to the $2 \times n$ case and a further restatement

First, we claim that it is enough to know that the theorem is true for $m = 2$, so suppose that it is and let $m \geqslant 2$ be arbitrary. Then, using dot product notation, $d(\mathbf{Ax}, \mathbf{Ay}) = \frac{(\mathbf{a}_i \cdot \mathbf{x})(\mathbf{a}_i \cdot \mathbf{y})^{-1}}{(\mathbf{a}_j \cdot \mathbf{x})(\mathbf{a}_j \cdot \mathbf{y})^{-1}}$ for some pair i, j with $1 \leqslant i, j \leqslant m$. If $\mathbf{A}'$ is the $2 \times n$ submatrix with first and second rows $\mathbf{a}_i$ and $\mathbf{a}_j$, then $d(\mathbf{A}'\mathbf{x}, \mathbf{A}'\mathbf{y}) = d(\mathbf{Ax}, \mathbf{Ay}) > 1$ and, therefore, $1 < d(\mathbf{A}') = d(\mathbf{a}_i, \mathbf{a}_j)$. But by our assumption and because τ increasing,

$$\begin{aligned} d(\mathbf{Ax},\mathbf{Ay}) &= d(\mathbf{A}'\mathbf{x},\mathbf{A}'\mathbf{y}) < d(\mathbf{x},\mathbf{y})^{\tau(d(\mathbf{A}'))} \\ &= d(\mathbf{x},\mathbf{y})^{\tau(d(\mathbf{a}_i,\mathbf{a}_j))} \leqslant d(\mathbf{x},\mathbf{y})^{\tau(d(\mathbf{A}))}. \end{aligned}$$

To prove the theorem for $m = 2$, we must show that given any positive $2 \times n$ matrix $\mathbf{A}$ with $d(\mathbf{A}) > 1$, we have for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$ with $d(\mathbf{x},\mathbf{y}) > 1$

$$d(\mathbf{Ax},\mathbf{Ay}) < d(\mathbf{x},\mathbf{y})^{\tau(d(\mathbf{a}_1,\mathbf{a}_2))}. \tag{1a}$$

Let us write $\mathbf{a}_1 = \mathbf{a}$ and $\mathbf{a}_2 = \mathbf{b}$ with $d(\mathbf{a},\mathbf{b}) = d(\mathbf{A}) > 1$. Then we must show for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$ with $d(\mathbf{x},\mathbf{y}) > 1$, that $d(\mathbf{Ax},\mathbf{Ay})$, which is the larger of $\frac{(\mathbf{a}\cdot\mathbf{x})(\mathbf{a}\cdot\mathbf{y})^{-1}}{(\mathbf{b}\cdot\mathbf{x})(\mathbf{b}\cdot\mathbf{y})^{-1}}$ and its reciprocal, is strictly exceeded by $d(\mathbf{x},\mathbf{y})^{\tau(d(\mathbf{a},\mathbf{b}))}$. But by switching $\mathbf{a}$ and $\mathbf{b}$, we get that reciprocal and do not change the proposed upper bound, so it is enough to show for all $\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y} \in \mathbf{R}^n_{++}$ with $d(\mathbf{a},\mathbf{b}), d(\mathbf{x},\mathbf{y}) > 1$, that $\frac{(\mathbf{a}\cdot\mathbf{x})(\mathbf{a}\cdot\mathbf{y})^{-1}}{(\mathbf{b}\cdot\mathbf{x})(\mathbf{b}\cdot\mathbf{y})^{-1}} < d(\mathbf{x},\mathbf{y})^{\tau(d(\mathbf{a},\mathbf{b}))}$ Now, recalling that multiplicative translation by $\mathbf{b}^{-1}$ is an isometry on $\mathbf{R}^n_{++}$, we can reframe inequality (1a) as follows. For all $\mathbf{a},\mathbf{b},\mathbf{x},\mathbf{y} \in \mathbf{R}^n_{++}$, with $d(\mathbf{a},\mathbf{b}), d(\mathbf{x},\mathbf{y}) > 1$,

$$\frac{(\mathbf{a}\cdot\mathbf{b}^{-1}\mathbf{x})(\mathbf{b}\cdot\mathbf{b}^{-1}\mathbf{y})^{-1}}{(\mathbf{b}\cdot\mathbf{b}^{-1}\mathbf{x})(\mathbf{b}\cdot\mathbf{b}^{-1}\mathbf{y})^{-1}} < d(\mathbf{b}^{-1}\mathbf{x},\mathbf{b}^{-1}\mathbf{y})^{\tau(d(\mathbf{a},\mathbf{b}))} = d(\mathbf{x},\mathbf{y})^{\tau(d(\mathbf{a},\mathbf{b}))}.$$

But, letting $\mathbf{r} = \mathbf{ab}^{-1}$ and $\mathbf{s} = \mathbf{xy}^{-1}$, both nonconstant since $d(\mathbf{a},\mathbf{b}), d(\mathbf{x},\mathbf{y}) > 1$,

$$\begin{aligned} \frac{(\mathbf{a}\cdot\mathbf{b}^{-1}\mathbf{x})(\mathbf{a}\cdot\mathbf{b}^{-1}\mathbf{y})^{-1}}{(\mathbf{b}\cdot\mathbf{b}^{-1}\mathbf{x})(\mathbf{b}\cdot\mathbf{b}^{-1}\mathbf{y})^{-1}} &= \frac{(\mathbf{ab}^{-1}\cdot\mathbf{x})(\mathbf{ab}^{-1}\cdot\mathbf{y})^{-1}}{(\mathbf{bb}^{-1}\cdot\mathbf{x})(\mathbf{bb}^{-1}\cdot\mathbf{y})^{-1}} = \frac{(\mathbf{ab}^{-1}\cdot\mathbf{x})(\mathbf{1}\cdot\mathbf{y})}{(\mathbf{ab}^{-1}\cdot\mathbf{y})(\mathbf{1}\cdot\mathbf{x})} \\ &= \frac{(\mathbf{r}\cdot\mathbf{sy})(\mathbf{1}\cdot\mathbf{y})}{(\mathbf{r}\cdot\mathbf{y})(\mathbf{1}\cdot\mathbf{sy})} = \frac{(\mathbf{rs}\cdot\mathbf{y})(\mathbf{1}\cdot\mathbf{y})}{(\mathbf{r}\cdot\mathbf{y})(\mathbf{s}\cdot\mathbf{y})}. \end{aligned}$$

Clearly, as $\mathbf{s}$ and $\mathbf{y}$ range through all of $\mathbf{R}^n_{++}$ with $\mathbf{s}$ nonconstant, so do $\mathbf{x}$ and $\mathbf{y}$ with $d(\mathbf{x},\mathbf{y}) > 1$. Now, $d(\mathbf{a},\mathbf{b}) = \|\mathbf{ab}^{-1}\| = \|\mathbf{r}\|$, and $d(\mathbf{x},\mathbf{y}) = \|\mathbf{s}\|$, so we define a function $F : \mathbf{R}^{3n}_{++} \to \mathbf{R}^n_{++}$, and restate the inequality once again. For all $\mathbf{r},\mathbf{s},\mathbf{y} \in \mathbf{R}^n_{++}$ with $\mathbf{r},\mathbf{s}$ nonconstant

$$F(\mathbf{r},\mathbf{s},\mathbf{y}) = \frac{(\mathbf{rs}\cdot\mathbf{y})(\mathbf{1}\cdot\mathbf{y})}{(\mathbf{r}\cdot\mathbf{y})(\mathbf{s}\cdot\mathbf{y})} < \|\mathbf{s}\|^{\tau(\|\mathbf{r}\|)}. \tag{1b}$$

In fact, we may as well assume that $F(\mathbf{r},\mathbf{s},\mathbf{y}) > 1$, for if not, since $\|\mathbf{s}\|^{\tau(\|\mathbf{r}\|)} > 1$, inequality (1b) is automatically true.

2.2. *Properties of F and reduction to the* 2×2 *case*

We shall broaden the domain of F slightly and consider inequality (1b) for all nonconstant $\mathbf{r},\mathbf{s} \in \mathbf{R}^n_{++}$ and $\mathbf{y} \in \mathbf{R}^n_{+}$. We observe that F is homogeneous of degree zero in all three variables, i.e., for all $a, b, c > 0$, $F(\mathbf{r},\mathbf{s},\mathbf{y}) = F(a\mathbf{r}, b\mathbf{s}, c\mathbf{y})$. By property (iv) of our norm, found in the introduction, $\|\mathbf{s}\|^{\tau(\|\mathbf{r}\|)}$ is also homogeneous of degree zero in both $\mathbf{r}$ and $\mathbf{s}$, so to prove inequality (1b) for a given $(\mathbf{r},\mathbf{s},\mathbf{y})$, it is enough to establish it for any triple of positive scalar multiples $(a\mathbf{r}, b\mathbf{s}, c\mathbf{y})$. Furthermore, simultaneously permuting the indices, $i = 1, \dots n$, of $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{y}$ does not change $\|\mathbf{r}\|$, $\|\mathbf{s}\|$, or the value of F and, therefore, does not alter the problem.

Let $\mathbf{r}, \mathbf{s} \in \mathbf{R}^n_{++}$ nonconstant, $\mathbf{y} \in \mathbf{R}^n_+$, and $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) > 1$. By scaling $\mathbf{r}$ and $\mathbf{s}$ suitably, we can assume that (i) $\mathbf{r} \cdot \mathbf{y} = 1$ and (ii) $\mathbf{s} \cdot \mathbf{y} = \mathbf{1} \cdot \mathbf{y}$, or $(\mathbf{s} - \mathbf{1}) \cdot \mathbf{y} = 0$. Under conditions (i) and (ii), $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) = \mathbf{rs} \cdot \mathbf{y}$, and so [18] $\mathbf{y}$ is a feasible solution to the linear program maximizing $\mathbf{rs} \cdot \mathbf{x}$ for $\mathbf{x} \in \mathbf{R}^n_+$ subject to the constraints $\mathbf{r} \cdot \mathbf{x} = 1$ and $(\mathbf{s} - \mathbf{l}) \cdot \mathbf{x} = 0$. (This $\mathbf{x}$ is just the dummy variable used for the statement of the programming problem and has nothing to do with $\mathbf{x} = \mathbf{sy}$ above.) Since the first constraint alone restricts the domain of $\mathbf{rs}\cdot$ to a compact (with respect to the usual metrics) subset of $\mathbf{R}^n_+$, it achieves a maximum. By the fundamental theorem of linear programming, it is maximized at a basic feasible solution $\mathbf{x}_0$, one with at most two nonzero coordinates. If $\mathbf{x}_0$ had only one nonzero coordinate x_i, then $F(\mathbf{r}, \mathbf{s}, \mathbf{x}_0) = F(r_i, s_i, x_i) = 1$, the latter F defined for $n = 1$. This means that there are $i \neq j$ and $x_i, x_j > 0$ satisfying $r_i x_i + r_j x_j = 1$ and $(s_i - 1)x_i + (s_j - 1)x_j = 0$ such that $1 < F(\mathbf{r}, \mathbf{s}, \mathbf{y}) \leqslant r_i s_i x_i + r_j s_j x_j = F((r_i, r_j), (s_i, s_j), (x_i, x_j))$, where the latter F is defined for $n = 2$. Then, (r_i, r_j) and (s_i, s_j) are nonconstant, or otherwise $F = 1$. Therefore, since $\|(r_i, r_j)\| \leqslant \|\mathbf{r}\|$ and $\|(s_i, s_j)\| \leqslant \|\mathbf{s}\|$ it is adequate to prove inequality (1b) for all nonconstant $\mathbf{r}, \mathbf{s} \in \mathbf{R}^2_{++}$, $\mathbf{y} \in \mathbf{R}^2_{++}$. We note that Birkhoff [2, Theorem 3, p. 384] obtains an analogous reduction to the case $n = 2$.

2.3. *The* 2×2 *case*

Let $\mathbf{r}, \mathbf{s} \in \mathbf{R}^2_{++}$, nonconstant, and $\mathbf{y} \in \mathbf{R}^2_{++}$. By switching indices if necessary and scaling $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{y}$, we may further suppose that $\mathbf{r} = (1, r)$ with $r > 1$, so $\|\mathbf{r}\| = r$, $\mathbf{s} = (1, s)$ with $s \neq 1$, and $\mathbf{y} = (1, y)$. Also assume that $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) > 1$. Then

$$\begin{aligned} F(\mathbf{r}, \mathbf{s}, \mathbf{y}) &= \frac{((1, rs) \cdot (1, y))((1, 1) \cdot (1, y))}{((1, r) \cdot (1, y))((1, s) \cdot (1, y))} = \frac{(1 + rsy)(1 + y)}{(1 + ry)(1 + sy)} \\ &= \frac{1 + rsy^2 + (1 + rs)y}{1 + rsy^2 + (r + s)y}. \end{aligned}$$

Now, if $s < 1$, then $0 > (r - 1)(s - 1) = (1 + rs) - (r + s)$, so the numerator of F is smaller than its denominator and $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) < 1$. Therefore, $s > 1$, which implies that $\|\mathbf{s}\| = s$. Since $0 \leqslant (1 - y\sqrt{rs})^2 = 1 + rsy^2 - 2y\sqrt{rs}$, so that $1 + rsy^2 \geqslant 2y\sqrt{rs}$, we have

$$F(\mathbf{r}, \mathbf{s}, \mathbf{y}) \leqslant \frac{2y\sqrt{rs} + (1 + rs)y}{2y\sqrt{rs} + (r + s)y} = \frac{2\sqrt{rs} + 1 + rs}{2\sqrt{rs} + r + s} = \left(\frac{1 + \sqrt{rs}}{\sqrt{r} + \sqrt{s}}\right)^2,$$

with equality when $y\sqrt{rs} = 1$ or $y = \sqrt{r^{-1}s^{-1}}$. It is, therefore, sufficient to show that

$$\left(\frac{1 + \sqrt{rs}}{\sqrt{r} + \sqrt{s}}\right)^2 < \|\mathbf{s}\|^{\tau(\|\mathbf{r}\|)} = s^{\tau(r)} = s^{\frac{\sqrt{r}-1}{\sqrt{r}+1}} \quad \text{or} \quad \frac{1 + \sqrt{rs}}{\sqrt{r} + \sqrt{s}} < (\sqrt{s})^{\frac{\sqrt{r}-1}{\sqrt{r}+1}}.$$

As this needs to be established for arbitrary $r, s > 1$, we may replace $\sqrt{r}$ by r and $\sqrt{s}$ by s, and are left with the task of proving that for all $r, s > 1$,

$$f_1(r,s) = \frac{1+rs}{r+s} < s^{\frac{r-1}{r+1}} = g_1(r,s).$$

We see immediately that $f_1(r,1) = g_1(r,1) = 1$, and so, by the mean value theorem, we shall be done if we show that for all $r, s > 1$, $\frac{\partial f_1}{\partial s}(r,s) < \frac{\partial g_1}{\partial s}(r,s)$, i.e., $\frac{r^2-1}{(r+s)^2} < \frac{r-1}{r+1} s^{\frac{-2}{r+1}}$. This is true if and only if

$$\left(\frac{r+1}{r+s}\right)^2 < s^{-\frac{2}{r+1}} \iff \frac{r+1}{r+s} < s^{-\frac{1}{r+1}}$$
$$\iff f_2(r,s) = \frac{r+s}{r+1} > s^{\frac{1}{r+1}} = g_2(r,s).$$

Again, $f_2(r,1) = g_2(r,1) = 1$, so it is enough to show that $\frac{\partial f_2}{\partial s}(r,s) > \frac{\partial g_2}{\partial s}(r,s)$, i.e., $\frac{1}{r+1} > \frac{1}{r+1} s^{\frac{-r}{r+1}}$ but this is clearly true whenever $r > 0$ and $s > 1$. This implies inequality (1b) for this case and finishes the proof that $\tau(d(\mathbf{A}))$ is a contraction coefficient.

2.4. $\tau(d(\mathbf{A}))$ is minimal

We continue working in the realm of F, $\mathbf{r}$, $\mathbf{s}$, and $\mathbf{y}$, leaving the translation back to that of d, $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{y}$ to the reader. We first consider the 2×2 case, so let $\mathbf{r} = (r_1, r_2) \in \mathbf{R}^2_{++}$ nonconstant but otherwise arbitrary. We shall assume that $r_2 > r_1$ and let $r = \frac{r_2}{r_1} > 1$. The argument for $r_2 < r_1$ is the same except for a coordinate switch at the end. Let $0 < \eta < \tau(\|\mathbf{r}\|) = \tau(r)$, so that $\eta = \tau(r')$, where $1 < r' < r$. From the calculation of $\frac{\partial f_1}{\partial s}$ and $\frac{\partial g_1}{\partial s}$ in the last section, we see that $\frac{\partial f_1}{\partial s}(\sqrt{r}, 1) = \frac{\sqrt{r}-1}{\sqrt{r}+1} = \tau(r) > \eta = \tau(r') = \frac{\partial g_1}{\partial s}(\sqrt{r'}, 1)$ and, since $f_1(\sqrt{r}, 1) = g_1(\sqrt{r'}, 1)$, we have $f_1(\sqrt{r}, s) > g_1(\sqrt{r'}, s)$ for all $s > 1$ and also close enough to 1. Therefore, for an even larger interval of $s > 1$ we deduce that $f_1(\sqrt{r}, \sqrt{s}) > g_1(\sqrt{r'}, \sqrt{s})$, i.e., $\frac{1+\sqrt{rs}}{\sqrt{r}+\sqrt{s}} > (\sqrt{s})^\eta$. As in Section 2.3, for such s this implies that $F(\mathbf{r}, (1,s), (1, \sqrt{r^{-1}s^{-1}})) = F((1,r),(1,s),(1,\sqrt{r^{-1}s^{-1}})) > \|(1,s)\|^\eta$, establishing the minimality of $\tau(d(\mathbf{A}))$ for $m = n = 2$.

Next, let $n \geqslant 3$ and let $\mathbf{r} \in \mathbf{R}^n_{++}$, nonconstant. Again without loss of generality, scaling $\mathbf{r}$ and permuting the indices of its coordinates, we may assume that $r_1 = \min_{1 \leqslant i \leqslant n} r_i = 1$ and $r_2 = \max_{1 \leqslant i \leqslant n} r_i = r > 1$, implying that $\|\mathbf{r}\| = r$. For $0 < \eta < \tau(r)$, we have just shown above that there are $s > 1$, such that if we choose $\mathbf{s} = (1, s, s_3, \ldots, s_n)$ with $1 \leqslant s_i \leqslant s$ for $i = 3, \ldots, n$ (implying that $\|\mathbf{s}\| = s$), and take $\mathbf{y} = (1, \sqrt{r^{-1}s^{-1}}, 0, \ldots, 0)$, then $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) > \|\mathbf{s}\|^\eta$. But F is continuous in $\mathbf{y}$ with respect to the usual metrics, so letting $y_i > 0$ but small enough for $i = 3, \ldots, n$, and letting $\mathbf{y} = (1, \sqrt{r^{-1}s^{-1}}, y_3, \ldots, y_n)$, we shall still have $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) > \|\mathbf{s}\|^\eta$.

Finally, if $\mathbf{A}$ is an arbitrary positive $m \times n$ matrix with $m, n \geqslant 2$, we simply apply the translation of the preceding argument to one of the $2 \times n$ submatrices with rows $\mathbf{a}_i$, and $\mathbf{a}_j$ such that $d(\mathbf{a}_i, \mathbf{a}_j)$ is maximal.

Remark 2. Birkhoff [1,2] proved Theorem 1 for a slightly wider choice of $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbf{R}^n_{+\frac{1}{2}} = \{\mathbf{x} \in \mathbf{R}^n_+ \mid x_i > 0 \text{ for at least two indices } i\}$. Two vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{+\frac{1}{2}}$ are said to be comparable if x_i and y_i are zero for exactly the same set of indices. We can define the norm of a vector $\mathbf{x} \in \mathbf{R}^n_{+\frac{1}{2}}$ by considering in the computation only indices for which $x_i \neq 0$, and use it to define the distance between any two comparable vectors in $\mathbf{R}^n_{+\frac{1}{2}}$. Looking back at our proof, it is not difficult to see that it generalizes to allow comparable pairs $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{+\frac{1}{2}}$.

Remark 3. Because F is symmetric in $\mathbf{r}$ and $\mathbf{s}$, i.e., $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) = F(\mathbf{s}, \mathbf{r}, \mathbf{y})$, we have also shown that $F(\mathbf{r}, \mathbf{s}, \mathbf{y}) < \|\mathbf{r}\|^{\tau(\|\mathbf{s}\|)}$ for all nonconstant $\mathbf{r}, \mathbf{s} \in \mathbf{R}^n_{++}$, $\mathbf{y} \in \mathbf{R}^n_{++}$. Applying this as we have above to any positive $m \times n$ matrix $\mathbf{A}$ with $d(\mathbf{A}) > 1$, we obtain for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_{++}$ with $d(\mathbf{x}, \mathbf{y}) > 1$, $d(\mathbf{Ax}, \mathbf{Ay}) < d(\mathbf{A})^{\tau(d(\mathbf{x},\mathbf{y}))}$. We can find no obvious use for this bonus result. Of course, it implies that $d(\mathbf{Ax}, \mathbf{Ay}) < d(\mathbf{A})$ if $d(\mathbf{A}) > 1$, which, in turn, implies that the image of $\mathbf{A}$ in $\mathbf{R}^n_{++}$ is bounded with respect to the Hilbert projective pseudometric, but one can give a direct proof of this weaker inequality. The argument is "dual" to the proof that $d(\mathbf{Ax}, \mathbf{Ay}) < d(\mathbf{x}, \mathbf{y})$ for $d(\mathbf{x}, \mathbf{y}) > 1$, which is a sharpening of [2, Theorem 2, p. 382] for $\mathbf{R}^n$, and which requires not much more than the fact that $\mathbf{A} : \mathbf{R}^n_+ \to \mathbf{R}^m_+$.

References

[1] G. Birkhoff, Extension of Jentzsh's theorem, Trans. Amer. Math. Soc. 85 (1957) 219–227.

[2] G. Birkhoff, Lattice Theory, third ed., Amer. Math. Soc. Colloq. Publ., vol. 25, American Mathematical Society, Providence, 1967.

[3] D. Hilbert, Neue Begründung der Bolyai–Lobatschefskyschen geometrie, Math. Ann. 57 (1903) 137–150.

[4] E. Seneta, Non-negative Matrices and Markov Chains, second ed., Springer-Verlag, New York, 1981.

[5] F.L. Bauer, An elementary proof of the Hopf inequality, Numen. Math. 7 (1965) 331–337.

[6] E. Hopf, An inequality for positive integral linear operators, J. Math. Mech. 12 (1963) 683–692.

[7] A.M. Ostrowski, Positive matrices and functional analysis, in: H. Schneider (Ed.), Recent Advances in Matrix Theory, University of Wisconsin Press, 1964, pp. 81–101.

[8] P.J. Bushnell, On the projective contraction ratio for positive linear mappings, J. London Math. Soc. 6 (1973) 256–258.

[9] F.L. Bauer, E. Deutsch, J. Stoer, Abschätzungen fur die Eigenwerte positiver linearer operatoren, Linear Algebra Appl. 2 (1969) 275–301.

[10] S. Sheridan, Inhomogeneous Products of Non-Negative Matrices, Honours Year Project, Department of Statistics, School of General Studies, Australian National University, Canberra, 1979 (Chapter 2).

[11] H. Caswell, Matrix Population Models, second ed., Sinauer Associates, Inc, Sunderland, MA, 2001.

[12] D.J. Hartfiel, Nonhomogeneous Matrix Products, World Scientific, Singapore, 2002.
[13] R. Cavazos-Cadena, An alternative derivation of Birkhoff's formula for the contraction coefficient of a positive matrix, Linear Algebra Appl. 375 (2003) 291–297.
[14] E. Kohlberg, J.W. Pratt, The contraction mapping approach to the Perron–Frobenius theory: why Hilbert's metric? Math. Oper. Res. 7 (2) (1980) 198–210.
[15] E. Kohlberg, A. Neyman, Convergence in Hilbert's metric and convergence in direction, J. Math. Anal. Appl. 93 (1) (1983) 104–108.
[16] E. Kohlberg, The Perron–Frobenius theorem without additivity, J. Math. Econom. 10 (2–3) (1982) 299–303.
[17] S. Gaubert, J. Gunawardena, The Perron–Frobenius theorem for homogeneous, monotone functions, Technical Report HPL-BRIMS-2001-12, Basic Research Institute in the Mathematical Science, HP Laboratories Bristol, 2001.
[18] D.G. Luenberger, Linear and Nonlinear Programming, second ed., Addison-Wesley, Menlo Park, 1984.