Proceedings of the 32nd Conference
on Decision and Control
San Antonio, Texas • December 1993

A Circuit Theory of the Kalman Filter

David W. Carter

Analysis and Software Department
The Charles Stark Draper Laboratory, Inc.
Cambridge, MA 02139
dcarter@draper.com

Introduction

This note gives a short heuristic derivation of the discrete Kalman filter by considering simple voltage measurement circuits. The idea is to obtain conclusions about the general estimation problem by reasoning about an electric analog. The usual Kalman filter and the formulation known as the information filter are both quickly derived by this method.

A basis for this approach is Nyquist's result[5] that any resistance R exhibits a thermal noise voltage — the Johnson noise in radio engineering — of constant power spectral density proportional to R. It follows that multivariate white noise may be realized as the thermal noise of a resistive n-port. We model the general noisy measurement problem as a measurement problem on a resistive circuit. Covariance matrices arise in a concrete fashion as resistance matrices.

Measurement Circuits

Imagine using an ideal volt-ohm-milliammeter (VOM) to observe an unknown voltage source x. Such a measurement consists of observing the current i which flows through a known resistance R and applying Ohm's Law $\hat{x} = Ri$. A surprising fact is that *this ideal measurement is noisy,* since any resistance R exhibits a thermal noise potential difference. The precise information we need is a consequence of thermodynamic arguments due to Nyquist[5], who showed:

Theorem 1 *The potential difference due to thermal agitation of electrons in a pure resistance R at temperature T is white noise of power spectral density $4kTR$.*

Here k is the Boltzmann constant. With the correct assumptions about the sampling time and the temperature we may infer:

Corollary *The discrete random process realized by fixed rate sampling of the voltage across a resistance R is white noise of covariance R.*

Write v_R to denote white noise of covariance R. In circuit diagrams, we show the thermal noise v_R inherent to each resistance R explicitly.

Nyquist's result holds no matter how the resistance R is realized. In particular, we may imagine R to be constructed from two resistances R_1 and R_2 in parallel, as shown in Figure 1.

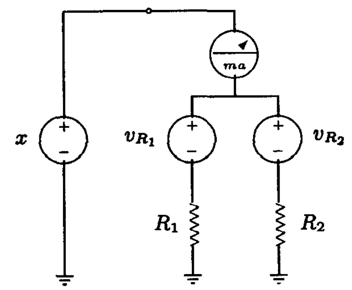

Figure 1: Measurement Noise is v_R if $R_1 \| R_2 = R$.

Adding a second ideal milliammeter to this circuit, as shown in Figure 2, clearly will not improve our knowledge of x. But this gives a strategy for combining independent measurements which is the basis for sequential estimation:

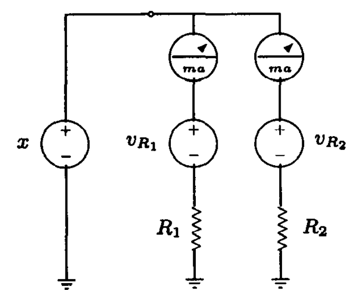

Figure 2: Two Independent Measurements of x.

Theorem 2 (Static Kalman Update Formulas) *Suppose that*

$$\begin{aligned}\hat{x}_1 &= x - v_{R_1}\\ \hat{x}_2 &= x - v_{R_2}\end{aligned}$$

are measurements of a deterministic quantity x corrupted by independent zero-mean random variables v_{R_1} and v_{R_2}

of covariance R_1 and R_2 respectively. The equivalent single measurement has covariance

$$R = R_1 \| R_2 \stackrel{\text{def}}{=} \left(R_1^{-1} + R_2^{-1}\right)^{-1}$$

and is given by

$$\hat{x} = R\left(R_1^{-1}\hat{x}_1 + R_2^{-1}\hat{x}_2\right)$$

Proof: Find the Thévenin equivalent of the subcircuit of resistors and thermal noise sources of Figure 1. It is helpful to find the Norton equivalent first. The Thévenin equivalent resistance is $R = R_1 \| R_2$ and the equivalent thermal noise v satisfies

$$v = R\left(R_1^{-1}v_{R_1} + R_2^{-1}v_{R_2}\right)$$

The result for $\hat{x} = x - v$ follows thanks to the identity $x = R\left(R_1^{-1}x + R_2^{-1}x\right)$.

Expressing the analysis in terms of parallel currents and conductances, we obtain the data processing expressions used in the *information filter* (*cf.* [1]). Note that inverse covariance is known as information; it is conductance in the electrical analog.

Corollary (Information Filter Update) *If $\hat{x}_1$ and $\hat{x}_2$ are independent measurements with covariances R_1 and R_2 respectively, the covariance R of the equivalent single measurement $\hat{x}$ satisifes*

$$R^{-1} = R_1^{-1} + R_2^{-1}$$

and $\hat{x}$ is determined by

$$R^{-1}\hat{x} = R_1^{-1}\hat{x}_1 + R_2^{-1}\hat{x}_2$$

It is important to understand that both the statement of Theorem 2 and its proof by electrical analog are valid for multivariate (vector) measurements. This is a consequence of a standard result in network synthesis (*cf.* [3]):

Theorem 3 *Every positive definite real symmetric matrix R arises as the resistance matrix of a network of ideal transformers and resistors.*

Corollary *Every positive definite real symmetric matrix R is the thermal noise covariance of a network of ideal transformers and resistors whose resistance matrix is R.*

In sequential estimation, update formulas are used iteratively — *a priori* estimates for the mean and the covariance of the state x are taken as the initial measurement. Each successive measurement of x is incorporated as it becomes available using an algorithm based on Theorem 2, but there is often an additional consideration: one or both of the measurements to be combined may be of *functions* of the state x to be estimated rather than observations of x itself.

Linear transformations of x may be represented using ideal transformers; Figure 3 shows an electrical realization of a noisy observation of $y \stackrel{\text{def}}{=} \alpha x$. The observed value is given by $\hat{y} = Ri' = \alpha x - v_R$.

Voltages and currents in this expression are understood to be vector quantities. The measurement noise v_R is the thermal noise inherent to the resistance matrix R, and $R = \text{cov}(v_R)$. The turns matrix α of the transformer is not assumed to be square.

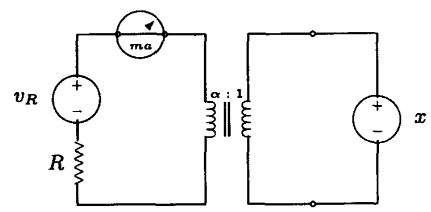

Figure 3: Observation of a Transformation of x

Theorem 4 *Suppose that $\hat{y} = \alpha x - v_R$ is a measurement of the image of a deterministic quantity x under the linear transformation α corrupted by the zero-mean random variable v_R of covariance R. The equivalent direct measurement of x has covariance pseudoinverse*

$$\tilde{R}^{+} = \alpha^T R^{-1} \alpha$$

and is given by

$$\hat{x} = \tilde{R}\alpha^T R^{-1}\hat{y}$$

Proof: We analyze the measurement device of Figure 4, determining its Norton and Thévenin equivalents.

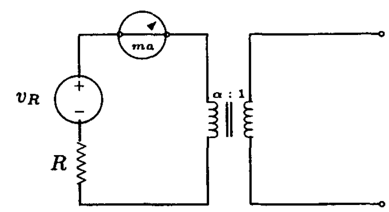

Figure 4: Transforming Volt-Ohm-Milliameter

Let i_0 denote the short circuit current, *i.e.* the current in the secondary when it is terminated by zero resistance, and let i'_0 denote the corresponding primary current. Since the voltage across the primary is zero, $Ri'_0 = v_R$, so

$$i_0 = \alpha^T R^{-1} v_R$$

Now suppose the secondary is terminated by an n-port whose resistance matrix is r. Let i' and i denote the primary and secondary currents, and let v' and v denote the voltages. Then

$$i = \alpha^T i' = \alpha^T R^{-1}(v_R - v')$$

so

$$r^{-1} v = \alpha^T R^{-1} v_R - \alpha^T R^{-1} \alpha v$$

from which

$$\left[\alpha^T R^{-1} \alpha + r^{-1}\right] v = \alpha^T R^{-1} v_R$$

But therefore

$$v = \left(\tilde{R} \parallel r\right) i_0$$

where $\tilde{R} = \left(\alpha^T R^{-1} \alpha\right)^+$ and the general parallel combination operator $\parallel$ is given by

$$A \| B \stackrel{\text{def}}{=} \left(A^+ + B^+\right)^+$$

It follows that $\left(\tilde{R}, \alpha^T R^{-1} v_R\right)$ is the Norton equivalent of the device in Figure 4 and that $\left(\tilde{R}, \tilde{R}\alpha^T R^{-1} v_R\right)$ is the n-port Thévenin equivalent.

Now consider the observation $\hat{y}$: it arises by measurement of transformer primary current $i' = R^{-1}\hat{y}$ as shown in Figure 3. But this implies a secondary current $i = \alpha^T R^{-1} \hat{y}$. Multiplying by the equivalent resistance $\tilde{R}$ of the measurement n-port, we infer the state measurement

$$\hat{x} = \tilde{R} \alpha^T R^{-1} \hat{y}$$

This completes the proof of the theorem.

Remark: Let α^+ denote the pseudoinverse of α. If α has a right inverse, as is the case for scalar measurements $\hat{y}$, then α^+ is such, and by application of the Penrose conditions (see [4]) it may be shown that

$$\tilde{R} = \left(\alpha^+ R \alpha^{+T}\right)$$

and therefore that

$$\hat{x} = \tilde{R}\alpha^T R^{-1} \hat{y} = \alpha^+ \hat{y}$$

Combining these results and Theorem 2, we obtain:

Theorem 5 (Dynamic Kalman Update Formulas) *Consider the linear system*

$$x_{n+1} = \Phi x_n$$

and assume the state transition matrix $\Phi \stackrel{\text{def}}{=} \Phi(n, n+1)$ *is invertible. Suppose* $\hat{x}_n$ *is an unbiased estimate of the state at time* n *of covariance* R_n *and that*

$$\hat{y}_{n+1} = \alpha x_{n+1} - v_S$$

is an unbiased estimate of the linear image αx_{n+1} *of the state at time* $n+1$ *of covariance* S. *If* α *has a right inverse, then the equivalent combined estimate* $\hat{x}_{n+1}$ *of the state at time* $n+1$ *has covariance*

$$R_{n+1} = \left(\Phi R_n \Phi^T\right) \parallel \left(\alpha^+ S \alpha^{+T}\right)$$

and is given by

$$\hat{x}_{n+1} = R_{n+1}\left(\Phi^{-T} R_n^{-1} \hat{x}_n + \alpha^T S^{-1} \hat{y}_{n+1}\right)$$

Proof: The result follows from Theorem 4 and Theorem 2. Note that $\hat{x}_n$ may be viewed as a noisy observation of a linear transformation of x_{n+1} as follows:

$$\hat{x}_n = \Phi^{-1} x_{n+1} - v_{R_n}$$

References

[1] G.J.Bierman, *Factorization Methods for Discrete Sequential Estimation*, Academic Press, New York (1977).

[2] R.E. Kalman, A new approach to linear filtering and prediction problems, *J. Basic Engineering* **82D**, (1960), 35–45.

[3] S. Karni, *Network Theory: Analysis and Synthesis*, Allyn and Bacon, Inc., (1966).

[4] C.L. Lawson and R.J.Hanson, *Solving Least Squares Problems*, Prentice-Hall, Englewood Cliffs, N.J. (1974).

[5] H. Nyquist, Thermal agitation of electric charge in conductors, *Phys. Rev.* **32**, (1928), 110–113.