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# Powers of matrices over distributive lattices—a review

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**Abstract**

In this paper we review the results on powers of matrices over a distributive lattice. We stress the significance of the graph-theoretical approach and show how several previous results for matrices of special types can be obtained in a unified way.
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## 1. Introduction

Powers of real matrices computed with at least one of the addition/multiplication operations replaced by maximum or minimum have been extensively studied, since they have applications in various types of discrete systems. In particular, for the description of the behaviour of fuzzy systems, whose states are expressed using fuzzy values taken from the real interval $\langle 0,1\rangle$ or an arbitrary distributive lattice, the powers of lattice matrices are of special importance.

Among the structures, over which the powers of matrices are computed, are max–addition (linear systems with synchronization [3,11,24,39,45]) max–multiplication [12,13], max–min (fuzzy systems, [9,16,17,18,20,35,36,53]), and their generalizations to max–t-norm [14,15] and to general sup–inf in a distributive lattice [52]. Results include proving convergence for special types of matrices [16,29,53], proving upper bounds [36] or computing the length of the oscillation period of the matrix power sequence [20], estimating its exponent, investigating connections between power sequence and eigenvectors [8,51], etc.

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In this paper we shall concentrate our attention to powers of matrices over distributive lattices and leave out other important and extensive areas, namely products of matrices over max-addition, max-multiplication algebras or products involving t-norms. (The reason is that in these structures, in contrary to the distributive lattices case, the 'multiplication' operation is not idempotent and so the behaviour of the powers of a matrix does not in general exhibit a periodic pattern. An interested reader is invited to study the relevant references, e.g. [3,11,24,38,39,45,54], for max-plus algebra, [12,13,41] for max-times algebra, [14,15] for products using t-norms and [54] for a general algebraic framework.) We would like to stress the significance of the graph-theoretical approach in the study of this topic, which was applied as early as in 1957 [43]. Graph theory is a powerful tool enabling many easy generalizations of results for binary matrices to the case of linear and general distributive lattices and giving unified proofs of many isolated results, scattered through the literature in linear algebra and in the world of fuzzy sets. Many authors are not aware of this possibility and of previous works, so the subsequent history of this topic is full of individualized terminology and rediscovering old results.

Let us also note here, that digraphs are used in a similar way for the study of nonnegative matrices, for which a good reference is the monograph [7].

## 2. Lattice matrices

A distributive lattice $(\mathscr{L}, \vee, \wedge, 0, 1)$ is an algebraic structure with two operations $\vee, \wedge$ called **join** and **meet**, that are commutative, associative, idempotent, distributive with respect to each other and posses neutral elements $0, 1$ for the operations $\vee$ and $\wedge$, respectively. For a detailed account of the lattice theory the reader is recommended to study e.g. the monograph [27].

Let us denote the set of all $n$ by $n$ matrices over $\mathscr{L}$ by $M_n(\mathscr{L})$ and the set of $n$-vectors by $V_n(\mathscr{L})$. The matrices of order $n$ with all entries equal to 0 and to 1, respectively, will be denoted by $\mathbf{0}_n$ and $\mathbf{J}_n$. The vector with all entries equal to 1 will be denoted by $\mathbf{e}$, the one with all entries equal to $\lambda \in \mathscr{L}$ by $\mathbf{e}_\lambda$ (the order of such vectors will be usually clear from the context).

We shall review the properties of powers of matrices over $\mathscr{L}$. These powers are computed formally in the same way as it is done in the classical linear algebra. It is common to use here a notation resembling the classical linear algebra, namely, the sign $\oplus$ is used for 'addition' and $\otimes$ for 'multiplication'. In the context of distributive lattices it is usually supposed that $\oplus = \vee$ and $\otimes = \wedge$, and so for $A, B \in M_n(\mathscr{L})$

$$A \otimes B = C \text{ with } c_{ij} = \sum_{k=1}^{n\oplus} a_{ik} \otimes b_{kj} = \bigvee_{k=1}^{n} a_{ik} \wedge b_{kj}.$$

An element $a \in \mathscr{L}$ is called **join-irreducible** if for any $x, y \in \mathscr{L}$, equality $a = x \vee y$ implies $a = x$ or $y$. The set of all nonzero join-irreducible elements of $\mathscr{L}$ will be denoted by $J(\mathscr{L})$.

It is well known that each finite distributive lattice can be embedded into a finite Boolean lattice $\mathscr{B}_k$ (see [27]) in the following way. For $x \in \mathscr{L}$ denote

$$r(x) = \begin{cases} \emptyset, & x = 0, \\ \{y \in J(\mathscr{L}); y \leqslant x\}, & x \neq 0. \end{cases} \tag{1}$$

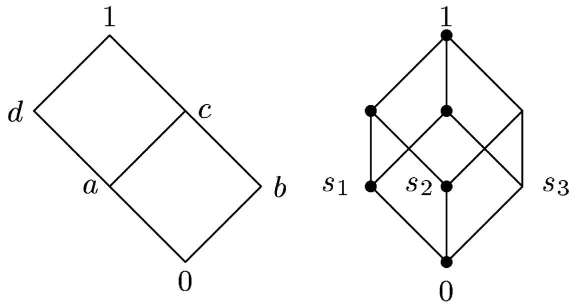


Fig. 1.

Then the set $\mathscr{P}(J(\mathscr{L}))$ of all subsets of $J(\mathscr{L})$ with $\vee$ equal union and $\wedge$ equal intersection is a finite Boolean lattice, $r$ is an embedding of $\mathscr{L}$ into $\mathscr{P}(J(\mathscr{L}))$. Atoms of this Boolean lattice will be denoted in this paper by letters s and they correspond to one-element subsets of $J(L)$.

As an illustration of this embedding take the lattice considered in [52], depicted in the left half of Fig. 1. Its join-irreducible elements are $a, b$ and $d$. In the right half of Fig. 1 one can see the corresponding Boolean lattice, with its atoms $s_1, s_2, s_3$. The embedding is $r(0) = 0 = \emptyset$, $r(a) = s_1 = \{a\}$, $r(b) = s_2 = \{b\}$, $r(c) = s_1 \vee s_2 = \{a, b\}$, $r(d) = s_1 \vee s_3 = \{a, d\}$, $r(1) = 1 = \{a, b, d\}$.

For a matrix $A \in M_n(\mathscr{L})$ the set of its different entries will be denoted by $k(A)$. Clearly, $k(A)$ is finite, so the sublattice $\mathscr{L}(A)$ of $\mathscr{L}$ generated by $k(A)$ is finite and therefore can also be embedded into a finite Boolean lattice. We shall denote this Boolean lattice by $\mathscr{B}(A)$ and call it the **Boolean lattice associated with** $A$. Let us denote the set of atoms of $\mathscr{B}(A)$ by $\sigma(A)$.

A matrix $A \in M_n(\mathscr{L})$ can be represented by binary matrices $A_s$, $s \in \sigma(A)$ of the form

$$(A_s)_{ij} = \begin{cases} 1 & \text{if } a_{ij} \geqslant s, \\ 0 & \text{otherwise,} \end{cases}$$

called the $s$th **constituent** of $A$ [34]. Matrix $A$ can be uniquely expressed as a linear combination of its constituents with coefficients $s, s \in \sigma(A)$ in the following way (compare [34,52]):

$$A = \sum_{s \in \sigma(A)}^{\oplus} s \otimes A_s. \tag{2}$$

For example, the following matrix, taken from [52]:

$$A = \begin{pmatrix} 0 & d & b \\ c & 0 & d \\ d & 1 & a \end{pmatrix} \tag{3}$$

over the lattice from Fig. 1 has three constituents

$$A_{s_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_{s_2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_{s_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{4}$$

An important special case of a distributive lattice is a bounded linearly ordered set; we shall consider without loss of generality the interval $\mathscr{F}=\langle 0,1\rangle$ (which, endowed with operations $\oplus=\max$ and $\otimes=\min$ is called by many authors **fuzzy algebra**). Here, the representation of matrices by the join-irreducible elements of $\mathscr{F}$ and hence its constituents is useless (in fact, each element of a linear lattice is join-irreducible), so instead binary **cut matrices** $A_\alpha$ for $\alpha\in\mathscr{F}$ defined by

$$(A_\alpha)_{ij}=\begin{cases}1 & \text{if } a_{ij}\geqslant\alpha,\\ 0 & \text{otherwise}\end{cases} \tag{5}$$

are used. (Cut matrices are called by some authors its **sections** or **zero patterns** [33] or **threshold matrices** [9,20] etc.) Notice, that although for $A\in M_n(\mathscr{F})$ theoretically infinitely many cut matrices exist, in practice only at most $n^2+1$ of them may be different, and the decisive changing points correspond to the entries of $A$. Matrix $A$ can again be uniquely expressed as a linear combination of its cut matrices:

$$A=\sum_{\alpha\in k(A)}^{\oplus}\alpha\otimes A_\alpha. \tag{6}$$

For example, the matrix

$$A=\begin{pmatrix}0.2 & 0.4 & 1\\ 0.5 & 0 & 0.2\\ 0.4 & 0.5 & 0.5\end{pmatrix}$$

over $\mathscr{F}$ has five different cut matrices, $A_0=\mathbf{J}_3$ and $A_{0.2}$, $A_{0.4}$, $A_{0.5}$, $A_1$ follow:

$$\begin{pmatrix}1&1&1\\1&0&1\\1&1&1\end{pmatrix},\begin{pmatrix}0&1&1\\1&0&0\\1&1&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&1\end{pmatrix},\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

Each power sequence of $A\in M_n(\mathscr{L})$ is ultimately periodic, since $\mathscr{L}(A)$ is finite and hence only a finite number of different matrices can be generated (compare [53,52]). Formally, there exist two integers $e,p$ such that

$$A^e=A^{e+p}. \tag{7}$$

The smallest $p$ such that (7) is satisfied, is called the **period** of $A$, denoted sometimes by $p(A)$. If $p(A)=1$ then $A$ is called **convergent**. The smallest $e$ such that $A^e=A^{e+p(A)}$ holds, is called the **exponent** of $A$.

Taking into account the representation of a matrix over a distributive lattice $\mathscr{L}$ or over the fuzzy algebra $\mathscr{F}$ in the form (2) and (6), respectively, for each $v$ we have

$$A^v=\sum_{s\in\sigma(A)}^{\oplus}s\otimes A_s^v,\ \text{or}\ A^v=\sum_{\alpha\in k(A)}^{\oplus}\alpha\otimes A_\alpha^v.$$

So the study of power sequences of a matrix over a distributive lattice can essentially be reduced to the study of binary matrices.

We shall also review the known results for some special types of matrices; their definitions are summarized in

**Definition 1.** A matrix $A \in M_n(\mathscr{L})$ is called

1. transitive, if $A^2 \leqslant A$,
2. increasing, if $A^2 \geqslant A$,
3. diagonally dominant, if $a_{jj} = \sum_{i \leqslant n}^{\oplus} (a_{ij} \oplus a_{ji})$ for $j = 1, \ldots, n$,
4. symmetric, if $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$,
5. nilpotent, if $A^m = \mathbf{0}_n$ for some integer $m$,
6. primitive, if $A^m = \mathbf{J}_n$ for some integer $m$,
7. circulant, if it has the following form:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix}.$$

## 3. Basic notions of graph theory

Now we shall review basic notions and facts of graph theory used in this paper. For a detailed introduction to this area refer to the monograph [5].

A **digraph** is a pair $G = (V, H)$, where $V$ is a finite set, called the **vertex set**, and $H$ is a subset of $V \times V$, called the **arc set**. A digraph $G' = (V', H')$ is a **subdigraph** of $G$, if $V' \subseteq V$ and $H' \subseteq H$. A sequence of vertices $p = (i_0, i_1, \ldots, i_m)$ is called a **path** from vertex $i_0$ to vertex $i_m$, if for all $j = 1, \ldots, m$ the pair $(i_{j-1}, i_j) \in H$; its **length** is $m$ and is denoted by $\ell(p)$. A path is called a **cycle** if $i_0 = i_m$, a cycle of length 1 is called a **loop**. Notice that on a path as well as on a cycle vertices may occur repeatedly; if this is not the case, the path and the cycle are called **elementary**.

Now, for the present theory the following obvious fact is crucial:

**Lemma 1.** *Every path of length at least $n$ in a digraph on $n$ vertices contains a cycle, i.e is not elementary.*

A digraph that does not contain any cycle is called **acyclic**. If for each pair of vertices $u, v$ in $G$ there is a path from $u$ to $v$ and a path from $v$ to $u$ in $G$, then $G$ is called **strongly connected**. A maximal (with respect to inclusion) strongly connected subdigraph of a given digraph is its **strongly connected component** (SCC for short). Every SCC can contain either several vertices (in that case it must contain at least one cycle) or a single vertex $u$. A SCC containing a single vertex $u$ but not the loop $(u, u)$ is called **acyclic**.

There exist efficient algorithms for computing the partition of a digraph on $n$ vertices into its strongly connected components with complexity $O(n^2)$; as a reference see e.g. [1].

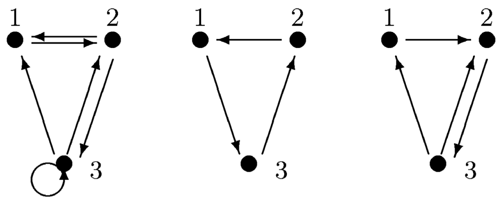


Fig. 2.

The greatest common divisor (gcd for short) of all cycle lengths in a non-acyclic strongly connected digraph $G$ is called its **period**; the period of any digraph is the least common multiple (lcm for short) of periods of its nonacyclic strongly connected components. Balcar and Vienott devised an algorithm with complexity $O(n^2)$ for computing the period of a strongly connected digraph [4]. This algorithm successively condenses all terminal vertices of arcs starting in one vertex, until it gets a digraph isomorphic to an elementary cycle. The number of arcs on this cycle is equal to the period of the digraph.

Now let us define the **associated digraph** $G(A)=(V,H)$ of a binary matrix $A$ of order $n$ to be the digraph on the set of vertices $V=\{1,2,\dots,n\}$ such that

$$(i,j)\in H \text{ if and only if } a_{ij}=1.$$

For example, the associated digraphs for matrices from (4) are in Fig. 2.

There is a well-known connection between the entries in powers of binary matrices and paths in associated digraphs: $(i,j)$th entry $a_{ij}^{(k)}$ in $A^k$ is equal to 1 if and only if in the associated digraph $G(A)$ of $A$ there exists a path of length $k$ from vertex $i$ to vertex $j$. A careful argument leads to the following assertion:

**Lemma 2.** *Let* $A\in M_n(\mathscr{L})$. *Then*

(i) $\sum_{k>n}^{\oplus} A^k \leqslant \sum_{k\leqslant n}^{\oplus} A^k$,
(ii) $A^n \leqslant \sum_{k>n}^{\oplus} A^k$.

**Proof.** (i) Due to Lemma 1, each path on at least $n+1$ vertices can be reduced to an elementary path with length at most $n$ by leaving out some of the cycles it contains, hence if $a_{ij}^{(k)}=1$ for some $k>n$ in a binary matrix, then $a_{ij}^{(k)}=1$ for some $k\leqslant n$. Since these observations hold for each constituent or cut matrix of a given matrix $A\in M_n(\mathscr{L})$, due to the expression of lattice matrices in the forms (2) and (6), inequality (i) holds for an arbitrary lattice matrix.

(ii) A path of length $n$ must contain a cycle and if we add this cycle into this path several times, we get another path between the same pair of vertices, which has its length greater than $n$. Hence, if $a_{ij}^{(n)}=1$, then there exists $k>n$ such that $a_{ij}^{(k)}=1$ and the second inequality follows. □

## 4. Binary matrices

In [31, Chapter 5, Theorem 5.4.25] a summary of the properties of binary matrices is given, including the following ones:

**Theorem 1.** *Let $A$ be a binary matrix of order $n$ and $G$ its associated digraph.*

1. *$A$ is nilpotent if and only if $G$ is acyclic.*
2. *$A$ converges to $\mathbf{J}_n$ if and only if $G$ is strongly connected with period 1.*
3. *The period of $A$ is equal to the period of its associated digraph.*
4. *The exponent of $A$ is at most $(n-1)^2+1$.*

Kim attributes the authorship of the individual assertions in this theorem to [43,47]. A detailed characterization of asymptotic forms of binary matrices is given in [46].

Kim [31, p. 226] gives an algorithm of complexity $O(n^3 \log n)$ for computing the period of a binary matrix. This algorithm first computes (by repeated squaring) the $p$th power of $A$, where $p$ is a prime number not smaller than $(n-1)^2+1$ (notice that $per(A) = per(A^p)$). Then a power $B$ of $I + A^p$ larger than the $n$th is computed (again by repeated squaring), since $B \circ B^T$, where $\circ$ is the elementwise product, is the matrix of the equivalence relation of strongly connected components of $A^p$. Rows in a submatrix $A_{kk}$ of $A^p$, corresponding to a SCCs of $A^p$ are either equal or have nonoverlapping sets of entries equal to 1. The period of $A_{kk}$ is the number of distinct vectors among them. The period of $A$ is then the lcm of periods of matrices $A_{kk}$.

Notice that a graph–theoretic approach (i.e. first computing the SCCs by the algorithm in [1] and then using Balcar–Veinott's algorithm for each one of them) yields an algorithm with complexity $O(n^3)$.

Now we show how to use the stated properties of binary matrices for deriving results about powers of fuzzy or lattice matrices.

## 5. Convergence and the period length

The most cited reference in the papers on powers of fuzzy matrices is [53] (often considered to be the origin of this topic), which proves convergence of fuzzy matrices fulfilling some relatively strong conditions. In [35] it was shown, using purely algebraic methods, that the period length of a matrix $A \in M_n(\mathscr{F})$ divides $[n]$, the least common multiple of all integers not greater than $n$. The author also gives an algorithm for computing the period length, which, in the worst case, involves computing the power of $A$ as high as the $[n]$th.

However, the graph–theoretical approach gives the following theorem, proved independently in [33,20,17] (the references are given chronologically).

**Theorem 2.** *The period of a fuzzy matrix $A$ is equal to the least common multiple of the periods of its cut matrices.*

For computing the period of a fuzzy matrix it is possible to use the algorithm of Kim for computing the period of each cut matrix, which will give an algorithm with complexity $O(n^5 \log n)$. However, for obtaining a good complexity bound, it is useful to realize that the associated digraphs of cut matrices with greater cut values are subdigraphs of those for smaller cut values. As the cut value decreases, the edges are added. Some of them are in existing SCCs or they merge previous SCCs; in both cases, the periods of new SCCs divide the periods of SCCs existing in digraphs with greater cut values. So what only really matters, are the periods of minimal (with respect to vertex inclusion) SCCs and the number of those is at most $n$ (see [20]). Hence using the Balcar–Veinott's algorithm for computing the period of each minimal SCC yields and algorithm with complexity $O(n^3)$.

In [37] it is rediscovered that in computing the period of a fuzzy matrix only minimal SCCs are relevant. The authors of the latter paper are, however, aware neither of Kim's nor of Balcar–Veinott's algorithm, so they propose to compute the period of an SCC by finding all its cycles, which is obviously computationally not tractable. Another paper using digraphs in the study of powers of fuzzy matrices in [30]. Here the authors use neither the standard graph-theoretical language nor the tools of graph theory, so the paper is difficult to read and the results, although equivalent to the previous ones, have no immediate algorithmic realization.

The author of the present paper is not aware of any work using digraphs in the study of powers of lattice matrices. To our knowledge the only paper on this topic is [52], where an analogy of [36] for distributive lattices has been proved, i.e. the period of a matrix $A \in M_n(\mathscr{L})$ divides $[n]$ and also a similar algorithm, using successive squaring of $A$ has been given. However, Theorem 2 has an obvious analogy for this case:

**Theorem 3.** *The period of a lattice matrix $A$ is equal to the least common multiple of the periods of its constituents. So, if the number of atoms of $\mathscr{B}(A)$ is $k$, then it is possible to compute it in $O(kn^3)$ time.*

Let us notice here, that to the author's knowledge, there does not exist a tight upper bound for the number of atoms of $\mathscr{B}(A)$. From Chapter 2, Section 2 in [27] it follows, that the number of elements in the lattice generated by a set of $n$ elements is bounded from above by $2^{2^n-2}$, which is clearly an exponential bound. However, at least in cases when the underlying lattice $\mathscr{L}$ has a simple structure, the approach based on Theorem 3 might lead to a better algorithm than those proposed so far.

We will show the advantages of the graph-theoretical approach by applying it to the example given in [52]. Let us consider the lattice $\mathscr{L}$ given in Fig. 1 and the matrix $A$ from (3). The constituents of $A$ are in (4) and their associated digraphs $G(A_{s_1}), G(A_{s_2})$ and $G(A_{s_3})$ are depicted in Fig. 2. We see that their periods are 1,3, and 1, respectively, so the period of $A$ is 3.

When the order of the matrix is greater, the computational advantages are even more striking. So suppose that $n=6$. Then the bound for the period length from [52] is $lcm\{1,2,3,4,5,6\}=60$, so in the worst case we would have to compute more than the 60th power of $A$. For the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & b \\ 0 & 0 & 0 & d & 0 & d \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & b & 0 & d \\ 1 & 0 & c & 0 & b & 0 \end{pmatrix}$$

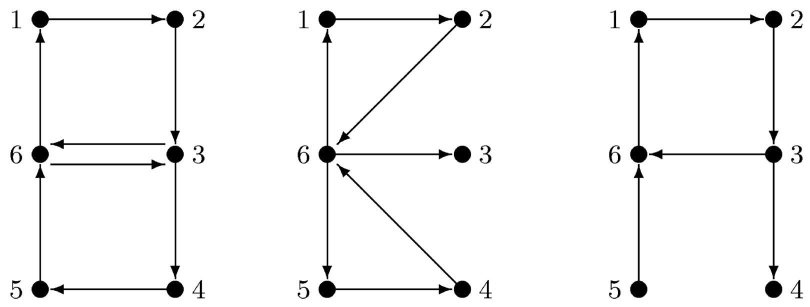


Fig. 3.

over the lattice from Fig. 1 we have three constituent matrices $A_{s_1}, A_{s_2}, A_{s_3}$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Their associated digraphs are depicted in Fig. 3 and we see, that $per(G(A_{s_1}) = 2$, $per(G(A_{s_2})) = 3$ and $per(G(A_{s_3})) = 4$, so $per(A) = 12$.

The graph-theoretical language provides a unified way of proving the convergence results for several types of special matrices.

**Theorem 4.** *If $A \in M_n(\mathscr{L})$ is symmetric then $d(A) = 1$ or $d(A) = 2$.*

**Proof.** For a symmetric matrix, the digraphs corresponding to each constituent contain arc $(i, j)$ if and only if $(j, i)$ is also an arc. Hence each strongly connected component has period 1 or 2. □

**Theorem 5.** *Each matrix, that is either diagonally dominant, or transitive or increasing, is convergent.*

**Proof.** The associated digraphs of constituents of a diagonally dominant matrix have the following property: if $(i, j)$ is an arc, then $(i, i)$ is also an arc. Hence each nonacyclic strongly connected component contains a loop, thus its period is 1.

In each associated digraph of a transitive matrix we have: if $(i, j)$ and $(j, k)$ are arcs, then $(i, k)$ is also an arc. Hence if a strongly connected component contains a cycle containing vertex $i$, it also contains the loop $(i, i)$ and therefore has period 1.

Property $A^2 \geqslant A$ means that for each arc $(i, j)$ in each associated digraph $G$ there exists a vertex $k$ such that $(i, k)$ and $(k, j)$ are also arcs in $G$. Hence if a strongly connected component of $G$ contains a cycle of length $m$, it also contains a cycle of length $m + 1$ and so its period is 1. □

Some nice nontrivial results have been obtained for circulants. To be able to formulate them, let us denote by $\{i_1, i_2, \ldots, i_t\}$ the set of indices of positions of maximal elements in the first row of a circulant. The following theorem was proved for binary matrices independently in [32,48]:

**Theorem 6.** *Binary circulant A of order n is primitive if and only if*

$$gcd(i_2 - i_1, i_3 - i_1, \ldots, i_t - i_1, n) = 1.$$

For circulants over the fuzzy algebra the following expression for the period length has been obtained in [2,22]:

**Theorem 7.** *For a circulant $A \in M_n(\mathscr{F})$, $per(A) = d/d'$, where*

$$d = gcd(i_2 - i_1, i_3 - i_1, \ldots, i_t - i_1, n),$$

$$d' = gcd(i_1, i_2, \ldots, i_t, n).$$

*Hence a circulant $A \in M_n(\mathscr{F})$ is convergent if and only if d divides $i_1$.*

A similar expression for the period of a circulant over a general distributive lattice is not known.

## 6. Exponent

Many results for the exponent of a matrix have been proved for primitive nonnegative matrices using the usual product. In this context, a matrix $A$ of order $n$ is primitive if some of its powers is a strictly positive matrix and the exponent is the smallest integer $e$ such that $A^e$ is strictly positive. This clearly corresponds to max–min product of Boolean matrices.

The first bound for the exponent of a Boolean matrix

$$e(A) \leqslant (n-1)^2 + 1 \tag{8}$$

has been proved by Rosenblatt [43] and generalized to the distributive lattice case in [52]. Other bounds, proved for primitive nonnegative matrices, include

$$e(A) \leqslant d^2 + 1, \tag{9}$$

where $d$ is the diameter of the associated digraph (in [50] and independently in [40]) and

$$e(A) \leqslant (m-1)^2 + 1, \tag{10}$$

where $m$ is the degree of the minimal polynomial of matrix $A$ in [49]. Another bound uses the Boolean rank $b$ of $A$, i.e. the minimum integer $b$ such that $A$ is equal to the Boolean product of an $n \times b$ and a $b \times n$ Boolean matrix,

$$e(A) \leqslant (b-1)^2 + 1 \tag{11}$$

see [28], where also inequality $d \leqslant b$ has been proved. For binary matrices of order $n$ with period $p$ inequality

$$e(A) \leqslant 2n^2 - 3n + 2 \tag{12}$$

was proved in [46], Theorem 3.20.

In [36] the bound $e(A) \leqslant (n-1)[n]$ was proved for a fuzzy matrix. For special types of matrices tighter bounds can be obtained. For example, $e(A) \leqslant n-1$ for transitive [29] and circulant [2] fuzzy matrices. Tan in [52] proves for a matrix $A \in M_n(\mathscr{L})$ that $e(A) \leqslant n$ if $A$ is transitive or increasing, $e(A) \leqslant n-1$ if $A$ is diagonally dominant and $e(A) \leqslant 2(n-1)$ if $A$ is symmetric.

However, the actual computation of the exponent of a Boolean matrix is NP-hard. This result was proved for max-plus matrices in [6], but it can easily be strengthened for the binary case. First, let $a_1 < a_2 < \cdots < a_k$ be relatively prime positive integers. Then there exists $N$ such that each integer $m \geqslant N$ can be expressed in the form $\mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_k a_k = m$ with $\mu_1, \mu_2, \ldots, \mu_k$ nonnegative integers. The smallest number $N$ is called the Frobenius–Shur index, is denoted by $\Phi(a_1, a_2, \ldots, a_k)$ and for $k \geqslant 3$ only some estimates of it are known. Moreover, Ramírez-Alfonsín [42] proved:

**Theorem 8.** *The computation of* $\Phi(a_1, a_2, \ldots, a_k)$ *for* $k \geqslant 3$ *is NP-hard.*

Using this result, one can derive

**Theorem 9.** *Computing the exponent of a binary matrix is NP-hard.*

**Proof.** For a given sequence $a_1 < a_2 < \cdots < a_k$ of relatively prime integers construct the digraph $G = (V, H)$ where the vertices are the integers $1, 2, \ldots, a_k$ and the edges are of the form $(j, j+1)$ for $j = 1, 2, \ldots, a_k - 1$ and $(a_i, 1)$ for all $i = 1, 2, \ldots, k$. The binary matrix associated to $G$ converges to $\mathbf{J}_{a_k}$, since this digraph is strongly connected and its period is 1. Notice that lengths of circuits in $G$ are of the form $\mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_k a_k = m$ with all $\mu_i$ nonnegative and recall that $a_{ij}^m = 1$ if and only if there exists a path from $i$ to $j$ in $G$ of length $m$. Notice that for each pair $i, j$ of indices there is a path of length at most $a_k - 1$ in $G$ from $i$ to $j$, and since each vertex lies on a cycle of length $m$ for each $m \geqslant \Phi(a_1, a_2, \ldots, a_k)$, we can insert this cycle into the path and get $a_{ij}^l = 1$ for each $l \geqslant \Phi(a_1, a_2, \ldots, a_k) + a_k - 1$. Hence $e(A) \leqslant \Phi(a_1, a_2, \ldots, a_k) + a_k - 1$. On the other hand, since there exists a path from any vertex to any other vertex with any length not smaller than $e(A)$ and at most $a_k - 1$ edges on it connect different vertices, it follows that the rest of this path is a cycle. Hence each integer not smaller than $e(A) - a_k + 1$ can be expressed in the form $\mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_k a_k$ with all $\mu_i$ nonnegative and therefore $\Phi(a_1, a_2, \ldots, a_k) \leqslant e(A) - a_k + 1$. It follows that $\Phi(a_1, a_2, \ldots, a_k) = e(A) - a_k + 1$ and hence the computation of $e(A)$ is also NP-hard. □

## 7. Powers of matrices, orbits and eigenvectors

If $\mathbf{x} \in V_n(\mathscr{L})$, then the **orbit** $orb(A, \mathbf{x})$ of the matrix $A$ generated by vector $\mathbf{x}$ is the sequence $\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \ldots$ . Each orbit must clearly either converge or oscillate. The length of its period is denoted by $per(A, \mathbf{x})$.

The basic observation concerning the relation between the asymptotic behaviour of the powers of $A$ and of its orbits was proved in [9] (for fuzzy algebra, but in fact this assertion is true over any distributive lattice):

**Theorem 10.** *The powers of a matrix $A$ converge if and only if each orbit of $A$ converges.*

Relations between powers of a matrix and its orbits have been further investigated by Gavalec, who has obtained several interesting results. Namely, the period of $A$ is the lcm of all its orbit periods [9] and an $O(n^4)$ algorithm for computing the period of an orbit was given in [24]. However, the problem to decide whether for a given matrix $A$ a vector $\mathbf{x}$ exists, such that $per(A) = per(A, \mathbf{x})$, is an NP-complete problem already in the binary case [21]. The matrix used in this NP-completeness proof has its period decomposable into the product of many primes, so it would be interesting to know, whether the reachability problem remains NP-complete also for matrices whose period decomposes into a product of two (three) primes. Other restricting conditions making the reachability problem polynomially solvable were given in [26].

A vector $\mathbf{x} \in V_n(\mathscr{L})$ is called an **eigenvector** of a matrix $A \in M_n(\mathscr{L})$, if there exists $\lambda \in \mathscr{L}$ such that $A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}$; in that case $\lambda$ is the associated **eigenvalue**. (Notice that some authors require the eigenvector to be nonzero, for example [34], and in this case the results would be slightly different.) For brevity, let us call eigenvectors associated with the eigenvalue $\lambda = 1$ the **standard** eigenvectors. The following assertions have been proved for the fuzzy algebra in [8,44] and for an arbitrary distributive lattice in [51].

**Theorem 11.** *Each matrix $A \in M_n(\mathscr{L})$ has a unique greatest standard eigenvector $\mathbf{x}^*(A)$ and $orb(A, \mathbf{e})$ converges to $\mathbf{x}^*(A)$ in at most $n$ steps.*

The previous theorem yields an algorithm for computing $\mathbf{x}^*(A)$ with complexity $O(n^3)$. However, for the fuzzy algebra an algorithm with time complexity $O(n^2 \log n)$ has been obtained in [10] based on the ideas of [44]. A similar improvement for the general distributive lattice case is not known.

Theorem 11 can be partly generalized to an arbitrary eigenvalue.

**Theorem 12.** *Let $A \in M_n(\mathscr{L})$. Then each $\lambda \in \mathscr{L}$ is an eigenvalue of $A$ and $orb(A, \mathbf{e}_\lambda)$ converges to an eigenvector with associated eigenvalue $\lambda$ in at most $n$ steps.*

**Proof.** We will show that $A^{n+1} \otimes \mathbf{e}_\lambda = \lambda \otimes A^n \otimes \mathbf{e}_\lambda$, which implies that $A^n \otimes \mathbf{e}_\lambda$ is an eigenvector of $A$ with the associated eigenvalue $\lambda$.

To this end, realize first that

$$A \otimes \mathbf{e}_\lambda \leqslant \lambda \otimes \mathbf{e}_\lambda = \mathbf{e}_\lambda$$

and so for each $i$:

$$A^{i+1} \otimes \mathbf{e}_\lambda = A^i \otimes (A \otimes \mathbf{e}_\lambda) \leqslant A^i \otimes \mathbf{e}_\lambda = \lambda \otimes A^i \otimes \mathbf{e}_\lambda. \tag{13}$$

Hence, in particular for $i = n$:

$$
\begin{aligned}
A \otimes (A^n \otimes \mathbf{e}_\lambda) &\leqslant \lambda \otimes A^n \otimes \mathbf{e}_\lambda \leqslant \quad \text{(due to Lemma 2)}\\
&\leqslant \lambda \otimes \sum_{k \geqslant n}^{\oplus} A^k \otimes \mathbf{e}_\lambda = \sum_{k > n}^{\oplus} A^k \otimes \mathbf{e}_\lambda \leqslant \text{(due to (13))}\\
&\leqslant A^{n+1} \otimes \mathbf{e}_\lambda.
\end{aligned}
$$

Hence everywhere equality must hold and we have the desired assertion. □

The structure of the set of all eigenvectors of a fuzzy or a lattice matrix has not been fully described yet. A first step in this direction is the work [25], where an $O(n^2)$ algorithm for computing the lower and the upper ends of the intervals consisting of all monotone eigenvectors of a fixed monotonicity type is described.

## 8. Conclusion

In this paper we gave an overview of the known results for the powers of matrices computed over a distributive lattice. We have stressed the unifying potential of the graph–theory language and pointed out several open problems in this area.

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