10 Circle Maps: Irrationally Winding

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In these lectures we shall discuss circle maps as an example of a physically interesting chaotic dynamical system with rich number-theoretic structure. Circle maps arise in physics in a variety of contexts. One setting is the classical *Hamiltonian mechanics;* a typical island of stability in a Hamiltonian 2-d map is an infinite sequence of concentric KAM tori and chaotic regions. In the crudest approximation, the radius can here be treated as an external parameter Ω, and the angular motion can be modelled by a map periodic in the angular variable[1, 2]. In *holomorphic dynamics* circle maps arise from the winding of the complex phase factors as one moves around the Mandelbrot cacti[3]. In the context of *dissipative* dynamical systems one of the most common and experimentally well explored routes to chaos is the two-frequency mode-locking route. Interaction of pairs of frequencies is of deep theoretical interest due to the generality of this phenomenon; as the energy input into a dissipative dynamical system (for example, a Couette flow) is increased, typically first one and then two of intrinsic modes of the system are excited. After two Hopf bifurcations (a fixed point with inward spiralling stability has become unstable and outward spirals to a limit cycle) a system lives on a two-torus. Such systems tend to mode-lock: the system adjusts its internal frequencies slightly so that they fall in step and minimize the internal dissipation. In such case the ratio of the two frequencies is a rational number. An irrational frequency ratio corresponds to a quasiperiodic motion - a curve that never quite repeats itself. If the mode-locked states overlap, chaos sets in. Typical examples[4] are dynamical systems such as the Duffing oscillator and models of the Josephson junction, which possess a natural frequency ω_1 and are in addition driven by an external frequency ω_2. Periodicity is in this case imposed by the driving frequency, and the dissipation confines the system to a low dimensional attractor; as the ratio ω_1/ω_2 is varied, the system sweeps through infinitely many mode-locked states. The likelyhood that a mode-locking occurs depends on the strength of the coupling of the internal and the external frequencies.

By losing all of the "island-within-island" structure of real systems, circle map models skirt the problems of determining the symbolic dynamics for a realistic Hamiltonian system, but they do retain some of the essential features of such systems, such as the golden mean renormalization[7, 1] and non-hyperbolicity in form of sequences of cycles accumulating toward the borders of stability. In particular, in such systems there are orbits that stay "glued" arbitrarily close to stable regions for arbitrarily long times. As this is a generic phenomenon in physically interesting dynamical systems, such as the Hamiltonian systems with coexisting elliptic islands of stability and hyperbolic homoclinic webs, development of good computational techniques is here of utmost practical importance.

We shall start by briefly summarizing the results of the “local” renormalization theory for transitions from quasiperiodicity to chaos. In experimental tests of this theory one adjusts the external frequency to make the frequency ratio as far as possible from being mode-locked. This is most readily attained by tuning the ratio to the “golden mean” $(\sqrt{5}-1)/2$. The choice of the golden mean is dictated by number theory: the golden mean is the irrational number for which it is hardest to give good rational approximants. As experimental measurments have limited accuracy, physicists usually do not expect that such number-theoretic subtleties as how irrational a number is should be of any physical interest. However, in the dynamical systems theory to chaos the starting point is the enumeration of asymptotic motions of a dynamical system, and through this enumeration number theory enters and comes to play a central role.

Number theory comes in full strength in the “global” theory of circle maps, the study of universal properties of the entire irrational winding set – the main topic of these lectures. We shall concentrate here on the example of a global property of the irrational winding set discovered by Jensen, Bak, and Bohr[5]: the set of irrational windings for critical circle maps with cubic inflection has the Hausdorff dimension $D_H = 0.870\ldots$, and the numerical work indicates that this dimension is *universal*. The universality (or even existence) of this dimension has not yet been rigorously established. We shall offer here a rather pretty explanation[8] of this universality in form of the explicit formula (39) which expresses this Hausdorff dimension as an average over the Shenker[2, 9, 10] universal scaling numbers. The renormalization theory of critical circle maps demands at present rather tedious numerical computations, and our intuition is much facilitated by approximating circle maps by number-theoretic models. The model that we shall use here to illustrate the basic concepts might at first glance appear trivial, but we find it very instructive, as much that is obscured by numerical work required by the critical maps is here readily number-theoretically accessible. Indicative of the depth of mathematics lurking behind physicists' conjectures is the fact that the properties that one would like to establish about the renormalization theory of critical circle maps might turn out to be related to number-theoretic abysses such as the Riemann conjecture, already in the context of the “trival” models.

The literature on circle maps is overwhelming, ranging from pristine Bourbakese[11, 12] to palpitating chicken hearts[13], and attempting a comprehensive survey would be a hopeless undertaking; the choice of topics covered here is of necessity only a fragment of what is known about the dipheomorphisms of the circle.

10.1 Mode Locking

The Poincaré section of a dynamical system evolving on a two-torus is topologically a circle. A convenient way to study such systems is to neglect the radial variation of the Poincaré section, and model the angular variable by a map of a circle onto itself. Both quantitatively and qualitatively this behavior is often well described[14, 15] by 1-dimensional circle maps $x \to x' = f(x)$, $f(x+1) = f(x)+1$ restricted to the circle, such as the *sine map*

$$x_{n+1} = x_n + \Omega - \frac{k}{2\pi}\sin(2\pi x_n) \qquad mod\ 1 \ . \tag{1}$$

$f(x)$ is assumed to be continuous, have a continuous first derivative, and a continuous second derivative at the inflection point. For the generic, physically relevant case (the only one considered here) the inflection is cubic. Here k parametrizes the strength of the mode-mode interaction, and Ω parametrizes the ω_1/ω_2 frequency ratio. For $k = 0$, the map is a simple rotation (the *shift map*)

$$x_{n+1} = x_n + \Omega \qquad mod\ 1\ , \tag{2}$$

and Ω is the winding number

$$W(k, \Omega) = \lim_{n\to\infty} x_n/n. \tag{3}$$

If the map is monotonically increasing ($k < 1$ in (1)), it is called *subcritical.* For subcritical maps much of the asymptotic behavior is given by the trivial (shift map) scalings[11, 12]. For invertible maps and rational winding numbers $W = P/Q$ the asymptotic iterates of the map converge to a unique Q-cycle attractor

$$f^Q(x_i) = x_i + P, \quad i = 0, 1, 2, \cdots, Q-1\ .$$

For any rational winding number, there is a finite interval of parameter values for which the iterates of the circle map are attracted to the P/Q cycle. This interval is called the P/Q *mode-locked* (or *stability*) interval, and its width is given by

$$\Delta_{P/Q} = Q^{-2\mu_{P/Q}} = \Omega^{right}_{P/Q} - \Omega^{left}_{P/Q}\ . \tag{4}$$

Parametrizing mode lockings by the exponent μ rather than the width Δ will be convenient for description of the distribution of the mode-locking widths, as the exponents μ turn out to be of bounded variation. The stability of the P/Q cycle is defined as

$$\Lambda_{P/Q} = \frac{\partial x_Q}{\partial x_0} = f'(x_0)f'(x_1)\cdots f'(x_{Q-1})$$

For a stable cycle $|\Lambda|$ lies between 0 (the superstable value, the "center" of the stability interval) and 1 (the $\Omega^{right}_{P/Q}$, $\Omega^{left}_{P/Q}$ ends of the stability interval in (4)). For the shift map, the stability intervals are shrunk to points. As Ω is varied from 0 to 1, the iterates of a circle map either mode-lock, with the winding number given by a rational number $P/Q \in (0, 1)$, or do not mode-lock, in which case the winding number is irrational. A plot of the winding number W as a function of the shift parameter Ω is a convenient visualization of the mode-locking structure of circle maps. It yields a monotonic "devil's staircase" of fig. 10.1 whose self-similar structure we are to unravel.

Fig. 10.1. The critical circle map ($k = 1$ in (1)) devil's staircase[5]; the winding number W as function of the parameter Ω.

Circle maps with zero slope at the inflection point x_c

$$f'(x_c) = 0, f''(x_c) = 0$$

($k = 1$, $x_c = 0$ in (1)) are called *critical*: they delineate the borderline of chaos in this scenario. As the non-linearity parameter k increases, the mode-locked intervals become wider, and for the critical circle maps ($k = 1$) they fill out the whole interval[16]. A critical map has a superstable P/Q cycle for any rational P/Q, as the stability of any cycle that includes the inflection point equals zero. If the map is non-invertable ($k > 1$), it is called supercritical; the bifurcation structure of this regime is extremely rich and beyond the scope of these (and most other such) lectures.

For physicists the interesting case is the critical case; the shift map is "easy" number theory (Farey rationals, continued fractions) which one uses as a guide to organization of the non-trivial critical case. In particular, the problem of organizing subcritical mode lockings reduces to the problem of organizing rationals on the unit interval. The self-similar structure of the devil's staircase suggests a systematic way of separating the mode lockings into hierarchies of levels. The set of rationals P/Q clearly possesses rich number-theoretic structure, which we shall utilize here to formulate three different partitionings of rationals:

1. Farey series
2. Continued fractions of fixed length
3. Farey tree levels

10.2 Farey Series Partitioning

Intuitively, the longer the cycle, the finer the tuning of the parameter Ω required to attain it; given finite time and resolution, we expect to be able to resolve cycles up to some maximal length Q. This is the physical motivation for partitioning[19] mode lockings into sets of cycle length up to Q. In number theory such set of rationals is called a *Farey series.*

(10.1) Definition. *The Farey series*[21] $\mathcal{F}_Q$ *of order* Q *is the monotonically increasing sequence of all irreducible rationals between 0 and 1 whose denominators do not exceed* Q. *Thus* P_i/Q_i *belongs to* $\mathcal{F}_Q$ *if* $0 < P_i \le Q_i \le Q$ *and* $(P_i|Q_i) = 1$.
For example

$$\mathcal{F}_5 = \left\{ \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}$$

A Farey sequence can be generated by observing that if P_{i-1}/Q_{i-1} and P_i/Q_i are consecutive terms of $\mathcal{F}_Q$, then

$$P_i Q_{i-1} - P_{i-1} Q_i = 1 .$$

The number of terms in the Farey series F_Q is given by

$$\Phi(Q) = \sum_{n=1}^{Q} \phi(Q) = \frac{3Q^2}{\pi^2} + O(Q \ln Q). \tag{5}$$

Here the Euler function $\phi(Q)$ is the number of integers not exceeding and relatively prime to Q. For example, $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$, $\ldots, \phi(12) = 4, \phi(13) = 12, \ldots$ As $\phi(Q)$ is a highly irregular function of Q, the asymptotic limits are not approached smoothly: incrementing Q by 1 increases $\Phi(Q)$ by anything from 2 to Q terms. We refer to this fact as the "Euler noise".

The Euler noise poses a serious obstacle for numerical calculations with the Farey series partitionings; it blocks smooth extrapolations to $Q \to \infty$ limits from finite Q data. While this in practice renders inaccurate most Farey-sequence partitioned averages, the finite Q Hausdorff dimension estimates exhibit (for reasons that we do not understand) surprising numerical stability, and the Farey series partitioning actually yields the *best* numerical value of the Hausdorff dimension (30) of any methods used so far; for example[19], the sine map (1) estimate based on $240 \le Q \le 250$ Farey series partitions yields $D_H = .87012 \pm .00001$. The quoted error refers to the variation of D_H over this range of Q; as the computation is not asymptotic, such numerical stability can underestimate the actual error by a large factor.

10.3 Continued Fraction Partitioning

From a number-theorist's point of view, the *continued fraction partitioning* of the unit interval is the most venerable organization of rationals, preferred already by Gauss. The continued fraction partitioning is obtained by deleting successively mode-locked intervals (points in the case of the shift map) corresponding to continued fractions of increasing length. The first level is obtained by deleting $\Delta_{[1]}, \Delta_{[2]}, \cdots, \Delta_{[a_1]}, \cdots$ mode-lockings; their complement are the *covering* intervals $\ell_1, \ell_2, \ldots, \ell_{a_1}, \ldots$ which contain all windings, rational and irrational, whose continued fraction expansion starts with $[a_1, \ldots]$ and is of length at least 2. The second level is obtained by deleting $\Delta_{[1,2]}, \Delta_{[1,3]}, \Delta_{[2,2]}, \Delta_{[2,3]}, \cdots, \Delta_{[n,m]}, \cdots$ and so on, as illustrated in fig. 10.2.

Fig. 10.2. Continued fraction partitioning of the irrational winding set[23]. At level n=1 all mode locking intervals $\Delta_{[a]}$ with winding numbers 1/1, 1/2, 1/3, …, $1/a$, … are deleted, and the cover consists of the complement intervals l_a. At level n=2 the mode locking intervals $\Delta_{[a,2]}, \Delta_{[a,3]}, \ldots$ are deleted from each cover l_a, and so on.

(10.2) Definition. *The* nth *level continued fraction partition* $\mathcal{S}_n = \{a_1 a_2 \cdots a_n\}$ *is the monotonically increasing sequence of all rationals* P_i/Q_i *between 0 and 1 whose continued fraction expansion is of length n:*

$$\frac{P_i}{Q_i} = [a_1, a_2, \cdots, a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots \cfrac{1}{a_n}}}$$

The object of interest, the set of the irrational winding numbers, is in this partitioning labeled by $\mathcal{S}_\infty = \{a_1 a_2 a_3 \cdots\}$, $a_k \in Z^+$, *ie.*, the set of winding numbers with infinite continued fraction expansions. The continued fraction labeling is particularly appealing in the present context because of the close connection of the Gauss shift to the renormalization transformation R, discussed below. The Gauss shift[24]

$$T(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & x \neq 0 \\ 0\ , & x = 0 \end{cases} \tag{6}$$

($[\cdots]$ denotes the integer part) acts as a shift on the continued fraction representation of numbers on the unit interval

$$x = [a_1, a_2, a_3, \ldots] \to T(x) = [a_2, a_3, \ldots] \ , \tag{7}$$

and maps "daughter" intervals $\ell_{a_1 a_2 a_3 \ldots}$ into the "mother" interval $\ell_{a_2 a_3 \ldots}$.

However natural the continued fractions partitioning might seem to a number theorist, it is problematic for an experimentalist, as it requires measuring infinity of mode-lockings even at the first step of the partitioning. This problem can be overcome both numerically and experimentally by some understanding of the asymptotics of mode-lockings with large continued fraction entries[23, 8]. Alternatively, a finite partition can be generated by the partitioning scheme to be described next.

10.4 Farey Tree Partitioning

The *Farey tree partitioning* is a systematic bisection of rationals: it is based on the observation that roughly halfways between any two large stability intervals (such as 1/2 and 1/3) in the devil's staircase of fig. 10.1 there is the next largest stability interval (such as 2/5). The winding number of this interval is given by the Farey mediant[21] $(P+P')/(Q+Q')$ of the parent mode-lockings P/Q and P'/Q'. This kind of cycle "gluing" is rather general and by no means restricted to circle maps; it can be attained whenever it is possible to arrange that the Qth iterate deviation caused by shifting a parameter from the correct value for the Q-cycle is exactly compensated by the Q'th iterate deviation from closing the Q'-cycle; in this way the two near cycles can be glued together into an exact cycle of length $Q+Q'$. The Farey tree is obtained by starting with the ends of the unit interval written as 0/1 and 1/1, and then recursively bisecting intervals by means of Farey mediants. This kind of hierarchy of rationals is rather new[26], and, as far as we are aware, not previously studied by number theorists. It is appealing both from the experimental and from the the golden-mean renormalization[30] point of view, but it has a serious drawback of lumping together mode-locking intervals of wildly different sizes on the same level of the Farey tree.

(10.3) Definition. *The* nth *Farey tree level* T_n *is the monotonically increasing sequence of those continued fractions* $[a_1, a_2, \ldots, a_k]$ *whose entries* $a_i \geq 1$, $i = 1, 2, \ldots, k-1$, $a_k \geq 2$, *add up to* $\sum_{i=1}^{k} a_i = n+2$. For example

$$T_2 = \{[4], [2,2], [1,1,2], [1,3]\} = \left(\frac{1}{4}, \frac{1}{5}, \frac{3}{5}, \frac{3}{4}\right).$$

The number of terms in T_n is 2^n. Each rational in T_{n-1} has two "daughters" in T_n, given by

$$\begin{array}{ccc} & [\cdots, a] & \\ [\cdots, a-1, 2] & & [\cdots, a+1] \end{array}$$

Iteration of this rule places all rationals on a binary tree, labelling each by a unique binary label[29]. The transcription from the binary Farey labels to the continued fraction labels follows from the mother-daughter relation above; each block $1 \cdots 0$ ("1" followed by $a-1$ zeros) corresponds to entry $[\cdots, a, \cdots]$ in the continued fraction label. The Farey tree has a variety of interesting symmetries (such as "flipping heads and tails" relations obtained by reversing the order of the continued-fraction entries) with as yet unexploited implications for the renormalization theory: some of these are discussed in ref. [29].

Fig. 10.3. The Farey tree in the continued fraction representation (from ref. [3]).

The smallest and the largest denominator in T_n are respectively given by

$$[n-2] = \frac{1}{n-2}, \quad [1,1,\ldots,1,2] = \frac{F_{n+1}}{F_{n+2}} \propto \rho^n \,, \tag{8}$$

where the Fibonacci numbers F_n are defined by $F_{n+1} = F_n + F_{n-1}; \quad F_0 = 0,\ F_1 = 1$, and ρ is the golden mean ratio

$$\rho = \frac{1+\sqrt{5}}{2} = 1.61803\ldots \tag{9}$$

Note the enormous spread in the cycle lengths on the same level of the Farey tree: $n \le Q \le \rho^n$. The cycles whose length grows only as a power of the Farey tree level will cause strong non-hyperbolic effects in the evaluation of various averages.

The Farey tree rationals can be generated by backward iterates of $1/2$ by the Farey presentation function[30]:

$$\begin{array}{rclr} f_0(x) & = & x/(1-x) & 0 \le x < 1/2 \\ f_1(x) & = & (1-x)/x & 1/2 < x \le 1\ . \end{array} \tag{10}$$

(the utility of the presentation function is discussed at length in ref. [30]). The Gauss shift (6) corresponds to replacing the binary Farey presentation function branch f_0 in (10) by an infinity of branches

$$
\begin{aligned}
f_a(x) &= f_1 \circ f_0^{(a-1)}(x) = \frac{1}{x} - a, \qquad \frac{1}{a-1} < x \le \frac{1}{a}, \\
f_{ab\cdots c}(x) &= f_c \circ \cdot \circ f_b \circ f_a(x)\,. \qquad (11)
\end{aligned}
$$

A rational $x = [a_1, a_2, \ldots, a_k]$ is "annihilated" by the kth iterate of the Gauss shift, $f_{a_1 a_2 \cdots a_k}(x) = 0$. The above maps look innocent enough, but note that what is being partitioned is not the dynamical space, but the parameter space. The flow described by (10) and by its non-trivial circle-map generalizations will turn out to be a *renormalization group* flow in the function space of dynamical systems, not an ordinary flow in the phase space of a particular dynamical system.

Having defined the three partitioning schemes, we now briefly summarize the results of the circle-map renormalization theory.

10.5 Local Theory: "Golden Mean" Renormalization

Possible trajectories of a dynamical system are of three qualitatively distinct types: they are either asymptotically unstable (positive Lyapunov exponent), asymptotically marginal (vanishing Lyapunov) or asymptotically stable (negative Lyapunov). The asymptotically stable orbits can be treated by the traditional integrable system methods. The asymptotically unstable orbits build up chaos, and can be dealt with using the machinery of the hyperbolic, "Axiom A" dynamical systems theory[31]. Here we shall concentrate on the third class of orbits, the asymptotically marginal ones. I call them the "border of order"; they lie between order and chaos, and remain on that border to all times.

The way to pinpoint a point on the border of order is to recursively adjust the parameters so that at the recurrence times $t = n_1, n_2, n_3, \cdots$ the trajectory passes through a region of contraction sufficiently strong to compensate for the accumulated expansion of the preceding n_i steps, but not so strong as to force the trajectory into a stable attracting orbit. The *renormalization operation* R implements this procedure by recursively magnifying the neighborhood of a point on the border in the dynamical space (by rescaling by a factor α), in the parameter space (by shifting the parameter origin onto the border and rescaling by a factor δ), and by replacing the initial map f by the nth iterate f^n restricted to the magnified neighboorhood

$$
f_p(x) \to Rf_p(x) = \alpha f^n_{p/\delta}(x/\alpha)
$$

There are by now many examples of such renormalizations in which the new function, framed in a smaller box, is a rescaling of the original function, *ie.* the fix-point function of the renormalization operator R. The best known is the period doubling renormalization, with the recurrence times $n_i = 2^i$. The simplest circle map example is the golden mean renormalization[2], with recurrence times $n_i = F_i$ given by the Fibonacci numbers (8). Intuitively, in this context a metric self-similarity arises because iterates of critical maps are themselves critical, *ie.* they also have cubic inflection points with vanishing derivatives.

The renormalization operator appropriate to circle maps[9, 10] acts as a generalization of the Gauss shift (11); it maps a circle map (represented as a pair of functions (g, f), see fig. 10.4) of winding number $[a, b, c, \ldots]$ into a rescaled map of winding number $[b, c, \ldots]$:

$$R_a \begin{pmatrix} g \\ f \end{pmatrix} = \begin{pmatrix} \alpha g^{a-1} \circ f \circ \alpha^{-1} \\ \alpha g^{a-1} \circ f \circ g \circ \alpha^{-1} \end{pmatrix}, \tag{12}$$

Acting on a map with winding number $[a, a, a, \ldots]$, R_a returns a map with the same winding number $[a, a, \ldots]$, so the fixed point of R_a has a quadratic irrational winding number $W = [a, a, a, \ldots]$. This fixed point has a single expanding eigenvalue δ_a. Similarly, the renormalization transformation $R_{a_p} \ldots R_{a_2} R_{a_1} \equiv R_{a_1 a_2 \ldots a_p}$ has a fixed point of winding number $W_p = [a_1, a_2, \ldots, a_{n_p}, a_1, a_2, \ldots]$, with a single[9, 10, 33] expanding eigenvalue δ_p.

Fig. 10.4. The golden-mean winding number fixed-point function pair (f, g) for critical circle maps with cubic inflection point. The symbolic dynamics dictates a unique framing such that the functions (f, g) are defined on intervals $(\bar{x} \leq x \leq \bar{x}/\alpha, \bar{x}/\alpha \leq x \leq \bar{x}\alpha)$, $\bar{x} = f^{-1}(0)$: in this framing, the circle map (f, g) has continuous derivatives across the f–g junctions (from ref. [32]).

For short repeating blocks, δ can be estimated numerically by comparing succesive continued fraction approximants to W. Consider the P_r/Q_r rational approximation to a quadratic irrational winding number W_p whose continued fraction expansion consists of r repeats of a block p. Let Ω_r be the parameter for which the map (1) has a superstable cycle of rotation number $P_r/Q_r = [p, p, \ldots, p]$. The δ_p can then be estimated by extrapolating from[2]

$$\Omega_r - \Omega_{r+1} \propto \delta_p^{-r}. \tag{13}$$

What this means is that the "devil's staircase" of fig. 10.4 is self-similar under magnification by factor δ_p around any quadratic irrational W_p.

The fundamental result of the renormalization theory (and the reason why all this is so interesting) is that the ratios of successsive P_r/Q_r mode-locked intervals converge to *universal* limits. The simplest example of (13) is the sequence of Fibonacci number continued fraction approximants to the golden mean winding number $W = [1, 1, 1, \ldots] = (\sqrt{5} - 1)/2$. For critical circle maps with a cubic inflection point $\delta_1 = -2.833612\ldots$; a list of values of δ_p's for the shortest continued fraction blocks p is given in ref. [8].

When the repeated block is not large, the rate of increase of denominators Q_r is not large, and (13) is a viable scheme for estimating δ's. However, for long repeating blocks, the rapid increase of Q_r's makes the periodic orbits hard to determine and better methods are required, such as the unstable manifold method employed in ref. [8]. This topic would take us beyond the space allotted here, so we merely record the golden-mean unstable manifold equation[35, 36, 37]

$$g_p(x) = \alpha g_{1+p/\delta}\left(\alpha g_{1+1/\delta+p/\delta^2}(x/\alpha^2)\right) \tag{14}$$

and leave the reader contemplating methods of solving such equations. We content ourself here with stating what the extremal values of δ_p are.

For a given cycle length Q, the *narrowest* interval shrinks with a power law[38, 5, 29]

$$\Delta_{1/Q} \propto Q^{-3} \tag{15}$$

This leading behavior is derived by methods akin to those used in describing intermittency[39]: $1/Q$ cycles accumulate toward the edge of $0/1$ mode-locked interval, and as the successive mode-locked intervals $1/Q$, $1/(Q-1)$ lie on a parabola, their differences are of order Q^{-3}. This should be compared to the subcritical circle maps in the number-theoretic limit (2), where the interval between $1/Q$ and $1/(Q-1)$ winding number value of the parameter Ω shrinks as $1/Q^2$. For the critical circle maps the $\ell_{1/Q}$ interval is *narrower* than in the k=0 case, because it is squeezed by the nearby broad $\Delta_{0/1}$ mode-locked interval.

For fixed Q the *widest* interval is bounded by $P/Q = F_{n-1}/F_n$, the nth continued fraction approximant to the *golden mean*. The intuitive reason is that the golden mean winding sits as far as possible from any short cycle mode-locking. Herein lies the suprising importance of the golden mean number for dynamics; it corresponds to extremal scaling in physical problems characterized by winding numbers, such as the KAM tori of classical mechanics[7, 1]. The golden mean interval shrinks with a universal exponent

$$\Delta_{P/Q} \propto Q^{-2\mu_1} \tag{16}$$

where $P = F_{n-1}$, $Q = F_n$ and μ_1 is related to the universal Shenker number δ_1 (13) and the golden mean (9) by

$$\mu_1 = \frac{\ln|\delta_1|}{2\ln\rho} = 1.08218\ldots \tag{17}$$

The closeness of μ_1 to 1 indicates that the golden mean approximant mode-lockings barely feel the fact that the map is critical (in the k=0 limit this exponent is $\mu = 1$).

To summarize: for critical maps the spectrum of exponents arising from the circle maps renormalization theory is bounded from above by the harmonic scaling, and from below by the geometric golden-mean scaling:

$$3/2 > \mu_{m/n} \geq 1.08218\cdots. \tag{18}$$

10.6 Global Theory: Ergodic Averaging

So far we have discussed the results of the renormalization theory for isolated irrational winding numbers. Though the local theory has been tested experimentally[40, 41], the golden-mean universality utilizes only a few of the available mode-locked intervals, and from the experimental point of view it would be preferable to test universal properties which are *global* in the sense of pertaining to a range of winding numbers. We first briefly review some of the attempts to derive such predictions using ideas from the ergodic number theory, and then turn to the predictions based on the thermodynamic foramlism.

The ergodic number theory[25, 42] is rich in (so far unfulfilled) promise for the mode-locking problem. For example, while the Gauss shift (6) invariant measure

$$\mu(x) = \frac{1}{\ln 2}\frac{1}{1+x} \tag{19}$$

was known already to Gauss, the corresponding invariant measure for the critical circle maps renormalization operator R has so far eluded description. It lies on a fractal set - computer sketches are given in refs. [43, 44] - and a general picture of what the "strange repeller" (in the space of limit functions for the renormalization operator (12)) might look like is given in refs. [45]. Rand *et al.*[10] have advocated ergodic explorations of this attractor, by sequences of renormalizations R_{a_k} corresponding to the digits of the continued fraction expansion of a "normal" winding number $W = [a_1, a_2, a_3, \ldots]$. A numerical implementation of this proposal [43, 44] by Monte Carlo generated strings $a_1, a_2, a_3, \ldots$ yields estimates of "mean" scalings $\bar{\delta} = 15.5 \pm .5$ and $\bar{\alpha} = 1.8 \pm .1$. $\bar{\delta}^n$ is the estimate of the mean width of an "average" mode-locked interval Δ_{P_n/Q_n}, where P_n/Q_n is the nth continued fraction approximation to a normal winding number $W = [a_1, a_2, a_3, \ldots]$. In this connection the following beautiful result of the ergodic number theory is suggestive:

(10.1) Theorem (Khinchin, Kuzmin, Levy[42]) *For almost all $W \in [0,1]$ the denominator Q_n of the n-th continued fraction approximant $W = P_n/Q_n + \epsilon_n$, $P_n/Q_n = [a_1, a_2, a_3, \ldots, a_n]$ converges asymptotically to*

$$\lim_{n\to\infty} \frac{1}{n} \ln Q_n = \frac{\pi^2}{12 \ln 2} \ . \tag{20}$$

In physics this theorem pops up in various guises; for example, $\pi^2/6\ln 2$ can be interpreted as the Kolmogorov entropy of "mixmaster" cosmologies[46]. In the present context this theorem has been used[44] to connect the ergodic estimate of $\bar{\delta}$ to $\hat{\delta}$ estimated[5] by averaging over all available mode-lockings up to given cycle length Q, but it is hard to tell what to make out of such results. The numerical convergence of ergodic averages is slow, if not outright hopeless, so we abandon henceforth the ergodic "time" averages (here the "time" is the length of a continued fraction) and turn instead to the "thermodynamic" averages (averages over all "configurations", here all mode lockings on a given level of a resolution hierarchy).

10.7 Global Theory: Thermodynamic Averaging

Consider the following average over mode-locking intervals (4):

$$\Omega(\tau) = \sum_{Q=1}^{\infty} \sum_{(P|Q)=1} \Delta_{P/Q}^{-\tau}. \tag{21}$$

The sum is over all irreducible rationals P/Q, $P < Q$, and $\Delta_{P/Q}$ is the width of the parameter interval for which the iterates of a critical circle map lock onto a cycle of length Q, with winding number P/Q.

The qualitative behavior of (21) is easy to pin down. For sufficiently negative τ, the sum is convergent; in particular, for $\tau = -1$, $\Omega(-1) = 1$, as for the critical circle maps the mode-lockings fill the entire Ω range[18]. However, as τ increases, the contributions of the narrow (large Q) mode-locked intervals $\Delta_{P/Q}$ get blown up to $1/\Delta_{P/Q}^{\tau}$, and at some critical value of τ the sum diverges. This occurs for $\tau < 0$, as $\Omega(0)$ equals the number of all rationals and is clearly divergent.

The sum (21) is infinite, but in practice the experimental or numerical mode-locked intervals are available only for small finite Q. Hence it is necessary to split up the sum into subsets $\mathcal{S}_n = \{i\}$ of rational winding numbers P_i/Q_i on the "level" n, and present the set of mode-lockings hierarchically, with resolution increasing with the level:

$$\bar{Z}_n(\tau) = \sum_{i \in \mathcal{S}_n} \Delta_i^{-\tau}. \tag{22}$$

The original sum (21) can now be recovered as the $z = 1$ value of a "generating" function $\Omega(z, \tau) = \sum_n z^n \bar{Z}_n(\tau)$. As z is anyway a formal parameter, and n is a rather arbitrary "level" in some *ad hoc* partitioning of rational numbers, we bravely introduce a still more general, P/Q weighted generating function for (21):

$$\Omega(q, \tau) = \sum_{Q=1}^{\infty} \sum_{(P|Q)=1} e^{-\nu_{P/Q} q} Q^{2\mu_{P/Q}\tau} . \tag{23}$$

The sum (21) corresponds to $q = 0$. Exponents $\nu_{P/Q}$ will reflect the importance we assign to the P/Q mode-locking, *ie.* the *measure* used in the averaging over all mode-lockings. Three choices of of the $\nu_{P/Q}$ hierarchy that we consider here correspond respectively to the Farey series partitioning (definition (10.1))

$$\Omega(q, \tau) = \sum_{Q=1}^{\infty} \Phi(Q)^{-q} \sum_{(P|Q)=1} Q^{2\mu_{P/Q}\tau} , \tag{24}$$

the continued fraction partitioning (definition (10.2))

$$\Omega(q, \tau) = \sum_{n=1}^{\infty} e^{-qn} \sum_{[a_1, \dots, a_n]} Q^{2\mu_{[a_1, \dots, a_n]}\tau} , \tag{25}$$

and the Farey tree partitioning (definition (10.3))

$$\Omega(q,\tau) = \sum_{k=n}^{\infty} 2^{-qn} \sum_{i=1}^{2^n} Q_i^{\tau\mu_i} \ , \quad Q_i/P_i \in T_n \ . \tag{26}$$

Other measures can be found in the literature, but the above three suffice for our purposes.

Sum (23) is an example of a "thermodynamic" average. In the thermodynamic formalism[31, 47] a function $\tau(q)$ is defined by the requirement that the $n \to \infty$ limit of generalized sums

$$Z_n(\tau,q) = \sum_{i \in \mathcal{S}_n} \frac{p_i^q}{\ell_i^\tau} \tag{27}$$

is finite. Thermodynamic formalism was originally introduced to describe measures generated by strongly mixing ergodic systems, and for most practitioners p_i in (27) is the probability of finding the system in the partition i, given by the "natural" measure. What we are using here in the Farey series and the Farey tree cases are the "equipartition" measures $p_i = 1/N_n$, where N_n is the number of mode-locking intervals on the nth level of resolution. In the continued fraction partitioning this does not work, as N_n is infinite - in this case we assign all terms of equal continued fraction length equal weigth. It is important to note that as the Cantor set under consideration is generated by scanning the parameter space, not by dynamical stretching and kneading, there is no "natural" measure, and a variety of equally credible measures can be constructed[5, 19, 47, 48]. Each distinct hierarchical presentation of the irrational winding set (distinct partitioning of rationals on the unit interval) yields a *different* thermodynamics. As far as I can tell, no thermodynamic function $q(\tau)$ considered here (nor any of the $q(\tau)$ or $f(\alpha)$ functions studied in the literature in other contexts) has physical significance, but their qualitative properties are interesting; in particular, all versions of mode-locking thermodynamics studied so far exhibit phase transitions.

We summarize by succintly stating what our problem is in a way suggestive to a number theorist, by changing the notation slightly and rephrasing (21) this way;

(10.4) Definition: The mode-locking problem. *Develop a theory of the following "zeta" function:*

$$\hat{\zeta}(s) = \sum_{n=1}^{\infty} \sum_{(m|n)=1} n^{-2\mu_{m/n} s} \ , \tag{28}$$

where μ is defined as in (4).

For the shift map (2), $\mu_{m/n} = 1$, and this sum is a ratio of two Riemann zeta functions

$$\hat{\zeta}(s) = \frac{\zeta(2s-1)}{\zeta(2s)} \ .$$

For critical maps the spectrum of exponents arising from the circle maps renormalization theory is non-trivial; according to (18) it is bounded from above by the harmonic scaling, and from below by the geometric golden-mean scaling.

Our understanding of the $\hat{\zeta}(s)$ function for the critical circle maps is rudimentary – almost nothing that is the backbone of the theory of number-theoretic zeta functions has been accompished here: no good integral representations of (28) are known, no functional equations (analogous to reflection formulas for the classical zeta functions) have been constructed, no Riemann-Siegel formulas, *etc.*. We summarize basically all that is known in the remainder of this lecture, and that is not much.

10.8 The Hausdorff Dimension of Irrational Windings

A finite cover of the set irrational windings at the "nth level of resolution" is obtained by deleting the parameter values corresponding to the mode-lockings in the subset $\mathcal{S}_n$; left behind is the set of complement *covering* intervals of widths

$$\ell_i = \Omega^{min}_{P_r/Q_r} - \Omega^{max}_{P_l/Q_l} \ . \tag{29}$$

Here $\Omega^{min}_{P_r/Q_r}$ ($\Omega^{max}_{P_l/Q_l}$) are respectively the lower (upper) edges of the mode-locking intervals Δ_{P_r/Q_r} (Δ_{P_l/Q_l}) bounding ℓ_i and i is a symbolic dynamics label, for example the entries of the continued fraction representation $P/Q = [a_1, a_2, ..., a_n]$ of one of the boundary mode-lockings, $i = a_1 a_2 \cdots a_n$. ℓ_i provide a finite cover for the irrational winding set, so one may consider the sum

$$Z_n(\tau) \ = \ \sum_{i \in \mathcal{S}_n} \ell_i^{-\tau} \tag{30}$$

The value of $-\tau$ for which the $n \to \infty$ limit of the sum (30) is finite is the *Hausdorff dimension*[49] D_H of the irrational winding set. Strictly speaking, this is the Hausdorff dimension only if the choice of covering intervals ℓ_i is optimal; otherwise it provides an upper bound to D_H. As by construction the ℓ_i intervals cover the set of irrational winding with no slack, we expect that this limit yields the Hausdorff dimension. This is supported by all numerical evidence, but a proof that would satisfy mathematicians is lacking.

Jensen *et al.*[5] have provided numerical evidence that this Hausdorff dimension is approximately $D_H = .870\ldots$ and that it is universal. It is not at all clear whether this is the optimal global quantity to test - a careful investigation[19] shows that D_H is surprisingly hard to pin down numerically. At least the Hausdorff dimension has the virtue of being independent of how one partitions mode-lockings and should thus be the same for the variety of thermodynamic averages in the literature[50].

10.9 A Bound on the Hausdorff Dimension

We start by giving an elementary argument that the Hausdorff dimension of irrational windings for critical circle maps is less than one. The argument depends on the reasonable, but so far unproven assumption that the golden mean scaling (17) is the extremal scaling.

In the crudest approximation, one can replace $\mu_{P/Q}$ in (28) by a "mean" value $\hat{\mu}$; in that case the sum is given explicitely by a ratio of the Riemann ζ-functions:

$$\Omega(\tau) = \sum_{Q=1}^{\infty} \phi(Q)\, Q^{2\tau\hat{\mu}} = \frac{\zeta(-2\tau\hat{\mu}-1)}{\zeta(-2\tau\hat{\mu})} \tag{31}$$

As the sum diverges at $-\tau =$ Hausdorff dimension, the "mean" scaling exponent $\hat{\mu}$ and D_H are related by the ζ function pole at $\zeta(1)$:

$$D_H\hat{\mu} = 1. \tag{32}$$

While this does not enable us to compute D_H, it does immediately establish that D_H for critical maps exists and is smaller than 1, as the μ bounds (18) yield

$$\frac{2}{3} < D_H < .9240\ldots \tag{33}$$

To obtain sharper estimates of D_H, we need to describe the distribution of $\mu_{P/Q}$ within the bounds (18). This we shall now attempt using several variants of the thermodynamic formalism.

10.10 The Hausdorff Dimension in Terms of Cycles

Estimating the $n \to \infty$ limit of (30) from finite numbers of covering intervals ℓ_i is a rather unilluminating chore. Fortunately, there exist considerably more elegant ways of extracting D_H. We have noted that in the case of the "trivial" mode-locking problem (2), the covering intervals are generated by iterations of the Farey map (10) or the Gauss shift (11). The nth level sum (30) can be approximated by $\mathcal{L}^n$, where $\mathcal{L}(y,x) = \delta(x - f^{-1}(y))|f'(y)|^{\tau}$; this amounts to approximating each cover width ℓ_i by $|df^n/dx|$ evaluated on the ith interval. By nothing much deeper than use of the identity *log det = tr log*, the spectrum of $\mathcal{L}$ can be expressed[31] in terms of stabilities of the prime (non-repeating) periodic orbits p of $f(x)$:

$$\begin{aligned} \det(1 - z\mathcal{L}) &= \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{z^{rn_p}}{r} \frac{|\Lambda_p^r|^{\tau}}{1 - 1/\Lambda_p^r}\right) \\ &= \prod_p \prod_{k=0}^{\infty} \left(1 - z^{n_p}|\Lambda_p|^{\tau}/\Lambda_p^k\right) . \end{aligned} \tag{34}$$

In the "trivial" Gauss shift (11) renormalization model, the Fredholm determinant and the dynamical zeta functions have been introduced and studied by Mayer[51] who has shown that the eigenvalues of the transfer operator are exponentially spaced, just as for the dynamical zeta functions[52] for the "Axiom A" hyperbolic systems.

The sum (30) is dominated by the leading eigenvalue of $\mathcal{L}$; the Hausdorff dimension condition $Z_n(-D_H) = O(1)$ means that $\tau = -D_H$ should be such that the leading eigenvalue is $z = 1$. The leading eigenvalue is determined by the $k = 0$ part of (34); putting all these pieces together, we obtain a pretty formula relating the Hausdorff dimension to the prime cycles of the map $f(x)$:

$$0 = \prod_p \left(1 - 1/|\Lambda_p|^{D_H}\right) \ . \tag{35}$$

For the Gauss shift (11) the stabilities of periodic cycles are available analytically[51, 23], as roots of quadratic equations: For example, the x_a fixed points (quadratic irrationals with $x_a = [a, a, a \ldots]$ infinitely repeating continued fraction expansion) are given by

$$x_a = \frac{-a + \sqrt{a^2 + 4}}{2}, \qquad \Lambda_a = -\left(\frac{a + \sqrt{a^2 + 4}}{2}\right)^2 \tag{36}$$

and the $x_{ab} = [a, b, a, b, a, b, \ldots]$ 2–cycles are given by

$$\begin{aligned} x_{ab} &= \frac{-ab + \sqrt{(ab)^2 + 4ab}}{2b} \\ \Lambda_{ab} &= (x_{ab}x_{ba})^{-2} = \left(\frac{ab + 2 + \sqrt{ab(ab + 4)}}{2}\right)^2 \end{aligned} \tag{37}$$

We happen to know beforehand that $D_H = 1$ (the irrationals take the full measure on the unit interval; the continuous Gauss measure (19) is invariant under the Gauss shift (6); the Pérron-Frobenius theorem), so is the infinite product (35) merely a very convoluted way to compute the number 1? Possibly so, but availability of this exact result provides a useful testing ground for trashing out the optimal methods for determining zeros of Fredholm determinants in presence of *nonhyperbolicities*. The Farey map (10) has one marginal stability fixed point $x_0 = 0$ which is excluded from the cycle expansion of (35), but its ghost haunts us as a nonhyperbolic "intermittency" ripple in the cycle expansion. One has to sum[23] infinities of cycles of nearly same stability

$$\prod_p (1 - |\Lambda_p|^\tau) = 1 - \sum_{a=1}^{\infty} |\Lambda_a|^\tau + (\text{curvatures}) \tag{38}$$

in order to attain the exponential convergence expected on the basis of the hyperbolicity[51] of this dynamical ζ function. We know from (36) that $|\Lambda_n| \propto n^2$, so the stability falls off only as a opower of the cycle length n, and these infinite sums pose a serious numerical headache for which we (as yet) know of no satisfactory cure. The sum (38) behaves essentially as the Riemann $\zeta(-2\tau)$, and the analytic number theory techniques might still rescue us.

Once the meaning of (35) has been grasped, the corresponding formula[8] for the *critical* circle maps follows immediately:

$$0 = \prod_p \left(1 - 1/|\delta_p|^{D_H}\right) \ . \tag{39}$$

This formula relates the Jensen *et al.* dimension of irrational windings to the universal Shenker parameter scaling ratios δ_p; its beauty lies in relating D_H to the universal scalings δ_p, thus rendering the universality of the Jensen *et al.* dimension manifest. As a practical formula for evaluating this dimension, (39) has so far yielded estimates of D_H of modest accuracy, but that can surely be improved. In particular, computations based on the (34) infinite products should be considerably more convergent[53, 54], but have not been carried out so far.

The derivation of (39) relies only on the following aspects of the "hyperbolicity conjecture" of refs. [29, 45, 55, 32]:

1) *limits* for Shenker δ's *exist* and are universal. This should follow from the renormalization theory developed in refs. [9, 10, 33], though a general proof is still lacking.
2) δ_p grow *exponentially* with n_p, the length of the continued fraction block p.
3) δ_p for $p = a_1 a_2 \dots n$ with a large continued fraction entry n grows as a *power* of n. According to (15), $\lim_{n\to\infty} \delta_p \propto n^3$. In the calculation of ref. [8] the explicit values of the asymptotic exponents and prefactors were not used, only the assumption that the growth of δ_p with n is not slower than a power of n.

Explicit evaluation of the spectrum was first attempted in ref. [23] – prerequsite for attaining the exponential (or faster[53, 54]) convergence of the cycle expansions are effective methods for summation of *infinite* families of mode-lockings. At present, those are lacking - none of the tricks from the Riemann-zeta function theory (integral representations, saddle-point expansions, Poisson resummations, *etc.* have not worked for us) so we have been forced to rely on the rather trecherous logarithmic convergence acceleration algorithms[56].

10.11 Farey Series and the Riemann Hypothesis

The *Farey series* thermodynamics (24) is obtained by deleting all mode-locked intervals $\Delta_{P'/Q'}$ of cycle lengths $1 \le Q' \le Q$. What remains are the irrational winding set covering intervals (29).

The thermodynamics of the Farey series in the number-theory limit (2) has been studied by Hall and others[57, 58]; their analytic results are instructive and are reviewed in ref. [19].

The main result is that $q(\tau)$ consists of two straight sections

$$q(\tau) = \begin{cases} \tau/2 & \tau \le -2 \\ 1 + \tau & \tau \ge -2 \end{cases} . \tag{40}$$

and the Farey arc thermodynamics undergoes a first order phase transition at $\tau = -2$. What that means is that almost all covering intervals scale as Q^{-2} (the $q = 1 + \tau$ phase); however, for $\tau \le -2$, the thermodynamics average is dominated by the handful of fat intervals which scale as Q^{-1}. The number-theoretic investigations[57, 58] also establish the rate of convergence as $Q \to \infty$;

at the phase transition point it is very slow, logarithmic[19]. In practice, the Euler noise is such numerical nuisance that we skip here the discussion of the $q(\tau)$ convergence althogether.

For the *critical* circle maps the spectrum of scales is much richer. The $1/Q$ mode-locked intervals which lie on a parabolic devil staircase[38, 5, 29] yield the broadest covering interval $\ell(1, Q) \simeq kQ^{-2}$, with the minimum scaling exponent $\mu_{min} = 1.$ and the narrowest covering interval $\ell(Q, Q-1) \approx kQ^{-3}$, with the exponent $\mu_{max} = 3/2$.

The Farey series thermodynamics is of a number theoretical interest, because the Farey series provide uniform coverings of the unit interval with rationals, and because they are closely related to the deepest problems in number theory, such as the Riemann hypothesis[60, 61] . The distribution of the Farey series rationals across the unit interval is suprisingly uniform - indeed, so uniform that in the pre-computer days it has motivated a compilation of an entire handbook of Farey series[62]. A quantitive measure of the non-uniformity of the distribution of Farey rationals is given by displacements of Farey rationals for $P_i/Q_i \in \mathcal{F}_Q$ from uniform spacing:

$$\delta_i = \frac{i}{\Phi(Q)} - \frac{P_i}{Q_i}, \quad i = 1, 2, \cdots, \Phi(Q)$$

The Riemann hypothesis states that the zeros of the Riemann zeta function lie on the $s = 1/2 + i\tau$ line in the complex s plane, and would seem to have nothing to do with physicists' real mode-locking widths that we are interested in here. However, there is a real-line version of the Riemann hypothesis that lies very close to the mode-locking problem. According to the theorem of Franel and Landau[59, 60, 61], the Riemann hypothesis is equivalent to the statement that

$$\sum_{Q_i \leq Q} |\delta_i| = o(Q^{\frac{1}{2}+\epsilon})$$

for all ϵ as $Q \to \infty$. The mode-lockings $\Delta_{P/Q}$ contain the necessary information for constructing the partition of the unit interval into the ℓ_i covers, and therefore implicitely contain the δ_i information. The implications of this for the circle-map scaling theory have not been worked out, and is not known whether some conjecture about the thermodynamics of irrational windings is equivalent to (or harder than) the Riemann hypothesis, but the danger lurks.

10.12 Farey Tree Thermodynamics

The narrowest mode-locked interval (16) at the n-th level of the Farey tree partition sum (26) is the golden mean interval

$$\Delta_{F_{n-1}/F_n} \propto |\delta_1|^{-n}. \tag{41}$$

It shrinks exponentially, and for τ positive and large it dominates $q(\tau)$ and bounds $dq(\tau)/d\tau$:

(42) $$q'_{max} = \frac{\ln|\delta_1|}{\ln 2} = 1.502642\ldots$$

However, for τ large and negative, $q(\tau)$ is dominated by the interval (15) which shrinks only harmonically, and $q(\tau)$ approaches 0 as

(43) $$\frac{q(\tau)}{\tau} = \frac{3\ln n}{n\ln 2} \to 0.$$

So for finite n, $q_n(\tau)$ crosses the τ axis at $-\tau = D_n$, but in the $n \to \infty$ limit, the $q(\tau)$ function exhibits a phase transition; $q(\tau) = 0$ for $\tau < -D_H$, but is a non-trivial function of τ for $-D_H \leq \tau$. This non-analyticity is rather severe - to get a clearer picture, we illustrate it by a few number-theoretic models (the critical circle maps case is qualitatively the same).

An cute version of the "trivial" Farey level thermodynamics is given by the "Farey model"[19], in which the intervals $\ell_{P/Q}$ are replaced by Q^{-2}:

(44) $$Z_n(\tau) = \sum_{i=1}^{2^n} Q_i^{2\tau}.$$

Here Q_i is the denominator of the ith Farey rational P_i/Q_i. For example (see (definition (10.3)),

$$Z_2(1/2) = 4 + 5 + 5 + 4.$$

Though it might seem to have been pulled out of a hat, the Farey model is as sensible description of the distribution of rationals as the periodic orbit expansion (34). By the "anihilation" property of the Gauss shift (11), the nth Farey level sum $Z_n(-1)$ can be written as the integral

(45) $$Z_n(-1) = \int dx\delta(f^n(x)) = \sum 1/|f'_{a_1\ldots a_k}(0)|\,,$$

with the sum restricted to the Farey level $\sum^{a_1+\ldots+a_k=n+2}$. It is easily checked that $f'_{a_1\ldots a_k}(0) = (-1)^k Q^2_{[a_1,\ldots,a_k]}$, so the Farey model sum is a partition generated by the Gauss map preimages of $x = 0$, *ie.* by rationals, rather than by the quadratic irrationals as in (34). The sums are generated by the same transfer operator, so the eigenvalue spectrum should be the same as for the periodic orbit expansion, but in this variant of the finite level sums we can can evaluate $q(\tau)$ *exactly* for $\tau = k/2$, k a nonnegative integer. First one observes that $Z_n(0) = 2^n$. It is also easy to check that[27] $Z_n(1/2) = \sum_i Q_i = 2\cdot 3^n$. More surprisingly, $Z_n(3/2) = \sum_i Q^3 = 54\cdot 7^{n-1}$. Such "sum rules", listed in the table 10.1, are consequence of the fact that the denominators on a given level are Farey sums of denominators on preceding levels[63, 19]. Regretably, we have not been able to extend this method to evaluating $q(-1/2)$, or to real τ.

A bound on D_H can be obtained by approximating (44) by

(46) $$Z_n(\tau) = n^{2\tau} + 2^n\rho^{2n\tau}.$$

In this approximation we have replaced all $\ell_{P/Q}$, except the widest interval $\ell_{1/n}$, by the narrowest interval ℓ_{F_{n-1}/F_n} (see (16)). The crossover from the harmonic dominated to the golden mean dominated behavior occurs at the τ value for which the two terms in (46) contribute equally:

$$D_n = \hat{D} + O\left(\frac{\ln n}{n}\right), \quad \hat{D} = \frac{\ln 2}{2 \ln \rho} = .72\ldots \tag{47}$$

For negative τ the sum (46) is the lower bound on the sum (30) , so $\hat{D}$ is a lower bound on D_H. The size of the level-dependent correction in (47) is ominous; the finite n estimates converge to the asymptotic value logarithmically. What this means is that the convergence is excruciatingly slow and cannot be overcome by any amount of brute computation.

τ	$2^{q(\tau)}$	$Z_n(\tau) =$
0	2	$2\, Z_{n-1}$
1/2	3	$3\, Z_{n-1}$
1	$(5+\sqrt{17})/2$	$5Z_{n-1} - 2Z_{n-2}$
3/2	7	$7\, Z_{n-1}$
2	$(11+\sqrt{113})/2$	$10Z_{n-1} + 9Z_{n-2} - 2Z_{n-3}$
5/2	$7+4\sqrt{6}$	$14Z_{n-1} + 47Z_{n-2}$
3	$26.20249\ldots$	$20Z_{n-1} + 161Z_{n-2} + 40Z_{n-3} - Z_{n-4}$
7/2	$41.0183\ldots$	$29Z_{n-1} + 485Z_{n-2} + 327Z_{n-3}$
$n/2$	ρ^n	ρ= golden mean

Table 10.1 Recursion relations for the Farey model partition sums (44) for $\tau = 1, 1/2, 1, \ldots, 7/2$; they relate the $2^{q(\tau)} = \lim_{n\to\infty} Z_{n+1}(\tau)/Z_n(\tau)$ to roots of polynomial equations.

10.13 Artuso Model

The Farey model (30) is difficult to control at the phase transition, but considerable insight into the nature of this non-analyticity can be gained by the following factorization approximation. Speaking very roughly, the stability $\Lambda \approx (-1)^n Q^2$ of a $P/Q = [a_1, \ldots, a_n]$ cycle gains a hyperbolic golden-mean factor $-\rho^2$ for each bounce in the central part of the Farey map (10), and a power-law factor for every a_k bounces in the neighborhood of the marginal fixed point $x_0 = 0$. This leads to an estimate of Q in $P/Q = [a_1, \ldots, a_n]$ as a product of the continued fraction entries[48]

$$Q \approx \rho^n a_1 a_2 \cdots a_n$$

In this approximation the cycle weights factorize, $\Lambda_{a_1 a_2 \ldots a_n} = \Lambda_{a_1}\Lambda_{a_2}\cdots\Lambda_{a_n}$, and the curvature corrections in the cycle expansion (38) vanish *exactly*:

$$1/\zeta(q,\tau) = 1 - \sum_{a=1}^{\infty} (\rho a)^{2\tau} z^a, \quad z = 2^{-q}$$

The $q = q(\tau)$ condition $1/\zeta(q,\tau) = 0$ yields

$$\rho^{-2\tau} = \Phi(-2\tau, z) \tag{48}$$

where Φ is the Jonquière function[64]

$$\Phi(s,x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \frac{t^{s-1}x}{e^t - x}$$

The sum (48) diverges for $z > 1$, so $q \geq 0$. The interesting aspect of this model, easy to check[48], is that the $q(\tau)$ curve goes to zero at $\tau = -D_H$, with all derivatives $d^n q/d\tau^n$ *continuous* at D_H, so the phase transition is of infinite order. We believe this to be the case also for the exact trivial and critical circle maps thermodynamics, but the matter is subtle and explored to more depth in ref. [30].

There is one sobering lesson in this: the numerical convergence acceleration methods of ref. [56] consistently yield finite gaps at the phase transition point; for example, they indicate that for the Farey model evaluated at $\tau = -D_H + \epsilon$, the first derivative converges to $dq/d\tau \to .64 \pm .03$. However, the phase transition is not of a first order, but logarithmic of infinite order[20], and the failure of numerical and heuristic arguments serves as a warning of how delicate such phase transitions can be.

10.14 Summary and Conclusions

The fractal set discussed here, the set of all parameter values corresponding to irrational windings, has no "natural" measure. We have discussed three distinct thermodynamic formulations: the *Farey series* (all mode-lockings with cycle lengths up to Q), the *Farey levels* (2^n mode-lockings on the binary Farey tree), and the *Gauss partitioning* (all mode-lockings with continued fraction expansion up to a given length). The thermodynamic functions are *different* for each distinct partitioning. The only point they have in common is the Hausdorff dimension, which does not depend on the choice of measure. What makes the description of the set of irrational windings considerably trickier than the usual "Axiom A" strange sets is the fact that here the range of scales spans from the marginal (harmonic, power-law) scalings to the the hyperbolic (geometric, exponential) scalings, with a generic mode-locking being any mixture of harmonic and exponential scalings. One consequence is that all versions of the thermodynamic formalism that we have examined here exhibit phase transitions. For example, for the continued fraction partitioning choice of weights t_p, the cycle expansions of ref. [22, 23] behave as hyperbolic averages only for sufficiently negative values of τ; hyperbolicity fails at the "phase transition"[19, 48] value $\tau = -1/3$, due to the power law divergence of the harmonic tails $\delta_{\dots n} \approx n^3$.

The universality of the critical irrational winding Hausdorff dimension follows from the universality of quadratic irrational scalings. The formulas used are formally identical to those used for description of dynamical strange sets[22], the deep difference being that here the cycles are not dynamical trajectories in the coordinate space, but renormalization group flows in the function spaces representing families of dynamical systems. The "cycle eigenvalues" are in present context the universal quadratic irrational scaling numbers.

In the above investigations we were greatly helped by the availability of the number theory models: in the $k = 0$ limit of (1) the renormalization flow is given by the Gauss map (6), for which the universal scaling δ_p reduce to quadratic irrationals. In retrospect, even this "trival" case seems not so trivial; and for the critical circle maps we are a long way from having a satisfactory theory. Symptomatic of the situation is the fact that while for the period doubling repeller D_H is known to 25 significant digits[54], here we can barely trust the first three digits.

The quasiperiodic route to chaos has been explored experimentaly in systems ranging from convective hydrodynamic flows[40] to semiconductor physics[41]. Such experiments illustrate the high precision with which the experimentalists now test the theory of transitions to chaos. It is fascinating that not only that the number-theoretic aspects of dynamics can be measured with such precision in physical systems, but that these systems are studied by physicists for reasons other than merely testing the renormalization theory or number theory. But, in all fairness, chaos via circle-map criticality is not nature's preferred way of destroying invariant tori, and the critical circle map renormalization theory remains a theoretical physicist's toy.

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