WEAK ERGODICITY AND PRODUCTS OF RANDOM MATRICES

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ABSTRACT

A representation for a weakly ergodic sequence of (non stochastic) matrices allows products of non-negative matrices which eventually become strictly positive to be expressed via products of some associated stochastic matrices and ratios of values of a certain function. This formula used in a random setup leads to a representation for the logarithm of a random matrix product. If the sequence of random matrices is in addition stationary then automatically almost all sequences are weakly ergodic, and the representation is expressed in terms of an one–dimensional stationary process. This permits properties of products of random matrices to be deduced from the latter. Second moment assumptions guarantee that central limit theorems and laws of the iterated logarithm hold for the random matrix products if and only if they hold for the corresponding stationary process. Finally, a central limit theorem for some classes of weakly dependent stationary random matrices is derived doing away with the restriction of boundedness of the ratios of column entries assumed by previous studies. Extensions beyond stationarity are discussed.

RANDOM MATRIX; STOCHASTIC MATRIX; WEAKLY ERGODIC; SPACE–TIME HARMONIC; STATIONARY; ERGODIC THEOREM; CENTRAL LIMIT THEOREM; STRONG MIXING; ρ–MIXING.

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0. INTRODUCTION Let $\{M_n\}$ be a sequence of finite order $p \times p$ nonnegative matrices, $M(i,j)$ the (i,j) entry of the matrix M and ${}^mM^n = M_m \cdots M_n$ for $m < n$. Assume that for each m all the entries of ${}^mM^n$ become strictly positive for n large enough.

We shall say (see e.g., Seneta [11]) that $\{M_n\}$ is *weakly ergodic* if for all m, i, j and l

$$\lim_{n\to\infty} \frac{{}^mM^n(i,j)}{{}^mM^n(l,j)} := \gamma_m(i,l). \tag{1}$$

(For stochastic $\{M_n\}$ weak ergodicity necessarily implies $\gamma_m(i,l) = 1$.) Simple manipulations show that for arbitrary k weak ergodicity yields that

$$h(m,i) := \lim_{n\to\infty} \frac{{}^mM^n(i,k)}{{}^0M^n(1,k)} = \left(\sum_{j=1}^{p} {}^0M^{m-1}(1,j)\, \gamma_m(j,i) \right)^{-1} \tag{2}$$

exists for all m and i, $0 < h(m,i) < \infty$, and

$$\sum_{j=1}^{p} M_n(i,j) h(n+1,j) = h(n,i). \tag{3}$$

A function h with the property (3) is said to be *space–time* harmonic for $\{M_n\}$. Up to a multiplicative constant, there is only one space–time harmonic function associated with a weakly ergodic $\{M_n\}$ (see [2]). It follows from (3) that the matrices $\{P_n\}$ with $P_n(i,j) = h(n+1,j)M_n(i,j)/h(n,i)$ are stochastic. Write ${}^mP^n = P_m \cdots P_n$. An important property implied by (3) is

$${}^mM^n(i,j) = \frac{h(m,i)}{h(n+1,j)}\, {}^mP^n(i,j) \tag{4}$$

for all m, n, i and j, which allows the study of products of nonnegative matrices to be carried out via stochastic matrices (see [2]).

Consider now a probability space $(\Omega, \mathcal{F}, P)$ with generic element ω and some $p \times p$ random matrices $\{X_n\}$. Denote the corresponding matrix products by $\{{}^mX^n\}$. Assume that almost all sequences $\{X_n\}$ are weakly ergodic and let $H(n,i)$ stand for the space–time harmonic function of $\{X_n\}$ which is also random, being a function of $X_0, X_1, \ldots$. Notice that

$$H(n,i)/H(n+1,j) = \lim_{s\to\infty} {}^nX^s(i,k)/{}^{n+1}X^s(j,k) \text{ a.s.} \tag{5}$$

Write $\{Y_n\}$ and $\{{}^mY^n\}$ for the attached random stochastic matrices. Simple manipulation of (4) in the random setup yield the representation

$$\log {}^0X^n(i,j) = \eta_0(i,l) + \sum_{m=0}^{n} \zeta_m(l) + \eta_{n+1}(l,j) + \log {}^0Y^n(i,j) \tag{6}$$

where $\eta_m(k,u) = \log(H(m,k)/H(m,u))$ and $\zeta_m(l) = \log(H(m,l)/H(m+1,l))$ for any l and $m = 0, 1 \ldots$. If we assume that $\{X_n\}$ is stationary then $\{\zeta_n(j)\}$ and $\{\eta_{n+1}(l,j)\}$ are also stationary. It will be seen that under suitable assumptions, η_n is negligible in limit properties obtained when dividing this representation by constants tending to ∞, and if $\{{}^0Y^n\}$ does not converge too rapidly to 0 then limit results for suitably normed $\{{}^mX^n\}$ turn out to be deducible from the stationary process $\{\zeta_n(j)\}$. For example, Kingman's subadditive ergodic theorem [9] implies

$$\lim_{n\to\infty} \log \|{}^0X^n\|/n = \gamma \text{ a.s.}$$

where the norm of the matrix $M = (M(i,j))$ is defined as $\|M\| = \max_{1\le i\le n} \sum_{j=1}^{p} |M(i,j)|$. This will be shown to follow from (6) and Birkhoff's ergodic theorem. We shall also prove central limit theorems for weakly dependent stationary random matrices which improves on the variants known in the literature.

1. WEAK ERGODICITY Before developing the main theme of the paper, we shall need to derive some criteria for weak ergodicity which may also be of independent interest.

Hajnal (see [4] or [9]) showed that for a sequence of strictly positive matrices $\{M_n\}$ weak ergodicity is guaranteed by

$$\sum_{n=1}^{\infty} \sqrt{\min_{i,j,k,l} \frac{M_n(i,k)\, M_n(j,l)}{M_n(i,l)\, M_n(j,k)}} = \infty. \tag{7}$$

We shall say (see e.g., Hajnal [4]) that the matrix M is *allowable* if each row and each column of it contains at least one strictly positive entry. A more general result of [4] asserts that for weak ergodicity of allowable $\{M_n\}$ it suffices to exist some n_0 such that

$$\sum_{n=1}^{\infty} \sqrt{\min_{i,j,k,l} \frac{{}^{n}M^{n+n_0}(i,k)\, {}^{n}M^{n+n_0}(j,l)}{{}^{n}M^{n+n_0}(i,l)\, {}^{n}M^{n+n_0}(j,k)}} = \infty. \tag{8}$$

Consider now the random matrices $\{X_n\}$. Then without assuming stationarity we get

PROPOSITION 1 *Suppose that $\{X_n\}$ are strictly positive with probabilty 1 and identically distributed. Then $\{X_n\}$ is a.s. weakly ergodic.*

Proof Choose an arbitrary positive ϵ. Then there must be some δ_1, δ_2 with $0 < \delta_1 < \delta_2 < \infty$ such that

$$P(\min_{i,j} X_n(i,j) > \delta_1, \max_{i,j} X_n(i,j) < \delta_2) > 1 - \epsilon.$$

It follows that

$$P(\min_{i,j} X_n(i,j) > \delta_1, \max_{i,j} X_n(i,j) < \delta_2 \text{ i.o.}) > 1 - \epsilon$$

where $\{A_n \text{ i.o.}\} = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$. To finish the proof notice that (7) holds on $\{\min_{i,j} X_n(i,j) > \delta_1, \max_{i,j} X_n(i,j) < \delta_2 \text{ i.o.}\}$ and that ϵ is arbitrary.

PROPOSITION 2 *Suppose that $\{X_n\}$ is a stationary sequence of nonnegative matrices and there exists a random index β such that ${}^{0}X^{\beta+m}$ is strictly positive with probability 1 for any $m \geq 0$. Then $\{X_n\}$ is a.s. weakly ergodic.*

Proof By stationarity, for any n there exists a random index β_n such that ${}^nX^{n+\beta_n+m}$ is a.s. strictly positive. This makes almost all $\{X_n\}$ allowable. Indeed, if X_n has a row of 0's so does ${}^nX^r$ for all r with $r \geq n$, and if X_r has a column of 0's so does ${}^nX^r$ for all n with $n \leq r$. Therefore on $\{X_n$ nonallowable i.o.$\}$ there are no strictly positive products ${}^nX^r$ and this in conjunction with stationarity prove that $\{X_n\}$ is almost surely allowable.

Fix now an $\epsilon > 0$ and choose a nonrandom m such that

$$P(\min_{i,j} {}^0X^m(i,j) > 0) > 1 - \epsilon/2.$$

This is possible for allowable $\{X_n\}$ since ${}^0X^n(i,j) > 0$ for all i, j implies ${}^0X^m(i,j) > 0$ for all i, j and $m \geq n$. Next choose $\delta_1,\ \delta_2$ with $0 < \delta_1 < \delta_2 < \infty$ such that

$$P(\min_{i,j} {}^0X^m(i,j) > \delta_1,\ \max_{i,j} {}^0X^m(i,j) < \delta_2) > 1 - \epsilon.$$

Now, by stationarity

$$P(\min_{i,j} {}^nX^{n+m}(i,j) > \delta_1,\ \max_{i,j} {}^nX^{n+m}(i,j) < \delta_2 \text{ i.o.}) > 1 - \epsilon$$

and an appeal to (8) finishes the proof.

2. THE REPRESENTATION AND SOME APPLICATIONS The main tool in this paper is the following representation for the logarithm of a random matrix product.

THEOREM 1 *Suppose that $\{X_n\}$ is a sequence of nonnegative random matrices such that for all m, i and j and almost all ω, ${}^mX^n(i,j)$ become strictly positive for n large enough (which may depend on ω and m). Assume also that almost all sequences $\{X_n\}$ are weakly ergodic. Then*

$$\log {}^0X^n(i,j) = \eta_0(i,l) + \sum_{m=0}^{n} \zeta_m(l) + \eta_{n+1}(l,j) + \log {}^0Y^n(i,j)$$

where $\eta_m(k,u)=\log(H(m,k)/H(m,u))$ *and* $\zeta_m(l)=\log(H(m,l)/H(m+1,l))$ *for any* l *and* $m=0,1\ldots$.

Proof It is easy to see that for all i,j,l, and $s>u_n$

$$\alpha(n)<{}^nX^s(i,j)/{}^nX^s(l,j)<(\alpha(n))^{-1} \tag{9}$$

where $\alpha(n)=\min_{i,j}{}^nX^{u_n}(i,j)/\max_{i,j}{}^nX^{u_n}(l,j)$ and u_n is a random integer such that $P(\min_{i,j}{}^nX^{u_n}(i,j)>0)=1$. In view of (9) and (5) H turns out to be a.s. strictly positive and finite and therefore the random variables η_n and ζ_n are well defined. Rearranging (4) (in the random setup) and taking the logarithm yields the representation.

LEMMA 1 *Suppose that* $\{X_n\}$ *is a stationary sequence satisfying the assumptions of Theorem 1 and* $E(|\log X_n(i,j)|^t)<\infty$ *for all* i *and* j *and some* $t\geq 1$.
Then $E(|\zeta_n(l)|^t)<\infty$.

Proof ¿From (9) with $\alpha(n)=\min_{i,j}X_n(i,j)/\max_{i,j}X_n(i,j)$ we get

$$(\max_l X_n(j,l))^{-1}\alpha(n+1)<{}^{n+1}X^s(i,k)/{}^nX^s(j,k)<(\min_l X_n(j,l))^{-1}(\alpha(n+1))^{-1}. \tag{10}$$

Notice that the bounds in (10) involve ratios of minima and maxima over a finite number of variables with finite t order moments of their logarithm. The logarithm of both numerators and denominators of such variables have finite t order moments. Using now (5) and the definition of ζ_n concludes the proof.

The following result is due to Kingman [9] who used subadditivity to improve on earlier work by Furstenberg and Kesten [3]. We shall derive a new proof of it based on (6).

THEOREM 2 *Suppose that* $\{X_n\}$ *is a stationary sequence of strictly positive random matrices and that* $E(|\log X_0(i,j)|)<\infty$ *for all* i *and* j. *Then there exists an a.s. finite shift–invariant random variable* γ, *which does not depend on* i *or* j, *such that*

$$\lim_{n\to\infty}\log{}^0X^n(i,j)/n=\gamma \text{ a.s.}$$

for all i and j.

Proof Weak ergodicity follows from Proposition 1. We shall divide the representation (6) by n and consider the four resulting terms. The first one being independent of n has no asymptotic contribution. In view of Lemma 1 and the ergodic theorem

$$\lim_{n\to\infty} n^{-1}\sum_{m=0}^{n}\zeta_m(l)=\gamma \text{ a.s.} \tag{11}$$

which takes care of the second term. As far as the third term is concerned, the convergent part of the Borell-Cantelli lemma applied to identically distributed variables $\{\eta_{n+1}(l,j)\}$ yields $\lim_{n\to\infty}\eta_{n+1}(l,j)/n=0$ a.s. . Choose now a sequence of random entries $\{x_n\}$ such that ${}^0Y^n(i,x_n)=\max_j {}^0Y^n(i,j)$. Then

$$\liminf_{n\to\infty}\log {}^0Y^n(i,j)/n \geq \liminf_{n\to\infty}(\log {}^0Y^{n-1}(i,x_{n-1})/n+\log Y_n(x_{n-1},j)/n). \tag{12}$$

Since $p^{-1}\leq {}^0Y^{n-1}(i,x_{n-1})\leq 1$, the first term on the right hand side of (12) has no asymptotic contribution. The second term is larger than $\min_i \log Y_n(i,j)/n$. But $\{\min_i \log Y_n(i,j)\}$ are identically distributed and by an argument based on (10), (4) and (5) have finite expectation. This term, when devided by n, tends a.s. to 0 as $n\to\infty$ again by the convergent part of the Borel–Cantelli lemma. Because $\{{}^0Y^n\}$ are stochastic $\limsup_{n\to\infty}\log {}^0Y^n(i,j)/n\leq 0$ a.s. holds and therefore $\lim_{n\to\infty}\log {}^0Y^n(i,j)/n=0$ a.s. Therefore a.s. convergence follows. Notice now that the limit in the Theorem does not depend on i or j as γ in (11) pertains only to $\{\zeta_n(l)\}$ where l is arbitrarily chosen.

The next result shows that under second moment assumptions limit properties involving norming constants of the order of $\sqrt{n}$ or higher are deducible from the corresponding results for the stationary process $\{\zeta_n(l)\}$.

THEOREM 3 *Suppose that $\{X_n\}$ is a stationary sequence of strictly positive random matrices and that $E(\log X_0(i,j))^2<\infty$ for all i and j. Then for all i, j and l*

$$\lim_{n\to\infty}|\log {}^0X^n(i,j)-\sum_{m=0}^{n}\zeta_m(l)|/(\sqrt{n})=0 \text{ a.s.} \tag{13}$$

Proof   It will suffice to show that when divided by $\sqrt{n}$ the two η-terms and the last term of the representation (6) has a.s. no asymptotic contribution. This can be done as in the proof of Theorem 2 taking into account that $\{U_n/\sqrt{n}\}$ converges a.s. to 0 as $n \to \infty$ for any sequence of identically distributed random variables $\{U_n\}$ with finite second moments.

As applications of Theorem 3 we refer to the central limit theorem which we shall consider under some mixing conditions in Section 4 and the law of the iterated logarithm.

3. MATRICES WITH NONNEGATIVE ENTRIES

Although in the above applications of Theorem 1 we assumed strictly positive and stationary random matrices, the representation (6) applies to arbitrary weakly ergodic nonnegative matrices. Consider for example the conditions of the type of Heyde [6] and Heyde and Cohen [7].

(A1)   There exists an integer n_0 such that any product $X_{j+1} \dots X_{j+n_0}$ of n_0 of the matrices has all its elements strictly positive with probability one.

(A2)   For some constant C, $1 < C < \infty$ and each matrix X_j

$$1 \leq M(X_j)/m(X_j) \leq C$$

with probability one, where $M(X)$ is the maximum of elements of X and $m(X)$ is the minimum of *strictly positive* elements of X.

(In fact Cohen and Heyde deal with reversed order products, which by a transposition becomes equivalent to the case considered here.) It is easy to see that (A1) and (A2) guarantee weak ergodicity at a geometric rate by the Coale-Lopez theorem (see [4] and [9]). The asymptotic representation of Heyde [5] for an ergodic stationary sequence $\{X_n\}$ follows easily from Theorem 1 and geometric weak convergence. As a matter of fact, we get a stronger statement without moment conditions or even stationarity as expressed by the following.

THEOREM 4 *Suppose that $\{X_n\}$ is a sequence of nonnegative matrices, and that (A1) and (A2) hold. Then there exist some finite, positive constants K_1 and K_2 such that for $n > n_0$*

$$\log {}^0X^n(i,j) = \sum_{m=0}^{n} \zeta_m(l) + \alpha_n(i,j)$$

with $K_1 \leq \alpha_n(i,j) \leq K_2$ for all i, j and l.

Proof It suffices to notice that the first and the last two terms in the right hand side of the representation (6) are bounded. Indeed, $C^{-n_0} \leq \eta_m(k,u) \leq C^{n_0}$ for m large enough and any k and u follows from (A1) and (A2). On the other hand, for n large enough, $C_1 \leq {}^0Y^n(i,j) \leq 1$ with $C_1 = (1/p)^{n_0} C^{n_0(n_0+1)}$ may be argued as in the proof of Theorem 4 of [2] in the light of the stochasticity of $\{{}^0Y^n\}$.

Theorem 4 improves on the asymptotic representation for $\log {}^0X^n(i,j)$ given in Heyde [6].

Next we show that conditions considerably less restrictive than (A2) may be considered for properties of the type given above. We choose to prove only a variant of Theorem 3.

THEOREM 5 *Suppose that $\{X_n\}$ is a stationary sequence of nonnegative random matrices, (A1) holds, $E((\log m(X_1))^2) < \infty$ and $E((\log M(X_1))^2) < \infty$. Then for all i, j and l*

$$\lim_{n\to\infty} |\log {}^0X^n(i,j) - \sum_{m=0}^{n} \zeta_m(l)|/(\sqrt{n}) = 0 \text{ a.s.} \tag{14}$$

Proof We shall show that (13) still holds arguing along the lines of the proof of Theorem 3. Weak ergodicity for $\{X_n\}$ follows from Proposition 2 applied to the sequence $\{X_n\}$. Notice that

$$p^{-n_0} \frac{m(X_n)}{M(X_n)} \cdots \frac{m(X_{n+n_0})}{M(X_{n+n_0})} \leq {}^nX^s(i,j)/{}^nX^s(l,j) \leq p^{n_0} \frac{M(X_n)}{m(X_n)} \cdots \frac{M(X_{n+n_0})}{m(X_{n+n_0})} \tag{15}$$

for all i, j, l and $s > n + n_0$. From this we also get

$$p^{-n_0} m(X_n) \frac{m(X_{n+1})}{M(X_{n+1})} \cdots \frac{m(X_{n+n_0})}{M(X_{n+n_0})} \leq {}^{n}X^{s}(i,j)/{}^{n+1}X^{s}(l,j)$$
$$\leq p^{n_0+1} M(X_n) \frac{M(X_{n+1})}{m(X_{n+1})} \cdots \frac{M(X_{n+n_0})}{m(X_{n+n_0})} \tag{16}$$

for all i, j, l and $s > n + n_0 + 1$. It is easy to see that in view of (15) and (16) the assertion of Lemma 1 with $t = 2$ holds if its assumptions are replaced by the moment conditions of Theorem 5. On the other hand if in the analogue of (12) for Theorem 5 (i.e., with n substituted by $\sqrt{n}$ in the denominators) we replace ${}^{0}Y^{n-1}$ by ${}^{0}Y^{n-n_0-1}$ and Y_n by ${}^{n-n_0}Y^{n}$, and take into account the inequality

$${}^{n-n_0}Y^{n}(i,j) \geq \frac{H(n-n_0,i)}{H(n-n_0,j)} \frac{H(n-n_0,j)}{H(n-n_0+1,j)} \cdots \frac{H(n+1,j)}{H(n,j)} m(X_{n-n_0}) \cdots m(X_n)$$

we deduce the finiteness of the second moment of $\log {}^{n-n_0}Y^{n}(i,j)$, which in conjunction with the Borel–Cantelli lemma and the stationarity of $\{\log {}^{n-n_0}Y^{n}(i,j)\}$ implies (14).

Theorem 5 is an improvement of the result given in Heyde [6] as it does away with condition (A2).

Notice that if (A2) holds the conditions $E(|\log M(X_1)|^t) < \infty$ and $E(|\log m(X_1)|^t) < \infty$ are equivalent, in which case the latter suffices for the assertion of Theorem 5.

4. WEAKLY DEPENDENT STATIONARY MATRICES

In this last section we shall prove a central limit theorem for products of stationary sequences of matrices satisfying one of two sets of mixing conditions. Consider a sequence $\{\alpha_n\}$ of nonnegative integers such that $\lim_{n\to\infty} \alpha_n = 0$. Write $\mathcal{M}_m^n$ for the σ–field generated by $X_m, \ldots, X_n$, $\mathcal{M}_{-\infty}^n$ for the σ–field generated by $\ldots, X_n$, and $\mathcal{M}_n^\infty$ for the σ–field generated by $X_n, \ldots$. We shall say that the sequence $\{X_n\}$ is strongly mixing if for each k and n, $A \in \mathcal{M}_{-\infty}^k$ and $B \in \mathcal{M}_{n+k}^\infty$

$$|P(A \cap B) - P(A)P(B)| \leq \alpha_n.$$

We shall say that the sequence $\{X_n\}$ is $\rho-$mixing if $\lim_{n\to\infty}\rho_n = 0$ where

$$\rho_n := \sup_{m, f\in L_2(\mathcal{M}_{-\infty}^m), g\in L_2(\mathcal{M}_{m+n}^\infty)} |\mathrm{Corr}(f,g)|.$$

Then for every $f \in L_2(\mathcal{M}_{-\infty}^m)$ and $g \in L_2(\mathcal{M}_{m+n}^\infty)$ we have

$$|\mathrm{Cov}(f,g)| \le \rho_n \|f\|_2 \|g\|_2 \tag{21}$$

and for every $f \in L_p(\mathcal{M}_{-\infty}^m)$ and $g \in L_q(\mathcal{M}_{m+n}^\infty)$ with $1 < p, q \le \infty$ such that $1/p+1/q < 1$ we have (see Hall and Heyde [5], Corollary A.2 p.278)

$$|\mathrm{Cov}(f,g)| \le 8\alpha_n^{1-1/p-1/q} \|f\|_p \|g\|_q. \tag{22}$$

In what follows $\xrightarrow{d}$ denotes convergence in distribution and $N(0,1)$ is the standard normal law. We shall prove the following.

THEOREM 6 *Suppose that $\{X_n\}$ is a strictly stationary sequence of positive $p\times p$ matrices such that one of the following conditions holds.*

(i) For some $\delta > 0$, $\sum_{n=1}^\infty \alpha_n^{\delta/(2+\delta)} < \infty$ and $E|\log(X_1(i,j))|^{2+\delta} < \infty$ for every i and j.

(ii) $\sum_{n=1}^\infty \rho_n < \infty$ and, $E|\log(X_1(i,j))|^2 < \infty$ for every i and j.

Then there are two constants, $\sigma \ge 0$ and γ such that for all i and j

$$n^{-1/2}\{\log {}^0X^n(i,j) - n\gamma\} \xrightarrow{d} N(0,\sigma).$$

where we interpret $N(0,0)$ as a point mass at 0.

This theorem improves on the results in Ishitani [8] in several directions. First, the condition (1) from his Theorem 2 (i.e., $1 \le \max_{i,j} X_1(i,j)/\min_{i,j} X_1(i,j) \le C$; $C > 0$) is removed. Second, our theorem uses a more general mixing coefficient ρ instead of ϕ so that a similar result holds true for a so called ϕ-mixing sequence. (It is well known that $\rho_n \le 2\phi_n^1/2$; see Bradley [1] for a survey of basic properties of mixing coefficients.)

For the proof of this result we shall need a theorem due to Gordin (see e.g., [5], Theorem 5.4, p.136).

THEOREM A *Let* $\{X_n = X_0(T^n\omega)\}$ *be a stationary and ergodic sequence with* $E|X_0| < \infty$ *and* $EX_0 = 0$. *Put* $S_n = \sum_{k=1}^n X_k$. *Suppose* $\mathcal{M}_0$ *is a sub* σ*-field of* $\mathcal{F}$ *with* $\mathcal{M}_0 \subset T^{-1}(\mathcal{M}_0)$, *and set* $\mathcal{M}_k = T^{-k}(\mathcal{M}_0)$. *If*

(i)

$$\sum_{k=0}^{\infty} E|E(X_0|\mathcal{M}_{-k})| < \infty$$

(ii)

$$\sum_{k=0}^{\infty} E|X_0 - E(X_0|\mathcal{M}_k)| < \infty$$

and

(iii)

$$\limsup_{n\to\infty} n^{-1/2} E|S_n| < \infty$$

hold, then $\lim_{n\to\infty} n^{-1/2}E|S_n| = \lambda$, *for some* λ *with* $0 \le \lambda < \infty$ *and*

$$n^{-1/2}S_n \xrightarrow{d} N(0, \pi\lambda^2/2).$$

We shall next consider the Birkhoff coefficient $b(M)$ (for an account see e.g., Seneta [11]) defined for a nonnegative matrix M by

$$b(M) = \frac{1 - \Phi^{1/2}(M)}{1 + \Phi^{1/2}(M)}$$

where $\Phi(M) = 0$ if $M(i,j) = 0$ for some i,j; and otherwise

$$\Phi(M) = \min_{i,j,k,l} \frac{M(i,k)\,M(j,l)}{M(i,l)\,M(j,k)}.$$

It can be easily concluded that for any sequence of identically distributed random variables $u_1, u_2, \ldots$ with $0 \le u_n < 1$, $u_1 \cdots u_n$ tends a.s. and in L_1 to 0 as $n \to \infty$, and as a consequence $b(X_1, \ldots, X_n) \to 0$ a.s. and in L_1.

It is known that b is a contraction coefficient i.e., $b(ST) \leq b(S)b(T)$ for any non–negative $p \times p$ matrices S and T. This property will turn out to be useful in the proof of the following.

LEMMA 2 *Let $M_0, \ldots, M_n$ be a sequence of strictly positive $p \times p$ matrices. Then*

$$|\log \Phi(M_0 \cdots M_n)| \leq 4|\log(1 - b(M_0))|b(M_1) \cdots b(M_n).$$

Proof Simple manipulations lead to

$$\Phi(M) = \frac{(1 - b(M))^2}{(1 + b(M))^2}. \tag{23}$$

Thus

$$\log \Phi(M_0 \cdots M_n) = 2(\log(1 - b(M_0 \cdots M_n)) - \log(1 + b(M_0 \cdots M_n))). \tag{24}$$

On the other hand, the elementary inequality

$$|\log(1 - x)| \geq \log(1 + x)$$

for $0 < x \leq 1$ applied to (24) yields

$$|\log \Phi(M_0 \cdots M_n)| \leq 4|\log(1 - b(M_0 \cdots M_n))|. \tag{25}$$

Also, it is easy to see that the inequality

$$|\log(1 - xy)| \leq |y \log(1 - x)|$$

for $0 < x, y \leq 1$ used in (25) in conjunction with the contraction property of b leads to

$$|\log(1 - b(M_0 \cdots M_n))| \leq |\log(1 - b(M_0))|b(M_1) \cdots b(M_n). \tag{26}$$

Combining (25) and (26) completes the proof.

Take for definiteness $l = 1$ and write ζ_m for $\zeta_m(1)$ and $\zeta_{0,n} = \log \frac{{}^0X^n(1,1)}{{}^1X^n(1,1)}$.

LEMMA 3 *Suppose that $\{X_n\}$ is a strongly mixing strictly stationary sequence of positive $p \times p$ matrices such that for some $\delta > 0$, $\sum_{n=1}^{\infty} \alpha_n^{\delta/(2+\delta)} < \infty$ and $\| \log \Phi(X_1)\|_{2+\delta} < \infty$. Then*

$$\sum_{n=1}^{\infty} E|\log \Phi(X_1 \cdots X_n)| < \infty.$$

Proof By Lemma 2, the Holder inequality and stationarity,

$$\begin{aligned} E|\log \Phi(X_1 \cdots X_n)| &\le 4E|\log(1 - b(X_1))b(X_2 \cdots X_n)| \\ &\le 4\|\log(1 - b(X_1))\|_{2+\delta} \|b(X_1 \cdots X_{n-1}\|_{(2+\delta)/(1+\delta)}. \end{aligned} \tag{27}$$

It is easy to see that $\| \log \Phi(X_1)\|_{2+\delta} < \infty$ is equivalent to $\| \log(1 - b(X_1))\|_{2+\delta} < \infty$. Now because $Eb(X_1 \cdots X_n) \to 0$, we can find and fix and integer $q \ge 1$ such that $E|b(X_1 \cdots X_q)| < 1/4$. Write $r = r(n) = [n/\log_2 n]$ and let l be the largest integer m with the property $m(q + r) \le n$. Notice that $l = 0(\log_2 n)$ as $n \to \infty$. Denote $Z_i = b(X_{i(q+r)+1} \cdots X_{i(q+r)+q})$ for $i = 0, 1, \ldots, l - 1$. Since b is contractive, positive and bounded by 1, we have

$$Eb(X_1 \cdots X_n)^{(2+\delta)/(1+\delta)} \le E(Z_1 \cdots Z_l).$$

Now we apply (22) to the case $p = q = \infty$ and get, by iterating and using stationarity

$$\begin{aligned} E(Z_1 \cdots Z_l) &\le E(Z_1 \cdots Z_{l-1})E(Z_l) + 8\alpha_r \\ &\le (EZ_1)^l + (EZ_1)^{l-1} 8\alpha_r + \cdots + (EZ_1)8\alpha_r + 8\alpha_r \\ &\le (EZ_1)^l + 8\alpha_r/(1 - EZ_1). \end{aligned} \tag{28}$$

whence by the definition of q, r and l

$$\begin{aligned} E(Z_1 \cdots Z_l) &\le (1/4)^{\log_2 n} + (32/3)\alpha_{[n/\log_2 n]} \\ &= n^{-2} + (32/3)\alpha_{[n/\log_2 n]}. \end{aligned} \tag{29}$$

By (28) and (29)

$$\|b(X_1\cdots X_n)\|_{(2+\delta)/(1+\delta)} \le n^{-2(1+\delta)/(2+\delta)} + ((4/3)\alpha_{[n/\log_2 n]})^{(1+\delta)/(2+\delta)}.$$

Because α_n is decreasing, from the summability condition imposed on $\{\alpha_n\}$ we get $\alpha_m = 0(m^{-(2+\delta)/\delta})$ as $m \to \infty$. Using now (27) we complete the proof.

LEMMA 4 *Suppose that $\{X_n\}$ is a strictly stationary ρ-mixing sequence of positive $p\times p$ matrices such that $E(\log\Phi(X_1))^2 < \infty$. Then, for every positive integer m, there is a constant K_m such that $E|\log\Phi(X_1\cdots X_n)| \le K_m n^{-m}$ for every $n \ge 1$.*

Proof Because $Eb(X_1\cdots X_n) \to 0$ and $\rho_n \to 0$, for a given integer m we can find p and q sufficiently large such that

$$\log(E(b(X_1\cdots X_p) + \rho_q) \le -2m. \tag{30}$$

Now for n large enough we take r to be the largest integer k having the property $2^k \le n/(p+q)$. Write $Z_{i+1} = b(X_{i(p+q)+1}\cdots X_{i(p+q)+p})$ for $i = 0, 1, 2, \ldots, 2^r - 1$. It is easy to see by (23) that $E(\log\Phi(X_1))^2 < \infty$ is equivalent to $E(\log(1 - b(X_1))^2 < \infty$. By applying Lemma 1 to the random matrices $\{X_n\}$ and the Cauchy-Schwartz inequality we get

$$\begin{aligned} E|\log\Phi(X_1\cdots X_n)| &\le 4E|\log(1-b(X_1))|b(X_2\cdots X_n) \\ &\le 4\|\log(1-b(X_1))\|_2\|b(X_2\cdots X_n)\|_2 \end{aligned}$$

whence by the contraction property of b and $0 \le b(M) \le 1$ for every M

$$E|\log\Phi(X_1\cdots X_n)| \le 4\|\log(1-b(X_1)\|_2(E(Z_1\cdots Z_{2^r}))^{1/2}. \tag{31}$$

Now by (21) and stationarity

$$\begin{aligned} E(Z_1\cdots Z_{2^r}) &\le E(Z_1\cdots Z_{2^{r-1}})E(Z_{2^{r-1}+1}\cdots Z_{2^r}) \\ &\quad +\rho_q\|Z_1\cdots Z_{2^{r-1}}\|_2\|Z_{2^{r-1}+1}\cdots Z^{2^r}\|_2 \\ &\le E(Z_1\cdots Z_{2^{r-1}})(E(Z_1\cdots Z_{2^{r-1}}) + \rho_q). \end{aligned}$$

By recurrence

$$E(Z_1 \cdots Z_{2^r}) \leq EZ_1(EZ_1 + \rho_q) \cdots (E(Z_1 \cdots Z_{2^{r-1}}) + \rho_q) \leq (EZ_1 + \rho_q)^r.$$

Now by (30) and the definition of r

$$E(Z_1 \cdots Z_{2^r}) \leq n^{-2m}$$

for n sufficiently large, and the desired result now follows from (31).

LEMMA 5 *Under the assumptions of Theorem 4*

$$|\zeta_0 - \zeta_{0,n}| \leq |(\log \Phi(X_1 \cdots X_n))|.$$

Proof Notice that

$$\min_j \frac{{}^0X^n(1,j)}{{}^1X^n(1,j)} \leq \frac{{}^0X^n(1,1)}{{}^1X^n(1,1)} \leq \max_j \frac{{}^0X^n(1,j)}{{}^1X^n(1,j)}.$$

On the other hand

$$\frac{H(0,1)}{H(1,1)} = \frac{\sum_j {}^0X^n(1,j)H(n+1,j)}{\sum_j {}^1X^n(1,j)H(n+1,j)}$$

which implies

$$\min_j \frac{{}^0X^n(1,j)}{{}^1X^n(1,j)} \leq \frac{H(0,1)}{H(1,1)} \leq \max_j \frac{{}^0X^n(1,j)}{{}^1X^n(1,j)}.$$

Thus

$$\begin{aligned}
&|\log \frac{H(0,1)}{H(1,1)} - \log \frac{{}^0X^n(1,1)}{{}^1X^n(1,1)}| \\
&\leq \log \left[\max_j \frac{{}^0X^n(1,j)}{{}^1X^n(1,j)} \Big/ \min_{j'} \frac{{}^0X^n(1,j')}{{}^1X^n(1,j')} \right] \\
&= \log \left[\max_j \sum_k X_0(1,k) \frac{{}^1X^n(k,j)}{{}^1X^n(1,j)} \Big/ \min_{j'} \sum_k X_0(1,k) \frac{{}^1X^n(k,j')}{{}^1X^n(1,j')} \right] \\
&\leq \log \max_{j,j',k} \frac{{}^1X^n(k,j)\,{}^1X^n(1,j')}{{}^1X^n(k,j')\,{}^1X^n(1,j)} \\
&\leq -\log \min_{i,j,j',k} \frac{{}^1X^n(k,j)\,{}^1X^n(i,j')}{{}^1X^n(k,j')\,{}^1X^n(i,j)} \\
&= -\log \Phi(X_1 \cdots X_n)
\end{aligned}$$

which finishes the proof.

LEMMA 6 *Suppose that the assumptions of Theorem 6 hold. Then there exists a finite constant $\sigma \geq 0$ such that*

$$\lim_{n\to\infty} \frac{Var\left(\sum_{i=0}^{n} \zeta_i\right)}{n} = \sigma^2.$$

Proof We first establish this Lemma under condition (i) of the Theorem 6. Notice first that (22) implies

$$|E(\zeta_{0,[j/3]})(\zeta_j-\gamma)| \leq \alpha_{[2j/3]}^{\delta/(2+\delta)} \|\zeta_{0,[j/3]}\|_{2+\delta}\|\zeta_j-\gamma\|_{2+\delta} \leq \|\zeta_j-\gamma\|_{2+\delta}(\|\zeta_j-\gamma\|_{2+\delta}+C)$$

where C is a constant to be identified bellow. Once again by (22) we get

$$\begin{aligned} E|\log\Phi(X_1\cdots X_{[j/3]})|\,|\zeta_j-\gamma| \leq E|\log\Phi(X_1\cdots X_{[j/3]})|E|\zeta_j-\gamma| \\ +\alpha_{[2j/3]}^{\delta/(2+\delta)}\|\log\Phi(X_1\cdots X_{[j/3]})\|_{2+\delta}\|\zeta_j-\gamma\|_{2+\delta}. \end{aligned}$$

By the integrability condition of Theorem 1, (ii) and Lemma 1 we get

$$\|\log\Phi(X_1\cdots X_{[j/3]})\|_{2+\delta} \leq 4\|\log(1-b(X_1))\|_{2+\delta} = C < \infty$$

and the result now follows by Lemma 4 and (22).

To prove (ii) we start by estimating the covariance of ζ_0 and ζ_j. Let j be an integer with $j \geq 6$. Denote $\gamma = E\zeta_0$. We have

$$\begin{aligned} &|\mathrm{Cov}(\zeta_0,\zeta_j)| = |E\zeta_0(\zeta_j-\gamma)| = \\ &\leq |E(\zeta_0-\zeta_{0,[j/3]})(\zeta_j-\gamma)| + |E(\zeta_{0,[j/3]})(\zeta_{-}\gamma)|. \end{aligned} \tag{32}$$

Notice that ζ_j is $\mathcal{M}_{j+1}^{\infty}$ measurable and $\zeta_{0,[j/3]}$ is $\mathcal{M}_0^{[j/3]}$ measurable. By (21) and stationarity we get

$$|E(\zeta_{0,[j/3]})(\zeta_j-\gamma)| \leq \rho_{[2j/3]}\sqrt{2\mathrm{Var}\zeta_0(\mathrm{Var}\zeta_0+C_1^2)}.$$

where C_1 will be further identified. By Lemma 5 and once again by (21)

$$\begin{aligned}|E(\zeta_0 - \zeta_{0,[j/3]})(\zeta_j - \gamma)| &\le E|\log\Phi(X_1\cdots X_{[j/3]})||\zeta_j - \gamma| \\ &\le E|\log\Phi(X_1\cdots X_{[j/3]})|E|\zeta_j - \gamma| \\ &+ \rho_{[j/3]}\|\log\Phi(X_1\cdots X_{[j/3]})\|_2\sqrt{\mathrm{Var}\zeta_0}.\end{aligned}$$

The integrability condition of Theorem 6(i) implies $\|\log(\Phi(X_1)\|_2 < \infty$ which in turn implies $\|\log(1-b(X_1))\|_2 < \infty$. The last relation and Lemma 2 give $\|\log\Phi(X_1\cdots X_{[j/3]}\|_2 \le C_1$ for every $j \ge 1$ where C_1 a positive constant. By Lemma 3 we can find another positive constant C_2 such that

$$E|\log\Phi(X_1\cdots X_{[j/3]})| \le C_2[j/3]^{-2}$$

for every $j \ge 1$. All these considerations and (32) lead to

$$|\mathrm{Cov}(\zeta_0, \zeta_j)| \le C_3(\rho_{[2j/3]} + [j/3]^{-2})$$

for a positive constant C_3 and for every $j \ge 1$. This prove that the series $\sigma^2 := \mathrm{Var}\zeta_0 + 2\sum_{j=1}^{\infty}\mathrm{Cov}(\zeta_0, \zeta_j)$ is absolutely convergent.

By stationarity

$$\mathrm{Var}(\sum_{i=0}^{n-1}\zeta_i) = n\mathrm{Var}(\zeta_0) + 2\sum_{i=0}^{n}(n-j)\mathrm{Cov}(\zeta_0, \zeta_j)$$

and therefore $n^{-1}\mathrm{Var}(\sum_{i=0}^{n-1}\zeta_i) \to \sigma^2$ as $n \to \infty$. This concludes the proof of Lemma 6 under (ii).

Proof of Theorem 6 We shall apply Theorem A to the stationary sequence $\{\zeta_n\}$. It is well known (see Hall and Heyde [5], p.139) that

$$E|E(\zeta_0 - \gamma)|\mathcal{M}_{-\infty}^{-k})| \le 6\alpha_k^{(1+\delta)/(2+\delta)}\|\zeta_0 - \gamma\|_{2+\delta}. \tag{33}$$

Similarly for the ρ-mixing processes we get

$$\|(\zeta_0 - \gamma|\mathcal{M}_{-\infty}^{-k})\|_2^2 \le \rho_k\|\zeta_0 - \gamma\|_2^2$$

and therefore

$$E|E(\zeta_0 - \gamma|\mathcal{M}_{-\infty}^{-k})| \le E\|(\zeta_0 - \gamma|\mathcal{M}_{-\infty}^{-k})\|_2 \le \rho_k\|\zeta_0 - \gamma\|_2. \tag{34}$$

The relations (33) and (34) show that under the conditions (i) and (ii) of Theorem 6, the condition (i) of Theorem A is satisfied.

In order to verify the condition (ii) of Theorem A we notice that

$$\begin{aligned} E|\zeta_0 - E(\zeta_0|\mathcal{M}_{-\infty}^{k})| &\le E|\zeta_0 - \zeta_{0,k}| + E|\zeta_{0,k} - E(\zeta_0|\mathcal{M}_{-\infty}^{k})| \\ &\le 2E|\zeta_0 - \zeta_{0,k}|. \end{aligned}$$

The condition (ii) of Theorem A is now verified by Lemma 5 in conjunction with Lemmas 3 and 4 whereas the condition (iii) is a direct consequence of Lemma 6. This concludes the proof.

Using Theorem 5 and the above Lemmas we can obtain central limit theorems for weakly dependent matrices satisfying the less restrective condition (A1).

THEOREM 7 *Suppose that $\{X_n\}$ is a strictly stationary sequence of nonnegative $p \times p$ matrices satisfying (A1), and one of the following conditions.*

(i) For some $\delta > 0$, $\sum_{n=1}^{\infty} \alpha_n^{\delta/(2+\delta)} < \infty$, $E|\log m(X_0)|^{2+\delta} < \infty$ and

$E|\log M(X_0)|^{2+\delta} < \infty$.

(ii) $\sum_{n=1}^{\infty} \rho_n < \infty$, $E|\log m(X_0)|^2 < \infty$ and $E|\log M(X_0)|^2 < \infty$.

Then there are two nonnegative constants, σ and $\gamma > 0$ such that for all i and j

$$n^{-1/2}\{\log {}^0X^n(i,j) - n\gamma\} \xrightarrow{d} N(0,\sigma).$$

Proof It is easy to see that the proof of the Theorem 6 can be adapted to this case. Indeed, we have already proved an analogue to Theorem 3 (Theorem 5). On the other hand, the validity of Lemma 3 with strict positivity replaced by (A1) can be seen from the fact that $\{{}^{n_0 n}X^{n_0(n+1)-1}\}$ satisfies the original conditions with the same ϕ–sequence, and the monotonicity in n of $\{\Phi(X_1, \ldots, X_n)\}$. Other components of the proof can be similarly handled.

REMARK Notice that Theorem 1 does not require any restriction on the structure of $\{X_n\}$. Cases other than stationary e.g., as independent but nonidentically distributed random matrices and various kinds of dependence may be considered as well.

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