Square ice, alternating sign matrices, and classical orthogonal polynomials

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Abstract

The six-vertex model with domain wall boundary conditions, or square ice, is considered for particular values of its parameters, corresponding to 1-, 2-, and 3-enumerations of alternating sign matrices (ASMs). Using Hankel determinant representations for the partition function and the boundary correlator of homogeneous square ice, it is shown how the ordinary and refined enumerations can be derived in a very simple and straightforward way. The derivation is based on the standard relationship between Hankel determinants and orthogonal polynomials. For the particular sets of parameters corresponding to 1-, 2-, and 3-enumerations of ASMs, the Hankel determinant can be naturally related to Continuous Hahn, Meixner-Pollaczek, and Continuous Dual Hahn polynomials, respectively. This observation allows for a unified and simplified treatment of ASMs enumerations. In particular, along the lines of the proposed approach, we provide a complete solution to the long standing problem of the refined 3-enumeration of AMSs.

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1. Introduction

The six-vertex model on a square lattice with domain wall boundary conditions (DWBC) was introduced in [1] and subsequently solved in [2], where a determinant formula for the partition function was obtained and proven (see also [3]). Analogous determinant formulae has been given also for the boundary one point correlation functions [4]. The model, in its inhomogeneous formulation, i.e., with position-dependent Boltzmann weights, was originally proposed within the theory of correlation functions of quantum integrable models, in the framework of the quantum inverse scattering method [5]. The model was later found to be deeply related with the problems of enumeration of alternating sign matrices (ASMs) [6–11] and domino tilings [12–15]. It should be mentioned that ASM enumerations appear to be in turn deeply related with quantum spin chains and some loop models, via Razumov-Stroganov conjecture [16]; for recent works, see for instance Ref. [17] and references therein.

In its homogeneous version, the six-vertex model with DWBC admits usual interpretation as a model of statistical mechanics with fixed boundary conditions, and it may be seen as a variation of the original model with periodic boundary conditions [18–21]. The latter was originally proposed as a model for two-dimensional ice (hence the alternative denomination: 'square ice'), and has been for decades a paradigmatic one in statistical mechanics [22]. Till now, specific results for the six-vertex model with DWBC at particular values of its parameters (in fact, mainly derived within the context of ASMs) were obtained from general results for the inhomogeneous version, first specializing the parameters to the considered case, and then performing the homogeneous limit at the very end, and hence once for each particular case. Each time, the homogeneous limit was an hard task on its own right, and a specific approach was devised to work it out in each single case [10, 23, 24].

The purpose of the present paper is to explain how one can proceed the other way around, first performing the homogeneous limit once for all for the model with generic vertex weights, and then specializing the result to the case of interest. The homogeneous limit has already been done in [3] for the partition function, and in [4] for the boundary one point correlation functions, resulting in Hankel determinant representations for these quantities. Here we show how, by specializing the parameters to some particular values, and exploiting the standard relationship between Hankel determinants and orthogonal polynomials, all previously known results concerning ordinary and refined enumerations can be derived in a very simple and straightforward way. It appears that for the particular sets of parameters corresponding to 1-, 2-, and 3-enumerations of ASMs, the Hankel determinant is naturally related to Continuous Hahn, Meixner-Pollaczek, and Continuous Dual Hahn polynomials, respectively. The approach which we propose here allows for a unified and simplified treatment of ASMs enumerations, including the problem of refined 3-enumeration.

The paper is organized as follows. In the next Section we recall the definition of the model and some related results, with particular attention to the Hankel determinant representations for the partition function and the one-point boundary correlator. In Section 3, known results on ASMs are reviewed, together with their connection with the square ice. In Section 4, we show how the Hankel determinant entering the partition function of square ice can be reinterpreted, in the three cases corresponding to 1-, 2- and 3-enumeration of ASMs, as the Gram determinant for an appropriate choice of Continuous Hahn, Meixner-Pollaczek, and Continuous Dual Hahn polynomials, respectively, thus providing a very simple and straightforward derivation of the known results for ASM enumerations. In Section 5 we exploit the fact that these polynomials, being of hypergeometric type, satisfy some finite difference equations, which can be translated into recurrence relations for the one-point boundary correlator. These recurrences in turn can be solved, thus giving the results for the refined 1-, 2- and, especially interesting, 3-enumeration of ASMs. We conclude in Section 6 with a discussion of the proposed approach.

2. Square ice with DWBC

Let us start with recalling the formulation of the model. The six-vertex model, which was originally proposed as a model of two-dimensional ice, is formulated on a square lattice with arrows lying on edges, and obeying the so called 'ice-rule', namely, the only admitted configurations are such that there are always two arrows pointing away from, and two arrows pointing into, each lattice vertex. An equivalent and graphically simpler description of the configurations of the model can be given in terms of lines flowing through the vertices: for each arrow pointing downward or to the left, draw a thick line on the corresponding link. The six possible vertex states and the Boltzmann weights w_i assigned to each vertex according to its state i ($i = 1, \dots, 6$) are shown in Fig. 1. As it has been already stressed in the Introduction only the homogeneous version of the model, where the Boltzmann weights are site independent, is

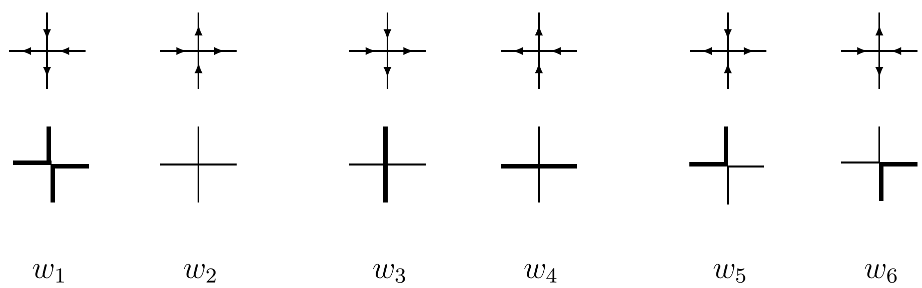

Figure 1: The six allowed types of vertices in terms of arrows (first row), in terms of lines (second row), and their Boltzmann weights (third row).

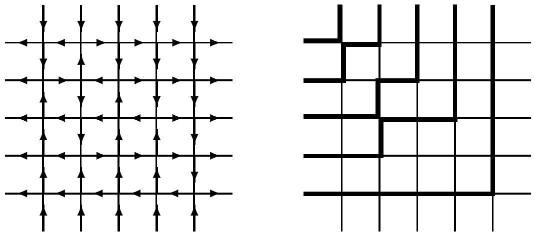

Figure 2: One of the possible configurations of the model with DWBC, in the case $N = 5$: (a) in terms of arrows; (b) in terms of lines.

considered here.

The DWBC are imposed on the $N \times N$ square lattice by fixing the direction of all arrows on the boundaries in a specific way. Namely, the vertical arrows on the top and bottom of the lattice point inward, while the horizontal arrows on the left and right sides point outward. Equivalently, a generic configuration of the model with DWBC can be depicted by N lines flowing from the upper boundary to the left one. This line picture (besides taking into account the 'ice rule' in an automated way) is intuitively closer to ASMs recalled in the next Section. A possible state of the model both in terms of arrows and of lines is shown in Fig. 2.

The partition function is defined, as usual, as a sum over all possible arrow configurations, compatible with the imposed DWBC, each configuration being assigned its Boltzmann weight, given as the product of all the corresponding vertex weights,

$$Z_N = \sum_{\substack{\text{arrow configurations} \\ \text{with DWBC}}} \prod_{i=1}^{6} w_i^{n_i} \; . \tag{2.1}$$

Here n_i denotes the number of vertices in the state i in each arrow configuration ($n_1 + \cdots + n_6 = N^2$).

The six-vertex model with DWBC can be considered, with no loss of generality, with its weights invariant under the simultaneous reversal of all arrows,

$$w_1 = w_2 =: a, \qquad w_3 = w_4 =: b, \qquad w_5 = w_6 =: c. \tag{2.2}$$

Under different choices of Boltzmann weights the six-vertex model exhibits different behaviors, according to the value of the parameter Δ defined as

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}. \tag{2.3}$$

It is well known that there are three physical regions or phases for the six-vertex model: the ferroelectric phase, $\Delta > 1$; the anti-ferroelectric phase, $\Delta < -1$; and, the disordered phase,

$-1 < \Delta < 1$. In the present paper we shall discuss only some particular cases, with values of the Boltzmann weights that correspond to the disordered phase. A convenient parametrization of the Boltzmann weights in this phase is

$$a = \sin(\lambda + \eta), \qquad b = \sin(\lambda - \eta), \qquad c = \sin 2\eta. \tag{2.4}$$

With this choice one has $\Delta = \cos 2\eta$. The parameter λ is the so-called spectral parameter and η is the crossing parameter. The physical requirement of positive Boltzmann weights, in the disordered regime, restricts the values of the crossing and spectral parameters to $0 < \eta < \pi/2$ and $\eta < \lambda < \pi/2 - \eta$.

An exact representation for the partition function was obtained in Ref. [3]. When the weights are parameterized according to (2.4) such representation reads

$$Z_N = \frac{[\sin(\lambda - \eta)\sin(\lambda + \eta)]^{N^2}}{\prod_{k=1}^{N-1} (k!)^2} \det \mathcal{Z} \tag{2.5}$$

where $\mathcal{Z}$ is an $N \times N$ matrix with entries

$$\mathcal{Z}_{jk} = \frac{\partial^{j+k}}{\partial \lambda^{j+k}} \frac{\sin 2\eta}{\sin(\lambda - \eta)\sin(\lambda + \eta)} \,. \tag{2.6}$$

Here and in the following we use the convention that indices of $N \times N$ matrices run over the values $j, k = 0, 1, \ldots, N - 1$.

This formula for the partition function has been obtained as the homogeneous limit of a more general formula for a partially inhomogeneous six-vertex model with DWBC. The inhomogeneous model, with site-dependent weights, is defined by introducing two sets of spectral parameters $\{\lambda_j\}_{j=1}^N$ and $\{\nu_k\}_{k=1}^N$, such that the weights of the vertex lying at the intersection of the j-th column with the k-th row depend on $\lambda_j - \nu_k$ rather than simply on λ, still through formulae (2.4). The inhomogeneous model, though apparently more complicate, can be fruitfully investigated through the Quantum Inverse Scattering Method, see papers [1–3] and book [5] for details. As a result, the partition function of the inhomogeneous model is represented in terms of certain determinant formula which, however, requires great caution in the study of its homogeneous limit, $\nu_k \to 0$ and $\lambda_j \to \lambda$, since in this limit the determinant possess $N^2 - N$ zeros that are cancelled by the same number of singularities coming from the pre-factor. A recipe for taking such a limit was explained in detail in Ref. [3] where formula (2.5) was originally obtained. Subsequently, formula (2.5) was used in papers [25, 26] to investigate the thermodynamic limit, $N \to \infty$, of the partition function. In these studies the Hankel nature of the determinant appearing in (2.5), a natural outcome of the homogeneous limit procedure, was exploited through its relation with the Toda chain differential equation or with the random matrix partition function.

The aim of this paper is to explain how this Hankel determinant formula (and its analogue for the boundary correlator, given below) can also be used in application to some well-known problems of combinatorics, such as enumerations (and the so-called 'refined' enumerations) of ASMs. These problems and the known results are reviewed in the next Section. It is to be mentioned that though in these combinatorial problems one deals in fact with the homogeneous six-vertex model with DWBC, the Hankel determinant formula for the partition function was never discussed previously in this context. Instead, the more complicated determinant formula for the inhomogeneous model partition function was used in the combinatorial proofs. In these proofs, the homogeneous limit can be regarded, in fact, as the most complicated part on the way to the result. Consequently, one can expect that extracting relevant information directly from the homogeneous model should be technically much simpler. We will show that this is indeed the case since some standard classical orthogonal polynomials can be naturally related (for some particular choices of the parameters) to the Hankel determinant in (2.5).

In addition to the partition function, we shall discuss here also boundary one point correlation functions. In general, two kinds of one point correlation functions can be considered in the six-vertex model: the first one ('polarization') is the probability to find an arrow on a given edge in a particular state, while the second one is the probability to find a given vertex in some state i. If one restricts to edges or vertices adjacent to the boundary, then such correlators are called boundary correlators. Following the notations of paper [4], where these boundary correlators were studied, let $G_N^{(r)}$ denote the probability that an arrow on the last column and between the r-th and the $r+1$-th rows (enumerated from the bottom) points upward (or, in the line language, that there is no thick line on this edge), and let $H_N^{(r)}$ denote the probability that the r-th vertex of the last column is in the state $i=5$ (or that the thick line flows from the top to the left), see Figs. 1 and 2. The first correlator, $G_N^{(r)}$, is, in fact, the boundary polarization, whose interpretation is more direct from a physical point of view, while the second one, $H_N^{(r)}$, is closely related to the refined enumerations of ASMs. It is easy to see that, due to DWBC, the two correlators are related to each other as follows

$$G_N^{(r)} = H_N^{(r)} + H_N^{(r-1)} + \cdots + H_N^{(1)}. \tag{2.7}$$

In Ref. [4] both correlators were computed using Quantum Inverse Scattering Method for the inhomogeneous six-vertex model. In the homogeneous limit, which is the situation we are interested in here, determinant formulae generalizing (2.5) were found for these correlators. For instance, for $H_N^{(r)}$, the following expression was derived

$$H_N^{(r)} = \frac{(N-1)!\,\sin 2\eta}{\big[\sin(\lambda-\eta)\big]^r\big[\sin(\lambda+\eta)\big]^{N-r+1}}\frac{\det\,\mathcal{H}}{\det\,\mathcal{Z}} \tag{2.8}$$

where the $N \times N$ matrix $\mathcal{H}$ differs from $\mathcal{Z}$ only in the last column,

$$\mathcal{H}_{jk} = \begin{cases} \mathcal{Z}_{jk} & \text{for} \quad k = 0, \dots, N-2 \\ \dfrac{\partial^j}{\partial \varepsilon^j} \dfrac{\big[\sin\varepsilon\big]^{r-1}\big[\sin(\varepsilon - 2\eta)\big]^{N-r}}{\big[\sin(\varepsilon + \lambda - \eta)\big]^{N-1}}\Bigg|_{\varepsilon=0} & \text{for} \quad k = N-1 \end{cases} . \tag{2.9}$$

A similar expression is valid for $G_N^{(r)}$ as well. In what follows we shall focus on $H_N^{(r)}$; the results for $G_N^{(r)}$ will follow immediately from relation (2.7). From the DWBC it immediately follows that $G_N^{(N)} = 1$, and therefore, from (2.7), correlator $H_N^{(r)}$ has to satisfy

$$\sum_{r=1}^{N} H_N^{(r)} = 1 \, . \tag{2.10}$$

In what follows this normalization condition will be used in application to the generating function of $H_N^{(r)}$.

3. Alternating Sign Matrices

An alternating sign matrix (ASM) is a matrix of 1's, 0's and -1's such that in each row and in each column *(i)* all nonzero entries alternate in sign, and *(ii)* the first and the last nonzero entries are 1. An example of such matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.1}$$

There are many nice results concerning ASMs, for a review, see book [11]. Many of these results have been first formulated as conjectures which were subsequently proved by different methods.

The most celebrated result concerns the total number $A(N)$ of $N \times N$ ASMs. It was conjectured in papers [6, 7] and proved in papers [9, 10] that

$$A(N) = \prod_{k=1}^{N} \frac{(3k-2)!\,(k-1)!}{(2k-1)!\,(2k-2)!} = \prod_{k=1}^{N} \frac{(3k-2)!}{(2N-k)!}. \tag{3.2}$$

Other results concern the weighted enumerations or the so-called x-enumerations of ASMs. In x-enumeration the matrices are counted with a weight x^k where k is the total number of -1 entries in a matrix (the number x here should not be confused with the variable x widely used in the following Sections). The number of x-enumerated ASMs is denoted traditionally as $A(N; x)$. The extension of the $x = 1$ result above to the case of generic x is not known, but for a few nontrivial cases, namely $x = 2$ and $x = 3$, closed expressions for x-enumerations are known (the case $x = 0$ is trivial since assigning a vanishing weight to each -1 entry restricts the enumeration to the sole permutation matrices: $A(n; 0) = n!$). The result in the case $x = 2$, related to the domino tilings of Aztec diamond [12,13,15], have been obtained in [7], and reads

$$A(N; 2) = 2^{N(N-1)/2}. \tag{3.3}$$

The answer in the case of 3-enumeration, again conjectured in [6, 7], was subsequently [10] proved to be

$$A(2m+1; 3) = 3^{m(m+1)} \prod_{k=1}^{m} \left[\frac{(3k-1)!}{(m+k)!}\right]^2, \qquad A(2m+2; 3) = 3^m \frac{(3m+2)!\, m!}{\left[(2m+1)!\right]^2} A(2m+1; 3). \tag{3.4}$$

Another class of results concerns the so-called refined enumerations of ASMs. In the refined enumeration one counts the number of $N \times N$ ASMs with their sole 1 of the last column at the r-th entry. The refined enumeration can be naturally extended to be also an x-enumeration. The standard notation for the refined x-enumeration is $A(N, r; x)$; in the case $x = 1$ one writes simply $A(N, r)$ just like $A(N)$ for the total number of ASMs. It was again conjectured in [6,7], and proved in [23] that refined enumeration of ASMs is given by

$$A(N, r) = \frac{\binom{N+r-2}{N-1}\binom{2N-1-r}{N-1}}{\binom{3N-2}{N-1}} A(N). \tag{3.5}$$

In the case of the refined 2-enumeration it has been shown in [7, 12, 13, 15] that

$$A(N, r; 2) = \frac{1}{2^{N-1}} \binom{N-1}{r-1} A(N; 2). \tag{3.6}$$

The case of the refined 3-enumeration appears to be much more complicate. Direct computer enumeration does not suggest any factorized form and no conjecture concerning $A(N, r; 3)$ has ever been proposed. Recently, Stroganov obtained a certain representation for the corresponding generating function [27], allowing in principle a recursive computation of the numbers $A(N, r; 3)$.

The most direct way to recover all the previous results is based on a nice bijective correspondence between ASMs and six-vertex model, which has been pointed out in [8, 12, 13], and applied for the first time in [10], namely, that to each single $N \times N$ ASM corresponds one and only one arrow configuration of the six-vertex model on the $N \times N$ square lattice with DWBC. The correspondence between matrix entries and vertices is depicted in Fig. 3. As an example,

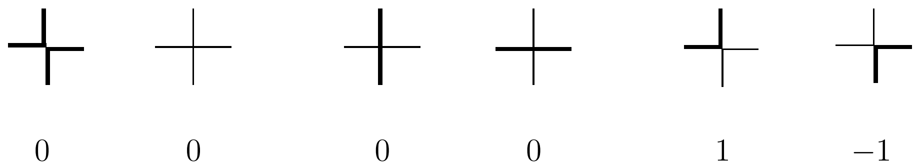

Figure 3: Vertex states—ASM's entries correspondence.

matrix (3.1) corresponds to the configuration of Fig. 2 and vice versa.

As an immediate consequence of this correspondence, ASM enumeration is exactly given by the partition function of square ice, when all vertex weights are set equal to unity. More generally, the number of -1's in a given ASM being equal to the number of vertices of type 6 (see Fig. 3), and the number of vertex of type 5 and 6 being constrained by the condition $n_5 - n_6 = N$, we readily get

$$A(N;x) = (1-x/4)^{-N^2/2}\, x^{-N/2}\, Z_N \Big|_{\substack{\lambda=\pi/2 \\ \eta=\arcsin(\sqrt{x}/2)}} . \tag{3.7}$$

Therefore, x-enumeration of ASM corresponds to the computation of the partition function of square ice on the subset of parameters space given by $a = b$. In this correspondence, values of x belonging to the interval $(0, 4)$ corresponds to the disordered regime of the model, $-1 < \Delta < 1$.

This nice correspondence can be further extended to the refined x-enumeration of ASMs. In the language of square ice, the ratio $A(N, r; x)/A(N; x)$ can be rephrased as the probability of finding the unique vertex of type 5 on the right boundary at the r-th site, which is exactly the definition of the boundary correlator $H_N^{(r)}$. Explicitly, one has

$$\frac{A(N,r;x)}{A(N;x)} = H_N^{(r)} \Big|_{\substack{\lambda=\pi/2 \\ \eta=\arcsin(\sqrt{x}/2)}} . \tag{3.8}$$

Thus, being able to compute the partition function and the boundary correlator for some particular choice of parameters, one immediately obtains x-enumerations and refined x-enumerations of ASMs respectively, for some corresponding values of x.

4. The partition function and enumerations of ASMs

4.1. Preliminaries

From representation (2.5), it is evident that the evaluation of the the partition function in the homogeneous case essentially reduces, modulo a trivial pre-factor, to the calculation of the determinant of the $N \times N$ matrix $\mathcal{Z}$, which is a Hankel matrix, whose entries do not depend on N. There is a standard method to treat such determinants, which is based on the theory of orthogonal polynomials [28, 29], and has proven to be quite powerful when some assumption are verified.

Let us assume that the entries of our Hankel matrix are written in the canonical form

$$\mathcal{Z}_{jk} = \int_{-\infty}^{\infty} x^{j+k} \mu(x)\, \mathrm{d}x. \tag{4.1}$$

Let us moreover suppose that there exist a complete set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ subject to the orthogonality condition

$$\int_{-\infty}^{\infty} p_j(x)\, p_k(x)\, \mu(x)\, \mathrm{d}x = h_j \delta_{jk}\ . \tag{4.2}$$

Then, denoting by κ_n the leading coefficient of $p_n(x)$,

$$p_n(x) = \kappa_n x^n + \dots\ , \qquad \kappa_n \neq 0\ , \tag{4.3}$$

and using standard properties of determinants and of orthogonal polynomials, one obtains for the determinant of the $N \times N$ matrix $\mathcal{Z}$ the following formula

$$\det \mathcal{Z} = \det \left[\int \frac{1}{\kappa_j \kappa_k} p_j(x) p_k(x) \mu(x) \mathrm{d}x \right]_{j,k=0}^{N-1} \tag{4.4}$$

$$= \prod_{n=0}^{N-1} \frac{h_n}{\kappa_n^2}\ . \tag{4.5}$$

Obviously, this formula may turn out useful provided that the set of polynomials which are orthogonal with respect to the weight $\mu(x)$ can be identified.

In the case of matrix $\mathcal{Z}$ with entries (2.6) one can easily obtain the weight $\mu(x)$ using the representation

$$\frac{\sin 2\eta}{\sin(\lambda - \eta) \sin(\lambda + \eta)} = \int_{-\infty}^{\infty} \mathrm{e}^{(\lambda - \pi/2)x}\, \frac{\sinh \eta x}{\sinh \frac{\pi}{2} x}\, \mathrm{d}x \tag{4.6}$$

which is valid for λ and η corresponding to the disordered regime [26]. Unfortunately, appropriate polynomials are not available in general and the previous scheme cannot be fulfilled for generic values of λ and η. However, for some very particular values of these parameters the appropriate orthogonal polynomials appear to be known, and have just to be suitably chosen in the framework of Askey scheme of hypergeometric orthogonal polynomials [30]. To be precise, there are essentially three cases, indicated in the phase diagram of the model, see Fig. 4, which fit into the scheme: *i)* the so-called 'free fermion' line $\eta = \pi/4$ and $\pi/4 < \lambda < 3\pi/4$; *ii)* the $\Delta = 1/2$ symmetric point (or 'ice point') $\eta = \pi/6$ and $\lambda = \pi/2$; and *iii)* the $\Delta = -1/2$ symmetric point $\eta = \pi/3$ and $\lambda = \pi/2$. With 'symmetric' here we mean that these points lie on the

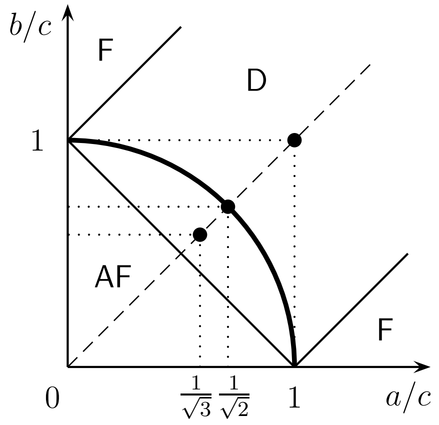

Figure 4: The phase diagram of the model, with ferroelectric (F), antiferroelectric (AF) and disordered (D) phases, separated by the solid lines. The three considered cases, all belonging to the disordered phase, are shown in bold: the free fermion line, and the three points corresponding to 1-, 2- and 3-enumeration of ASMs.

line $a = b$. The polynomials corresponding to these three cases are Meixner-Pollaczek polynomials, Continuous Hahn polynomials and Continuous Dual Hahn polynomials, respectively. In the rest of this Section we give details of computation for each of these cases.

It is worth mentioning that all previously listed choices of parameters, for which the Hankel determinant appearing in (2.5) turns out to be related to some set of classical orthogonal polynomials, exactly cover 1-, 2- and 3-enumerations of ASMs. On the other hand, the fact that no set of polynomials in the Askey scheme corresponds to some other choice of the parameter η (even with λ set equal to $\pi/2$) is likely to be deeply related to the lack of factorizable formulae for x-enumerations of ASMs other than for $x = 1, 2, 3$.

4.2. The free fermion line

We shall start with the $\eta = \pi/4$ case which is technically the simplest one, though the parameter λ is not fixed, $\pi/4 < \lambda < 3\pi/4$. In this case the orthogonality weight $\mu(x)$ can be written as

$$\mu(x) = \mathrm{e}^{(\lambda-\pi/2)x} \frac{\sinh \frac{\pi}{4}x}{\sinh \frac{\pi}{2}x} = \frac{\mathrm{e}^{(\lambda-\pi/2)x}}{2\cosh \frac{\pi}{4}x} = \frac{\mathrm{e}^{(\lambda-\pi/2)x}}{2\pi} \left| \Gamma\left(\frac{1}{2} + \mathrm{i}\frac{x}{4}\right)\right|^2. \tag{4.7}$$

Comparing this formula with the orthogonality condition for Meixner-Pollaczek polynomials [30]

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} P_m^{(\alpha)}(x;\phi)\, P_n^{(\alpha)}(x;\phi)\, |\Gamma(\alpha + \mathrm{i}x)|^2\, \mathrm{e}^{(2\phi-\pi)x}\, \mathrm{d}x = \frac{\Gamma(n+2\alpha)}{(2\sin\phi)^{2\alpha}\, n!}\, \delta_{nm} \tag{4.8}$$

where

$$P_n^{(\alpha)}(x;\phi) = \frac{(2\alpha)_n}{n!}\, \mathrm{e}^{\mathrm{i}n\phi}\, {}_2F_1\left(\begin{matrix} -n\,,\ \alpha + \mathrm{i}x \\ 2\alpha \end{matrix} \middle| 1 - \mathrm{e}^{-2\mathrm{i}\phi} \right) \tag{4.9}$$

we find that the polynomials related to the Hankel determinant in this case are

$$p_n(x) = P_n^{(1/2)}\left(\frac{x}{4}; 2\lambda - \frac{\pi}{2}\right). \tag{4.10}$$

Using, for convenience, the parameter $\phi = 2\lambda - \pi/2 \in (0, \pi)$ instead of λ, we have

$$h_n = \frac{2}{\sin\phi}, \qquad \kappa_n = \frac{(\sin\phi)^n}{2^n\, n!}. \tag{4.11}$$

Inserting these expressions into (4.5), we straightforwardly obtain the following result for the partition function

$$Z_N\Big|_{\eta=\pi/4} = \frac{(\sin\phi)^{N^2}}{2^{N^2}\prod_{k=0}^{N-1}(k!)^2}\prod_{n=0}^{N-1}\frac{2}{\sin\phi}\left[\frac{2^n n!}{(\sin\phi)^n}\right]^2 = 1. \tag{4.12}$$

Using Eqn. (3.7), we immediately recover formula (3.3) for 2-enumeration of ASMs.

4.3. The ice point and the total number of ASMs

At the ice point, or $\Delta = 1/2$ symmetric point, the computation is very similar. For $\eta = \pi/6$ and $\lambda = \pi/2$ the orthogonality weight reads

$$\mu(x) = \frac{\sinh\frac{\pi}{6}x}{\sinh\frac{\pi}{2}x} = \frac{1}{4\pi^2}\left|\Gamma\left(\frac{1}{3}+i\frac{x}{6}\right)\Gamma\left(\frac{2}{3}+i\frac{x}{6}\right)\right|^2. \tag{4.13}$$

where the triplication formula for the Γ-function

$$\Gamma(3x) = \frac{3^{3x-1/2}}{2\pi}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{3}\right)\Gamma\left(x+\frac{2}{3}\right) \tag{4.14}$$

has been used. The orthogonality condition for Continuous Hahn polynomials is

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} p_n(x;a,b,c,d)p_m(x;a,b,c,d)\,\Gamma(a+\mathrm{i}x)\,\Gamma(b+\mathrm{i}x)\,\Gamma(c-\mathrm{i}x)\,\Gamma(d-\mathrm{i}x)\,\mathrm{d}x \\ = \frac{\Gamma(n+a+c)\,\Gamma(n+a+d)\,\Gamma(n+b+c)\,\Gamma(n+b+d)}{(2n+a+b+c+d-1)\,\Gamma(n+a+b+c+d-1)\,n!}\,\delta_{nm}\end{aligned} \tag{4.15}$$

and the polynomials are given by

$$p_n(x;a,b,c,d) = \mathrm{i}^n\frac{(a+c)_n(a+d)_n}{n!}\,{}_3F_2\left(\begin{matrix}-n\,,\ n+a+b+c+d-1\,,\ a+\mathrm{i}x\\ a+c\,,\ a+d\end{matrix}\middle|1\right). \tag{4.16}$$

Orthogonality condition (4.15) is valid if the parameters a,b,c,d satisfy $\mathrm{Re}\,(a,b,c,d) > 0$, $a = \bar{c}$ and $b = \bar{d}$. Comparing (4.13) with (4.15) we are naturally led to the choice of parameters

$a = c = 1/3$ and $b = d = 2/3$. Hence, the appropriate polynomials to be associated to the Hankel determinant in this case are

$$p_n(x) = p_n\left(\frac{x}{6}; \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) = \mathrm{i}^n (2/3)_n \; {}_3F_2\left(\begin{matrix} -n\,,\ n+1\,,\ 1/3 + \mathrm{i}x/6 \\ 2/3\,,\ 1 \end{matrix}\middle| 1\right). \tag{4.17}$$

For the normalization constant and the leading coefficient we have the expressions

$$h_n = \frac{2(3n+1)!}{(2n+1)\, 3^{3n+1/2}\, n!}, \qquad \kappa_n = \frac{(2n)!}{6^n\, (n!)^2} \tag{4.18}$$

where the triplication formula, Eqn. (4.14) has been used to simplify the normalization constant h_n.

Substituting the obtained values of h_n and κ_n in expression (4.5) for the determinant, taking into account the value of the prefactor in (2.5) for $\lambda = \pi/2$ and $\eta = \pi/6$, and cancelling whatever possible, we arrive to the following value for the ice point partition function

$$Z_N\Big|_{\substack{\lambda=\pi/2 \\ \eta=\pi/6}} = \left(\frac{\sqrt{3}}{2}\right)^{N^2} \prod_{n=0}^{N-1} \frac{(3n+1)!\, n!}{(2n)!\,(2n+1)!}. \tag{4.19}$$

The product expression here gives exactly the total number of ASMs, $A(N)$, since by formula (3.7) the first factor relates $A(N)$ with the partition function,

$$A(N) = (3/4)^{-N^2/2}\, Z_N\Big|_{\substack{\lambda=\pi/2 \\ \eta=\pi/6}}. \tag{4.20}$$

Thus, we have easily recovered the celebrated result, Eqn. (3.2), for ASMs enumeration directly from the Hankel determinant formula (2.5).

4.4. The $\Delta = -1/2$ symmetric point and 3-enumeration of ASMs

In this case, $\lambda = \pi/2$ and $\eta = \pi/3$, the weight $\mu(x)$ can be rewritten, using (4.14), in the form

$$\mu(x) = \frac{\sinh \frac{\pi}{3}x}{\sinh \frac{\pi}{2}x} = \frac{1}{8\pi^2}\left|\frac{\Gamma(\mathrm{i}\frac{x}{6})\Gamma(\frac{1}{3} + \mathrm{i}\frac{x}{6})\Gamma(\frac{2}{3} + \mathrm{i}\frac{x}{6})}{\Gamma(\mathrm{i}\frac{x}{3})}\right|^2. \tag{4.21}$$

This expression recalls the weight for Continuous Dual Hahn polynomials $S_n(x^2; a, b, c)$, which are defined by

$$S_n(x^2; a, b, c) = (a+b)_n (a+c)_n \, {}_3F_2\left(\begin{matrix} -n\,,\ a + \mathrm{i}x\,,\ a - \mathrm{i}x \\ a+b\,,\ a+c \end{matrix}\middle| 1\right), \tag{4.22}$$

and, for real and nonnegative values of parameters a, b, c, satisfy the orthogonality condition

$$\frac{1}{2\pi}\int_0^\infty S_m(x^2; a, b, c) S_n(x^2; a, b, c) \left|\frac{\Gamma(a+\mathrm{i}x)\Gamma(b+\mathrm{i}x)\Gamma(c+\mathrm{i}x)}{\Gamma(2\mathrm{i}x)}\right|^2 \mathrm{d}x$$
$$= \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)\, n!\, \delta_{nm}\,. \tag{4.23}$$

At a first glance, however, Continuous Dual Hahn polynomials do not seem appropriate for the evaluation of the Hankel determinant: on one hand, these are polynomials in x^2 rather than in x, and on the other hand the integration domain in (4.23) is restricted to the positive half-axis. These inconvenience may nevertheless easily be circumvented, thanks to the following nice feature of matrix $\mathcal{Z}$, holding for $\lambda = \pi/2$ and generic η: whenever the sum of indices $j + k$ is odd, the corresponding entry $\mathcal{Z}_{jk}$ vanishes. In other words, the matrix $\mathcal{Z}$ at $\lambda = \pi/2$ exhibit a chessboard pattern of vanishing and non vanishing entries. As can be seen from the Laplace formula for the determinant of the sum of two matrices, this chessboard structure immediately implies that the determinant of such matrix always factorizes onto two determinants of smaller matrices with no vanishing entries.

To be precise, let us denote

$$D_N := \det \mathcal{Z}\Big|_{\substack{\lambda=\pi/2\\ \eta=\pi/3}} = \det \left[\int_{-\infty}^{\infty} x^{j+k}\,\mu(x)\,\mathrm{d}x\right]_{j,k=0}^{N-1} \tag{4.24}$$

where $\mu(x)$ is given by (4.21). Since the orthogonality weight is even, $\mu(x) = \mu(-x)$, one has

$$\int_{-\infty}^{\infty} x^{j+k}\,\mu(x)\,\mathrm{d}x = \begin{cases} 2\displaystyle\int_0^{\infty} x^{j+k}\,\mu(x)\,\mathrm{d}x & \text{if } j+k \text{ is even} \\ 0 & \text{if } j+k \text{ is odd} \end{cases} \tag{4.25}$$

and consequently the following factorization arises

$$D_{2m} = D_m^{(0)} D_m^{(1)}, \qquad D_{2m+1} = D_{m+1}^{(0)} D_m^{(1)}. \tag{4.26}$$

Here $D_m^{(0)}$ and $D_m^{(1)}$ are determinants of $m \times m$ Hankel matrices, built only from the even moments of the weight $\mu(x)$,

$$D_m^{(\sigma)} = \det \left[\int_0^{\infty} x^{2(j+k)}\,\mu^{(\sigma)}(x)\,\mathrm{d}x\right]_{j,k=0}^{m-1}, \qquad \sigma = 0,1 \tag{4.27}$$

where

$$\mu^{(0)}(x) = 2\mu(x), \qquad \mu^{(1)}(x) = 2x^2\mu(x). \tag{4.28}$$

Let $p_n^{(\sigma)}(x)$ be the polynomials subject to the orthogonality condition

$$\int_0^{\infty} p_j^{(\sigma)}(x^2) p_k^{(\sigma)}(x^2) \mu^{(\sigma)}(x)\,\mathrm{d}x = h_j^{(\sigma)} \delta_{jk}, \tag{4.29}$$

then, in analogy with Eqn. (4.5), we have the following formula

$$\begin{aligned} D_m^{(\sigma)} &= \det \left[\int_0^{\infty} \frac{1}{\kappa_j^{(\sigma)}\kappa_k^{(\sigma)}}\, p_j^{(\sigma)}(x^2)\, p_k^{(\sigma)}(x^2)\, \mu^{(\sigma)}(x)\,\mathrm{d}x\right]_{j,k=0}^{m-1} \\ &= \prod_{n=0}^{m-1} \frac{h_n^{(\sigma)}}{[\kappa_n^{(\sigma)}]^2}, \end{aligned} \tag{4.30}$$

where $\kappa_n^{(\sigma)}$ is the leading coefficient of the polynomial $p_n^{(\sigma)}(x^2)$. Thus, Continuous Dual Hahn polynomials can be related to the Hankel determinants $D_m^{(\sigma)}$ by properly specializing the parameters (a, b, c) in each of the two cases, $\sigma = 0$ and $\sigma = 1$.

Comparing Eqns. (4.28) and (4.21) with the orthogonality weight (4.23), we are led to specialize the parameters (a, b, c) to the values $(0, 1/3, 2/3)$ for $\sigma = 0$ and $(1, 1/3, 2/3)$ for $\sigma = 1$. Thus, we have

$$p_n^{(0)}(x^2) = S_n\left(\frac{x^2}{36}; 0, \frac{1}{3}, \frac{2}{3}\right) = \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n {}_3F_2\left(\begin{matrix} -n\,,\ \mathrm{i}x/6\,,\ -\mathrm{i}x/6 \\ 1/3\,,\ 2/3 \end{matrix}\middle| 1\right), \tag{4.31}$$

and

$$p_n^{(1)}(x^2) = S_n\left(\frac{x^2}{36}; 1, \frac{1}{3}, \frac{2}{3}\right) = \left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n {}_3F_2\left(\begin{matrix} -n\,,\ 1+\mathrm{i}x/6\,,\ 1-\mathrm{i}x/6 \\ 4/3\,,\ 5/3 \end{matrix}\middle| 1\right). \tag{4.32}$$

For the normalization constant and the leading coefficient we have

$$h_n^{(0)} = 2\frac{n!(3n)!}{3^{3n-1/2}}, \qquad \kappa_n^{(0)} = \left(-\frac{1}{36}\right)^n, \tag{4.33}$$

and

$$h_n^{(1)} = 8\frac{n!(3n+2)!}{3^{3n-1/2}}, \qquad \kappa_n^{(1)} = \left(-\frac{1}{36}\right)^n, \tag{4.34}$$

respectively. The Γ-function triplication formula, Eqn. (4.14) has again been used to simplify the expressions for the normalization constants $h_n^{(0)}$ and $h_n^{(1)}$. Using (4.30) we obtain

$$\begin{aligned} D_m^{(0)} &= 2^{2m^2-m}\, 3^{m^2/2} \prod_{k=0}^{m-1} k!\,(3k)!\,, \\ D_m^{(1)} &= 2^{2m^2+m}\, 3^{m^2/2} \prod_{k=0}^{m-1} k!\,(3k+2)!\,. \end{aligned} \tag{4.35}$$

Substituting these values into (4.26), taking into account the pre-factor of Eqn. (2.5), and cancelling whatever possible, we obtain the following expression for the partition function at $\lambda = \pi/2$ and $\eta = \pi/6$,

$$Z_{2m}\Big|_{\substack{\lambda=\pi/2\\ \eta=\pi/3}} = \left(\frac{1}{2}\right)^{4m^2} 3^{m^2+m}\, \frac{m!}{(3m)!} \prod_{k=0}^{m-1} \left[\frac{(3k+2)!}{(m+k)!}\right]^2 \tag{4.36}$$

$$Z_{2m+1}\Big|_{\substack{\lambda=\pi/2\\ \eta=\pi/3}} = \left(\frac{1}{2}\right)^{(2m+1)^2} 3^{m^2+2m+1/2} \prod_{k=0}^{m-1} \left[\frac{(3k+2)!}{(m+k+1)!}\right]^2 \tag{4.37}$$

Recalling (3.7) which now reads

$$A(N;3) = 2^{N^2}\left(\frac{1}{\sqrt{3}}\right)^N Z_N\Big|_{\substack{\lambda=\pi/2\\ \eta=\pi/3}}. \tag{4.38}$$

formulae (3.4) for 3-enumeration of ASMs are readily recovered.

In conclusion, in this Section we have given explicit formulae for the partition function of the six-vertex model with DWBC for some particular values of parameters λ and η. Analogous, essentially equivalent, expressions were already known from the investigations of ASMs x-enumeration. It is worth emphasizing that the approach presented here allows to recover these results in a very simple and straightforward way. The keystone of the whole approach is the nice connection with known classical orthogonal polynomials that will now be further exploited to explore the boundary correlation function in application to the refined x-enumerations of ASMs.

5. The boundary correlator and refined enumerations of ASMs

5.1. Preliminaries

In this Section we shall show how the knowledge of the suitable set of classical orthogonal polynomials associated to each of the three considered cases, can be further exploited to derive explicit answers for the one point boundary correlation function. As a consequence the results for the refined x-enumerations of ASMs can be obtained; the result for the refined 3-enumeration of ASMs is of primary interest since it was not conjectured previously.

In what follows, for the sake of simplicity and clarity, we shall often ignore the overall normalization of the boundary correlator. As already discussed at the end of Section 2, the proper normalization can always be restored trough the use of Eqn. (2.10).

The basic idea is very simple, stemming from the fact, see Refs. [28,29], that the polynomials associated to a given orthogonality weight $\mu(x)$ can be in turn represented as determinants,

$$p_{N-1}(x) = \text{const} \times \det \mathcal{W}, \tag{5.1}$$

where $N \times N$ matrix $\mathcal{W}$ differs from $\mathcal{Z}$, defined by (4.1), just only in the elements of the last column,

$$\mathcal{W}_{jk} = \begin{cases} \mathcal{Z}_{jk} & \text{for} \quad k = 0, \ldots, N-2, \\ x^j & \text{for} \quad k = N-1 \end{cases}. \tag{5.2}$$

The boundary one point correlator, see Eqn. (2.8), therefore reads

$$H_N^{(r)} = \text{const} \times \left[\frac{\sin(\lambda+\eta)}{\sin(\lambda-\eta)}\right]^r \left\{ p_{N-1}(\partial_\varepsilon) \frac{(\sin\varepsilon)^{r-1}[\sin(\varepsilon-2\eta)]^{N-r}}{[\sin(\varepsilon+\lambda-\eta)]^{N-1}} \right\} \Bigg|_{\varepsilon=0}. \tag{5.3}$$

This representation is completely general, however it can be fruitfully exploited only when the explicit form of the polynomial entering it is known, which happens precisely in each of the three cases under consideration, as explained in the previous Section.

Indeed, in these three cases, the polynomials of interest, being of hypergeometric type, are known to satisfy the following finite difference equations with respect to their variable, see, e.g., Ref. [30]. Denoting $y(x) = P_n^{(\alpha)}(x;\phi)$ for Meixner-Pollaczek polynomials one has

$$\mathrm{e}^{\mathrm{i}\phi}(\alpha - \mathrm{i}x)y(x+\mathrm{i}) + 2\mathrm{i}[x\cos\phi - (n+\alpha)\sin\phi]y(x) - \mathrm{e}^{-\mathrm{i}\phi}(\alpha + \mathrm{i}x)y(x-\mathrm{i}) = 0. \tag{5.4}$$

The Continuous Hahn polynomials satisfies

$$B(x)y(x+\mathrm{i}) - [B(x) + D(x) + n(n+a+b+c+d-1)]y(x) + D(x)y(x-\mathrm{i}x) = 0 \tag{5.5}$$

where $y(x) = p_n(x;a,b,c,d)$ and

$$B(x) = (c - \mathrm{i}x)(d - \mathrm{i}x), \qquad D(x) = (a + \mathrm{i}x)(b + \mathrm{i}x). \tag{5.6}$$

For the Continuous Dual Hahn polynomials one has

$$B(x)y(x+\mathrm{i}) - [B(x) + D(x) + n]y(x) + D(x)y(x-\mathrm{i}x) = 0 \tag{5.7}$$

where $y(x) = S_n(x^2;a,b,c)$ and

$$B(x) = \frac{(a-\mathrm{i}x)(b-\mathrm{i}x)(c-\mathrm{i}x)}{2\mathrm{i}x(2\mathrm{i}x-1)}, \qquad D(x) = \frac{(a+\mathrm{i}x)(b+\mathrm{i}x)(c+\mathrm{i}x)}{2\mathrm{i}x(2\mathrm{i}x+1)}. \tag{5.8}$$

The approach we shall apply here to compute the boundary correlator is based on the fact that each of these finite-difference equations for the polynomials can be directly translated into a recurrence relation for the boundary correlator which in turn can be solved explicitly.

The derivation of the recurrence relations for the boundary correlator in the three cases is quite similar and will be explained below in detail in each case. The general idea underlying the procedure to obtain the recurrence relation is based on the simple relation $y(\partial_\varepsilon \pm \mathrm{i}) = \mathrm{e}^{\mp\mathrm{i}\varepsilon} y(\partial_\varepsilon)\mathrm{e}^{\pm\mathrm{i}\varepsilon}$ which allows us to derive from each finite difference equation a relation of the form

$$y(\partial_\varepsilon)K_\varepsilon f(\varepsilon)\Big|_{\varepsilon=0} = 0. \tag{5.9}$$

Here K_ε is some linear differential operator whose form is determined by the finite difference equation, and $f(\varepsilon)$ is a trial function. Under special choice of this function (5.9) becomes a linear recurrence relation for the boundary correlator.

To find this function we shall use also the fact that the formula (5.3) can be further rewritten as follows

$$H_N^{(r)} = \text{const} \times \left\{ p_{N-1}(\partial_\varepsilon)[g(\varepsilon)]^{N-1}[\omega(\epsilon)]^{r-1} \right\}\Big|_{\varepsilon=0} \tag{5.10}$$

where $\omega(\varepsilon)$ and $g(\varepsilon)$ are given by

$$\omega(\epsilon) := \frac{\sin(\lambda+\eta)\sin\varepsilon}{\sin(\lambda-\eta)\sin(\varepsilon-2\eta)}\,, \qquad g(\epsilon) := \frac{\sin(\lambda-\eta)\sin(\varepsilon-2\eta)}{\sin(2\eta)\sin(\varepsilon+\lambda-\eta)}\,, \tag{5.11}$$

and related to each other as

$$g(\varepsilon) = \frac{1}{\omega(\varepsilon) - 1}. \tag{5.12}$$

It is clear that in order to rewrite Eqn. (5.9) as a recurrence relation for $H_N^{(r)}$ the function $f(x)$ should be chosen of the form

$$f(\varepsilon) = [g(\varepsilon)]^{N-1}[\omega(\epsilon)]^{r-1}\tau(\varepsilon) \tag{5.13}$$

where $\tau(\varepsilon)$ is still arbitrary. Then, rewriting the operator K_ε in terms of the differential operator

$$D_\varepsilon := \omega\partial_\omega = -\frac{\sin\epsilon\sin(\epsilon - 2\eta)}{\sin 2\eta}\,\partial_\epsilon. \tag{5.14}$$

and reexpressing all quantities in terms of the variable ω, the expression for $\tau(\varepsilon)$ can be easily chosen in such a way that Eqn. (5.9) turns into a recurrence relation for the boundary correlation function. The actual procedure becomes apparent after turning to the examples.

As a last comment here it should be mentioned that the recurrence relation for $H_N^{(r)}$ can equivalently be viewed as an ordinary differential equation for the generating function

$$H_N(z) := \sum_{r=1}^{N} H_N^{(r)}\, z^{r-1}, \qquad H_N(1) = 1. \tag{5.15}$$

Thus the problem of solving the obtained recurrence relation can also be regarded as that of finding a polynomial solution to the corresponding differential equation. In all cases considered below, such differential equations appear to be at most of the second order.

5.2. The free fermion line

We start with reminding that this is the case when $\eta = \pi/4$, with the spectral parameter free to vary within the interval $\pi/4 < \lambda < 3\pi/4$, or, using $\phi = 2\lambda - \pi/2$ we have $0 < \phi < \pi$. The value $\lambda = \pi/2$ (or $\phi = \pi/2$) corresponds to 2-enumerated ASMs.

The boundary correlator in this case is given by the formula (5.10) where

$$\omega(\varepsilon) = -\cot(\phi/2)\tan\varepsilon, \qquad g(\varepsilon) = -\frac{\sin(\phi/2)\cos\varepsilon}{\sin(\varepsilon + \phi/2)}, \tag{5.16}$$

and $p_{N-1}(x)$ is given by the formula (4.10), i.e., being the particular case of the Meixner-Pollaczek polynomial. The finite-difference equation (5.4) in this case reads

$$\mathrm{e}^{\mathrm{i}\phi}\left(\frac{1}{2} - \frac{\mathrm{i}x}{4}\right)p_{N-1}(x+4\mathrm{i}) + 2\mathrm{i}\left[\frac{x}{4}\cos\phi - \left(N - \frac{1}{2}\right)\sin\phi\right]p_{N-1}(x) \\ - \mathrm{e}^{-\mathrm{i}\phi}\left(\frac{1}{2} + \frac{\mathrm{i}x}{4}\right)p_{N-1}(x-4\mathrm{i}) = 0. \tag{5.17}$$

Using $p_{N-1}(\partial_\varepsilon \pm 4\mathrm{i}) = \mathrm{e}^{\mp 4\mathrm{i}\varepsilon} p_{N-1}(\partial_\varepsilon) \mathrm{e}^{\pm 4\mathrm{i}\varepsilon}$ we derive the condition

$$p_{N-1}(\partial_\varepsilon)\ K_\varepsilon\ [g(\epsilon)]^{N-1} [\omega(\epsilon)]^{r-1} \tau(\epsilon) \Big|_{\varepsilon=0} = 0 \tag{5.18}$$

where K_ε is the first order linear differential operator

$$K_\varepsilon = -\frac{\sin\varepsilon\cos\varepsilon}{\sin\phi}\ \partial_\varepsilon\ \sin(2\varepsilon+\phi) + N - 1 \tag{5.19}$$

and $\tau(\epsilon)$ is an arbitrary function which will be suitably chosen below to turn equation (5.18) into a recurrence relation for the boundary correlator. Our aim now is to explain how this function can be found and the recurrence relation can be obtained.

First, using the operator

$$D_\varepsilon = \sin\varepsilon\cos\varepsilon\,\partial_\varepsilon = \omega\partial_\omega \tag{5.20}$$

and changing to the variable ω using

$$\frac{\sin(2\varepsilon+\phi)}{\sin\phi} = -\frac{(\omega-1)(\alpha\omega+1)}{\alpha\omega^2+1}\,, \qquad \alpha := \tan^2(\phi/2), \tag{5.21}$$

we find

$$K_\varepsilon = \Big\{ D_\varepsilon(\omega-1)(\alpha\omega+1) + (N-1)(\alpha\omega^2+1) \Big\} \frac{1}{\alpha\omega^2+1}. \tag{5.22}$$

Next, let us define the operator $\widetilde{K}_\varepsilon$ by

$$K_\varepsilon g^{N-1} = g^{N-1} \widetilde{K}_\varepsilon \tag{5.23}$$

where g is given by (5.16). Using

$$g^{-1}(D_\varepsilon g) = -\frac{\omega}{\omega-1} \tag{5.24}$$

we find

$$\widetilde{K}_\varepsilon = \Big\{ D_\varepsilon(\alpha\omega+1) - (N-1) \Big\} \frac{\omega-1}{\alpha\omega^2+1}. \tag{5.25}$$

Now the choice of the function $\tau(\varepsilon)$ is evident, since in terms of the operator $\widetilde{K}_\varepsilon$ the relation (5.18) reads

$$p_{N-1}(\partial_\varepsilon)\, g^{N-1} \widetilde{K}_\varepsilon\, \omega^{r-1} \tau \Big|_{\varepsilon=0} = 0 \tag{5.26}$$

and therefore if we choose $\tau(\varepsilon)$ to cancel the factor standing outside the braces in (5.25) then we immediately obtain

$$p_{N-1}(\partial_\varepsilon)\, g^{N-1} \Big\{ D_\varepsilon(\alpha\omega+1) - (N-1) \Big\}\, \omega^{r-1} \Big|_{\varepsilon=0} = 0. \tag{5.27}$$

Finally, reminding that $D_\varepsilon = \omega\partial_\omega$ this last equation directly leads to the recurrence relation

$$\alpha r H_N^{(r+1)} - (N-r) H_N^{(r)} = 0 \tag{5.28}$$

where we have used (5.10). Note, that actual choice of the function $\tau(\varepsilon)$ have been governed to have all coefficients in the braces in (5.25) to be polynomials in ω. Note also the 'anti-normal' ordering of the differential operator there. Below we shall proceed in other cases exactly in the same way.

Solving the recurrence (5.28) we find

$$H_N^{(r)} = \binom{N-1}{r-1} \frac{\left[\tan^2(\phi/2)\right]^{N-r}}{\left[1+\tan^2(\phi/2)\right]^{N-1}} \,. \tag{5.29}$$

Here the proper normalization is achieved by directly satisfying condition (2.10).

Thus, the result of Ref. [4] is readily recovered. Moreover, recalling (3.8), and specializing $\lambda = \pi/2$, that is $\phi = \pi/2$ or $\tan(\phi/2) = 1$, the expression (3.6) for the refined 2-enumeration, obtained in [7, 12, 13, 15], is immediately reproduced.

5.3. The ice point and the refined enumeration of ASMs

Now we turn to the interesting case of the ice point, which corresponds to the values $\lambda = \pi/2$ and $\eta = \pi/6$. In this case the boundary correlator $H_N^{(r)}$ is equivalent to the refined 1-enumeration of ASMs.

In this case the boundary correlator is given by formula (5.10) where

$$\omega(\epsilon) = \frac{\sin\varepsilon}{\sin(\varepsilon-\pi/3)}, \qquad g(\epsilon) = \frac{\sin(\varepsilon-\pi/3)}{\sin(\varepsilon+\pi/3)}, \tag{5.30}$$

and $p_{N-1}(x)$ is Continuous Hahn polynomial, see (4.17). The finite difference equation (5.5) reads

$$\left(\frac{1}{3}-\frac{\mathrm{i}x}{6}\right)\left(\frac{2}{3}-\frac{\mathrm{i}x}{6}\right) p_{N-1}(x+6\mathrm{i}) + \left[\frac{x^2}{18}-\frac{4}{9}-N(N-1)\right] p_{N-1}(x) \\ + \left(\frac{1}{3}+\frac{\mathrm{i}x}{6}\right)\left(\frac{2}{3}+\frac{\mathrm{i}x}{6}\right) p_{N-1}(x-6\mathrm{i}) = 0 \tag{5.31}$$

Similarly to the previous case, employing $p_{N-1}(\partial_\varepsilon \pm 6\mathrm{i}) = \mathrm{e}^{\mp 6\mathrm{i}\varepsilon} p_{N-1}(\partial_\varepsilon)\mathrm{e}^{\pm 6\mathrm{i}\varepsilon}$ we obtain

$$p_{N-1}(\partial_\varepsilon)\; K_\varepsilon\; [g(\epsilon)]^{N-1}[\omega(\epsilon)]^{r-1}\tau(\epsilon)\Big|_{\varepsilon=0} = 0 \tag{5.32}$$

where K_ε is the second order differential operator

$$K_\varepsilon = \frac{1}{9}\sin 3\varepsilon\, \partial_\varepsilon^2\, \sin 3\varepsilon + \frac{1}{9}\sin^2 3\varepsilon - N(N-1). \tag{5.33}$$

Now our aim is to obtain the recurrence relation for $H_N^{(r)}$ and find its solution.

The operator D_ε in this case reads

$$D_\varepsilon =: \omega\partial_\omega = -\frac{2}{\sqrt{3}}\sin\varepsilon\sin(\varepsilon-\pi/3)\partial_\varepsilon. \tag{5.34}$$

Taking into account the identity

$$\sin 3\varepsilon = -4\sin\varepsilon\sin(\varepsilon+\pi/3)\sin(\varepsilon-\pi/3) \tag{5.35}$$

and

$$\sin\varepsilon = -\frac{\sqrt{3}}{2}\frac{\omega}{\sqrt{\omega^2-\omega+1}}, \qquad \sin(\varepsilon+\pi/3) = -\frac{\sqrt{3}}{2}\frac{\omega-1}{\sqrt{\omega^2-\omega+1}}, \tag{5.36}$$

we reexpress the operator K_ε in terms of ω and the operator D_ε, with the latter acting from the very left,

$$K_\varepsilon = \Big[D_\varepsilon^2(\omega-1)^2 - D_\varepsilon(\omega^2-1) - N(N-1)(\omega^2-\omega+1)\Big]\frac{1}{\omega^2-\omega+1}. \tag{5.37}$$

Taking into account that

$$g^{-1}(D_\varepsilon g) = -\frac{\omega}{\omega-1}, \tag{5.38}$$

for the operator $\widetilde{K}_\varepsilon := g^{-N+1}K_\varepsilon g^{N-1}$ we obtain

$$\widetilde{K}_\varepsilon = \Big\{D_\varepsilon^2(\omega-1) - D_\varepsilon[(2N-1)\omega+1] + N(N-1)\Big\}\frac{\omega-1}{\omega^2-\omega+1}. \tag{5.39}$$

Choosing $\tau(\varepsilon) = (\omega^2-\omega+1)/(\omega-1)$ in (5.32) allows us to write

$$p_{N-1}(\partial_\varepsilon)\, g^{N-1}\Big\{D_\varepsilon^2(\omega-1) - D_\varepsilon[(2N-1)\omega+1] + N(N-1)\Big\}\,\omega^{r-1}\Big|_{\varepsilon=0} = 0 \tag{5.40}$$

which, recalling that $D_\varepsilon = \omega\partial_\omega$, and Eqn. (5.10), immediately gives us the following recurrence relation

$$r(r-2N+1)H_N^{(r+1)} - (r-N)(N+r-1)H_N^{(r)} = 0. \tag{5.41}$$

This recurrence can be easily solved modulo a normalization constant

$$H_N^{(r)} = \text{const} \times \frac{(N+r-2)!\,(2N-1-r)!}{(r-1)!\,(N-r)!}. \tag{5.42}$$

A possible way to fit the normalization condition (2.10), is to consider the generating function $H_N(z)$ defined via Eqn. (5.15). The result reads

$$H_N(z) = \frac{(2N-1)!\,(2N-2)!}{(N-1)!\,(3N-2)!}\ {}_2F_1\left(\begin{matrix}1-N\,,\ N\\ 2-2N\end{matrix}\,\middle|\, z\right) \tag{5.43}$$

where the proper normalization is easily determined through Chu-Vandermonde identity

$$ {}_2F_1\left(\begin{matrix}-m\,,\ b\\ c\end{matrix}\middle|\,1\right)=\frac{(c-b)_m}{(c)_m};\qquad (a)_m:=a(a+1)\cdots(a+m-1). \tag{5.44}$$

Inspecting the coefficient of z^{r-1} in (5.43) we finally obtain

$$H_N^{(r)}=\frac{\binom{N+r-2}{N-1}\binom{2N-1-r}{N-1}}{\binom{3N-2}{N-1}}. \tag{5.45}$$

The refined 1-enumeration of ASMs, Eqn. (3.5), immediately follows from the last formula and the relation (3.8).

It is worth to noting that the proof presented here for the refined 1-enumeration of ASMs is considerably simpler in comparison to that of Refs. [23, 24], which were based on the inhomogeneous square ice partition function formula of Ref. [2].

5.4. The $\Delta=-1/2$ symmetric point and the refined 3-enumeration of ASMs

We shall now apply the same approach to compute the boundary correlator $H_N^{(r)}$ in the case of the $\Delta=-1/2$ symmetric point, i.e., when $\eta=\pi/3$ and $\lambda=\pi/2$. This case corresponds to the refined 3-enumeration of ASMs.

As shown in Section 4.4, in this case there are two sets of Continuous Dual Hahn polynomials, with differently specified parameters, see Eqns. (4.31) and (4.32), which are related to the determinant of the Hankel matrix $\mathcal{Z}$, see Eqn. (2.6). The appearance of two sets of polynomials is due to the factorization of the Hankel determinant $\det\mathcal{Z}$. The explicit form of this factorization in turn depends on whether N is odd or even, see Eqn. (4.26). Such factorization occurs also for the determinant of matrix $\mathcal{W}$, defined by (5.2). Denoting $D_N(x)=\det\mathcal{W}$, similarly to (4.26) we have

$$D_{2m}(x)=D_m^{(0)}\,x\,D_m^{(1)}(x^2),\qquad D_{2m+1}(x)=D_{m+1}^{(0)}(x^2)\,D_m^{(1)}, \tag{5.46}$$

where $D_m^{(\sigma)}(x^2)$ stands for the determinants of the matrices built from the even moments of measures $\mu^{(\sigma)}$, with entries of the last column replaced by x^{2j}. These determinants are precisely the Continuous Dual Hahn polynomials, $p_{m-1}^{(\sigma)}(x^2)$, specified by Eqns. (4.31) and (4.32). Therefore up to overall constants determined by $D_m^{(1)}$ and $D_m^{(0)}$, we have

$$p_{N-1}(x)=\begin{cases}\text{const}\times p_m^{(0)}(x^2) & \text{if } N \text{ is odd, } N=2m+1\\ \text{const}\times x\,p_m^{(1)}(x^2) & \text{if } N \text{ is even, } N=2m+2\end{cases}. \tag{5.47}$$

In this way the polynomials appearing in (5.3) are expressed in terms of Continuous Dual Hahn polynomials.

Hence, in the considered case representation (5.10) acquires the form

$$H_{2m+2}^{(r)} = \text{const} \times p_m^{(1)}(\partial_\varepsilon^2)\, \partial_\varepsilon\, g^{2m+1} \omega^{r-1} \Big|_{\varepsilon=0} \tag{5.48}$$

$$H_{2m+3}^{(r)} = \text{const} \times p_{m+1}^{(0)}(\partial_\varepsilon^2)\, g^{2m+2} \omega^{r-1} \Big|_{\varepsilon=0} \tag{5.49}$$

where

$$\omega = \omega(\varepsilon) = -\frac{\sin \varepsilon}{\sin(\varepsilon + \pi/3)}, \qquad g = \frac{1}{\omega - 1}. \tag{5.50}$$

Here we have shifted $m \to m+1$ for the N odd case for later convenience.

Starting from expressions (5.48) and (5.49), one can derive the recurrence relation for $H_N^{(r)}$ in the cases of N even and odd, respectively. Here instead we shall proceed differently using the fact that it is possible to express the boundary correlator in terms of a single set of polynomials, and thus treat both cases in a unified and simplified way. Indeed, denoting

$$u_{2m}^{(\sigma)}(x) := p_m^{(\sigma)}(x^2), \qquad \sigma = 0, 1, \tag{5.51}$$

let us consider the polynomials

$$\tilde{u}_{2m}(x) = S_m\left(\frac{x^2}{36}; \frac{1}{2}, -\frac{1}{6}, \frac{1}{6}\right). \tag{5.52}$$

These polynomials arise when one studies the action of the 'forward shift operator' for the Continuous Dual Hahn polynomials (see, e.g., Ref. [30], Eqn. (1.3.7)),

$$\tilde{u}_{2m+2}(x + 3\mathrm{i}) - \tilde{u}_{2m+2}(x - 3\mathrm{i}) = -\mathrm{i}(m+1)\frac{x}{3}\, u_{2m}^{(1)}(x). \tag{5.53}$$

Surprisingly enough, changing the sign in LHS of this relation gives us again Continuous Dual Hahn polynomials, which are exactly those defined above as $u_{2n}^{(0)}(x)$, i.e.,

$$\tilde{u}_{2m}(x + 3\mathrm{i}) + \tilde{u}_{2m}(x - 3\mathrm{i}) = 2u_{2m}^{(0)}(x). \tag{5.54}$$

This relation can be easily proven by expressing all polynomials on both sides as truncated hypergeometric series. It is worth to note that this relation is not a specialization of some general relation for the Continuous Dual Hahn polynomials, but is instead specific for the particular choice of parameters of the polynomials.

As a direct consequence of relations (5.53) and (5.54) we can rewrite the correlator as

$$H_{2m+2}^{(r)} = \text{const} \times \tilde{u}_{2m+2}(\partial_\varepsilon)\ \sin 3\varepsilon\, g^{2m+1} \omega^{r-1} \Big|_{\varepsilon=0}, \tag{5.55}$$

$$H_{2m+3}^{(r)} = \text{const} \times \tilde{u}_{2m+2}(\partial_\varepsilon)\ \cos 3\varepsilon\, g^{2m+2} \omega^{r-1} \Big|_{\varepsilon=0}. \tag{5.56}$$

Noticing that

$$\sin 3\varepsilon = -\frac{3\sqrt{3}}{2}\frac{\omega(\omega+1)}{(\omega^2+\omega+1)^{3/2}}, \qquad \cos 3\varepsilon = -\frac{(\omega-1)(2\omega+1)(\omega+2)}{2(\omega^2+\omega+1)^{3/2}} \tag{5.57}$$

and recalling that $g = 1/(\omega - 1)$ it can be easily seen that the correlator possesses the structure

$$H^{(r)}_{2m+2} = \frac{B^{(r-1)}_{2m} + B^{(r-2)}_{2m}}{2}, \tag{5.58}$$

$$H^{(r)}_{2m+3} = \frac{2B^{(r-1)}_{2m} + 5B^{(r-2)}_{2m} + 2B^{(r-2)}_{2m}}{9}, \tag{5.59}$$

where the quantities $B^{(r)}_{2m}$ are defined as

$$B^{(r)}_{2m} := b_{2m}\tilde{u}_{2m+2}(\partial_\varepsilon)\frac{g^{2m+1}\omega^{r+2}}{(\omega^2+\omega+1)^{3/2}}\bigg|_{\varepsilon=0}; \qquad r = 0, 1, \ldots, 2m. \tag{5.60}$$

Here, b_{2m} is some normalization constant; we assume that

$$\sum_{r=0}^{2m} B^{(r)}_{2m} = 1. \tag{5.61}$$

This condition, together with Eqns. (5.58) and (5.59), ensures that normalization condition (2.10) is satisfied. As in previous cases, the proper normalization will be restored at the end of computation, according to condition (5.61).

Thus, instead of studying the correlator for N even and odd separately it is enough to consider the quantity $B^{(r)}_{2m}$ which is defined by an essentially similar formula, Eqn. (5.60). The procedure developed previously will be applied now to derive a recurrence relation for $B^{(r)}_{2m}$. The finite-difference equation (5.7) reads

$$(1+x^2)\Big[\tilde{u}_{2m+2}(x+6\mathrm{i}) - \tilde{u}_{2m+2}(x-6\mathrm{i})\Big] - 24\mathrm{i}(m+1)x\tilde{u}_{2m+2}(x) = 0. \tag{5.62}$$

Using this equation, like in the previous cases, we can write

$$\tilde{u}_{2m+2}(\partial_\varepsilon)K_\varepsilon f(\varepsilon)\Big|_{\varepsilon=0} = 0 \tag{5.63}$$

where K_ε is the second order differential operator

$$K_\varepsilon = \frac{1}{\sqrt{3}}\Big[\sin 3\varepsilon \cos 3\varepsilon \left(\partial^2_\varepsilon + 1\right) - 6(m+1)\partial_\varepsilon\Big] \tag{5.64}$$

while $f(\varepsilon)$ is some arbitrary function. Obviously, to obtain a recurrence relation for the quantities $B^{(r)}_{2m}$, the function $f(\varepsilon)$ is to be chosen of the form

$$f = \tau\,\frac{g^{2m+1}\omega^{r+2}}{(1+\omega+\omega^2)^{3/2}} \tag{5.65}$$

where again τ is some function to be chosen later.

Taking into account that the differential operator D_ε in the considered case reads

$$D_\varepsilon = \omega\partial_\omega = \frac{2}{\sqrt{3}}\sin\varepsilon\sin(\varepsilon+\pi/3)\partial_\varepsilon \tag{5.66}$$

and passing to the variable ω, we obtain

$$\begin{aligned} K_\varepsilon = D_\varepsilon^2 &\frac{(2\omega^2+5\omega+2)(\omega^2-1)}{\omega(\omega^2+\omega+1)} \\ &+ D_\varepsilon\left[\frac{(\omega^2+4\omega+1)(2\omega^2+5\omega+2)(\omega-1)^2}{\omega(\omega^2+\omega+1)^2} + 4m\frac{(\omega^2+\omega+1)}{\omega}\right] \\ &- \left[\frac{(8\omega^4+4\omega^3+57\omega^2+4\omega+8)(2\omega^2+5\omega+2)(\omega^2-1)}{4\omega(\omega^2+\omega+1)^3} + 4m\frac{\omega^2-1}{\omega}\right]. \end{aligned} \tag{5.67}$$

Defining operator $\widetilde{K}_\varepsilon$ by the formula

$$K_\varepsilon\,\frac{g^{2m+1}}{(\omega^2+\omega+1)^{3/2}} = \frac{g^{2m+1}}{(\omega^2+\omega+1)^{3/2}}\,\widetilde{K}_\varepsilon \tag{5.68}$$

we may rewrite condition (5.63)

$$\tilde{u}_{2m+2}(\partial_\varepsilon)\frac{g^{2m+1}}{(\omega^2+\omega+1)^{3/2}}\,\widetilde{K}_\varepsilon\tau\omega^{r+2}\Big|_{\varepsilon=0} = 0, \tag{5.69}$$

where

$$\begin{aligned} \widetilde{K}_\varepsilon = \Big\{ &D_\varepsilon^2\,(2\omega^2+5\omega+2)(\omega^2-1) \\ &- D_\varepsilon\Big[(2\omega^2+5\omega+2)(7\omega^2-2\omega-1) + 4m(\omega^4+5\omega^3+4\omega^2-1)\Big] \\ &+ 2(2\omega^2+5\omega+2)(5\omega^2-2\omega+1) + 2m(4\omega^4+26\omega^3+17\omega^2+5\omega+2) \\ &+ 4m^2\omega(3\omega^2+4\omega+2)\Big\}\frac{1}{\omega(\omega^2+\omega+1)}. \end{aligned} \tag{5.70}$$

Choosing $\tau = \omega+1+\omega^{-1}$ we therefore obtain

$$\begin{aligned} \tilde{u}_{2m+2}(\partial_\varepsilon)\frac{g^{2m+1}}{(\omega^2+\omega+1)^{3/2}}\Big\{ &D_\varepsilon^2\,(2\omega^4+5\omega^3-5\omega-2) \\ &- D_\varepsilon\Big[(14\omega^4+31\omega^3+2\omega^2-9\omega-2) + 4m(\omega^4+5\omega^3+4\omega^2-1)\Big] \\ &+ 2(10\omega^4+21\omega^3+2\omega^2+\omega+2) + 2m(4\omega^4+26\omega^3+17\omega^2+5\omega+2) \\ &+ 4m^2(3\omega^3+4\omega^2+2\omega)\Big\}\omega^r\Big|_{\varepsilon=0} = 0. \end{aligned} \tag{5.71}$$

This equation leads immediately to the recurrence relation for $B_{2m}^{(r)}$. The latter enjoy the symmetry $B_{2m}^{(r)} = B_{2m}^{(2m-r)}$ which obviously guarantees $H_N^{(r)} = H_N^{(N-r+1)}$ (this is also known as the top-bottom or left-right symmetry of the set of ASMs). To make this, symmetry more apparent it is convenient to write this recurrence relation in terms of $E_m^{(r)} := B_{2m}^{(m+r)}$, so that $E_m^{(r)} = E_m^{(-r)}$, with $r = -m, \dots, m$. The recurrence relation reads

$$\begin{aligned} 2(r-m-2)(r+m+1)\,E_m^{(r-2)} + (5r^2+10rm+r-3m^2-9m-6)\,E_m^{(r-1)} \\ + 2(1+8m)r\,E_m^{(r)} - (5r^2-10rm-r-3m^2-9m-6)\,E_m^{(r+1)} \\ - 2(r-m-1)(r+m+2)\,E_m^{(r+2)} = 0. \end{aligned} \quad (5.72)$$

Using this relation one can find recursively all $E_m^{(r)}$'s for any given m assuming that $E_m^{(r)} = 0$ if $|r| > m$. However, since the recurrence relation is five-term it can hardly be solved, e.g., by guessing its solution. The remaining of this Section is an exposition of a possible way to solve it explicitly. As we shall show now this can be done by successive transformations of the generating function for $B_{2m}^{(r)}$.

Consider the generating function

$$E_m(z) := \sum_{r=-m}^{m} E_m^{(r)}\, z^r = z^{-m} \sum_{r=0}^{2m} B_{2m}^{(r)}\, z^r, \qquad E_m(z) = E_m(z^{-1}). \quad (5.73)$$

which satisfies, as a direct consequence of recurrence relation (5.72), the following homogeneous second order linear differential equation:

$$\begin{aligned} \Big\{ (z-z^{-1})(2z+5+2z^{-1})(z\partial_z)^2 \\ + \Big[(2z+5+2z^{-1})(3z-2+3z^{-1}) + 2m(5z+8+5z^{-1})\Big] z\partial_z \\ - (z-z^{-1})\Big[m(6z-1+6z^{-1}) + m^2(2z+3+2z^{-1})\Big]\Big\} E_m(z) = 0\,. \end{aligned} \quad (5.74)$$

The solution of this equation we are interested in is a polynomial of degree $2m$ times factor z^{-m}, see Eqn. (5.73). Let us now consider the substitution

$$z = -\frac{x-q}{qx-1}, \qquad q = \exp(i\pi/3) \quad (5.75)$$

which maps the six singularities of this equation lying on the real axis of the complex z-plane, at points $z = -1, 0, 1/2, 1, 2, \infty$, onto the six roots of equation $x^6 = 1$, i.e., $x = 1, q, q^2, q^3, q^4, q^5$, lying on the unit circle of the complex x-plane. Note moreover the trivial but useful identity $1 + q^2 = q$. Taking into account symmetry (5.73) enjoyed by $E_m(z)$, and the identity $(qx - 1)(x - q) = (qx)(x - 1 + x^{-1})$, it is easily seen that the function

$$V_m(x) := (x-1+x^{-1})^m\, E_m\left(-\frac{x-q}{qx-1}\right) \quad (5.76)$$

is again of the form x^{-m} times a polynomial of order $2m$ in x, symmetric under $x \to 1/x$, i.e., the function $V_m(\mathrm{e}^{\mathrm{i}\varphi})$ is an even trigonometric polynomial of degree m in φ. Equation (5.74) translates into the following equation:

$$\Big\{(x^3 - x^{-3})(x\partial_x)^2 + \Big[(x+1+x^{-1})(x^2-5x+12-5x^{-1}+x^{-2}) - 2m(x+x^{-1})(x^2-5+x^{-2})\Big]x\partial_x - (x-x^{-1})\Big[m(x^2-4x+13-4x^{-1}+x^{-2}) - m^2(x^2-7+x^{-2})\Big]\Big\}V_m(x) = 0. \quad (5.77)$$

At the first glance there is no advantage in considering the function $V_m(x)$ since the underlying recurrence relation for expansion coefficients of $V_m(x)$ is even worse than that for those of $E_m(z)$: it is a seven-term relation. However, if one considers instead the function,

$$h_m(x) = (x - x^{-1})^{2m+1}\,(x + 2 + x^{-1})V_m(x) \quad (5.78)$$

then it appears that the differential equation satisfied by $h_m(x)$ contains only integer powers of x^3 in their coefficients

$$\Big\{(x^3-x^{-3})(x\partial_x)^2 - \Big[3(2m+1)(x^3+x^{-3})-6\Big]x\partial_x + (3m+1)(3m+2)(x^3-x^{-3})\Big\}h_m(x) = 0, \quad (5.79)$$

or, using the variable ϕ, related to x as $x = \exp(\mathrm{i}\varphi)$, we have

$$\Big\{\partial_\varphi^2 - 3\Big[(2m+1)\cot 3\varphi - \frac{1}{\sin 3\varphi}\Big]\partial_\varphi - (3m+1)(3m+2)\Big\}h_m(\mathrm{e}^{\mathrm{i}\varphi}) = 0. \quad (5.80)$$

It is easy to see that the underlying recurrence relation now is just a three-term and it appears to be solvable explicitly.

The particular solution of Eqn. (5.80) which we are looking for is the function $h_m(\mathrm{e}^{\mathrm{i}\varphi})$ being an antisymmetric trigonometric polynomial of degree $3m+2$ in φ, see Eqn. (5.78). Hence, we are led to use the substitution of the form

$$h(\mathrm{e}^{\mathrm{i}\varphi}) = \sum_{k=0}^{2m+1} \gamma_k \sin[(3m+2-3k)\varphi]. \quad (5.81)$$

The coefficients γ_k have to satisfy the recurrence relation

$$(3k+2)(3k+3)\gamma_{k+1} + (3m+2-3k)\gamma_k - \big[3(2m-k)+7\big]\big[3(2m-k)+6\big]\gamma_{k-1} = 0. \quad (5.82)$$

Inspecting the explicit form of the first few γ_k's, allows us to guess that the solution of the recurrence relation is

$$\gamma_{2l} = \gamma_0\,(-1)^l\,\frac{(-m-2/3)_l}{(1/3)_l}\binom{m}{l}, \qquad \gamma_{2l+1} = \gamma_0\,(-1)^l\,\frac{(-m-2/3)_{l+1}}{(1/3)_{l+1}}\binom{m}{l}, \quad (5.83)$$

that can be easily verified directly. Here $(a)_l$ stands for Pochhammer symbol, defined previously in Eqn. (5.44). Thus, we have obtained for the function $h_m(x)$ the following expression:

$$h_m(x) = c_m\left(g_m(x) + \frac{3m+2}{3m+1}\, f_m(x)\right) \tag{5.84}$$

where the functions $f_m(x)$ and $g_m(x)$ are given by

$$g_m(x) := \sum_{k=0}^{m} \binom{m+2/3}{k}\binom{m-2/3}{m-k}\left(x^{3m+2-6k} - x^{-3m-2+6k}\right) \tag{5.85}$$

$$f_m(x) := \sum_{k=0}^{m} \binom{m+1/3}{k}\binom{m-1/3}{m-k}\left(x^{3m+1-6k} - x^{-3m-1+6k}\right) \tag{5.86}$$

and c_m is some constant such that $V_m(1) = E_m(1) = 1$; this normalization follows from the normalization condition (5.61) for $B_{2m}^{(r)}$.

It is worth noticing that function $h_m(x)$ as given in Eqns. (5.84), (5.85), (5.86), in connection with the problem of refined 3-enumeration has also been found by Stroganov in Ref. [27], within a different approach, using a certain functional equation satisfied by the inhomogeneous square ice partition function. This functional equation had already been investigated in [31] as the Baxter T-Q equation for the ground state of XXZ Heisenberg spin-$1/2$ chain at $\Delta = -1/2$ and with odd number of sites $N = 2m + 1$.

The problem we are facing now is to reconstruct, from the explicit knowledge of function $h_m(x)$, the generating function $E_m(t)$. This amounts essentially to divide out the factors $(x - x^{-1})^{2m+1}$ and $(x+2+x^{-1})$ to find function $V_m(x)$ first, and next to change back to the original variable $z = (x - q)/(1 - qx)$ to recover $E_m(z)$. We shall follow the line of our previous paper [32] where this procedure was fulfilled.

To undertake the first step in this program it is useful to note that the functions $f_m(x)$ and $g_m(x)$ can also be written as follows

$$g_m(x) = \frac{\Gamma(m+1/3)}{m!\,\Gamma(1/3)}\left[x^{3m+2}\,{}_2F_1\left(\left.\begin{matrix}-m\,,\ -m-2/3\\ 1/3\end{matrix}\right| x^{-6}\right) - x^{-3m-2}\,{}_2F_1\left(\left.\begin{matrix}-m\,,\ -m-2/3\\ 1/3\end{matrix}\right| x^{6}\right)\right], \tag{5.87}$$

$$f_m(x) = \frac{\Gamma(m+4/3)}{m!\,\Gamma(4/3)}\left[x^{-3m+1}\,{}_2F_1\left(\left.\begin{matrix}-m\,,\ -m+1/3\\ 4/3\end{matrix}\right| x^{6}\right) - x^{3m-1}\,{}_2F_1\left(\left.\begin{matrix}-m\,,\ -m+1/3\\ 4/3\end{matrix}\right| x^{-6}\right)\right]. \tag{5.88}$$

Since the parameters of the hypergeometric functions entering these expressions differ by integers one can expect that $g_m(x)$ and $f_m(x)$ are connected by some three-term relations via

Gauss relations (see, e.g., §2.8 of [29]). Indeed, using Gauss relations it can be shown that

$$
{}_2F_1\left(\begin{matrix}-m\,,\ -m-2/3\\ 1/3\end{matrix}\middle|\zeta\right) = \frac{3m+4}{2}\ {}_2F_1\left(\begin{matrix}-m-1\,,\ -m-2/3\\ 4/3\end{matrix}\middle|\zeta\right) - \frac{3m+2}{2}\,(1+\zeta)\,{}_2F_1\left(\begin{matrix}-m\,,\ -m+1/3\\ 4/3\end{matrix}\middle|\zeta\right) \tag{5.89}
$$

and therefore one can express the function $g_m(x)$ in terms of $f_m(x)$ and $f_{m+1}(x)$:

$$
g_m(x) = \frac{3m+2}{2(3m+1)}\,(x^3+x^{-3})\,f_m(x) - \frac{3(m+1)}{2(3m+1)}\,f_{m+1}(x). \tag{5.90}
$$

Substituting (5.90) into (5.84) we obtain an analogous formula for function $h_m(x)$:

$$
h_m(x) = c_m\,\frac{3m+2}{2(3m+1)}\left[(x^3+2+x^{-3})\,f_m(x) - \frac{3m+3}{3m+2}\,f_{m+1}(x)\right]. \tag{5.91}
$$

Introducing now the function $Q_m(x)$, implicitly defined by

$$
f_m(x) = (x-x^{-1})^{2m+1}Q_m(x) \tag{5.92}
$$

the factor $(x-x^{-1})^{2m+1}(x+2+x^{-1})$ can be formally extracted in expression (5.91) for $h_m(x)$, thus giving us a representation for $V_m(x)$ in terms of $Q_m(x)$ and $Q_{m+1}(x)$,

$$
V_m(x) = c_m\,\frac{3m+2}{2(3m+1)}\left[(x-1+x^{-1})^2\,Q_m(x) - \frac{3m+3}{3m+2}\,(x-2+x^{-1})\,Q_{m+1}(x)\right]. \tag{5.93}
$$

The meaning of this procedure becomes apparent by noticing that the function $Q_m(x)$ can be found explicitly from expression (5.88) for the function $f_m(x)$ in virtue of the so-called cubic transformation for the Gauss hypergeometric function. The details of this derivation are given in Appendix A. For function $Q_m(x)$ the following explicit formula is valid

$$
Q_m(x) = \frac{(2m)!}{3^m\,(m!)^2}\left(\frac{qx^{-1}-q^{-1}x}{q-q^{-1}}\right)^m {}_2F_1\left(\begin{matrix}-m\,,\ m+1\\ -2m\end{matrix}\middle|\frac{qx-q^{-1}x^{-1}}{qx^{-1}-q^{-1}x}\right). \tag{5.94}
$$

Hence, function $V_m(x)$ can be found by inserting this expression into (5.93), thus giving

$$
\begin{aligned}
V_m(x) = \frac{(2m)!\,(2m+1)!}{m!\,(3m+1)!}
&\times\Bigg[(x-1+x^{-1})^2\left(\frac{qx^{-1}-q^{-1}x}{q-q^{-1}}\right)^m {}_2F_1\left(\begin{matrix}-m\,,\ m+1\\ -2m\end{matrix}\middle|\frac{qx-q^{-1}x^{-1}}{qx^{-1}-q^{-1}x}\right)\\
&\quad-\frac{2(2m+1)}{3m+2}\,(x-2+x^{-1})\left(\frac{qx^{-1}-q^{-1}x}{q-q^{-1}}\right)^{m+1}\\
&\quad\times\,{}_2F_1\left(\begin{matrix}-m-1\,,\ m+2\\ -2m-2\end{matrix}\middle|\frac{qx-q^{-1}x^{-1}}{qx^{-1}-q^{-1}x}\right)\Bigg].
\end{aligned} \tag{5.95}
$$

Here we have written the expression for $V_m(x)$ taking into account also the proper normalization of this function, $V_m(1) = 1$. The normalization can be verified by virtue of Chu-Vandermonde identity (5.44) and it is equivalent to the choice

$$c_m = (3m+1)\,\frac{3^{m+1} m!\,(2m+2)!}{(3m+3)!}\,. \tag{5.96}$$

of this constant in Eqns. (5.84) and (5.93). It is worth to mention that functions $Q_m(x)$, Eqn. (5.94), and $V_m(x)$, Eqn. (5.95), are the first and the second solution, respectively, of Baxter T-Q equation for the ground state of XXZ Heisenberg $\Delta = -1/2$ spin chain with odd number of sites $N = 2m+1$, see Refs. [24, 27, 31, 33].

To obtain function $E_m(z)$ from the given expression for $V_m(x)$ one can use the formula

$$E_m(z) = 3^{-m}(z+1+z^{-1})^m V_m\left(\frac{z+q}{zq+1}\right) \tag{5.97}$$

which is the inverse of (5.76). Applying this transformation to (5.95) we obtain

$$\begin{aligned} E_m(z) = \frac{(2m)!\,(2m+1)!}{3^m\, m!\,(3m+1)!}\,\frac{1}{(z+1+z^{-1})^2}\Bigg[9(z+2)^m\,{}_2F_1\left(\left.\begin{matrix}-m\,,\ m+1\\ -2m\end{matrix}\,\right|\frac{z^{-1}+2}{z+2}\right) \\ +\frac{2(2m+1)}{3m+2}\,(z-2+z^{-1})(z+2)^{m+1}\,{}_2F_1\left(\left.\begin{matrix}-m-1\,,\ m+2\\ -2m-2\end{matrix}\,\right|\frac{z^{-1}+2}{z+2}\right)\Bigg]. \end{aligned} \tag{5.98}$$

This expression is however not the final answer yet, since the factor $(z+1+z^{-1})^2$ in the denominator is to be cancelled explicitly (recall that $E_m(z)$ is to be z^{-m} times a polynomial of degree $2m$ in z, hence the expression in the brackets contains implicitly the proper factor $(z+1+z^{-1})^2$). This can be achieved using again Gauss relations; for details, see Appendix B. The final result is

$$\begin{aligned} E_m(z) = \frac{(2m)!\,(2m+2)!}{3^m\,(m+1)!\,(3m+2)!}\Bigg[(2m+1)(z+2)^m\ {}_2F_1\left(\left.\begin{matrix}-m\,,\ m+2\\ -2m-1\end{matrix}\,\right|\frac{1+2z}{z(z+2)}\right) \\ -3m\,(z+2)^{m-1}\,{}_2F_1\left(\left.\begin{matrix}-m+1\,,\ m+2\\ -2m\end{matrix}\,\right|\frac{1+2z}{z(z+2)}\right)\Bigg]. \end{aligned} \tag{5.99}$$

This completes the derivation of function $E_m(x)$.

We can now extract from this expression closed formulae for the coefficients $B_{2m}^{(r)}$, which are related to $E_m(z)$ through Eqn. (5.73). To find these coefficients we expand (5.99) in power series in z, thus expressing $E_m(z)$, as a triple sum, and next we apply Chu-Vandermonde formula (5.44) to make the sum with respect to the index defining the hypergeometric series in (5.99), thus expressing $E_m(z)$ as a double sum. These two summations can be rearranged in such a way that one of them becomes with respect to r while the other one defines the coefficients of power expansion in z. We obtain

$$B_{2m}^{(r)} = \frac{(2m+1)!\,m!}{3^m\,(3m+2)!} \sum_{\ell=\max(0,r-m)}^{[r/2]} (2m+2-r+2\ell) \binom{3m+3}{r-2\ell} \times \binom{2m+\ell-r+1}{m+1} \binom{m+\ell+1}{m+1} 2^{r-2\ell}. \quad (5.100)$$

Here $[r/2]$ denotes integer part of $r/2$. This expression for $B_{2m}^{(r)}$ indeed solves the five terms recurrence relation (5.72) (recall that $E_m^{(r)} = B_{2m}^{(m+r)}$).

We would like also to mention that formula (5.100) can also be written in terms of terminating hypergeometric series, for instance, as follows:

$$B_{2m}^{(r)} = \frac{2^r \binom{3m+3}{r} \binom{2m+1-r}{m+1}}{3^m \binom{3m+2}{m+1}} \left[2\,{}_4F_3\left(\begin{matrix} -(r-1)/2\,,\ -r/2\,,\ m+2\,,\ 2m+2-r \\ (3m+4-r)/2\,,\ (3m+5-r)/2\,,\ m-r+1 \end{matrix} \middle| \frac{1}{4} \right) - \frac{r}{m+1}\,{}_4F_3\left(\begin{matrix} -(r-1)/2\,,\ -r/2+1\,,\ m+2\,,\ 2m+2-r \\ (3m+4-r)/2\,,\ (3m+5-r)/2\,,\ m-r+1 \end{matrix} \middle| \frac{1}{4} \right) \right]. \quad (5.101)$$

This formula is valid for $r = 0, 1, \ldots, m$ (a similar expression for $r = m+1, m+2, \ldots, 2m$ can be simply obtained through the replacement $r \to 2m - r$ in RHS of (5.101). The two ${}_4F_3$ in (5.101) can be further combined into a single ${}_5F_4$. Analogous formulae for $B_{2m}^{(r)}$ in terms of terminating hypergeometric series of argument 4 may be written down as well. Analyzing these expressions, however it seems to be hard to perform the sum in (5.100) in a closed form, even if very suggestive similarities can be found with known summation formulae, see §§7.5 and 7.6, especially §7.6.4, of Ref. [34].

The complete expression for the boundary correlator $H_N^{(r)}$ can be readily obtained by inserting (5.100) or (5.101) into (5.58) and (5.59). Finally, the refined 3-enumeration of ASMs, $A(N, r; 3)$, follows from multiplying the result for $H_N^{(r)}$ by the total number of 3-enumerated ASMs, $A(N, 3)$, see (3.8) and (3.4).

6. Conclusion

The main purpose of the present paper was to point out the close relation between ASM enumerations and some classical orthogonal polynomials. In particular, in Section 4 we have shown an alternative way to recover known results for the partition function of the six-vertex model with DWBC, in the three cases of the $\Delta = 0$ line, and the symmetric $\Delta = 1/2$ and $\Delta = -1/2$ points, corresponding to 2-, 1- and 3-enumeration of ASMs, respectively. The derivation we have presented in Section 4 is in our opinion extremely simple and straightforward. It is based on the fact that the Hankel determinant entering the representation for the partition function can be naturally related in these three cases to Meixner-Pollaczek, Continuous Hahn, and Continuous Dual Hahn polynomials, respectively.

It is to emphasize that the three considered cases, $x = 1, 2, 3$, are the only ones in which 'factorized' answers for ASMs' x-enumerations are known to exist. The fact that no set of polynomials corresponding to other choices of the parameters can be found in the Askey scheme strongly suggests that no 'factorized' answer exist for other values of x. However, one might speculate that for some values of the crossing parameter η, the partition function of the six-vertex model with DWBC might still admit a 'factorized' form, but in terms of q-numbers (with q related to η through $q = \mathrm{e}^{2\mathrm{i}\eta}$), and obtained via some suitable q-polynomial, possibly of Askey-Wilson type. Concerning this, it is worth mentioning a recent paper [35] which contains rather promising preliminary results in this direction. Speculating further, an interesting problem which we would like just to hint here concerns a possible relation of such q-polynomials as q tends to a cubic, quartic or sixth root of unity, with the three sets of classical polynomials mentioned above. In this respect, recall that in paper [23] q-Legendre polynomials were used while in the present paper Continuous Hahn polynomials have shown to play an analogous role.

In Section 5 we have used the knowledge of appropriate orthogonal polynomials for the cases under investigation to derive recurrence relations for the boundary one point correlator $H_N^{(r)}$. In the first two cases, such recurrences are trivially solved, and known result for the closely related problems of ASMs'refined 1- and 2-enumerations are easily reproduced. The same approach is applied to the symmetric $\Delta = -1/2$ point, but the resulting recurrence relation for $H_N^{(r)}$ appears very intricate, not being two-term (like in cases of 2- and 1-enumerations) but rather five-term. Even though the differential equation for the corresponding generating function is not of hypergeometric type, it has been shown to be solvable in terms of a suitable linear combination of hypergeometric functions. The so-called cubic transformation for the Gauss hypergeometric function has been applied to work out an explicit formula for the refined 3-enumeration of ASMs, $A(N, r; 3)$. The latter appears not to be writable as a single hypergeometric term, i.e., not to be 'round' (or 'smooth'), contrarily to other known expressions for enumerations of ASMs, and it hardly could have been conjectured on the basis, e.g., of computer experiments.

Acknowledgments

We acknowledge financial support from MIUR PRIN programme (SINTESI 2004). One of us (A.G.P.) was also supported in part by Russian Foundation for Basic Research, under RFFI grant No. 04-01-00825, and by the programme Mathematical Methods in Nonlinear Dynamics of Russian Academy of Sciences. This work was been partially done within the European Community network EUCLID (HPRN-CT-2002-00325).

Appendix A

We reproduce here for the sake of completeness the proof of Eqn. (5.94) given in a previous paper [32].

The key identity which is to be used here is the so-called cubic transformation of Gauss hypergeometric function [29] which in its most symmetric form reads:

$$\frac{\Gamma(a)}{\Gamma(2/3)}\;{}_2F_1\!\left(\begin{matrix}a+1/3\,,\;a\\2/3\end{matrix}\middle|\zeta^3\right)-\omega^{-1}\zeta\,\frac{\Gamma(a+2/3)}{\Gamma(4/3)}\;{}_2F_1\!\left(\begin{matrix}a+1/3\,,\;a+2/3\\4/3\end{matrix}\middle|\zeta^3\right)$$
$$=3^{-3a+1}\left(\frac{1-\zeta}{1-\omega}\right)^{-3a}\frac{\Gamma(3a)}{\Gamma(2a+2/3)}\;{}_2F_1\!\left(\begin{matrix}a+1/3\,,\;3a\\2a+2/3\end{matrix}\middle|\omega\frac{\zeta-\omega}{1-\zeta}\right). \quad \text{(A.1)}$$

Here ω is a primitive cubic root of unity, $\omega=\exp(\pm 2\mathrm{i}\pi/3)$, and a is arbitrary parameter. To show that indeed the cubic transformation is relevant to our case, let us rewrite (5.88) in the form consistent with LHS of (A.1). Taking into account that

$${}_2F_1\!\left(\begin{matrix}-m\,,\;-m+1/3\\4/3\end{matrix}\middle|\zeta\right)=\frac{\Gamma(1/3)\,\Gamma(4/3)}{\Gamma(-m+1/3)\,\Gamma(m+4/3)}(-\zeta)^m\,{}_2F_1\!\left(\begin{matrix}-m\,,\;-m-1/3\\2/3\end{matrix}\middle|\zeta^{-1}\right) \quad \text{(A.2)}$$

and

$$\frac{\Gamma(1/3)}{\Gamma(m+4/3)}=(-1)^{m+1}\frac{\Gamma(-m-1/3)}{\Gamma(2/3)}\,,\qquad \frac{\Gamma(m+4/3)}{\Gamma(-m+1/3)}=(-1)^m\frac{(3m+1)!}{3^{3m+1}\,m!} \quad \text{(A.3)}$$

it is easy to see that (5.88) can be rewritten in the form

$$f_m(x)=\frac{(-1)^{m+1}(3m+1)!}{3^{3m+1}\,(m!)^2}\;x^{3m+1}\left[\frac{\Gamma(-m-1/3)}{\Gamma(2/3)}\;{}_2F_1\!\left(\begin{matrix}-m\,,\;-m-1/3\\2/3\end{matrix}\middle|x^{-6}\right)\right.$$
$$\left.+\,x^{-2}\frac{\Gamma(-m+1/3)}{\Gamma(4/3)}\;{}_2F_1\!\left(\begin{matrix}-m\,,\;-m+1/3\\4/3\end{matrix}\middle|x^{-6}\right)\right]. \quad \text{(A.4)}$$

Clearly, both terms in the brackets are the same as in LHS of (A.1) provided the parameter a is specialized to the value $a=-m-1/3$.

To apply the cubic transformation to (A.4) we first define

$$W(a;\zeta):=\frac{\Gamma(a)}{\Gamma(2/3)}\;{}_2F_1\!\left(\begin{matrix}a+1/3\,,\;a\\2/3\end{matrix}\middle|\zeta^3\right)+\zeta\,\frac{\Gamma(a+2/3)}{\Gamma(4/3)}\;{}_2F_1\!\left(\begin{matrix}a+1/3\,,\;a+2/3\\4/3\end{matrix}\middle|\zeta^3\right) \quad \text{(A.5)}$$

so that

$$f_m(x)=\frac{(-1)^{m+1}(3m+1)!}{3^{3m+1}\,(m!)^2}\;x^{3m+1}W(-m-1/3;x^{-2}). \quad \text{(A.6)}$$

Next, we note that for a sum of two terms one can always write

$$X + Y = \frac{q}{q - q^{-1}} \left(X - q^{-2}Y\right) - \frac{q^{-1}}{q - q^{-1}}\left(X - q^{2}Y\right) \tag{A.7}$$

and if $q = \exp(\mathrm{i}\pi/3)$, which is exactly the case, one can set $\omega = q^2$ for the first pair of terms and $\omega = q^{-2}$ for the second one. This recipe allows one to apply the cubic transformation, that gives

$$\begin{aligned} W(a;\zeta) = \frac{3^{-3a+1}\,\Gamma(3a)}{\Gamma(2a+2/3)}\, \frac{(1-\zeta)^{-3a}}{q(1-q^2)^{-3a+1}} \Bigg[{}_2F_1\left(\begin{matrix} a+1/3\,,\ 3a \\ 2a+2/3 \end{matrix}\middle| q^2\frac{\zeta - q^2}{1-\zeta}\right) \\ + q^{3a+1}\, {}_2F_1\left(\begin{matrix} a+1/3\,,\ 3a \\ 2a+2/3 \end{matrix}\middle| q^{-2}\frac{\zeta - q^{-2}}{1-\zeta}\right)\Bigg]. \end{aligned} \tag{A.8}$$

To obtain a new formula for $f_m(x)$ via (A.6) we have to evaluate now the limit $a \to -m - 1/3$ of (A.8). The limit of the pre-factor can be easily found due to

$$\lim_{a\to -m-1/3} \frac{\Gamma(3a)}{\Gamma(2a+2/3)} = \frac{2}{3}\, \frac{(-1)^{m+1}(2m)!}{(3m+1)!}\,. \tag{A.9}$$

To find the limit of the expression in the brackets in (A.8) we note that

$$q^2\frac{\zeta - q^2}{1-\zeta} =: s \qquad q^{-2}\frac{\zeta - q^{-2}}{1-\zeta} = 1 - s \tag{A.10}$$

and hence the following formula can be used

$$\begin{aligned} \lim_{a\to -m-1/3} \Bigg[{}_2F_1\left(\begin{matrix} a+1/3\,,\ 3a \\ 2a+2/3 \end{matrix}\middle| s\right) + q^{3a+1}\, {}_2F_1\left(\begin{matrix} a+1/3\,,\ 3a \\ 2a+2/3 \end{matrix}\middle| 1-s\right)\Bigg] \\ = \frac{3}{2}\, {}_2F_1\left(\begin{matrix} -m\,,\ -3m-1 \\ -2m \end{matrix}\middle| s\right). \end{aligned} \tag{A.11}$$

Formula (A.11) can be proved, for instance, by virtue of standard analytic continuation formulae for the hypergeometric function (see, e.g., Eqns. (1) and (2) in §2.10 of [29]). Collecting formulae we arrive to the expression

$$W(-m-1/3;\zeta) = -\frac{3^{2m+1}\,(2m)!}{(3m+1)!}\frac{(1-\zeta)^{3m+1}}{(q-q^{-1})^m}\, {}_2F_1\left(\begin{matrix} -m\,,\ -3m-1 \\ -2m \end{matrix}\middle| q^2\frac{\zeta-q^2}{1-\zeta}\right). \tag{A.12}$$

Finally, substituting this expression into (A.6) and using the identity

$${}_2F_1\left(\begin{matrix} a\,,\ b \\ c \end{matrix}\middle| \zeta\right) = (1-\zeta)^{-a}\, {}_2F_1\left(\begin{matrix} a\,,\ c-b \\ c \end{matrix}\middle| \frac{\zeta}{\zeta-1}\right) \tag{A.13}$$

we obtain

$$f_m(x) = \frac{(2m)!}{3^m\,(m!)^2}\, (x - x^{-1})^{2m+1} \left(\frac{qx^{-1} - q^{-1}x}{q-q^{-1}}\right)^m {}_2F_1\left(\begin{matrix} -m\,,\ m+1 \\ -2m \end{matrix}\middle| \frac{qx - q^{-1}x^{-1}}{qx^{-1} - q^{-1}x}\right). \tag{A.14}$$

Obviously, this expression leads directly to (5.94) which is thus proved.

Appendix B

Here we explain how the final expression for function $E_m(z)$, Eqn. (5.99), can be obtained from Eqn. (5.98).

We begin with noting that the hypergeometric functions (polynomials) which enters expression (5.98) belong to the class of the functions

$$ {}_2F_1\left(\begin{matrix} a\,,\; b \\ a-b+1 \end{matrix}\middle|\,\zeta\right) \tag{B.1}$$

which are, in the case of a being a negative integer, possess the symmetry

$$ {}_2F_1\left(\begin{matrix} a\,,\; b \\ a-b+1 \end{matrix}\middle|\,\zeta\right) = \zeta^{-a}\, {}_2F_1\left(\begin{matrix} a\,,\; b \\ a-b+1 \end{matrix}\middle|\,\frac{1}{\zeta}\right). \tag{B.2}$$

Obviously, since $E_m(z)$ is symmetric with respect to $z \to z^{-1}$, it is natural to deal only with the functions of the form (B.1) when transforming the expression for $E_m(z)$. It is convenient to use the notation

$$\Psi_m^{(k)}(u) := (z+2)^m\, {}_2F_1\left(\begin{matrix} -m\,,\; k+1 \\ -m-k \end{matrix}\middle|\,\frac{z^{-1}+2}{z+2}\right)\,, \qquad u := z+1+z^{-1}. \tag{B.3}$$

Note that $\Psi_m^{(k)}(u)$ is a polynomial of degree m in u. Using Gauss relations the following identities can be proven

$$\Psi_{m+1}^{(k)}(u) = (u+3)\,\Psi_m^{(k)}(u) - \frac{m(m+2k+1)}{(m+k+1)(m+k)}\,(2u+3)\,\Psi_{m-1}^{(k)}(u) \tag{B.4}$$

$$\Psi_m^{(k+1)}(u) = \frac{m+2k+2}{2(m+k+1)}\,\Psi_m^{(k)}(u) + \frac{m}{2(m+k+1)}\,(u+3)\,\Psi_{m-1}^{(k+1)}(u)\,. \tag{B.5}$$

Returning to (5.98) we note that in terms of $\Psi_m^{(k)}$'s the generating function $E_m(z)$ reads

$$E_m(z) = \frac{(2m)!\,(2m+1)!}{3^m\, m!\,(3m+1)!}\,\frac{1}{u^2}\left\{9\,\Psi_m^{(m)}(u) + \frac{2(2m+1)}{3m+2}\,(u-3)\,\Psi_{m+1}^{(m+1)}(u)\right\}. \tag{B.6}$$

Applying (B.4), with $k = m+1$, gives us

$$\begin{aligned} E_m(z) = \frac{(2m)!\,(2m+1)!}{3^m\, m!\,(3m+1)!}\Bigg\{ &\frac{2(2m+1)}{3m+2}\,\Psi_m^{(m+1)}(u) - \frac{6m}{3m+2}\,\Psi_{m-1}^{(m+1)}(u) \\ &+ \frac{9}{u^2}\left[-\frac{2(2m+1)}{3m+2}\,\Psi_m^{(m+1)}(u) + \frac{m}{3m+2}\,(u+3)\,\Psi_{m-1}^{(m+1)}(u) + \Psi_m^{(m)}(u)\right]\Bigg\}. \end{aligned} \tag{B.7}$$

Now, the relation (B.5) with $k = m$ shows that the expression in the brackets is zero. Hence we find

$$E_m(z) = \frac{(2m)!\,(2m+2)!}{3^m\,(m+1)!\,(3m+2)!}\left[(2m+1)\,\Psi_m^{(m+1)}(u) - 3m\,\Psi_{m-1}^{(m+1)}(u)\right]. \tag{B.8}$$

Restoring the original notations, we arrive to expression (5.99).

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