A Survey of the Minkowski ?(x) Function

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Abstract

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Minkowski defined the question mark function ?(x) in 1904 as a map from the rational numbers in the interval [0,1] onto the dyadics in [0,1]. The map can be extended over the entire interval. In providing an historical survey of this function, I review basic results concerning Farey fractions and continued fractions. Then I examine Minkowski's definition and its extension, and I give a new proof of a basic fact about the extension. By stepping through the literature, I outline the known properties of ?(x) and state some generalizations of the function. Finally, I present an interesting connection to a simple combinatorial problem.

In Memoria

Dr. John Randolph
(1939-2000)

My first collegiate mathematics professor and the first to spur me towards my eventual pursuit of higher mathematics; an inspirational teacher whose dedication both to his students and to his family remained steadfast in the twilight of his life; a bright mathematician, a dedicated and professional teacher, a fellow (Harrison County) West Virginian, and a fine man.

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Introduction

Minkowski introduced his question mark function $?(x)$ in 1904 as a map from rational numbers in the interval [0,1] to dyadics in [0,1]. He hinted at its extension to all real numbers in [0,1] but showed no interest in the details. His function received only a little attention for the next half century, but its basic properties were established during this time. Greater attention has been paid to $?(x)$ more recently, especially in the decade preceding this writing. Because of its natural definition, $?(x)$ has been independently discovered several times. Some authors do not demonstrate awareness of all earlier publications, work that spans three languages and several countries. This paper seeks to aid this situation as an historical overview of Minkowski's function. First, some background theory prefaces Minkowski's definition and its extension to real numbers. Then a new proof is given for one of the basic properties of the (extended) function, and the various publications on and generalizations of $?(x)$ are presented in approximately chronological order. Finally, a somewhat surprising connection to a simple combinatorial problem appears at the end of this survey.

Background

Farey fractions

We begin by defining *Farey fractions*. (Credit for this sequence of fractions does not actually belong to Farey; see [2].) Consider the following procedure for listing rational numbers on the interval [0, 1]. Begin the list with the endpoints $\frac{0}{1}$ and $\frac{1}{1}$. In the second step, insert between them the number $\frac{1}{2}$. Call $\frac{1}{2}$ the *Farey mediant* of $\frac{0}{1}$ and $\frac{1}{1}$. In the third step, insert $\frac{1}{3}$ and $\frac{2}{3}$ such that the list is written in increasing order. Continue in like matter so that after the n^{th} step, the list contains (in increasing order) all reduced fractions with denominator at most n. Said another way, suppose $\frac{a}{b}$ and $\frac{a'}{b'}$ are adjacent fractions after n-1 steps. Then in step n, insert between them the mediant $\frac{a+a'}{b+b'}$ iff $b+b'=n$. Thus the first few steps are

$$\frac{0}{1} \qquad \frac{1}{1}$$

$$\frac{0}{1} \qquad \frac{1}{2} \qquad \frac{1}{1}$$

$$\frac{0}{1} \qquad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \qquad \frac{1}{1}$$

$$\frac{0}{1} \qquad \frac{1}{4} \quad \frac{1}{3} \qquad \frac{1}{2} \qquad \frac{2}{3} \quad \frac{3}{4} \qquad \frac{1}{1}$$

$$\frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{1}{1}$$

Figure 1: Farey Fractions

It can be shown (by induction) that this process generates all rational numbers in [0,1]. (See [13, p. 297 ff.] for details.) We shall use the common term *Farey fractions* for rational numbers developed in this way, as well as the term (*Farey*) *mediant* or *average* for $\frac{a+a'}{b+b'}$ when it is inserted in the list.

Continued Fractions

A s*imple continued fraction* is a number x in the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

Or, more compactly,

$$x = \langle a_0, a_1, a_2, a_3, ...\rangle$$

where a_i is a non-negative integer for all i, except that a_0 may be zero (and a_0 of course will be zero inside the interval (0, 1)).

It is convenient to describe types of continued fractions with the following terminology:

- A continued fraction representation of x is *finite* if $x = \langle a_0, a_1, \cdots, a_n\rangle$ for some $n \in \mathbb{N}$.
- A continued fraction representation of x is *infinite* if it is not finite.
- A continued fraction representation of x is *repeating* if

$$x = \langle a_0, a_1, \cdots, a_n, a_{n+1}, \cdots, a_{n+h}, a_{n+1}, \cdots, a_{n+h}, a_{n+1}, \cdots, a_{n+h}, \cdots\rangle = \langle a_0, a_1, \cdots, a_n, \overline{a_{n+1}, \cdots, a_{n+h}}\rangle$$

For example, by carrying out the arithmetic of fractions, we see for a finite case that $1 + \cfrac{1}{2 + \cfrac{1}{3}} = 1 + \cfrac{1}{7/3} = \cfrac{10}{7}$. Moving from right to left is possible using the division algorithm. Other examples are given below. Details may be found in [13, p. 325 ff.].

Although more general continued fractions may have numerators other than 1, simple continued fractions are sufficiently common that they are usually called just *Continued Fractions.* Successive *convergents* of a continued fraction are the values $\langle a_0\rangle$, $\langle a_0, a_1\rangle$, $\langle a_0, a_1, a_2\rangle$, … (This is analogous to partial sums of series. It can be shown that the sequence of convergents does indeed converge.)

Here is a well-known fact about continued fractions.

Theorem 1

a. A number x is rational iff its continued fraction representation is finite.
b. A number x is a quadratic irrational number iff its continued fraction representation is repeating.

Part (a) is trivial, and Lagrange showed part (b) in 1770. Quadratic irrational numbers are also called quadratic surds.

A rational number may have two distinct continued fraction representations. The second form of a given finite representation is the replacement of a_n by the two terms a_{n-1} and 1. (This is somewhat analogous to the familiar $0.\overline{9} = 1$.) This duality poses no problems for what follows in this paper.

Examples

$$\frac{2}{5} = \langle 0, 2, 2 \rangle \qquad \sqrt{2} = \langle 1, \bar{2} \rangle \qquad \phi = \frac{1+\sqrt{5}}{2} = \langle \bar{1} \rangle .$$

(As an aside, while the third identity is easily derived, the representation of the golden ratio as an infinite repetition of unity magnifies the mystical aura surrounding the number. Loosely speaking, its continued fraction notation causes ϕ to appear to be the simplest or most fundamental of all nontrivial quadratic irrational numbers and not coincidentally recalls the equation to which it is a solution: $x^2 = x + 1$. For more on continued fractions, see [13].)

Minkowski

In 1904 Hermann Minkowski defined the Question Mark function ?(x) as a map from the interval [0,1] onto itself. His original treatment is found on p. 50-51 of [12], and a fine English translation by Harris Hancock[1] exists in Article 196 on p. 754-755 of [9].

In his paper, Minkowski defines ?(x) as a map from the rational numbers in the interval [0,1] to the purely dyadic numbers in [0,1], that is, to numbers in [0,1] whose denominator is a power of two. He appears to think of his definition as taking place in the unit square, hence he gives the intuitive initial values of $?\left(\frac{0}{1}\right) = \frac{0}{1}$ and $?\left(\frac{1}{1}\right) = \frac{1}{1}$. He then maps the Farey Mediant of two numbers to the arithmetic mean of the images of those two numbers. Although Minkowski did not do so, his definition may be stated succinctly in equation form as

$$?\left(\frac{a+a'}{b+b'}\right) = \frac{?\left(\frac{a}{b}\right) + ?\left(\frac{a'}{b'}\right)}{2} \qquad \textbf{I}$$

This form allows calculation of ?(x) only for rational x, and it obviously requires knowing the "Farey parents" of x and their images under the function. We may see this more

[1] For my knowledge of Harris' translation, I am indebted to a parenthetical remark in the paper by Beaver and Garrity, discussed later. Though acquired late in my own research, the translation has enabled a more complete discussion of Minkowski's original definition.

clearly in the following table, where step *i* introduces the appropriate Farey averages for *x* and the corresponding arithmetic averages for ?(*x*). (The initial step is simply the endpoints of the interval.)

Step 2		Step 3		Step 4		Step 5			Step 8	
x	**?(x)**	**x**	**?(x)**	**x**	**?(x)**	**x**	**?(x)**		**x**	**?(x)**
0/1	0/1	0/1	0/1	0/1	0/1	0/1	0/1	…	0/1	0/1
								…	**1/8**	**1/128**
								…	1/7	1/64
								…	1/6	1/32
						1/5	**1/16**	…	1/5	1/16
				1/4	**1/8**	1/4	1/8	…	1/4	1/8
								…	2/7	3/16
		1/3	**1/4**	1/3	1/4	1/3	1/4	…	1/3	1/4
						2/5	**3/8**	…	2/5	3/8
								…	3/7	7/16
1/2	**1/2**	1/2	1/2	1/2	1/2	1/2	1/2	…	1/2	1/2
								…	4/7	9/16
						3/5	**5/8**	…	3/5	5/8
		2/3	**3/4**	2/3	3/4	2/3	3/4	…	2/3	3/4
								…	5/7	13/16
				3/4	**7/8**	3/4	7/8	…	3/4	7/8
						4/5	**15/16**	…	4/5	15/16
								…	5/6	31/32
								…	6/7	63/64
								…	**7/8**	**127/128**
1/1	1/1	1/1	1/1	1/1	1/1	1/1	1/1	…	1/1	1/1

Table 1: Construction of ?(x)

Minkowski provides a single graph of his function that consists of approximately 30 ordered pairs in the unit square. His plot suggests the shape that is more clearly visible in a computer-generated graph of ?(*x*).

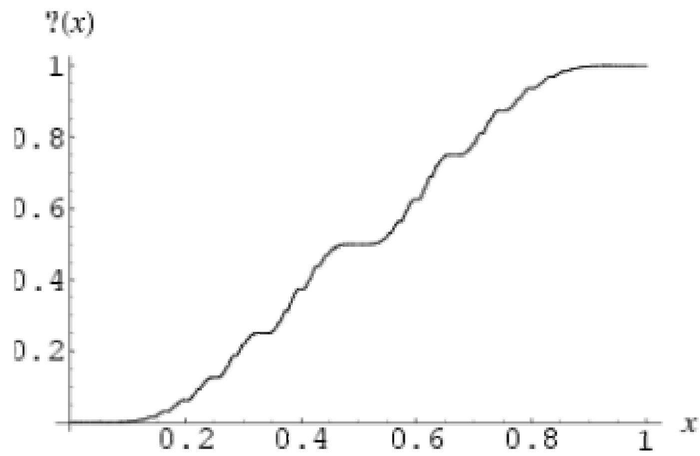

Figure 2: Graph of ?(x)

To extend his function, Minkowski says, in Harris' translation, that this

> arrangement of simultaneously constructed abscissae and ordinates offers the picture of a continuous increasing function[2] y=?(x) *first* for all rational values of x and *then* extended through the requirement of continuity to arbitrary real arguments in the interval $0 \le x \le 1$, while at the same time y transverses this interval at pleasure. [9]

He states without elaboration that because a quadratic surd has a representation as a periodic continued fraction, it maps to a rational number. He concludes with this result:

Theorem 2

a. x is a rational number iff ?(x) is a purely dyadic number
b. x is a quadratic surd iff ?(x) is a non-dyadic rational number

Part (a) is obvious from the construction, but (b) requires a detour into continued fractions, and we consider it shortly.

Before moving on, we may note a symmetry about x=1/2 that should be fairly clear from the construction, and a few other values of the function follow.

Symmetry of ?(*x*)

$$?(1-x) = 1 - ?(x) \text{ for all } x \in [0,1] \qquad \textbf{II}$$

[2] Harris makes the following intriguing footnote here, although I do not see it in Minkowski's paper: "The question mark is employed as an interrogation of the function that connects x and y."

$$?\left(1-\frac{\sqrt{2}}{2}\right)=\frac{1}{5} \qquad ?\left(\frac{\sqrt{2}}{2}\right)=\frac{4}{5} \qquad ?(2-\phi)=\frac{1}{3} \qquad ?(\phi-1)=\frac{2}{3}$$

(As a second aside, the golden ratio is again involved in mappings to the thirds, the two numbers that might be called the simplest of the non-dyadic rational numbers in [0,1].)

An Explicit Extension: Denjoy and Salem

The Extension

As seen in the preceding quotation, Minkowski demonstrates little interest in an explicit extension of his function. In 1932 Arnoud Denjoy defined the same function in [6], without explicitly mentioning Minkowski's work. Denjoy gives a more general definition, and this work was repeated in greater detail by R. Salem in [19] with knowledge of Minkowski's paper. Denjoy and Salem extend Minkowski's definition to the following:

If x is a real number such that $x=\langle 0,a_1,a_2,\cdots\rangle$, where this expansion may terminate in some n^{th} term or continue indefinitely, then

$$?(x)=\sum_{k\geq 1}\frac{(-1)^{k-1}}{2^{(a_1+\cdots+a_k)-1}} \qquad \textbf{III}$$

It is immediate that this definition is identical to Minkowski's for rational x. In this case, the summation terminates with the $k=n$ term. As Salem comments, this summation is precisely what is needed to establish Theorem 2 (b). Salem does not specify that $?(x)$ is non-dyadic for irrational x, although it is implicit. Salem gives no proof, and it is easy to convince oneself that it is true. Yet in writing a direct formal proof of (b), I found the finer details to be less than obvious.

Because the detail may obscure the simplicity of the main idea, an example is in order. Following the example I give the forward direction proof but hope that a more elegant argument will be found. To the best of my knowledge, no such proof has appeared in the literature.

Example

We calculate the image of $\sqrt{3}-1=\langle 0,\overline{1,2}\rangle$.

By (**III**),

$$?(\sqrt{3}-1)=\frac{1}{2^{1-1}}-\frac{1}{2^{1+2-1}}+\frac{1}{2^{1+2+1-1}}-\frac{1}{2^{1+2+1+2-1}}+\cdots=\sum_{i=0}^{\infty}\left(\frac{1}{2^3}\right)^i-\left(\frac{1}{2^2}\right)\cdot\sum_{j=0}^{\infty}\left(\frac{1}{2^3}\right)^j$$

Thence,

$$?(\sqrt{3}-1)=\frac{2^3}{2^3-1}-\left(\frac{1}{2^2}\right)\cdot\frac{2^3}{2^3-1}=\frac{6}{7} \qquad \textbf{IV}$$

Note how the image could be cleverly decomposed as an illustration of the reverse direction of Theorem 2(b).

Proof of Theorem 2(b)

First we establish that $?(x)$ is rational (the easy part), and then we show it is not dyadic. Let x be a quadratic surd. By the Lagrange condition (Theorem 1 (b)), the continued fraction representation of x is (eventually) repeating. Thus if h is the length of the period, then in the expansion of the series, every h^{th} term has denominator exponent incremented by the fixed sum of the h repeating digits (except possibly for a finite number of initial terms). An appropriate permutation of the terms of (**III**) yields a (finite sum of) convergent geometric series. Therefore, $?(x)$ is rational.

At this point, we could appeal to part (a) to finish the proof of (b), but to show that (b) stands on its own merits, we proceed with the direct proof of the forward direction. The reverse direction would use basically the same ideas.

We are about to show that whatever infinite series arise from (**III**) have a value such that an odd factor remains in the denominator of (**III**) after simplifying. We will consider a few different cases and in places argue more generally to avoid an excessive number of cases.

Let $x = \langle 0, a_1, \cdots, a_n, \overline{a_{n+1}, \cdots, a_{n+h}} \rangle$, so h is the length of the (purely) periodic part and n is the length of the pre-periodic part. Let $\alpha = a_{n+1} + a_{n+2} + \cdots + a_{n+h}$ be the sum of the repeating digits. Now consider the form of the sum when the geometric series are evaluated.All numerators of the pre-periodic terms are ± 1. There is at least one geometric series, and any series may have a dyadic coefficient with a numerator of ± 1. In general, this appears as

$$?(x) = \sum_{k_0=1}^{n} \left(\frac{(-1)^{k_0-1}}{2^{(a_1+\cdots+a_{k_0})-1}} \right) + \frac{(-1)^n}{2^{(a_1+\cdots+a_{n+1})-1}} \sum_{k_1=0}^{\infty} \left(\frac{1}{2^\alpha} \right)^{k_1} + \frac{(-1)^{n+1}}{2^{(a_1+\cdots+a_{n+2})-1}} \sum_{k_2=0}^{\infty} \left(\frac{1}{2^\alpha} \right)^{k_2} + \cdots$$
$$\cdots + \frac{(-1)^{n+h-1}}{2^{(a_1+\cdots+a_{n+h})-1}} \sum_{k_h=0}^{\infty} \left(\frac{1}{2^\alpha} \right)$$

Since $\sum_{k_0=1}^{n} \left(\frac{(-1)^{k_0-1}}{2^{(a_1+\cdots+a_{k_0})-1}} \right) = \frac{N}{2^s}$ is the sum of the finitely many summands arising from the non-repeating digits, where N and s are non-negative integers, $s = a_1 + \cdots + a_n - 1$, the first term is dyadic. Since the sum of a dyadic and a non-dyadic rational is non-dyadic, it suffices to show that the remaining terms do not sum to a dyadic. The remaining terms are

$$\frac{(-1)^n}{2^{(a_1+\cdots+a_{n+1})-1}} \sum_{k_1=0}^{\infty} \left(\frac{1}{2^\alpha} \right)^{k_1} + \frac{(-1)^{n+1}}{2^{(a_1+\cdots+a_{n+2})-1}} \sum_{k_2=0}^{\infty} \left(\frac{1}{2^\alpha} \right)^{k_2} + \cdots + \frac{(-1)^{n+h-1}}{2^{(a_1+\cdots+a_{n+h})-1}} \sum_{k_h=0}^{\infty} \left(\frac{1}{2^\alpha} \right)^{k_h}$$

Note that each series is evaluated as $\sum \frac{1}{2^\alpha} = \frac{1}{1-2^{-s}} = \frac{2^s}{2^s - 1}$, for some positive integer s (and therefore each denominator is odd). Substituting into the previous expression,

$$\frac{(-1)^n}{2^{(a_1+\cdots+a_{n+1})-1}} \cdot \frac{2^\alpha}{2^\alpha - 1} - \frac{(-1)^{n+1}}{2^{(a_1+\cdots+a_{n+2})-1}} \cdot \frac{2^\alpha}{2^\alpha - 1} + \cdots + \frac{(-1)^{n+h-1}}{2^{(a_1+\cdots+a_{n+h})-1}} \cdot \frac{2^\alpha}{2^\alpha - 1}$$

Obtain a common denominator of $2^\alpha - 1$, so the numerator is a sum of distinct powers of two, which have alternating signs. Let this sum be denoted σ. So its summands are given by the sequence $S = \{2^{\alpha - a_1 + 1}, 2^{\alpha - a_1 - a_2 + 1}, \cdots, 2^{\alpha - a_1 - a_2 - \cdots - a_{n+h} + 1}\}$ such that alternating signs are imposed (and some exponents may be negative). It suffices to show that σ is not $2^\alpha - 1$. One could show σ is not a multiple of $2^\alpha - 1$ with a similar argument.

To have $\sigma = 2^{\alpha} - 1$, one of the summands in *S* must be ±1. Consider first that a summand is (-1). In the first sub-case, suppose $a_1 = 1$ and that the first element of *S* is positive, 2^{α}. By way of contradiction, suppose $\sigma = 2^{\alpha} - 1$. This means the other terms (all but these two) may be collected by sign so that $\sum 2^i - \sum 2^j = 0$. Because all elements of *S* are distinct, this gives two distinct binary representations of a positive integer, which is a contradiction. In the second sub-case, suppose $a_1 = 1$ and the first element of *S* is negative. In the third sub-case, suppose $a_1 > 1$ so that the first element of *S* is at most half 2^{α} (of either sign). In both the second and third sub-cases, the sum of the remaining terms is assuredly less than 2^{α}. (The reader may observe why.)

Now consider that a summand in *S* is +1. Then no summand can be +2. By way of contradiction, suppose $\sigma = 2^{\alpha} - 1$, so $\sigma + 1 = 2^{\alpha}$. The new left-hand side has a summand of 2, so divide both sides of the equation by 2. This gives $\sigma' = 2^{\alpha-1}$ for σ' of the same form as σ, except that all exponents (other than the +1 term) are reduced by one. Except for the trivial case of $\alpha = 1$, an odd number (or nontrivial fraction) equals an even number, which is a contradiction.

Thus the forward direction of Theorem #(b) is established.

Although this argument is somewhat cumbersome, the fundamental idea that makes it work is the Lagrange condition. Note that just as the Lagrange condition for continued fractions and part (a) of Theorem 2 provide a "short" proof of part (b), so too do the Lagrange condition and the full Theorem 2 imply the following:

Corollary 1

The number *x* is a real number but is not a rational or a quadratic irrational number iff ?(*x*) is an irrational real number

Other Early Work by Denjoy and by Salem

Denjoy's Second Paper

In Denjoy's second paper, [7], he recalls the Minkowski function and then defines a more general function. The rest of the paper (about 40 pages) is devoted to the development of this and other functions and their properties.

For example, his first generalization is what he calls $\chi(x,\alpha)$. Provided that $pq' - qp' = 1$, in his notation but my translation,

> the number $\chi\left(\frac{p+p'}{q+q'},\alpha\right)$ divides the segment $\left[\chi\left(\frac{p'}{q'}\right),\chi\left(\frac{p}{q}\right)\right]$, $\frac{p'}{q'}<\frac{p}{q}$, in a constant ratio, that of $1-\alpha$ and α. α will be called the *base* of the function χ. Thus,
>
> $$\chi\left(\frac{p+p'}{q+q'},\alpha\right)=\alpha\chi\left(\frac{p'}{q'},\alpha\right)+(1-\alpha)\chi\left(\frac{p}{q},\alpha\right)$$
>
> [Equation (**I**), Minkowski's definition] corresponds to $\alpha = 1-\alpha = \frac{1}{2}$. But the function that I consider is defined on the field of all real numbers. It suffices…to regard any whole number $\frac{n}{1}$ as the mediant of two fractions: $\frac{n}{1}$ is the mediant of $\frac{n-1}{1}$ and of $\frac{1}{0}$, and also of $\frac{-1}{0}$ and of $\frac{n+1}{1}$. (p.106-107)

Denjoy's *base* might also be called *weight* in English, since it appeals to what is commonly called a weighted average. Denjoy goes on to suggest initial conditions $\chi\left(\frac{0}{1},\alpha\right)=1,\quad \chi\left(\frac{1}{0},\alpha\right)=0$ for any α. From the case considered previously, he deduces that by letting $p=1,\quad q=0,\quad p'=n-1,$ and $q'=1$, that he has $\chi(1,\alpha)=\alpha$ and $\chi(n,\alpha)=\alpha^n$. Upon making a few further observations, Denjoy gives the identity on [0, 1] of

$$?(x) = 2-2\chi\left(x,\frac{1}{2}\right)$$

We may quickly verify this identity for test values:

$$?\left(\frac{1}{3}\right)=2-2\chi\left(\frac{0+1}{1+2},\frac{1}{2}\right)=2-2\left(\frac{1}{2}\chi\left(\frac{0}{1},\frac{1}{2}\right)+\frac{1}{2}\chi\left(\frac{1}{2},\frac{1}{2}\right)\right)$$

Thence

$$\chi\left(\frac{1}{2},\frac{1}{2}\right)=\frac{1}{2}\chi\left(\frac{0}{1},\frac{1}{2}\right)+\frac{1}{2}\chi\left(\frac{1}{1},\frac{1}{2}\right)=\frac{1}{2}(1)+\frac{1}{2}\left(\frac{1}{2}\chi\left(\frac{0}{1},\frac{1}{2}\right)+\frac{1}{2}\chi\left(\frac{1}{0},\frac{1}{2}\right)\right)$$

So as expected,

$$?\left(\frac{1}{3}\right)=2-2\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}+0\right)\right)\right)=\frac{1}{4}$$

In the rest of his paper, he considers his broader class of functions and derives other properties, including singularity. His work relies heavily on the theory of continued fractions.

Other Results of Salem

After deriving (**III**), R. Salem summaries the basic properties of $?(x)$, which are Theorem 2 above and the fact that $?(x)$ is strictly increasing (by its construction). He then proves that $?(x)$ is a singular function. (Recall that a singular function is one whose derivative is zero almost everywhere, or everywhere except on a set of measure zero; see an analysis textbook for details.) His proof relies on tools of analysis with continued fraction convergents. He shows that the derivative of $?(x)$ is forced to be zero almost everywhere on a given set of measure 1, hence it is zero on a set of measure 1, so $?(x)$ is singular.

To give Salem's other results, we must repeat one of his definitions. In Salem's development of Farey Fractions, he fills in all possible gaps without regard to size of denominator, and he takes the endpoints as given. So, for example, his "steps 1 and 2" are the same as above except for index, but his "step 3" inserts four, not two, new fractions: 1/4, 2/5, 3/5, and 3/4. After his "step *p*" there are a total of 2^p+1 fractions. He calls these the *Minkowski sequence of order p*, denoted by $\mathfrak{M}_p$.

Salem establishes by induction the following lemma, then proves the theorem on the Lipschitz condition for ?(x). Finally he gives the Fourier-Stieltjes coefficients for the question mark function. Before stating these, we recall the definition of a Lipschitz condition of order α.

Definition of Lipschitz Condition of order α for a function

The function $f(x)$ satisfies a *Lipschitz Condition of order* α if

$$\left|f(x)-f(y)\right| \le B\left|x-y\right|^{\beta}$$

with $\beta > 0$ and α as an upper bound for all β for which a finite B exists.

Lemma

Let $\frac{p_n}{q_n} = \langle 0, a_1, \cdots, a_n \rangle$. We have the inequality $q_n < \phi^{a_1+\cdots+a_n}$. This upper bound is the best possible.

Theorem 3 (Salem)

The function ?(x) satisfies a Lipschitz condition of order $\alpha = \frac{\log 2}{2\log\phi}$. This α is the best possible exponent for the Lipschitz condition.

Fourier-Stieltjes coefficients of ?(*x*/2π)

Let $c_n = \int_0^{2\pi} e^{ni\pi}\, d\,?(x/2\pi)$. Then $c_n = \lim_{p\to\infty}\left(\frac{1}{2^p}\sum_{\rho\subset\mathfrak{M}_p} e^{2\pi ni\rho}\right)$

A Survey of More Recent Publications on ?(x)

The Analysis of J. R. Kinney

In 1959 J. R. Kinney published a seven-page "Note" [10] on ?(x) in which he uses detailed analysis to establish two properties regarding a Lipschitz condition and fractional dimension. The exact statement of the properties requires several definitions, and since they are not of primary interest here, I have placed them, essentially verbatim, in the Appendix. He thinks of ?(x) as defining a completely additive probability measure, with

$?(A) = \int_A d\,?(x)$, and some of his work has a statistical flavor. He establishes the Lipschitz condition as being of order $\alpha = \left(2\int_0^1 \log_2(1+x)\,d\,?(x)\right)^{-1}$. We will see below that another paper ([20]) tries to extend Kinney's work to a more general family of sets.

Ramharter: a New Proof

Ramharter's short paper [15] from 1987 recalls a known result and reformulates it so as to give a different proof. It is easily restated here. Let $?(x) = y = \langle 0, a_1, a_2, \ldots \rangle$. Consider the dyadic expansion of *x* as 0's and 1's: $x = 0_{a_1-1}1_{a_2}0_{a_3}1_{a_4}0_{a_5}\cdots$, where subscripts denote string length. It is known (Ramharter indirectly cites Salem, for example) that

i. $\langle a_1+1, 2_{a_2-1}, a_3+2, 2_{a_4-1}, a_5+2, \cdots \rangle = ?^{-1}(0_{a_1-1}1_{a_2}0_{a_3}1_{a_4}0_{a_5}\cdots)$, where again subscripts denote string length

ii. ? is singular

Ramharter then presents a "unified approach to both properties by splitting up (i) into the following inequalities, the second one making (ii) evident"

1. $\langle a_1+1, 2_{a_2-1}, a_3+2, 2_{a_4-1}, a_5+2, \cdots \rangle = \langle 0, a_1, a_2, a_3, a_4, a_5, \cdots \rangle$
2. $\langle 0, a_1, a_2, a_3, a_4, a_5, \cdots \rangle = ?^{-1}(0_{a_1-1}1_{a_2}0_{a_3}1_{a_4}0_{a_5}\cdots)$

He then quickly proves his claim. He works with slightly non-standard notation (in comparison to the other literature I have reviewed), and the reader who is deeply interested in continued fractions and Minkowski's function may wish to review his paper.

An Extension by Tichy and Uitz

In 1995 R. F. Tichy and J. Uitz [20] extended Kinney's work. They recall that Kinney "showed that *d*? is concentrated on a set *X* of Hausdorff dimension $\alpha \approx 0.875$ in the following sense: there is a set *X* of Hausdorff dimension α for which ?(*X*) has Lebesgue measure 1, while for any set *A* of Hausdorff dimension less than α the set ?(*A*) has measure zero." They extend this to a family $g_\lambda(x)$, where, as in Denjoy, the case of $\lambda=1/2$ is $?(x)$. This family is of functions that are singular, continuous, and strictly increasing on [0,1].

Interestingly, Tichy and Uitz do not mention any of Denjoy's work. They appear to share some thought about the generalization of $?(x)$. I leave as an open question exactly what connection there is between Tichy and Uitz's family and Denjoy's. (I know of no translations of Denjoy, and this question does not sufficiently interest me to warrant translating about 40 pages into English. But for starters, one might consider the similarity in form of Tichy and Uitz's (2.5) and Denjoy's definition of the chi function, given above.)

Historical notes aside, Tichy and Uitz build their family $g_\lambda(x)$ from a tree of Farey fractions. They work with this family and give results concerning the Hausdorff dimension of the set of points where its functions have nonzero derivative.

An Extension by Girgensohn

In a remarkable coincidence, Roland Girgensohn published a paper [8] generalizing $?(x)$ within a few months of the paper by Tichy and Uitz, and not too far away geographically: Tichy and Uitz were in Austria, Girgensohn in Germany. Like Denjoy and like Tichy and Uitz, Girgensohn develops from continued fractions a more general class of functions with a parameter such that a value of 1/2 gives $?(x)$. However, whereas Tichy and Uitz begin by citing Kinney but make no reference to Denjoy, Girgensohn refers to Denjoy but seems unaware of Kinney's paper. (This leads me to the self-serving conclusion that a survey of work on $?(x)$ is needed!)

Girgensohn follows Denjoy's idea to show the singularity of $?(x)$ by considering two systems of functional equations:

$$r\left(\frac{x}{x+1}\right)=tr(x),\quad r\left(\frac{1}{2-x}\right)=t+(1-t)r(x) \tag{R}$$

And

$$f\left(\frac{x}{x+1}\right)=tf(x),\quad f\left(\frac{1}{x+1}\right)=1-(1-t)f(x) \tag{F}$$

Both systems are defined for all $x \in [0,1]$. The first system is due to Georges de Rham (who gives only t=1/2, which corresponds to the Minkowski ?(x); see [16]). Girgensohn shows that (F) has a unique bounded solution which is precisely ?(x) for t=1/2. He uses Farey fractions to give a general construction of singular functions, and the solutions to (R) and (F) are specific cases within this construction. (The singularity of these solutions comes as a mere corollary to another theorem.) He views his work as a generalization of Salem's, except that unlike Salem he does not use continued fractions explicitly. As he works from a slightly modified development of Farey fractions, he prefers "to work from a real-variable point of view." In his concluding remarks, he posits that that a generalization of Farey fractions could lead, by generalizing his work, to an even larger class of singular functions, although he does not pursue the idea far.

Two Significant Papers by Viader, Paradís, and Bibiloni

Pelegrí Viader, Jaume Paradís, and Lluís Bibiloni published two papers on ?(x), the first in 1997 [21]and the second in 2001 [14] (I follow the order of names in [21] with "Viader et al.".) The first paper presents ?(x) as an asymptotic distribution function (a.d.f.) of a sequence of rational numbers and then gives two different proofs of the singularity of ?(x). The second paper gives criteria for determining whether $?'(x) = 0$ or $?'(x) = \infty$ and also gives some secondary results.

In the first paper, Viader et al. give an enumeration of the rational numbers in the open interval (0,1). They recall the definition of an a.d.f.:

Definition of an Asymptotic Distribution Function

Consider the sequence $\{q_n\}, \quad 0 \le q_n \le 1$. A function $F(x)$ is the a.d.f. of the sequence if $F(x) = \lim_{n \to \infty} \frac{\#(q_i \le x, \quad i = 1, 2, \cdots, n)}{n}$

They then carefully show that the a.d.f. of their sequence is precisely ?(x). For their first proof of singularity, they show that $?'(x) = 0$ everywhere on a set of measure 1. They point out that besides using a different set than Salem, their method is subtly different as well. They show the derivative to be zero on their entire set, whereas Salem gets a derivative of zero and says that the underlying set has measure zero. Their choice of set appears clever, but describing it is beyond my present scope. The second proof of singularity given by Viader et al. provides a vanishing set. In other words, they give a set S of measure 1 such that both the image of S and the inverse image of S under ? have measure zero. In so doing, they deal carefully with the set of alternated dyadic numbers (obviously used in (**III**)) and "normal" numbers. Overall their paper weaves together several different ideas with care and clarity.

The second paper recalls more explicitly Salem's method of proving singularity and refines it. Viader et al. find an analytic expression for $?'(x)$ using continued fractions. Their main result is what is already know: the derivative of ?(x) must be zero or infinite. They also give this interesting result about the derivative:

Theorem 4 (Viader et al.)

a. If, in the continued fraction representation of x, the average of partial quotients is greater than $\overline{k} \approx 5.31972$ and $?'(x)$ exists, then $?'(x) = 0$. The exact value of $\overline{k}$ is the solution of $2\log_2(1+x) - x = 0$.
b. If the average in (a) is less than $\underline{k} = 2\log_2 \phi$, where ϕ is the golden ratio, then $?'(x) = \infty$.

They conclude their second paper with some miscellaneous results about metric properties of alternated dyadic expressions and about continued fractions.

A Generalization into Two Dimensions: Beaver and Garrity

In late 2002 Olga R. Beaver and Thomas Garrity [1] published a marvelous generalization of Minkowski's ?(x). Their direction seems obvious, yet they appear to be the first to move in it: they generalize ?(x) and some of its properties to two dimensions.

Even outlining all their results would exceed my scope, but I consider the direction in which they head.

Beaver and Garrity provide a relatively lengthy review and bibliography for known properties of ?(*x*), even citing the work of Viader et al. as inspiring one of their results. One of their goals is to address an unsolved problem posed by Hermite, which in their words is: "Find methods for expressing real numbers as sequences of positive integers so that the sequence is eventually periodic precisely when the initial number is a cubic irrational." This is obviously an analog to the Lagrange condition (Theorem 1 above). Beaver and Garrity state that the only attempt at the problem using ?(*x*) was a dissertation by Louis Kollros, whom Minkowski's paper also cites. Minkowski very briefly discusses cubic irrationals after introducing ?(*x*). Beaver and Garrity do not solve the Hermite problem.

They do, however, introduce "Farey sums," and they partition the unit triangle $\Delta = \{(x, y) : 0 \le y \le x \le 1\}$ in two ways, Farey (F) and *barycentric* or *Bary* (B). From here they begin to define their two-dimensional generalization of ?(*x*), which they call $\delta : \Delta_F \to \Delta_B$. This function turns out to be continuous, but they build it in stages, much as ?(*x*) did not begin as a map on the real numbers but only on rational numbers. (I have been particularly impressed by the numerous diagrams of the unit triangle. The diagrams enhance the already beautiful mathematics behind them.) In order to return to work on Minkowski's function itself, I give Beaver and Garrity's own summary of the remainder of their paper. They establish:

- "that the Farey iteration [as developed in their paper] can be viewed as a multidimensional continued fraction [so that] periodicity of the Farey iteration corresponds to a class of cubic irrationals"
- "that periodicity for the barycentric iterations corresponds to a class of rational points"
- "an analog of singularity…the area of image triangles in the barycentric partitioning approaches zero far more quickly than the area of the domain triangles in the Farey partitioning." (p.4)

They conclude with several questions that could carry their work in different directions.

Independent Discoveries of ?(*x*)

My work up to this point has followed the historical development of the Minkowski function and its generalizations. As several authors comment, it has a rather natural definition. So perhaps it is not surprising that the function has been rediscovered numerous times. While I suspect that many rediscoveries go undocumented (who publishes what is known to be redundant?), I want to mention a few examples here. This section will have less mathematical rigor and be even more historical in nature.

Denjoy, Reconsidered

In his first paper [6], published in 1932, Denjoy gives an equation resembling (**III**) in connection with Farey fractions. He does not seem to state Minkowski's definition *per se*, but he is obviously in the same ballpark. His opening remarks in his second paper [7] indicate that he was presenting work from 1931 later the same year when a certain Mr. Hadamard introduced Denjoy to ?(x). "The subject interested me immediately," Denjoy says. "I have been very busy since."

For sake of any future historians of ?(x), I must make another note about Denjoy's first paper from 1932. It is commonly cited by the title "Sur une function de Minkowski" in the journal abbreviated *C. R. Acad. Sci. Paris*. Although Denjoy himself says he learned of ?(x) in December 1931, there is no direct evidence of this in the 1932 paper of which I have a copy. "*C. R.*" stands for *Comptes Rendus*, which is a generic term for work from the Academy of Sciences, somewhat like the English "Proceedings of" or "Notes from." The 1932 paper that I have is from a *comptes rendus* that deals with a broad range of papers in the mathematical sciences. (Library personnel at West Virginia University helped me find the article under the full journal title of *Comptes rendus hebdomadaires des séances de l'Académie des sciences*.) Curiously, the article does not bear the title mentioned above, but rather "Sur quelques points de la théorie des fonctions." Yet the volume, year, and page numbers of the common citation match precisely with this title.

It is possible that another paper from Denjoy exists from 1932 and mentions ?(x), but I have not been able to locate it. I can only speculate that an error has been made and propagated in the literature. The title bearing Minkowski's name appears only after the basic properties were known, and Denjoy's first paper is not central to the development of the properties. I suspect therefore that unfamiliarity with French Science journals from the 1930s has led to this confusion, and later authors have (very understandably) passed on the citation without reading Denjoy in the original. At any rate, there is ambiguity surrounding Denjoy's earliest paper that relates to ?(x).

An Early Discovery by Ryde

Folke Ryde discovered Minkowski's function in 1926 [17] without knowledge of Minkowski's definition. In 1981, more than half a century later, Ryde wrote a short paper [18] connecting his work to Minkowski's. (The paper appeared shortly after Ryde's death in 1981.) In this he restates his function, L(x), and its inverse, P(x). He shows that $?(x) = P(x) - 1$. His original definitions, besides being easily generalized to a larger domain than [0,1], are of interest mainly for historical purposes. Ryde's definition in 1981 involves continued fractions and dyadics, which is not surprising. He indicates that the 1926 definition was a bit more general. The interested reader may review Ryde's works for details.

Conway: Making a Game of ?(x)

In his popular book *On Numbers and Games* (ONAG) [5, p.82 ff], John Conway describes a "game" in which each of two players must modify a given number in a certain way. Both players, say Left and Right, must move to a fraction with smaller denominator (or smaller in absolute value if already an integer; irrationals are defined to have denominator infinity). Left must decrease the number; right must increase it. Left wants the largest legal move, right the smallest. He defines the curious-looking $\boxed{x}$ (which may be read as "x in a box" or perhaps "box (of) x") to represent the game position x. So, as Conway notes, from $\boxed{2/5}$, Left should move to $\boxed{1/3}$ (although $\boxed{1/4}$, $\boxed{0}$, $\boxed{-22/7}$, etc., are also legal). Right should move to $\boxed{1/2}$ (although $\boxed{2/3}$, $\boxed{7}$, etc., are also legal). This situation is expressed as $\boxed{2/5} = \left\{ \boxed{1/3} \mid \boxed{1/2} \right\}$

Prior to this setup, Conway gives a precise meaning to the term *simplicity*. By this he means that the simplest number is 0, followed by 1 and –1, then 2 and –2, and so forth through the integers. Next come the dyadics by denominator: first those with denominator 2, then 4, then 8, and so on. Finally come all other reals. After introducing $\boxed{x}$, he notes that it provides a sort of "distorted notion of simplicity," where being simple is having a smaller denominator (so powers of 2 are no longer special).

He quickly uses continued fractions and binary expansions[3] to look at $\boxed{x}$ as a function of x and then gives a few sample values. With the turn of a page (literally), he states that this relation, which he derived in the context of game theory, is more broadly known as Minkowski's $?(x)$, and he gives the standard graph of $?(x)$ and wraps up his discussion of $\boxed{x}$. (As best I can tell, he never mentions it again in ONAG.)

One of Conway's contributions is that he expounds a method to work with $\boxed{x}$, or $?(x)$, algebraically. The idea is not new: see Ramharter (lines (i) or (2) in the section on Ramharter above), for example. But with a few examples, Conway puts a little flesh on the bones. To find $?^{-1}(x)$, one must know the binary expansion of the argument (or equivalently, the dyadic expansion). Conway finds $?^{-1}\left(1/5\right)$, or in his notation solves $\boxed{x} = 1/5$, observing that

$$1/5 = 0 \cdot \underbrace{00}_{a_1 - 1}\underbrace{11}_{a_2}\underbrace{00}_{a_3}\underbrace{11}_{a_4}\cdots.$$

By counting digits as shown, we have immediately that

$$x = \langle 0,3,2,2,2,\cdots \rangle.$$

Expanding in the usual way,

$$x = \left(2 + \sqrt{2}\right)^{-1}.$$

[3] As has been done by others, such as Ramharter. The interested reader should review Conway's details.

In similar fashion I note that

$$\frac{1}{7} = \sum_{i=1}^{\infty} \left(\frac{1}{2^3} \right)^i = 0 \cdot 001001001 \cdots$$

Thus

$$?^{-1}\left(\frac{1}{7} \right) = \left\langle 0, 3, \overline{1, 2} \right\rangle = 2 - \sqrt{3} \, .$$

Similar calculations could find any value in the interval $\left(0, \frac{1}{2}\right)$ and the corresponding values in $\left(\frac{1}{2}, 1\right)$ could be found using the symmetry relation **II**. For instance, compare our result in **IV**. Anyway, Conway provides a connection between $?(x)$ and game theory. He connects $?(x)$ to something else too, but we will see this later.

Minkowski's Definition Repeated by Mays

My final example of an independent discovery of ?(x)—though almost certainly there are many more in the world—originates closer to home. Michael Mays introduced me to the question mark function (and while I do find it interesting, my love grew more slowly than that of Denjoy!). Based on [11], I state one observation he makes.

Mays finds $?(x)$ exactly like Minkowski. As Mays develops the function, he notes how the Farey fractions fill in the interval [0,1]. Specifically, they tend to cluster *away from* 0, 1/2, and 1. This is seen in the figure below that he generates with *Mathematica*. It is a plot of Farey fractions along a (invisible) line segment of unit length and represents approximately 16 steps of the Farey development.

Figure 3: Farey Distribution

This contrasts sharply with the corresponding picture for the images of these points, where the values *do* cluster around 0, 1/2, and 1.

Figure 4: ?(x) Distribution

This is not surprising; one need only reflect on the development shown much earlier in Table 1. In some sense, these pictures bring our treatment full circle to the title of Minkowski's paper wherein he introduced ?(x): *On the Geometry of Numbers*. We have only one thing left to discuss.

Combinatorics, anyone?

In a beautiful paper [3] published in 2000, Neil Calkin and Herbert Wilf give a fairly simple, explicit construction of the positive rational numbers. Succinctly put, they create a tree, beginning with the vertex $\frac{1}{1}$, such that each vertex $\frac{i}{j}$ has two "children" written below it. The "left child" is $\frac{i}{i+j}$, and the "right child" is $\frac{i+j}{j}$. Consider the sequence of rationals formed by reading successive rows of the tree. Consider the sequence of numerators of these fractions (which is actually just a shift of the sequences of the denominators). Here is a list of values, indexed by N, whose numerator is $f(N)$. I also give the fraction $g(N)$ explicitly for later reference.

N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
f(N)	1	1	2	1	3	2	3	1	4	3	5	2	5	3	4	1	5
g(N)	1/1	1/2	2/1	1/3	3/2	2/3	3/1	1/4	4/3	3/5	5/2	2/5	5/3	3/4	4/1	1/5	5/4

Table 2: The Calkin and Wilf Enumeration

Calkin and Wilf show that this sequence is governed by

$$f(0)=1, \quad f(2N+1)=f(N), \quad f(2N+2)=f(N)+f(N+1)$$

They offer the following definition

Definition of Hyperbinary Representations of *n*

> A *hyperbinary representation of n* is one such that powers of two may be used at most twice (instead of once, as in the usual binary). Denote the number of such representations of *n* by *b*(*n*). Thus, for example, $b(5) = 2$ because $5 = 4 + 1 = 2 + 2 + 1$.

The rather amazing fact is that $f(n) = b(n)$. Calkin and Wilf show this easily by matching recurrence relations for the two functions.

More amazing still is an observation made by Conway [4]. Adapting his notation slightly, and temporarily restricting the domain to $\{N : N = 2^s, s \in \mathbb{Z}^+ \cup 0\}$,

$$?^{-1}\left(\frac{1}{N}\right) = \frac{f(N-1)}{f(N)} = g(N-1)$$

This can be seen for test values using Table 2. I suppose that an inductive argument, based on Calkin and Wilf's tree, and the recurrence relations would extend this domain. These details do not strike me as very interesting compared to what is already seen.

Is it surprising that Minkowski's function would tie into the Calkin and Wilf enumeration? Probably not. The function was based on Farey fractions, which are of course an enumeration of the rationals in [0,1]. So one enumeration is essentially the same as the other, but scrambled.

The ultimate point of this section is this connection between ?(*x*) and the fairly simple combinatorial problem of counting hyperbinary representations of an integer. When the result is cast in this language, the connection is somewhat surprising.

Remarks

We have seen how ?(*x*) is developed from a very natural bridge between the Farey average and the arithmetic average to a real-valued function with various analytic properties and generalizations. It is possible that ?(*x*) could be part of a solution to the Hermite problem quoted by Beaver and Garrity, but this remains an open question. Although Minkowski mentions the question mark function almost as an afterthought and little work was done right away, there has been renewed interest. The literature reviewed here comes from Germany, France, Austria, Spain, Sweden, and the United States. The discoveries surrounding $?(x)$ span 98 years, including at least a half dozen citations from within a decade of this writing. We might wonder whether a collective mathematical consciousness is awaking and taking interest in $?(x)$.

"Math mysticism" aside, we may consider some outstanding questions about the question mark function:

- Does $?(x)$ or its generalizations lead to a solution to the Hermite problem?
- Is there any way to represent the real-valued $?(x)$ without recourse to continued fractions? What might we gain from an alternate representation?
- Is there an interesting extension of $?(x)$ to the field of complex numbers?
- In terms of actual graphs and visual representations, are there geometric pictures of $?(x)$ and its generalizations that have not been considered? (Minkowski, Mays, and Beaver and Garrity each offer a few pictures—are there any others? Do theirs generalize in any interesting way?)
- From a more historical point of view, are there any translations of Denjoy? Has his work on $?(x)$ been used in any other context?

While there will always be unanswered questions, the treatment here may provide a springboard from which to address these and other questions. For a handy review of the basic properties of ?(x)—the two definitions (I) and (III), Theorem 2, singularity, and sample values—see [22].

Appendix

The following is taken from [10]:

Kinney gives his result in terms of a Lipschitz condition and fractional dimension, which he defines as follows:

Let $C_\mu = \{I_j\}$ be a set of intervals covering E where $\mu \geq \max_j |I_j|$. We call C_μ a covering of E of norm μ, and define $\Gamma(\alpha, C_\mu, E) = \sum |I_i|^\alpha$. The α–dimensional measure of E is $\lim_{\mu \to 0} \operatorname{glb}_{c_\mu} \Gamma(\alpha, C_\mu, E)$.

The Hausdorff-Besicovitch dimensional number $\beta(E)$ is

$$\beta(E) = \inf(\beta \mid \Gamma(\beta, E) = 0) = \sup(\beta \mid \Gamma(\beta, E) = \infty).$$

We say that $f(x) \in \mathrm{Lip}\,\alpha$ at x if for every $\varepsilon > 0$, there is an $h(\varepsilon)$ such that, for $h < h(\varepsilon)$, $(2h)^{\alpha+\varepsilon} < |f(x+h) - f(x-h)| < (2h)^{\alpha-\varepsilon}$.

We let $a = \left[2\int_0^1 \log_2(1+x)\, d\,?(x)\right]^{-1}$. We wish to show

Theorem (Kinney)

There is a set A with $?(A) = ?[(0,1)] = 1$, for which

1. $\beta(A \cap B) = \alpha \quad if \quad ?(B) > 0$,
2. $?(x) \in \mathrm{Lip}\,\alpha \quad at \quad x \quad for \quad x \in A$

Addendum

As I was giving an oral defense of this thesis, another connection between the question mark function and a (seemingly) different subject surfaced. John Goldwasser, one of my thesis committee members, was struck by the familiarity of the recurrence relations from the Calkin and Wilf paper [3]. As it turns out, he and Chip Klostermeyer had used a certain class of Fibonacci polynomials, with coefficients taken from the integers modulo 2, defined by

$$g(0)=1 \quad g(1)=x \quad g(n)=xg(n-1)+g(n-2) \text{ for } n>2$$

to count the number of even dominating sets in the m by n grid graph. This graphs is defined by Goldwasser as

> the number of ways to select turned on lights in an m by n array of lights such that each light (whether it is off or on) has an even number of on lights among itself and its immediate neighbors (in the interior of the grid there are itself and 4 immediate neighbors). [A2]

Recurrences for said polynomials include two that do indeed appear familiar:

$$g(2n+1)=x[g(n)]^2 \quad g(2n+2)=g(n)^2+g(n+1)^2$$

There is more: if $h(n)$ counts the number of terms in $g(n)$, then $h(n)=b(n)=f(n)$, where *b* and *f* are as defined by Calkin and Wilf.

Goldwasser poses the question of whether a similar relationship holds for the following:

- Fibonacci polynomials with coefficients from the integers modulo *p*, for *p* prime,
- hyperbinary representations allowing *p* occurrences of a given power of *p*
- a tree with *p* children from each vertex
- and a function like the question mark that *p*-sects the interval [0,1] (or perhaps some generalization of it).

The complete answer to this question, as well as any other implications of this connection made by Goldwasser, are not known as of the completion of this thesis in mid-May 2003, although not all of the relationships appear to generalize so nicely.

Goldwasser has made another observation. The degrees of nonzero terms of the n^{th} Fibonacci polynomial (reduced mod 2) are related to the hyperbinary representations of n. The degrees are found by reducing the hyperbinary representations mod 2, that is, by deleting the numeral 2 from the representations. For example, here are the first few Fibonacci polynomials, followed by a "fast" calculation of the powers for the final polynomial $f_4(x)$. Angle brackets are used for hyperbinary representations, and S denotes the set of powers to use for the polynomial $f_n(x)$:

$$\begin{gathered} f_0(x)=1 \\ f_1(x)=x \\ f_2(x)=x^2+1 \\ f_3(x)=x^3 \\ f_4(x)=x^4+x^2+1 \end{gathered}$$

$$b(4)=3 \Rightarrow |S|=h(4)=3$$

$$4=\langle 1\quad 0\quad 0\rangle=\langle 0\quad 2\quad 0\rangle=\langle 0\quad 1\quad 2\rangle$$

$$\left\{\begin{aligned} \langle 1\quad 0\quad 0\rangle &\equiv \langle 1\quad 0\quad 0\rangle (\operatorname{mod} 2)=4 \\ \langle 0\quad 2\quad 0\rangle &\equiv \langle 0\quad 0\quad 0\rangle (\operatorname{mod} 2)=0 \\ \langle 0\quad 1\quad 2\rangle &\equiv \langle 0\quad 1\quad 0\rangle (\operatorname{mod} 2)=2 \end{aligned}\right\} \Rightarrow S=\{0,2,4\}$$

The information in this addendum is from [A2, A3]. For citations of the research by Goldwasser and Klostermeyer, see [A1] and Goldwasser's publication list (below).

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