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# PROBABILITY MEASURES ON INDEFINITE INNER PRODUCT SPACES

*by*

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1. An indefinite inner product space is a real or complex vector space together with a symmetric (in the complex case hermitian) bilinear form prescribed on it so that the corresponding quadratic form assumes both positive and negative values.

In the development of probability theory on indefinite inner product space, and in the study of solutions of random equations on these spaces, the construction of induced probability measures on concrete Banach spaces, Hilbert spaces or Fréchet spaces and the study of their properties is of fundamental importance. Indeed the application of the general duality theory of topological vector spaces to inner product spaces leads to partial majorants, majorants, admissible or intrinsic topologies which are, in many applications, Banach, Hilbert or Fréchet topologies [3].

In this Note we consider probability measures on an indefinite inner product space using the method of Kampé de Fériet [5], D. Kannan and A. T. Bharucha-Reid [6] and N. Vakhania [9].

2. We first introduce some concepts and notations.

Let $(X, [.,.])$ be an indefinite inner product space. If $[x, y] = 0$ or, equivalently, $[y, x] = 0$ for a pair of vectors $x$, $y$, we say that $x$ and $y$ are orthogonal to each other and write $x \perp y$. Two sets $A, B \subset X$ are said to be orthogonal if $x \perp y$ for every $x \in A$, $y \in B$. For any set $A \subset X$ we put

$$A^{\perp} = \{x \in X : x \perp A\}$$

The subspace $X^{\circ} = X^{\perp}$ is called the isotropic part and if $X^{\circ} \neq \{0\}$ the space $X$ is called degenerate.

Let $\{e_i\}_{i \in I}$ be a system of elements of an inner product space $(X, [. , .])$. If $[e_i, e_j] = \delta_{ij}$, $i,j \in I$, (Kronecker delta) we say that the system $\{e_i\}_{i \in I}$ is orthonormal.

*A* topology $\tau$ on $(X, [. , .])$ is a partial majorant (majorant) of the inner product if $\tau$ is locally convex and for any fixed $y \in X$ the function $\varphi_y(x) = [x, y]$ is $\tau$-continuous on $X$ (the inner product is jointly $\tau$-continuous).

As is well known a sequence $\{e_n\}$ of elements of a linear topological space is called a topological basis for $X$ if for each $x \in X$ there corresponds a unique sequence $\{\xi_n\}$ of scalars such that the series

$$\sum_{n=0}^{\infty} \xi_n e_n \tag{1}$$

converges to $x$ in $X$.

If $\{e_n\}$ is a topological basis for $X$ then there is a corresponding sequence $\{e_n^*\}$ of linear functionals, called coefficient functionals, such that

$$x = \sum_{n=0}^{\infty} e_n^*(x) e_n \tag{2}$$

for all $x \in X$. The sequences $\{e_n\}$ and $\{e_n^*\}$ are biorthogonal in the sense that $e_n^*(e_m) = \delta_{nm}$.

A topological basis for $X$ is called a Schauder basis for $X$ if all the associated coefficient functionals are continuous on $X$. If $X$ is a Fréchet space then every topological basis is a Schauder basis [7].

It is known [3, Theorem IV 3.3] that if $\tau$ is a separable majorant topollogy on non-degenerate inner product space $X$, then every orthonormal system in $X$ is countable and there is a $\tau$-complete orthonormal system in $X$.

Let $(X, [.\,,.])$ be a non-degenerate indefinite inner product space with a Fréchet majorant topology $\tau$ and a basis $\{e_n\}_{n \in N}$.

We will be concerned primarily with the Fréchet topology since there exists [3] a uniqueness condition for this class of topologies on $(X, [.\,,.])$.

Let $C$ be the space of complex numbers and let $C^{\infty} = \{\xi : \xi = (\xi_1, \ldots, \xi_n, \ldots),\ \xi_n \in C\}$. Since $X$ has a basis, there is a linear bijective map $T$ between $X$ and a subset $\widetilde{X} \subset C^{\infty}$,

$$x = T\xi, \quad \xi = T^{-1}x, \quad x \in X, \quad \xi \in X \tag{3}$$

i.e., for each $x \in X$ there corresponds a unique sequence $\{\xi_n\} \subset C$ with the property that the sequence $\{x_n\}$, where $x_n = \sum_{i=1}^{n} \xi_i e_i$, converges to zero in the metric $d$ induced by $\tau$ on $X$; that is $\lim_{n\to\infty} d(x, x_n) = 0$.

We now define the sequences $\{\psi_n\}$ and $\{\varphi_{n,k}\}$ of functions as follows

$$\psi_n(\xi_1, \ldots, \xi_n) = d(x_n, 0) = d(\xi_1 e_1 + \ldots + \xi_n e_n, 0) \tag{4}$$

and

$$\varphi_{n,k}(\xi_{n+1}, \ldots, \xi_{n+k}) = d(x_{n+k} - x_n, 0) = \psi_{n+k}(0, \ldots, 0, \xi_{n+1}, \ldots, \xi_{n+k}). \tag{5}$$

The following properties of $\psi_n$ are immediate

a) $\psi_n(\xi_1, \ldots, \xi_n) > 0$ if $\sum_{i=1}^{n} |\xi_i| > 0$,

b) $\psi_n(0, \ldots, 0) = 0$,

c) $|\psi_n(\xi_1', \ldots, \xi_n') - \psi_n(\xi_1'', \ldots, \xi_n'')| \leqslant d((\xi_1' - \xi_1'')e_1 + \ldots +$

$+ (\xi_n' - \xi_n'')e_n, 0) = \psi_n(\xi_1' - \xi_1'', \ldots, \xi_n' - \xi_n'') \leqslant d((\xi_1' - \xi_1'')e_1, 0) +$

$+ \ldots + d((\xi_n' - \xi_n'')e_n, 0)$

since $d$ is a translation-invariant metric on $X$; i.e., $\psi_n$ are continuous functions on $C^n$, $n \in N$, for a fixed basis on $X$.

Clearly the sequence $\{x_n\}$ converges if and only if

$$\inf_{n \geqslant 0} \sup_{k \geqslant 1} d(x_{n+k}, x_n) = 0 \tag{6}$$

hence the space $\widetilde{X} \subset C^\infty$ which is a bijective correspondence to $X$, can be defined as follows:

$$\widetilde{X} = \{\xi = \{\xi_n\} \in C^\infty : \inf_{n \geqslant 0} \sup_{k \geqslant 1} \varphi_{n,k}(\xi_{n+1}, \ldots, \xi_{n+k}) = 0\}. \tag{7}$$

If we define

$$d'(\xi, \eta) = d'(\xi - \eta, 0) = \sup_{n \geqslant 0} d(x_n - y_n, 0) = \tag{8}$$

$$= \sup_{n \geqslant 0} \psi_n(\xi_1 - \eta_1, \ldots, \xi_n - \eta_n)$$

then $(X, d')$ becomes a Fréchet space. On the other hand it follows that the above defined map $T : (\widetilde{X}, d') \to (X, d)$ is a continuous operator since

$$d'(\xi, 0) = \sup_{n \geqslant 0} d(x_n, 0) \geqslant \lim_{n \to \infty} d(x_n, 0) = d(x, 0) = d(T\xi, 0). \tag{9}$$

Hence, by the open mapping theorem, we have that $T$ is a topological isomorphism.

Because of the isomorphism $T$ between $X$ and $\widetilde{X} \subset C^\infty$, a measure $\tilde{\nu}$ on $\widetilde{X}$ will induce a measure $\nu$ on $X$. So, it is enough if we construct a measure on $X$, and we do this by using Kolmogorov's consistency theorem [7], [6].

Let $C^1 = C$, $C^n = \{(\xi_1, \ldots, \xi_n), \ \xi_i \in C, \ 1 \leqslant i \leqslant n\}$, and we define maps $\pi_n$ and $\pi_{n,m}$, $1 \leqslant m \leqslant n$, as follows

$$\pi_n : C^\infty \to C^n; \ \pi_n(\xi_1, \ldots, \xi_n, \ldots) = (\xi_1, \ldots, \xi_n) \tag{10}$$

$$\pi_{n,m} : C^n \to C^m; \ \pi_{n,m}(\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_m) \tag{11}$$

If $\mathscr{B}_n$ denotes the Borel $\sigma$-algebra in $C^n$, then the Borel space $(C^n, \mathscr{B}_n)$ is a separable standard Borel space for $n \geqslant 1$. Let $\tilde{\nu}_1, \tilde{\nu}_2, \ldots,$ be a sequence of measures defined on $\mathscr{B}_1, \mathscr{B}_2, \ldots$, respectively, satisfying the following consistency conditions

$$\tilde{\nu}_n(C^n) = 1, \ \tilde{\nu}_m(B) = \tilde{\nu}_n(\pi_{n,m}^{-1}(B)), \text{ for all } B \in \mathscr{B}_m. \tag{12}$$

From Kolmogorov's consistency theorem for standard separable Borel space [8], we obtain a unique measure $\tilde{\nu}$ on $(C^\infty, \mathscr{B}_\infty)$ such that

$$(13) \qquad \tilde{\nu}(C^\infty) = 1, \quad \tilde{\nu}_m(B) = \tilde{\nu}_n(\pi_n^{-1}(B)), \quad \text{for all} \quad B \in \mathscr{B}_n.$$

We note that the $\pi_n$'s and $\pi_{n,m}$'s are measurable and the $\psi_n$'s and $\varphi_{n,k}$'s are continuous. Thus $\varphi_{n,k}$ is $\tilde{\nu}_{n+k}$-measurable. $\tilde{X}$ is defined by the functions $\varphi_{n,k}$ under countable operations, hence from the properties (13) of $\tilde{\nu}$ we obtain

**Lemma** 1. *The set* $\tilde{X} \subset C^\infty$ *is* $\nu$*-measurable.*

If $\tilde{\mathscr{B}}$ denotes the restriction of $\mathscr{B}_\infty$ to $\tilde{X}$, that is, $\tilde{\mathscr{B}} = \tilde{X} \cap \tilde{\mathscr{B}}_\infty$, and if for all $B \subset X$ such that $B = T\tilde{B}$, $\tilde{B} \in \tilde{\mathscr{B}}$, put

$$\nu(B) = \tilde{\nu}(\tilde{B})$$

we have that

$$(X, \mathscr{B}, \nu) = T(\tilde{X}, \tilde{\mathscr{B}}, \tilde{\nu}) \quad \text{and} \quad (\tilde{X}, \tilde{\mathscr{B}}, \nu) = T^{-1}(X, \mathscr{B}, \nu)$$

are measure spaces.

It is clear that $0 \leqslant \nu(X) = \tilde{\nu}(\tilde{X}) \leqslant 1$. For to give a condition such that $\nu(X) = \tilde{\nu}(\tilde{X}) = 1$, i.e., that $(X, \mathscr{B}, \nu)$ to be a probability space, we observe that if, for a given $\varepsilon > 0$, we consider the cylinder set

$$C_{n,k}(\varepsilon) = \{\xi : (\xi_1, \ldots, \xi_{n+k}) \in B_{n,k}(\varepsilon), \; \xi_{n+k+i} \in C, \; i = 1, 2, \ldots\}.$$

with base

$$B_{n,k}(\varepsilon) = \bigcap_{i=1}^{k} \{(\xi_1, \ldots, \xi_{n+k}) : \varphi_{n,i}(\xi_{n+1}, \ldots, \xi_{n+i}) < \varepsilon\}.$$

We have

$$\tilde{X} = \bigcap_{\varepsilon>0} \bigcup_{n\geqslant 1} \bigcap_{k\geqslant 1} C_{n,k}(\varepsilon) \quad \text{and} \quad \tilde{\nu}(\tilde{X}) = \inf_{\varepsilon>0} \sup_{n\geqslant 1} \inf_{k\geqslant 1} \tilde{\nu}(C_{n,k}(\varepsilon))$$

**Theorem 1.** *Let* $\{\tilde{\nu}\}$ *be a sequence of measures defined on* $\{\mathscr{B}_n\}$ *satisfying conditions* (12) *and let* $\tilde{\nu}$ *be the measure on* $C^\infty$ *obtained via Kolmogorov's theorem from* $\{\tilde{\nu}_n\}$. *Then by the isomorphism* (3), $\tilde{\nu}$ *induces a measures* $\nu$ *on* $X$.

*The measure space* $(X, \mathscr{B}, \nu)$ *is a probability space if and only if, for any* $\varepsilon > 0$, *the sequence* $\{\tilde{\nu}_n\}$ *of measures satisfies in addition to consistency conditions* (12), *the condition*

$$\inf_{\varepsilon>0} \sup_{n\geqslant 1} \inf_{k\geqslant 1} \tilde{\nu}_{n+k}(B_{n,k}(\varepsilon)) = 1.$$

3. **Definition 1.** Let $X$ be a Fréchet space. A probability measure $\nu$ on $X$ is called an $L$-measure if all continuous linear functionals on $X$ are $\nu$-measurable.

**Theorem 2.** *The measure* $\nu$ *on the Fréchet space* $X$ *with the basis* $\{e_n\}$ *is an* $L$*-measure.*

*Proof.* The reasoning is similar to [6] since the projection map $P_n: C^\infty \to C$, $P_n(\xi) = \xi_n$ is clearly $\tilde{\nu}$-measurable and

$$e_n^*(x) = \xi_n = P_n(\xi) = (P_n T^{-1})(x), \ x \in X,$$

and for $x^* \in X^*$ we have

$$x^*(x) = \sum_{n \in N} e_n^*(x) x^*(e_n).$$

**Lemma 2.** *Every open set $G$ of the Fréchet space $X$ is a $\nu$-measurable set.*

*Proof.* Since $X$ is a separable topological space then every open set $G \subset X$ is a countable reunion of neighbourhoods. On the other hand every neighbourhood is a $\nu$-measurable set since for all $a \in X$ we have that $d(x, a) = \lim_{n\to\infty} d(x_n, a_n) = \lim_{n\to\infty} \psi_n(\xi_1 - a_1, \ldots, \xi_n - a_n)$ where $\psi_n$ are continuous functions on $C^n$.

Also we have

**Lemma 3.** *Every continuous functional on $X$ is a complex random variable on $(X, \mathscr{B}, \nu)$.*

We obtain

**Theorem 3.** *If $(X, [\cdot, \cdot])$ is a non-degenerate indefinite inner product space with a majorant topology $\tau$ and a basis $\{e_n\}$, then the indefinite inner product induces two complex random variables $\varphi_y(x) = [x, y]$ and $\varphi_x(y) = [x, y]$ on $(X, \mathscr{B}, \nu)$.*

**Corollary 1.** *The weak topology $\tau_0$ of the indefinite inner product space $(X, [\cdot,\cdot])$ defined by the family $\{\psi_y\}_{y\in X}$ were $\psi_y(x) = |[x, y]|$, $x \in X$, is a locally convex topology defined by a family of complex random variables on $(X, \mathscr{B}, \nu,)$.*

Also the characteristic functional of a probability measure on $(X, [\cdot,\cdot])$, the concepts of mean and covariance operator of a probability measure and results as theorems of Bochner or Lévy type can be obtained.

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