THE ALGEBRA OF PROBABLE INFERENCE

The Algebra of Probable Inference

by Richard T. Cox
PROFESSOR OF PHYSICS
THE JOHNS HOPKINS UNIVERSITY

BALTIMORE:
The Johns Hopkins Press

Distributed in Great Britain by Oxford University Press, London

Printed in the United States of America by Horn-Shafer Co., Baltimore

Library of Congress Catalog Card Number: 61-8039

to my wife Shelby

Preface

This essay had its beginning in an article of mine published in 1946 in the *American Journal of Physics*. The axioms of probability were formulated there and its rules were derived from them by Boolean algebra, as in the first part of this book. The relation between expectation and experience was described, although very scantily, as in the third part. For some years past, as I had time, I have developed further the suggestions made in that article. I am grateful for a leave of absence from my duties at the Johns Hopkins University, which has enabled me to bring them to such completion as they have here.

Meanwhile a transformation has taken place in the concept of entropy. In its earlier meaning it was restricted to thermodynamics and statistical mechanics, but now, in the theory of communication developed by C. E. Shannon and in subsequent work by other authors, it has become an important concept in the theory of probability. The second part of the present essay is concerned with entropy in this sense. Indeed I have proposed an even broader definition, on which the resources of Boolean algebra can be more strongly brought to bear. At the end of the essay, I have ventured some comments on Hume's criticism of induction.

Writing a preface gives a welcome opportunity to thank my colleagues for their interest in my work, especially Dr. Albert L. Hammond, of the Johns Hopkins Department of Philosophy, who was good enough to read some of the manuscript, and Dr. Theodore H. Berlin, now at the Rockefeller Institute in New York but recently with the Department of Physics at Johns Hopkins. For help with the manuscript it is a pleasure to thank Mrs. Mary B.

Rowe, whose kindness and skill as a typist and linguist have aided members of the faculty and graduate students for twenty-five years.

I have tried to indicate my obligations to other writers in the notes at the end of the book. Even without any such indication, readers familiar with *A Treatise on Probability* by the late J. M. Keynes would have no trouble in seeing how much I am indebted to that work. It must have been thirty years or so ago that I first read it, for it was almost my earliest reading in the theory of probability, but nothing on the subject that I have read since has given me more enjoyment or made a stronger impression on my mind.

The Johns Hopkins University
BALTIMORE, MARYLAND

R. T. C.

Contents

Preface vii

I. *Probability* 1

1. Axioms of Probable Inference 1
2. The Algebra of Propositions 4
3. The Conjunctive Inference 12
4. The Contradictory Inference 18
5. The Disjunctive Inference 24
6. A Remark on Measurement 29

II. *Entropy* 35

7. Entropy as Diversity and Uncertainty and the Measure of Information 35
8. Entropy and Probability 40
9. Systems of Propositions 48
10. The Entropy of Systems 53
11. Entropy and Relevance 58
12. A Remark on Chance 65

III. *Expectation* 69

13. Expectations and Deviations 69
14. The Expectation of Numbers 74
15. The Ensemble of Instances 79
16. The Rule of Succession 82
17. Expectation and Experience 87
18. A Remark on Induction 91

Notes 99

Index 109

THE ALGEBRA OF PROBABLE INFERENCE

I

Probability

1. Axioms of Probable Inference[1]

A probable inference, in this essay as in common usage, is one entitled on the evidence to partial assent. Everyone gives fuller assent to some such inferences than to others and thereby distinguishes degrees of probability. Hence it is natural to suppose that, under some conditions at least, probabilities are measurable. Measurement, however, is always to some extent imposed upon what is measured and foreign to it. For example, the pitch of a stairway may be measured as an angle, in degrees, or it may be reckoned by the rise and run, the ratio of the height of a step to its width. Either way the stairs are equally steep but the measurements differ because the choice of scale is arbitrary. It is therefore reasonable to leave the measurement of probability for discussion in later chapters and consider first what principles of probable inference will hold however probability is measured. Such principles, if there are any, will play in the theory of probable inference a part like that of Carnot's principle in thermodynamics, which holds for all possible scales of temperature, or like the parts played in mechanics by the equations of Lagrange and Hamilton, which have the same form no matter what system of coordinates is used in the description of motion.

It has sometimes been doubted that there are principles valid over the whole field of probable inference. Thus Venn wrote in his *Logic of Chance:*[2]

> "In every case in which we extend our inferences by Induction or Analogy, or depend upon the witness of others, or trust to our own memory of the past, or come to a conclusion through conflicting arguments, or even make a long and complicated deduction by mathematics or logic, we have a result of which we can scarcely feel as certain as of the premises from which it was obtained. In all these cases then we are conscious of varying quantities of belief, but are the laws according to which the belief is produced and varied the same? If they cannot be reduced to one harmonious scheme, if in fact they can at best be brought to nothing but a number of different schemes, each with its own body of laws and rules, then it is vain to endeavour to force them into one science."

In this passage, the first of three sentences distinguishes types of inference which common usage calls probable, the second asks whether inferences of these different kinds are subject to the same laws and the third implies that they are not. Nevertheless, if we look for them, we can find likenesses among these examples and likenesses also between these and others which would be accepted as proper examples of probability by all the schools of thought on the subject. Venn himself belonged to the school of authors who define probability in statistical terms and restrict its meaning to examples in which it can be so defined.[3] By their definition, they estimate the probability that an event will occur under given circumstances from the relative frequencies with which it has occurred and failed to occur in past instances of the same circumstances. Every instance in which it has occurred strengthens the argument that it will occur in a new instance and every contrary instance strengthens the contrary argument. Thus, whenever they estimate a probability in the restricted sense their definition allows and the way their theory prescribes, they "come to a conclusion through conflicting arguments," as do the advocates of other definitions and theories. The argument, moreover, which makes one inference more probable makes the contradictory inference less probable and thus the two probabilities stand in a mutual relation. In this all schools can agree and it may be taken as an axiom on any definition of probability that:

> The probability of an inference on given evidence determines the probability of its contradictory on the same evidence. (1.i)

Continuing with Venn's list of varieties of probable inference, let us consider the probability of the right result in "a long and complicated deduction in mathematics" and compare it with the probability of a long run of luck at cards or dice, a classical example in the theory of probability. In any game of chance, a long run of luck is, of course, less probable than a short one, because the run may be broken by a mischance at any single toss of a die or drawing of a card. Similarly, in a commonplace example of mathematical deduction, a long bank statement is less likely to be right at the end than a short one, because a mistake in any single addition or subtraction will throw it out of balance. Clearly we are concerned here with one principle in two examples. A mathematical deduction involving more varied operations in its successive steps or a chain of reasoning in logic would provide only another example of the same principle.

The uncertainties of testimony and memory, also cited by Venn, come under this principle as well. Consider, for example, the probability of the assertion, made by Sir John Maundeville in his *Travels*, that Noah's Ark may still be seen on a clear day, resting where it was left by the receding waters of the Flood, on the top of Mount Ararat. For this assertion to be probable on Sir John's testimony, it must first of all be probable that he made it from his recollection rather than his fancy. Then, on the assumption that he wrote as he remembered what he saw or heard told, it must be probable also that his memory could be trusted against a lapse such as might have occurred during the long years after he left the region of Mount Ararat and before he found in his writing a solace from his "rheumatic gouts" and his "miserable rest." Finally, on the assumption that his testimony was honest and his memory sound, it must be probable that he or those on whom he depended could be sure that they had truly seen Noah's

Ark, a matter made somewhat doubtful by his other statement that the mountain is seven miles high and has been ascended only once since the Flood.

Every assertion which, like this one, involves the transmission of knowledge by a witness or its retention in the memory is, on this account, a conjunction of two or more assertions, each of which contributes to the uncertainty of the joint assertion. For this reason, it comes under the same principle which we saw involved in the probability of a run of luck at cards and which can be stated in the following axiom:

> The probability on given evidence that both of two inferences are true is determined by their separate probabilities, one on the given evidence, the other on this evidence with the additional assumption that the first inference is true. (1.ii)

Thus the uncertainties of testimony and memory, of long and complicated deductions and conflicting arguments—all the specific examples in Venn's list—have traits in common with one another and with the classical examples provided by games of chance.

The more general subjects of induction and analogy, also mentioned in the quotation from Venn, must be reserved for discussion in later chapters, but the examples already considered may serve to launch an argument that all kinds of probable inference can be "reduced to one harmonious scheme."[4]

For this reduction, the argument will require only the two axioms just given, when they are implemented by the logical rules of Boolean algebra.[5]

2. *The Algebra of Propositions*

Ordinary algebra is the algebra of quantities. In our use of it here, quantities will be denoted by italic letters, as a, b, A, B. Boolean algebra is the algebra, among other things, of propositions. Propositions will be denoted here by small boldface letters, as **a**, **b**, **c**. The meaning of a proposition in Boolean algebra corresponds to the value of a quantity in ordinary algebra. For example, just as, in ordinary algebra, a certain quantity may have a constant value throughout a given calculation or a variable one, so, in Boolean algebra, a proposition may have a fixed meaning throughout a given discourse or its meaning may vary according to the context within the discourse. Thus "Socrates is a man" is a familiar proposition of constant meaning in logical discourse, whereas the proposition, "I agree with all that the previous speaker has said," has a meaning variable according to the occasion. For another example of the same correspondence, just as an ordinary algebraic equation, such as

$$(a + b)c = ac + bc,$$

states that two quantities, although different in form, are nevertheless the same in value, so a Boolean equation states that two propositions of different form are the same in meaning.

Of the signs used for operations peculiar to Boolean algebra, we shall need only three, $\sim$, $\cdot$ and $\vee$, which denote respectively *not*, *and* and *or*.[6] Thus the proposition *not* **a**, called the *contradictory* of **a**, is denoted by $\sim$**a**. The relation between **a** and $\sim$**a** is a mutual one, either being the other's contradictory. To deny $\sim$**a** is therefore to affirm **a**, so that

$$\sim\sim\mathbf{a} = \mathbf{a}.$$

The proposition **a** *and* **b**, called the *conjunction* of **a** and **b**, is denoted by **a**$\cdot$**b**. The order of propositions in the conjunction is the order in which they are stated. In ordinary speech and writing, if propositions describe events, it is customary to state them in the chronological order in which the events take place. So the nursery jingle runs, "Tuesday we iron and Wednesday we mend." It would have the same meaning, however, if it ran, "Wednesday we mend and Tuesday we iron." In this example, therefore, and also in general,

$$\mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b}.$$

Similarly the expression **a·a** means only that the proposition **a** is stated twice and not that an event described by **a** has occurred twice. Rhetorically it is more emphatic than **a**, but logically it is the same. Thus

$$\mathbf{a}\cdot\mathbf{a} = \mathbf{a}.$$

Parentheses are used in Boolean as in ordinary algebra to indicate that the expression they enclose is to be treated as a single entity in respect to an operation with an expression outside. They designate an order of operations, in that any operations indicated by signs in the enclosed expression are to be performed before those indicated by signs outside. The parentheses are unnecessary if the order of operations is immaterial. Thus $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ denotes the proposition obtained by first conjoining **a** with **b** and then conjoining **a·b** with **c**, whereas $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ denotes the proposition obtained by first conjoining **b** with **c** and then conjoining **a** with **b·c**, but the propositions obtained in these two sequences of operations have the same meaning and the parentheses may therefore be omitted. Accordingly,

$$(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}\cdot\mathbf{c}.$$

The proposition **a** *or* **b**, called the *disjunction* of **a** and **b**, is denoted by $\mathbf{a} \vee \mathbf{b}$. It is to be understood that *or* is used here in the sense intended by the notice, "Anyone hunting or fishing on this land will be prosecuted," which is meant to include persons who both hunt and fish along with those who engage in only one of these activities. This is to be distinguished from the sense intended by the item, "coffee or tea," on a bill of fare, which is meant to offer the patron either beverage but not both. Thus $\vee$ has the meaning which the form *and/or* is sometimes used to express.

Let us now consider expressions involving more than one of the signs, $\sim$, $\cdot$ and $\vee$. In this consideration it should be kept in mind that $\sim\mathbf{a}$ is not some particular proposition meant to contradict **a** item by item. For example, if **a** is the proposition, "The dog is small, smooth-coated, bob-tailed and white all over except for black ears," $\sim\mathbf{a}$ is not the proposition, "The dog is large, wire-haired, long-tailed and black all over except for white ears." To assert $\sim\mathbf{a}$ means nothing more than to say that **a** is false at least in some part. If **a** is a conjunction of several propositions, to assert $\sim\mathbf{a}$ is not to say that they are all false but only to say that at least one of them is false. Thus we see that

$$\sim(\mathbf{a}\cdot\mathbf{b}) = \sim\mathbf{a} \vee \sim\mathbf{b}.$$

From this equation and the equality of $\sim\sim\mathbf{a}$ with **a**, there is derived a remarkable feature of Boolean algebra, which has no counterpart in ordinary algebra. This characteristic is a duality according to which the exchange of the signs, $\cdot$ and $\vee$, in any equation of propositions transforms the equation into another one equally valid.[7] For example, exchanging the signs in this equation itself, we obtain

$$\sim(\mathbf{a} \vee \mathbf{b}) = \sim\mathbf{a}\cdot\sim\mathbf{b},$$

which is proved as follows:

$$\mathbf{a} \vee \mathbf{b} = \sim\sim\mathbf{a} \vee \sim\sim\mathbf{b} = \sim(\sim\mathbf{a}\cdot\sim\mathbf{b}).$$

Hence

$$\sim(\mathbf{a} \vee \mathbf{b}) = \sim\sim(\sim\mathbf{a}\cdot\sim\mathbf{b}) = \sim\mathbf{a}\cdot\sim\mathbf{b}.$$

From the duality in this instance and the mutual relation of **a** and $\sim\mathbf{a}$, the duality in other instances follows by symmetry. We have, accordingly, from the equations just preceding,

$$\mathbf{b} \vee \mathbf{a} = \mathbf{a} \vee \mathbf{b},$$

$$\mathbf{a} \vee \mathbf{a} = \mathbf{a}$$

and

$$(\mathbf{a} \vee \mathbf{b}) \vee \mathbf{c} = \mathbf{a} \vee (\mathbf{b} \vee \mathbf{c}) = \mathbf{a} \vee \mathbf{b} \vee \mathbf{c}.$$

The propositions $(\mathbf{a} \vee \mathbf{b})\cdot\mathbf{c}$ and $\mathbf{a} \vee (\mathbf{b}\cdot\mathbf{c})$ are not equal. For, if **a** is true and **c** false, the first of them is false but the second is true. Therefore the form **a** $\vee$ **b·c** is ambiguous. In verbal expressions the ambiguity is usually prevented by the meaning of the words. Thus, in a weather forecast, "rain or snow and high winds," would be understood to mean "(rain or snow) and high winds," whereas "snow or rising temperature and rain" would mean "snow or (rising temperature and rain)." In symbolic expressions, on the other hand, the meaning is not given and parentheses are therefore necessary.

When we assert (**a** $\vee$ **b**)·**c**, we mean that at least one of the propositions, **a** and **b**, is true, but **c** is true in any case. This is the same as to say that at least one of the propositions, **a·c** and **b·c**, is true and thus

$$(\mathbf{a} \vee \mathbf{b})\cdot\mathbf{c} = (\mathbf{a}\cdot\mathbf{c}) \vee (\mathbf{b}\cdot\mathbf{c}).$$

The dual of this equation is

$$(\mathbf{a}\cdot\mathbf{b}) \vee \mathbf{c} = (\mathbf{a} \vee \mathbf{c})\cdot(\mathbf{b} \vee \mathbf{c}).$$

If, in either of these equations, we let **c** be equal to **b** and substitute **b** for its equivalent, **b·b** in the first equation or **b** $\vee$ **b** in the second, we find that

$$(\mathbf{a} \vee \mathbf{b})\cdot\mathbf{b} = (\mathbf{a}\cdot\mathbf{b}) \vee \mathbf{b}.$$

In this equation, the exchange of the signs, · and $\vee$, has only the effect of transposing the members; the equation is dual to itself. Each of the propositions, (**a** $\vee$ **b**)·**b** and (**a·b**) $\vee$ **b**, is, indeed, equal simply to **b**. Thus to say, "He is a fool or a knave and he is a knave," or "He is a fool and a knave or he is a knave," sounds perhaps more uncharitable than to say simply, "He is a knave," but the meaning is the same.

In ordinary algebra, if the value of one quantity depends on the values of one or more other quantities, the first is called a function of the others. Similarly, in Boolean algebra, we may call a proposition a function of one or more other propositions if its meaning depends on theirs. For example, **a** $\vee$ **b** is a Boolean function of the propositions **a** and **b** as $a + b$ is an ordinary function of the quantities a and b.

It may be remarked that the operations of Boolean algebra generate functions of infinitely less variety than is found among the functions of ordinary algebra. In ordinary algebra, because $a \times a = a^2, a \times a^2 = a^3, \ldots$ and $a + a = 2a, a + 2a = 3a, \ldots,$ there is no end to the functions of a single variable which can be generated by repeated multiplications and additions. By contrast, in Boolean algebra, $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{a} \vee \mathbf{a}$ are both equal simply to **a**, and thus the signs, $\cdot$ and $\vee$, when used with a single proposition, generate no functions.

The only Boolean functions of a single proposition are itself and its contradictory. In form there are more; thus $\mathbf{a} \vee \sim\mathbf{a}$ has the form of a function of **a**, but it is a function only in the trivial sense in which $x - x$ and x/x are functions of x. In Boolean algebra, $\mathbf{a} \vee \sim\mathbf{a}$ plays the part of a constant proposition, because it is a truism and remains a truism through all changes in the meaning of **a**. To assert a truism in conjunction with a proposition is no more than to assert the proposition alone. Thus

$$(\mathbf{a} \vee \sim\mathbf{a}) \cdot \mathbf{b} = \mathbf{b}$$

for every meaning of **a** or **b**. On the other hand, to assert a truism in disjunction with a proposition is only to assert the truism; $\mathbf{a} \vee \sim\mathbf{a} \vee \mathbf{b}$, being true for every meaning of **a** or **b**, is itself a truism, so that

$$\mathbf{a} \vee \sim\mathbf{a} \vee \mathbf{b} = \mathbf{a} \vee \sim\mathbf{a}.$$

Each of these equations has its dual and thus

$$(\mathbf{a} \cdot \sim\mathbf{a}) \vee \mathbf{b} = \mathbf{b}$$

and

$$\mathbf{a} \cdot \sim\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \sim\mathbf{a}.$$

The proposition $\mathbf{a} \cdot \sim\mathbf{a}$ is an absurdity for every meaning of **a** and is thus another constant proposition. These two constant propositions, the truism and the absurdity, are mutually contradictory.

It will be convenient for future reference to have the following collection of the equations of this chapter.

$$\sim\sim\mathbf{a} = \mathbf{a}, \tag{2.1}$$

$\mathbf{a}\cdot\mathbf{a} = \mathbf{a},$ (2.2 I)	$\mathbf{a} \vee \mathbf{a} = \mathbf{a},$ (2.2 II)
$\mathbf{b}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{b},$ (2.3 I)	$\mathbf{b} \vee \mathbf{a} = \mathbf{a} \vee \mathbf{b},$ (2.3 II)
$\sim(\mathbf{a}\cdot\mathbf{b}) = \sim\mathbf{a} \vee \sim\mathbf{b},$ (2.4 I)	$\sim(\mathbf{a} \vee \mathbf{b}) = \sim\mathbf{a}\cdot\sim\mathbf{b},$ (2.4 II)
$(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}\cdot\mathbf{c},$ (2.5 I)	$(\mathbf{a} \vee \mathbf{b}) \vee \mathbf{c} = \mathbf{a} \vee (\mathbf{b} \vee \mathbf{c}) = \mathbf{a} \vee \mathbf{b} \vee \mathbf{c},$ (2.5 II)
$(\mathbf{a} \vee \mathbf{b})\cdot\mathbf{c} = (\mathbf{a}\cdot\mathbf{c}) \vee (\mathbf{b}\cdot\mathbf{c}),$ (2.6 I)	$(\mathbf{a}\cdot\mathbf{b}) \vee \mathbf{c} = (\mathbf{a} \vee \mathbf{c})\cdot(\mathbf{b} \vee \mathbf{c}),$ (2.6 II)
$(\mathbf{a} \vee \mathbf{b})\cdot\mathbf{b} = \mathbf{b},$ (2.7 I)	$(\mathbf{a}\cdot\mathbf{b}) \vee \mathbf{b} = \mathbf{b},$ (2.7 II)
$(\mathbf{a} \vee \sim\mathbf{a})\cdot\mathbf{b} = \mathbf{b},$ (2.8 I)	$(\mathbf{a}\cdot\sim\mathbf{a}) \vee \mathbf{b} = \mathbf{b},$ (2.8 II)
$\mathbf{a} \vee \sim\mathbf{a} \vee \mathbf{b} = \mathbf{a} \vee \sim\mathbf{a},$ (2.9 I)	$\mathbf{a}\cdot\sim\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\sim\mathbf{a}.$ (2.9 II)

Each of these equations after the first is dual to the equation on the same line in the other column, from which it can be obtained by the exchange of the signs, $\cdot$ and $\vee$. In the preceding discussion, the equations on the left were taken as axioms and those on the right were derived from them and the first equation. If, instead, the equations on the right had been taken as axioms, those on the left would have been their consequences. Indeed any set which includes the first equation and one from each pair on the same line will serve as axioms for the derivation of the others.

More equations can be derived from these by mathematical induction. For example, it can be shown, by an induction from Eq. (2.4 I), that

$$\sim(\mathbf{a}_1\cdot\mathbf{a}_2\cdot\ldots\cdot\mathbf{a}_m) = \sim\mathbf{a}_1 \vee \sim\mathbf{a}_2 \vee\ldots\vee \sim\mathbf{a}_m, \tag{2.10 I}$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m$ are any propositions.

We first assume provisionally, for the sake of the induction, that this equation holds when m is some number k and thence prove that it holds also when m is $k + 1$ and consequently when it is any number greater than k.

Replacing $\mathbf{a}$ in Eq. (2.4 I) by $\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k$ and $\mathbf{b}$ by $\mathbf{a}_{k+1}$, we have

$$\sim[(\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k) \cdot \mathbf{a}_{k+1}] = \sim(\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k) \vee \sim\mathbf{a}_{k+1}.$$

By the provisional assumption just made,

$$\sim(\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k) = \sim\mathbf{a}_1 \vee \sim\mathbf{a}_2 \vee \ldots \vee \sim\mathbf{a}_k,$$

and thus

$$\sim[(\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k) \cdot \mathbf{a}_{k+1}] = (\sim\mathbf{a}_1 \vee \sim\mathbf{a}_2 \vee \ldots \vee \sim\mathbf{a}_k) \vee \sim\mathbf{a}_{k+1}.$$

Therefore, by Eqs. (2.5 I) and (2.5 II)

$$\sim(\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_k \cdot \mathbf{a}_{k+1}) = \sim\mathbf{a}_1 \vee \sim\mathbf{a}_2 \vee \ldots \vee \sim\mathbf{a}_k \vee \sim\mathbf{a}_{k+1}.$$

Thus Eq. (2.10 I) is proved when m is $k + 1$ if it is true when m is k. By Eq. (2.4 I), it is true when m is 2. Hence it is proved when m is 3 and thence when m is 4 and when it is any number, however great.

By exchanging the signs, $\cdot$ and $\vee$, in Eq. (2.10 I), we obtain its dual, also valid:

$$\sim(\mathbf{a}_1 \vee \mathbf{a}_2 \vee \ldots \vee \mathbf{a}_m) = \sim\mathbf{a}_1 \cdot \sim\mathbf{a}_2 \cdot \ldots \cdot \sim\mathbf{a}_m, \quad (2.10\ \text{II})$$

an equation which can also be derived by mathematical induction from Eq. (2.4 II).

A mathematical induction from Eq. (2.6 I) gives:

$$(\mathbf{a}_1 \vee \mathbf{a}_2 \vee \ldots \vee \mathbf{a}_m) \cdot \mathbf{b} = (\mathbf{a}_1 \cdot \mathbf{b}) \vee (\mathbf{a}_2 \cdot \mathbf{b}) \vee \ldots \vee (\mathbf{a}_m \cdot \mathbf{b}). \quad (2.11\ \text{I})$$

By an exchange of signs in this equation or an induction from Eq. (2.6 II), we obtain

$$(\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \ldots \cdot \mathbf{a}_m) \vee \mathbf{b} = (\mathbf{a}_1 \vee \mathbf{b}) \cdot (\mathbf{a}_2 \vee \mathbf{b}) \cdot \ldots \cdot (\mathbf{a}_m \vee \mathbf{b}). \quad (2.11\ \text{II})$$