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Mean Field Theories and Models of Statistical Physics

A dissertation submitted in partial satisfaction
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by

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PUBLICATIONS

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ABSTRACT OF THE DISSERTATION

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In this dissertation, we treat three problems with a diverse array of techniques. The simple and powerful forbidden gap method, introduced by M. Biskup and L. Chayes, is used to prove first order transitions for some long range classical spin systems. We prove, for the first time, a first order transition occurs in the three state Potts model in three dimensions with exponentially decaying interaction strengths. Next we treat the mean field transverse Ising model. Using natural path integral expansions, this quantum model reduces via a large deviation principle to a variational problem. Applying of correlation inequalities, *i.e.* the FKG and GHS inequalities, and M. Aizenman's random current representation, we obtain precise control of the mean field theory of that model. Lastly we apply various arguments from the mathematical theory of classical Ising spin glasses along with the Lie-Trotter expansion to study the quenched free energy of mean field quantum spin glasses. Classical methods used include the interpolation technique introduced by F. Guerra and Gaussian concentration of measure phenomena, investigated extensively by M. Ledoux and M. Talagrand among others.

CHAPTER 1

Introduction

1.1 Contents Summary

This dissertation is comprised of five chapters. In the introduction technical terms are used without definition. The remainder of Chapter 1 is devoted to an explanation of some terminology and an exposition of the main techniques that feature in our work. Moreover, we will review what we consider to be some of the fundamental results in the literature related to these techniques.

The subsequent four chapters each treat one problem. Chapter 2 is devoted to the study of first order transitions in models whose mean field theory predicts such behavior. A general principle, first demonstrated by Biskup and Chayes [21] and subsequently extended in joint work with the author, is that one can produce lattice models of spin systems which have first order transitions as the inverse temperature parameter is varied whenever the corresponding mean field theory exhibits such behavior. Moreover, one obtains control over the location of the transition temperature in terms of the mean field value. This result has multiple implications. First, its proof provides an explicit connection between lattice models and their corresponding mean field theories. Second, when applicable the principle reduces the study of first order phase transitions in lattice systems to the (arguably) simpler analysis of the corresponding mean field model, which is often a glorified calculus problem. The driving force behind this principle is reflection positivity, which we review in Chapter 1.

While Chapter 2 deals exclusively with classical spin systems, in the next three chapters we consider quantum many body problems. The basic setup and representation techniques appear in Chapter 1. Chapter 3 considers in detail the mean field Ising model in a transverse external field. We present joint work with D. Ioffe and A. Levit which, among other things, computes the critical curve explicitly in terms of the inverse temperature β and transverse field strength λ. While the result itself is rather modest, the techniques used are interesting. In particular, the stochastic geometric representation of quantum spin systems introduced by Aizenman and B. Nachtergaele [7] transforms this problem into a product of continuum models of FK percolation on N circles of length β with interactions between components of circles mediated by classical Gibbs-Boltzman weighting. Standard large deviation techniques turn this into a variational problem which is solved with the help of correlation inequalities. The representation we derive, *i.e.* the random cluster picture, has the added benefit of hinting at generalizations which are not obviously realizable as quantum spin systems.

A more novel aspect of this work regards the applicability of this mean field theory to lattice spin systems. The theory does not predict a first order transition in temperature in a Hamiltonian with quadratic interaction, so lattice systems are not amenable to the techniques developed in Chapter 2. To prove phase transitions for lattice models via the mean field theory, it is important for us to obtain estimates on the stability of optimizers in the variational problem (here we have in mind adapting Kac model arguments [32, 38]). We apply Aizenman's random current representation to obtain lower bounds on the deviation of the free energy of a magnetization trajectory perturbed away from these optimizers.

In Chapter 4 we turn our attention to spin glasses. Because of the recent success of the analysis of the Sherrington-Kirkpatrick model [72, 73, 74, 126], it is mathematically reasonable and turns out to be physically relevant to ask the extent to which these techniques are applicable within the realm of quantum spin glasses. Here our main technical tool, besides the generous use of ideas from the classical theory, is the Lie-Trotter product formula (which lies at the bottom of [7] as well). We prove two theorems which apply very generally: First of all, we obtain concentration inequalities for quenched free energies of finite volume mean field quantum spin glasses around their mean. Second we show that the free energy for a mean field quantum spin glass (if it exists) is a universal quantity in the sense that it does not depend on the choice of disorder couplings under some mild regularity assumptions (*i.e.* that the random variables which mediate the couplings have third moments). In light of these results, to prove that the quenched free energy exists for any particular mean field quantum spin glass model we may assume that the couplings are independent Gaussian random variables with mean zero and variance one and need only show that the mean quenched free energy converges. Our final result in this chapter shows that the quenched free energy exists for the mean field transverse Ising spin glass.

1.2 Review of Techniques and Literature

1.2.1 Basic Terminology

In this chapter we review basic mathematical concepts in equilibrium statistical mechanics as a way of precisely introducing terminology and fixing notation. At the moment we consider classical spin systems on finite graphs. Let $\mathcal{G}$ denote a graph with vertex set Λ labeled by $x \in \Lambda$ and edge set $\mathcal{E}$ labeled by pair of vertices. Let $\mathbb{V}, \cdot$ be a fixed inner product space (not necessarily finite dimensional but always separable) and $\Omega \subset \mathbb{V}$ be a closed, bounded subset (not necessarily compact). An element $\mathbf{S} \in \Omega^\Lambda$ will be referred to as a spin configuration on the graph $\mathcal{G}$ and each coordinate is referred to as a spin. We assume that the spins S_x are random variables, distributed identically and independently from one another according to some *a priori* probability measure ν. Let us denote by $\mathcal{B}_\Lambda$ the Borel σ-algebra generated by the product topology on $\mathbb{V}^\Lambda$. Often we refer to $\prod_{x\in\Lambda}\nu(\,\mathrm{d}S_i)$ as the *a priori* measure on spin configurations and denote the integration of functions on $\mathbb{V}^\Lambda$ with respect to $\prod_{x\in\Lambda}\nu(\,\mathrm{d}S_x)$ by $\langle\cdot\rangle_0$

The Hamiltonian $H_\Lambda(\mathbf{S})$ determines the interactions between spins. For this discussion and most of the dissertation, we assume the interaction between spins is quadratic, see [121] for a general formulation. Thus we define the function $H_\Lambda(\mathbf{S})$ by

$$H_\Lambda(\mathbf{S}) = -\sum_{x,y\in\mathcal{G}} J_{x,y} S_x \cdot S_y.$$

The $J_{x,y}$ are called the coupling constants and the collection of coupling constants is often referred to as the interaction. Though they depend on pairs of vertices, they do not necessarily reflect the underlying graph structure. We also refer to $H_\Lambda(\mathbf{S})$ as the energy of the spin configuration $\mathbf{S}$. When $\Lambda\subset\mathbb{Z}^d$ a particularly important example is the *nearest neighbor* interaction defined as

$$J_{x,y} = \begin{cases} \frac{1}{2d} & \text{if } \|x-y\|_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\|\cdot\|_2$ is the usual Euclidean norm.

We modify the *a priori* measure on spin configurations via the Gibbs-Boltzmann weight $\mathrm{e}^{-\beta H_\Lambda(\mathbf{S})}$. Here the parameter β is 'the inverse temperature'. On the algebra of bounded measurable functions $f : \mathbb{V}^\Lambda \to \mathbb{R}$ we define the Gibbs state at inverse temperature β with respect to H_Λ by

$$\langle f\rangle_{\beta,\Lambda} = Z_\Lambda^{-1}\int f(\mathbf{S})\mathrm{e}^{-\beta H_\Lambda(\mathbf{S})}\prod_{x\in\Lambda}\nu(\,\mathrm{d}S_x).$$

We refer to the normalizing factor $Z_\Lambda = Z_\Lambda(\beta) = \int \mathrm{e}^{-\beta H_\Lambda(\mathbf{S})}\prod_{x\in\Lambda}\nu(\,\mathrm{d}S_x)$ as the partition function of the system and

$$F_\Lambda(\beta) = \frac{-1}{\beta|\Lambda|}\log Z_\Lambda(\beta)$$

as the free energy of the system. Note that, by taking the *a priori* measure to be weighted by $e^{S \cdot h}$, we can incorporate a magnetic field into this description without modifying our basic Hamiltonian.

Obviously the particulars of the behavior of a spin system depend entirely on the choices of Ω, *a priori* measure ν and coupling constants $J_{x,y}$. Below are a number of examples to orient the reader. For now we content ourselves to use these examples to point out the flexibility of the formalism and to introduce the reader to the main models of interest in this dissertation.

Example 1.2.1. [The q-state Potts Model on the Complete Graph [113]]

Consider $\Lambda = \{1, \dots N\}$. Let $\mathbb{V} = \mathbb{R}^{q-1}$ with the standard inner product and $\Omega = \{v_1, \dots, v_q\} \subset \mathbb{R}^{q-1}$ be the extreme points of a hyper-tetrahedron such that $v_i \cdot v_i = 1$ and $v_i \cdot v_j = \frac{-1}{q-1}$. This determines Ω up to rotations about the origin. The *a priori* measure in this case is the counting measure $\sum_{i=1}^{q} \delta_{v_i}$. We choose a Hamiltonian of the form

$$-H_N(\mathbf{S}) = \frac{1}{N} \sum_{x,y=1}^{N} S_x \cdot S_y,$$

i.e. $J_{x,y} = \frac{1}{N}$.

Example 1.2.2. [Ferromagnetic Potts Model with boundary conditions [113]]

Let $\mathbb{V}$, Ω, and ν be as in the previous example. Choose $J_{x,y} = J(x - y)$ for some function $J : \mathbb{Z}^d \to [0, \infty)$ satisfying $J(0) = 0$, $J(x) = J(-x)$ and $\sum_{x \in \mathbb{Z}^d} J(x) = 1$. Let $L \in \mathbb{N}$ be fixed and let $\Lambda = \Lambda_L = \{-L+1, \dots, L\}^d \subset \mathbb{Z}^d$. Let $\mathbf{S}_\Lambda$ and $\mathbf{S}_{\Lambda^c}$ denote spin configurations on Λ and Λ^c respectively. We refer to $\mathbf{S}_{\Lambda^c}$ as a boundary condition for the cube Λ. With respect to $\mathbf{S}_{\Lambda^c}$, we define the Hamiltonian $H_N(\mathbf{S}_\Lambda, \mathbf{S}_{\Lambda^c})$ by

$$-H_N(\mathbf{S}_\Lambda, \mathbf{S}_{\Lambda^c}) = \frac{1}{2} \sum_{x,y \in \Lambda} J_{x,y} S_x \cdot S_y + \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} J_{x,y} S_x \cdot S_y$$

Example 1.2.3. [Ferromagnetic Potts Model on the Torus]

Let $\mathbb{V}, \Omega, \nu$ and J be as in the previous example. Let $T_L = {}^{\mathbb{Z}^d}/_{L\mathbb{Z}^d}$ denote the d-dimensional torus of side length L. For $y' \in T_L$, let $\tilde{J}_L(y') = \sum_{x \in L \cdot \mathbf{Z}^d} J(x+y)$ where y is any element in the pre-image of y' under the natural projection map of $\mathbb{Z}^d$ onto T_L. For a spin configuration $\mathbf{S}$ on T_L let the Hamiltonian $H_L(\mathbf{S})$ be defined by

$$-H_L(\mathbf{S}) = \frac{1}{2} \sum_{x', y' \in T_L} \tilde{J}(x' - y') S_{x'} \cdot S_{y'}.$$

From a probabilistic point of view, there is no particular reason that we need the coupling constants $J_{x,y}$ to be deterministic. Moreover, it turns out there are quite a number of real world problems which are best modeled by systems with built in frustration, see for example [99]. To this end, we may consider coupling constants which are themselves random.

Example 1.2.4. [The Sherrington-Kirkpatrick Spin Glass Model [119]] To simplify the discussion, let $\Omega = \{-1, 1\}, \nu = \delta_{-1} + \delta_1$. Suppose that $\mathbf{J} = (J_{x,y})_{x,y \in \{1,\dots,N\}}$ is a sequence of i.i.d. standard normal random variables. Let $H_N(\mathbf{J})$ be defined by

$$-H_N(\mathbf{J}) = \frac{1}{2\sqrt{N}} \sum_{x,y=1}^{N} J_{x,y} S_x S_y.$$

Note, in comparison with Example 1.2.1, that here the coupling strengths are individually on the order of $\frac{1}{\sqrt{N}}$ rather than $\frac{1}{N}$. As such, a basic issue, solved only recently [72, 73, 74], is convergence of the free energy as N goes to infinity.

The final example concerns a probabilistic representation of a quantum statistical physics model. We postpone the explanation of how one arrives at this representation to the discussion in Section 1.2.4.

Example 1.2.5. [Transverse Ising Model on the Complete Graph]

Let $\mathbb{S}_\beta$ be the circle of length β and $\mathbb{V} = L^2(\mathbb{S}_\beta)$. Let $\Omega \subset L^2(\mathbb{S}_\beta)$ be the subset of piecewise constant, right continuous functions (with respect to the counterclockwise orientation) on $\mathbb{S}_\beta$ taking values in $\{-1, 1\}$. Let ν_λ be a probability measure, supported on Ω defined by the following two step process. Suppose $\mathbb{P}_\lambda$ is the measure corresponding to a homogeneous Poisson point process on $\mathbb{S}_\beta$ with arrival rate 2λ. Any realization of this process partitions $\mathbb{S}_\beta$ into a finite collection of intervals $\{I_j\}$. Orienting the circles counterclockwise we include the left endpoint of I_j in that interval. Next, we independently label I_j as -1 or $+1$ with probability $1/2$ to obtain an element of the configuration space Ω. Then ν_λ is the induced measure on Ω. The Hamiltonian of the system is defined using the inherited inner product on $L^2(\mathbb{S}_\beta)$, *i.e.* $f \cdot g = \int_0^\beta f(x)g(x)\, \mathrm{d}x$ and

$$-H_N(\mathbf{S}) = \frac{1}{N} \sum_{x,y=1}^{N} \mathbf{S}_x \cdot \mathbf{S}_y$$

It is noted that while this Hamiltonian is formally equivalent to the quantum Hamilitonian associated with the quantum versions of these problems, the mathematical content is quite different. Indeed in the quantum context, the S_x's represent operators on a vector space, rather than the vectors themselves. Reduction to a classical problem, albeit at the expense of a complicated spin space Ω, isis the subject of Section 1.2.4 of this dissertation.

1.2.1.1 Infinite Volume States and Phase Coexistence

Statistical physics models defined on $\mathbb{Z}^d$ (or some other lattice inside $\mathbb{R}^d$) are the central motivation of our study. Since we are interested in macroscopic properties of models of materials, the subsets of interest to us have large (infinite) volumes. For this reason we consider the notion of infinite volume Gibbs states on $\mathbb{Z}^d$.

Let us assume for the following discussion that Ω is a compact subset of $\mathbb{V}$, ν is some fixed finite measure, and the interaction $J_{x,y}$ satisfies the requirements

$$\begin{aligned} J_{x,y} =& J_{0,y-x} = J_{0,x-y} \\ \sum_{x\in\mathbb{Z}^d} & |J_{0,x}| = 1. \end{aligned} \tag{1.2.1}$$

For any finite volume Λ, let $\mathbf{S}_{\Lambda^c} \in \Omega^{\Lambda^c}$ denote the specification of a spin configuration on Λ^c. Analogous to Example 1.2.2, for each $\mathbf{S}_{\Lambda^c}$ we introduce the Hamiltonian

$$-H_\Lambda(\mathbf{S}_\Lambda, \mathbf{S}_{\Lambda^c}) = \frac{1}{2}\sum_{x,y\in\Lambda} J_{x,y} S_x \cdot S_y + \sum_{\substack{x\in\Lambda \\ y\in\Lambda^c}} J_{x,y} S_x \cdot S_y$$

and define the corresponding finite volume Gibbs state by

$$\langle F\rangle_{\beta,\mathbf{S}_{\Lambda^c}} = Z^{-1}_{\beta,\mathbf{S}_{\Lambda^c}} \int_{\Omega^\Lambda} F(\mathbf{S}_\Lambda) e^{-\beta H_N(\mathbf{S}_\Lambda,\mathbf{S}_{\Lambda^c})} \prod_{x\in\Lambda} \mathrm{d}\nu(S_x)$$

for each bounded measurable function $F : \Omega^\Lambda \to \mathbb{R}$. Thus each $\mathbf{S}_{\Lambda^c}$ naturally defines a boundary condition for the spin system in the finite volume Λ.

Because we assumed that $\Omega \subset \mathbb{V}$ is compact, standard Cantor diagonalization arguments allow us to define infinite volume Gibbs states on the algebra of local functions (functions which depend on finitely many vertices of $\mathbb{Z}^d$). This process is achieved by taking weak subsequential limits on the collection of all finite volume Gibbs states with arbitrary boundary conditions as the finite volume fills out $\mathbb{Z}^d$.

Each such limit point defines positive bounded linear functional on a dense subset of the vector space of functions continuous in the product topology on $\Omega^{\mathbb{Z}^d}$. Using continuity arguments, we may extend these linear functionals to states on the closure of the set of local functions. The Riesz representation theorem implies such a limit can be realized as a probability measure on $\Omega^{\mathbb{Z}^d}$. Moreover it must satisfy a collection of consistency conditions known as the Dobrushin-Lanford-Ruelle, or DLR, equations which we shall now formulate.

Definition 1.2.6. Let μ be a probability measure on the measure space $\left(\Omega^{\mathbb{Z}^d}, \mathcal{B}\right)$ where $\mathcal{B}$ denotes the σ-algebra generated by the product topology on $\Omega^{\mathbb{Z}^d}$. We say that μ satisfies the DLR conditions with respect to the interaction $J_{x,y}$ and inverse temperature β if for any fixed finite volume Λ, the restriction of μ to the σ-algebra $\mathcal{B}_\Lambda$ satisfies

$$\mu(A) = \int \langle \mathbf{1}_A \rangle_{\beta, \mathbf{S}_{\Lambda^c}} \, \mathrm{d}\mu(\mathbf{S}_{\Lambda^c}) \tag{1.2.2}$$

for all $A \in \mathcal{B}_\Lambda$.

In other words, whenever an infinite volume measure μ satisfies the DLR equations, its value on subsets of $\mathcal{B}_\Lambda$ is determined by its restriction to the complement subalgebra $\mathcal{B}_{\Lambda^c}$.

Let us endow the space of probability measures on $\Omega^{\mathbb{Z}^d}$ with the topology of weak convergence with respect to functions continuous in the product topology on $\Omega^{\mathbb{Z}^d}$. Suppose that we modify the *a priori* measure by the factor $\mathrm{e}^{S\cdot\mathbf{h}}$ for some external field $\mathbf{h} \in \mathbb{V}$. Using the notation of [121], let $\mathcal{M}(\beta, \mathbf{h})$ denote the closed convex hull of the collection of Gibbs states obtained by the above limiting process, taking limits over all boundary conditions. All such measures satisfy the DLR equations. The main problem of interest in Chapter 3 is the exploration of how $\mathcal{M}(\beta, \mathbf{h})$ or more accurately, how the *translation invariant* Gibbs states, denoted by $\mathcal{M}^I(\beta, \mathbf{h})$, behave in the weak topology as we vary the inverse temperature parameter β and field $\mathbf{h}$.

The following theorem characterizes $\mathcal{M}^I(\beta, \mathbf{h})$ for high temperature (which corresponds to small values of β).

Theorem 1.2.7 (Dobrushin's Uniqueness Theorem, [47, 48, 121])**.** *Let $J_{x,y}$ be a given collection of coupling constants satisfying conditions* (1.2.1). *Then for all β small enough, there is a unique translation invariant infinite volume Gibbs state.*

We remark that this formulation of Dobrushin's Uniqueness Theorem is quite restrictive. In particular, we may define Hamiltonians for spin systems using collections of interactions supported on arbitrary finite subsets of vertices of $\mathbb{Z}^d$. Under certain absolute summability conditions on the interaction terms, an analogous version of the Uniqueness Theorem holds. The question then is whether, as β and perhaps also an external magnetic field parameter $\mathbf{h}$ vary, $|\mathcal{M}^I(\beta, \mathbf{h})| > 1$. In $(\beta, \mathbf{h})$ parameter space we refer to this occurrence as a point of phase co-existence.

A precise way to study the variation of the set $\mathcal{M}^I(\beta, \mathbf{h})$ is through the infinite volume free energy, defined as

$$F(\beta, \mathbf{h}) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{-1}{|\Lambda|\beta} \log Z_\Lambda(\beta, h).$$

This limit exists independent of boundary conditions for all models under consideration, assuming for simplicity that the volumes Λ are taken to be hypercubes which fill out $\mathbb{Z}^d$. A general statement on the convergence of this limit is given in [121].

It is not too difficult, via Hölders inequality, to show that the free energy is jointly convex the variables β and $\mathbf{h}$. Phase transitions occur when some derivative of the free energy is discontinuous. In particular, we say that a first order phase transition occurs when some first partial derivative has a jump (this is the only possible type of discontinuity due to convexity). A second order, or continuous, transition occurs when the first derivative is continuous, but some higher order derivative (usuallly the second) is not derivative is not.

In many ways the paradigm for phase transitions is the Ising model, which in our language is the q=2-state Potts model.

Example 1.2.8 (The 2-d Ising Model)**.** In 1944, Onsager [107] explicitly calculated the infinite volume free energy for the 2-d Ising model. This formula showed that the free energy developed 'non-analyticities' as the parameter β varies from zero to infinity. More precisely, the derivative of the free energy with respect to β is continuous but not differentiable, showing that if we consider a system with zero magnetic field, a second order transition occurs at a unique critical value β_c.

This example is a bit misleading in that systems generically seem to exhibit first order, rather than second order transitions. To this end, we present the following:

Example 1.2.9 (The $q \geq 3$-state Potts Model in high dimensions)**.** The analysis of the nearest neighbor q-state Potts model is amenable to a number of the general methods available in statistical mechanics, [21, 92]. The method and results presented in [21] provide a clean path to results on this model. The following theorem appears there.

Theorem 1.2.10. *[21] Let $q \geq 3$ be fixed. Then there exists $d(q) \in \mathbb{N}$ such that for all $d \geq d(q)$ the nearest neighbor q-State Potts model on $\mathbb{Z}^d$ in zero magnetic field undergoes a first order phase transition as the parameter β varies.*

Parenthetically we note that our description of the previous theorem is only a small part of the picture: $d(q)$ can in principle be calculated explicitly (it is presumably not optimal though) and explicit estimates on the critical value of β may also be obtained. Moreover, the techniques presented there may in principle be applied to systems with continuous spins.

1.2.2 Mean Field Theory

Mean field theory has its origins in the physics literature as far back as P. Weiss' work in the 1900's [129] and appeared even earlier in the work of P. Currie. Its goal is to simplify the analysis of many body problems in general and phase transitions in particular by replacing the complexity of interactions between spins by averaged effective fields acting on individual spins. In the mean field ensemble, spin variables are independently distributed and under homogeneity and regularity assumptions on the interactions, identically distributed. Thus the behavior of an order parameter, *e.g.* the magnetization of the system, is reduced to a single spin problem.

The starting point for all discussions relevant to this dissertation is the notion that mean field theories give insight into the behavior of actual lattice spin systems. Making this statement carries a number of interesting questions with it. First and foremost, if one hopes to analyze lattice systems one should attempt to first control the model in the mean field approximation. From a mathematically rigorous point of view, this means defining the model on the complete graph of N vertices and studying the behavior which results as N tends to infinity while varying the parameters of the system. (An alternative is to formulate the model on a tree or 'the Bethe lattice' which often, but not always, leads to similar predictions.) Sometimes this simplification is readily solvable, as in the case of the ferromagnetic Ising and Potts models [88, 130]. At other times it is not, as in the case of the Sherrington-Kirkpatrick model of spin glasses [119]. Indeed, the study of the latter model has been going on for quite some time, has lead to much interesting mathematics [125] and has only recently seen significant mathematical progress [8, 36, 72, 73, 74, 126].

One may also ask, especially if the mean field theory is tractable (and even when it is not, see for example C. Newman-D. Stein [105] who argue against the heuristic predictions of the Sherrington-Kirkpatrick model), to what extent the simplification provides an accurate picture of the model on the lattice. A comparison of simple models, for example the one dimensional nearest neighbor Ising model with its complete graph counterpart, shows that the predictions can be misleading. On the other hand, the sophisticated work of M. Aizenman [2] shows that if one is willing to consider lattice Ising systems in dimension $d > 4$, then mean field predictions are accurate. Another approach to this problem was taken nearly simultaneously by J. Fröhlich [61], who used certain random walk representations, along with correlation inequalities and infrared bounds (which will be discussed below) in his analysis. Further, A. Sakai [117] recently extended Aizenman's work to include Ising models with spread out interactions.

There is a significant body of work exploring the connections between mean field theory and actual systems. Besides the aforementioned work of Aizenman, we mention the techniques of lace expansion [76, 77], Kac models [32, 33, 38] and aspects of chessboard estimates as examples connecting the mean field heuristics with physical systems of interest. We shall be particularly concerned in this dissertation with the many results which apply a technique related to chessboard estimates known as reflection positivity.

Another point worth mentioning is that the mathematical formulation of mean field theory is finding applications in a wide variety of settings in its own right. In particular, motivations for its study have emerged in computer science and models of neural networks [106, 125, 133].

We owe the following heuristic explanation of mean field theory for spin systems to M. Biskup and L. Chayes, though according to them it has been known for time immemorial. Other resources include the book by Simon [121] and a nice discussion by Glimm and Jaffe [69]. Let $\Omega \subset \mathbb{V}$ be fixed as a spin space with an associated single vertex *a priori* measure ν. For simplicity, we assume that the spins take values in some compact subset of $\mathbb{V}$ and let $\langle \cdot \rangle_\mu$ be an infinite volume translation invariant Gibbs state on $\mathbb{Z}^d$ corresponding to the nearest neighbor Hamiltonian

$$H_\Lambda = -\frac{1}{2d} \sum_{\|x-y\|_2=1} S_x \cdot S_y$$

and infinite volume measure μ. We are interested in possible values of the magnetization of the system at the origin, $\langle S_0 \rangle_\mu$. Let

$$m_0 = \frac{1}{2d} \sum_{\|x\|_2=1} S_x.$$

Conditioning on the value of m_0, we may apply the DLR equations (see [121]):

$$\mathbb{E}_\mu\left[S_0 \Big| \frac{1}{2d} \sum_{\|x\|_2=1} S_x = m_0\right] = \frac{\int S_0 \mathrm{e}^{\beta(S_0, m_0)} \,\mathrm{d}\nu(S_0)}{\int \mathrm{e}^{\beta(S_0, m_0)} \,\mathrm{d}\nu(S_0)} \quad \mu \text{ a.s.}$$

Making the at present unjustified assumption that under μ, the random variable m_0 is highly concentrated around a deterministic value m_*, we have

$$\langle S_0 \rangle_\mu \approx \frac{\int S_0 \mathrm{e}^{\beta(S_0, m_*)} \, \mathrm{d}\nu(S_0)}{\mathrm{e}^{\beta(S_0, m_*)} \, \mathrm{d}\nu(S_0)}.$$

On the other hand, by translation invariance of the measure μ, S_0 is equal to the spatial average $\frac{1}{2d}\langle \sum_{\|x\|_2=1} S_x \rangle_\mu$ so that the translation invariance of μ implies

$$\langle S_0 \rangle_\mu \approx m_*.$$

Thus we are thus lead to the single vertex equation

$$\mathbf{m} = \frac{\int S_0 \mathrm{e}^{\beta(S_0, \mathbf{m})} \, \mathrm{d}\nu(S_0)}{\int \mathrm{e}^{\beta(S_0, \mathbf{m})} \, \mathrm{d}\nu(S_0)}. \tag{1.2.3}$$

Much of the mean field analysis of particular systems amounts to the study of (1.2.3) as the parameters β and $\mathbf{h}$ vary.

Unfortunately, it is not usually possible to recover the phase structure of an actual system from the mean field equation. This is the case even on the complete graph where one would expect the concentration argument to be most applicable. For example, one can treat the mean field $q \geq 3$-state Potts model exactly to find that the corresponding mean field equation develops nontrivial solutions well below the transitional value of β, see Chapter 3 and [88].

The problem presented by the mean field $q \geq 3$-state Potts model is typical of systems characterized by a *first* order transition. The mean field equation will then have multiple solutions well below the transitional value β_c, but most of these solutions do not correspond to equilibrium states of the corresponding system. To correct for this, we introduce an auxiliary function of magnetization called the mean field free energy. We shall not attempt to give physical motivation for this function although it turns out this describes the thermodynamics of the complete graph models.

Foremost an entropy function is defined by means of the Legendre transform

$$S(\mathbf{m}) = \inf_{\mathbf{b}\in\mathbb{R}^n}\big\{G(\mathbf{b}) - (\mathbf{b},\mathbf{m})\big\} \tag{1.2.4}$$

of the cumulant generating function

$$G(\mathbf{b}) = \log\int\big(\,\mathrm{e}^{(\mathbf{b},S)}\big)\;\mathrm{d}\nu(S_0). \tag{1.2.5}$$

The mean-field free-energy function is then defined as the difference between the energy function, $E(\mathbf{m}) = -\frac{\beta}{2}|\mathbf{m}|^2$, and the entropy $S(\mathbf{m})$:

$$\Phi_\beta(\mathbf{m}) = -\frac{\beta}{2}|\mathbf{m}|^2 - S(\mathbf{m}). \tag{1.2.6}$$

As some justification for this terminology note that, formally at least, stationary points of the mean-field free-energy, $\nabla\Phi_\beta(\mathbf{m}) = 0$, are equivalent to solutions of the equation

$$\mathbf{m} = \nabla G(\mathbf{m}) = \frac{\int S_0\mathrm{e}^{\beta(S_0,\mathbf{m})}\;\mathrm{d}\nu(S_0)}{\int \mathrm{e}^{\beta(S_0,\mathbf{m})}\;\mathrm{d}\nu(S_0)}.$$

The introduction of the mean field free energy function is that it allows us to isolate relevant solutions to the mean field equations as minimizers of this function.

To return to the mean field Potts model, we consider the mean field free energy function in addition to the mean field equation. Kesten and Schonmann [88] show that the additional consideration of the free energy function implies that the relevant solutions to the mean field equation occur in a symmetric state: for some k, $x_k \geq x_j$ for all $j \neq k$ and $x_i = x_j$ otherwise. When we express $m \in \mathrm{conv}\,(\Omega)$ as $m = \sum_{i=1}^{q} x_i v_i$ for a unique collection $x_i \geq 0, \sum_{i=1}^{q} x_i = 1$. Considering those m such that $x_1 = \frac{1}{q}+\theta$ the mean field free energy is minimized when

$$\theta = \frac{\mathrm{e}^{\beta\theta} - 1}{\mathrm{e}^{\beta\theta} + q - 1}. \tag{1.2.7}$$

This is the scalar version of the mean field equation. If $q \geq 3$, these statements imply that the mean field theory predicts a jump in θ_{MF} as β varies, *i.e.* a first order transition occurs.

As mentioned previously, the mean field free energy function can be recovered from the complete graph as the thermodynamic limit of the free energy per unit volume. Given a spin space Ω with single vertex *a priori* measure $\mathrm{d}\nu$, define the Hamiltonian $-H_N(\mathbf{S})$ analogous to (1.2.1). Let $\mathrm{conv}(\Omega)$ denote the closed convex hull of the sample space Ω and $B_\epsilon(\mathbf{m})$ denote the ball of radius ϵ around $\mathbf{m} \in \mathbb{R}^n$. For the system of N vertices, denote the spatial average of a spin configuration $\mathbf{S} = (S_i)_{i=1}^N$ by

$$\mathbf{m}_N = \frac{1}{N}\sum_{i=1}^{N} S_i.$$

The following result appears in [21]:

Theorem 1.2.11. *[21] For each $\mathbf{m} \in conv(\Omega)$,*

$$\lim_{\epsilon \downarrow 0} \lim_{N\to\infty} \frac{1}{N} \log\left\langle e^{-\beta H_N(\mathbf{S})} \, \mathbf{1}_{\{\mathbf{m}_N(\mathbf{S}) \in B_\epsilon(\mathbf{m})\}} \right\rangle_0 = -\Phi_{J,\mathbf{b}}(\mathbf{m}), \tag{1.2.8}$$

where $\Phi_\beta(\mathbf{m})$ is as defined in (1.2.6)*. Moreover, if ν_N denotes the Gibbs measure obtained by normalizing $\mathrm{e}^{-\beta H_N(\mathbf{S})}$ and if $F_{MF}(J)$ denotes the infimum of $\Phi_\beta(\mathbf{m})$ over $\mathbf{m} \in conv(\Omega)$, then*

$$\lim_{N\to\infty} \nu_N\big(\Phi_\beta(\mathbf{m}_N(\mathbf{S})) \geq F_{MF}(\beta) + \epsilon\big) = 0$$

for every $\epsilon > 0$.

Another way of phrasing this result is that the mean field free energy function coincides with the large deviation rate function for a sequence of product measures on N spins measures weighted by the Gibbs-Boltzmann factor $\mathrm{e}^{-\beta H_N(\mathbf{S})}$.

1.2.3 Reflection Positivity

Reflection positivity (hereafter referred to as RP) originated as an axiom of quantum field theory in the work Osterwalder and Schrader [108], see also [69] for an introduction. Ideas motivating those presented below appeared also in Glimm, Jaffe and Spencer [70].

From the point of view of rigorous proofs, a review of the literature makes clear that this concept facilitates a number of elegant, useful arguments in the exploration of phase coexistence and phase transitions. There are a number of sources which set out the theory of RP. On the one hand the reader may consult the original papers by the 'old masters', see [57, 62, 63, 65]. On the other hand, we refer the reader to recent lecture notes [13], which gives an excellent review of available techniques and current developments. Thus what follows reviews its consequences rather than presenting a technical introduction.

In the nearest neighbor Ising model on the torus T_L, RP takes the following form. For this description, let us assume that L is an even integer. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ denote a pair of parallel hyperplanes each splitting $\mathbb{Z}^d$ at the midway point of a collection of lattice bonds and whose distance is $L/2$, let θ denote the lattice preserving map which reflects across one of these hyperplanes and let θ^* denote the induced map on T_L. The pair of hyperplanes naturally defines two subsets $T_L(r), T_L(l)$ of vertices $x \in T_L$, and θ^* is a reflection of these subsets onto one another. In the canonical way θ^* induces a map on Ω^{T_L}, which, by slight abuse of notation, is also denoted by θ^*. Let $\mathcal{F}_l$ denote the collection of bounded, measurable, real valued functions on Ω^{T_L} which depend only on spin configurations inside $T_L(l)$ and analogously for $\mathcal{F}_r, T_L(r)$. By any number of techniques, e.g. by expanding the Gibbs weight as a Taylor series, one may check directly in the case of the nearest neighbor Ising model that $\left\langle f(\mathbf{S})f(\theta^*(\mathbf{S}))\right\rangle_{T_L} \geq 0$ for any real valued function $f \in \mathcal{F}_l$.

Recall from Example 1.2.3 that any absolutely summable interaction $J_{x,y}$ induces an interaction $\tilde{J}_{x',y'}$ on T_L via

$$\tilde{J}_{x',y'} = \sum_{(z_1,z_2)\in L\cdot\mathbf{Z}^d} J_{x+z_1,y+z_2} \tag{1.2.9}$$

where (x, y) is any point in the inverse image of the projection onto T_L. To give some indication of RP for pair interactions we present the following definition based on the above discussion.

Definition 1.2.12. Consider a classical lattice spin system on $\mathbb{Z}^d$ with pair interaction satisfying (1.2.1). The interaction $J_{x,y}$ is said to be reflection positive if for any hyperplane reflection θ on $\mathbb{Z}^d$, the induced the quadratic form $\big\langle f(\mathbf{S})g(\theta^*(\mathbf{S}))\big\rangle_{T_L}$ is positive semi-definite and

$$\big\langle f(\mathbf{S})g(\theta^*(\mathbf{S}))\big\rangle_{T_L} = \big\langle f(\theta^*(\mathbf{S}))g(\mathbf{S})\big\rangle_{T_L}$$

for any $L \in \mathbb{N}$ and for all $f, g \in \mathcal{F}_r$

The original success of RP was the paper by Fröhlich, Simon, and Spencer [65]. In it the authors treated two models: The $(\phi\cdot\phi)_3^2$ quantum field model on $\mathbb{R}^d$ and the nearest neighbor classical isotropic Heisenberg model on the lattice $\mathbb{Z}^d$ where $d \geq 3$ in each case. The former model will not concern us here. The latter fits quite well within our formalism: Let $n \in \mathbb{N}$ be fixed. The classical n-vector isotropic Heisenberg ferromagnet is the spin system on $\mathbb{Z}^d$ with $\Omega = S^{n-1} \subset \mathbb{R}^n$, the standard inner product, ν taken as the normalized Haar measure and nearest neighbor coupling constants.

One way of detecting phase transitions is to study the correlations of spins. We shall say that a system exhibits absence of clustering at inverse temperature β if

$$\liminf_{\|x\|_2\to\infty} \Big[\big\langle S_0\cdot S_x\big\rangle_\mu - \langle S_0\rangle_\mu \big\langle S_x\big\rangle_\mu\Big] > 0. \tag{1.2.10}$$

for some infinite volume Gibbs state $\mu \in \mathcal{M}^I(\beta, \mathbf{h})$. As it turns out, absence of clustering necessarily implies the presence of multiple Gibbs states. In that event the model has internal symmetries which allow us to conclude that $\langle S_x \rangle_\mu = 0$ for each $x \in \mathbb{Z}^d$, absence of clustering is equivalent to the statement that

$$\liminf_{\|x\|_2 \to \infty} \langle S_0 \cdot S_x \rangle_\mu > 0.$$

This is, for example true for the $O(n)$ models in the absence of an external field.

The following theorem is taken from [65].

Theorem 1.2.13. *[65] Let $d \geq 3$ be fixed. Consider the classical Heisenberg ferromagnet on $\mathbb{Z}^d$ with nearest neighbor interaction.*

- *In zero external magnetic field, for all β sufficiently large the nearest neighbor classical Heisenberg model exhibits absence of clustering.*
- *The classical Heisenberg spin system exhibits non-zero spontaneous magnetization for all β sufficiently large. i.e. If $\mathbf{u} \in S^2$, then*

  $$\lim_{h \to 0^+} \frac{d}{dh} F(\beta, h\mathbf{u}) \neq 0$$

  for any β sufficiently large.

This theorem stands in sharp contrast to the result of Mermin-Wagner [97], with extensions by Dobrushin-Shlosman [52]. Those results say in particular that spontaneous symmetry breaking does not occur in short-range two dimensional systems with continuous symmetries.

There are two main ingredients of the proof of Theorem (1.2.13):

- Infrared Bounds for Finite Volume Torus States - Nearest Neighbor Case: For each L, let T_L^* denote the dual to the torus T_L. For each $k \in T_L^*$ let

  $$\widehat{S}_k = \sum_{x \in T_L} \frac{1}{L^{\frac{d}{2}}} S_x \mathrm{e}^{-ik \cdot x}.$$

The bound

$$\left\langle \widehat{S}_k \widehat{S}_{-k} \right\rangle_{T_L} \leq \frac{n}{2\beta d} \frac{1}{1 - \frac{1}{d}\sum_{i=1}^{d} \cos(k_i)} \tag{1.2.11}$$

holds for all $k \in T_L^* \backslash \{0\}$ and all $\beta \geq 0$.

- Secondly, by Parsival's identity,

$$1 = \left\langle S_0 \cdot S_0 \right\rangle_{T_L} = \frac{1}{L^d} \sum_{k \in T_L^*} \left\langle \widehat{S}_k \cdot \widehat{S}_{-k} \right\rangle_{T_L}. \tag{1.2.12}$$

Since the function on the righthand side of (1.2.11) is (Riemann) integrable for $d \geq 3$,

the first inequality implies

$$\frac{1}{L^d} \sum_{k \in T_L^* \backslash \{0\}} \left\langle \widehat{S}_k \widehat{S}_{-k} \right\rangle_{T_L} \leq \frac{C}{\beta} \tag{1.2.13}$$

where C does not depend on L. In light of the second identity, for all β large enough the zero mode of the spin wave vector must be giving a macroscopic contribution to the sum on the righthand side. This proves absence of clustering for the classical Heisenberg ferromagnet when the formula for the zero mode is unraveled. Further arguments imply that the spontaneous magnetization is non-zero.

Of course the infrared bound is the crux of the matter in the classical Heisenberg models. Proving this bound is where RP is crucial. All one really needs from (1.2.12) to make this argument work in general is uniform boundedness of the expected length of the spin vector. This will be satisfied if Ω is a bounded set, but becomes a nontrivial requirement otherwise, and interestingly presents problems in some cases (for example in the other model considered by Frohlich *et.al.* [65]).

The paper [65] sets out a blueprint for proving that phase transitions occur. Slightly later, Dyson, Lieb and Simon [57] extended the method to isotropic and anisotropic quantum Heisenberg spin systems, where one replaces the random variables S_x by spin operators from quantum mechanics, see Subsection 1.2.4 for a discussion of this formalism. Simultaneously, Frohlich and Lieb [64] introduced a technique known as chessboard estimates, which combined with RP extended the flexibility of the Peirls contour estimation argument. They also addressed anisotropic classical and quantum Heisenberg models, producing proofs of long range order at low temperatures in many cases.

These authors had ambitions to use their respective methods to prove long range order for ferromagnetic quantum Heisenberg models. Each paper addressed anti-ferromagnetic versions in full, but could not apply RP to the ferromagnetic case. There is good reason for this; a 1985 paper by Speer [124] showed that the quantum Heisenberg ferromagnet is *not* reflection positive in a number of anisotropic cases and in the isotropic case on the lattice $\mathbb{Z}$.

The papers by Frohlich *et al* [62, 63] were written in part to clarify this confusion and also due to the inapplicability of the methods in [65] to general non-nearest neighbor finite range interactions. Here the general theory of RP, infrared bounds, and chessboard estimates was elucidated. We summarize their general theory below:

They developed a unified characterization of RP as the positivity of a quadratic form defined by symmetries of the system. It should be noted that, though the literature is rather sparse on the idea, there should be applications in other systems with 'lots' of $\mathbb{Z}_2$ symmetries. Further, they characterized those quadratic interactions which satisfy RP completely, ruling out the possibility of long but finite range interaction potentials. Next, they proved the existence of infrared bounds as a consequence of RP and chessboard estimates. They further extended the abstract theory of chessboard estimates and isolated the idea away from RP. Finally they applied the methods to a wide array of models. In particular, they proved the occurrence of phase transitions in a large collection of spin systems with long range coupling constants, in Coloumb systems, *i.e.* systems in which the interaction decays as the inverse of the distance between points, and the borderline cases for finite range RP systems.

Besides the aforementioned papers, Heilmann and Lieb [78] provide an interesting early result using chessboard estimates. Here they prove a version of 'long range order' for hardcore dimer (in two dimensions) and fourmer (in three dimensions) models of liquid crystals. This application is notable in that these models have Hamiltonians with interactions between occupied *bonds*, and do not fit within our present set up. Nevertheless, they are able to formulate a notion of RP.

In 1981 and 1982, a pair of related contributions to the theory was made by Dobrushin and Shlosman, [53] and Koteckỳ and Shlosman, [92]. The former paper provides a criteria for proving first order transitions. In the latter paper, the authors use this criteria along with a novel notion of contours and chessboard estimates to prove that first order phase transitions occur in 'large entropy models', *i.e.* q state Potts models with $q \gg 1$, so that there are a large but discrete number of minimal energy states. This was the first demonstration of first order transitions in q state Potts models on $\mathbb{Z}^d$ for $d \geq 3$.

Besides the original contributors, we would like mention recent results by Biskup and Chayes (and their co-authors) [20, 21, 22, 24, 25, 26, 27] as examples of modern contributions to the literature. In particular, their method of comparison to mean field theory [21], which shall be described in Chapter 3 of the present paper, provides a good deal more control over the phase diagram than was originally available. Besides the papers mentioned above, interesting modern applications of the technology described can be found in the work [28, 39, 40, 59, 98].

1.2.4 Quantum Spin Systems and Path Integral Expansions

In this subsection, we address the final fundamental concept of this dissertation. Path integral representations have a long history of use since the inception of quantum mechanics. Indeed, quantum field theory was presented in this fashion to the author in the UCLA physics department. Feynman's work on quantum electrodynamics was facilitated by the introduction of a particular expansion, now refereed to as the Feynman-Kac expansion, which we shall discuss below. Moreover Feynman's diagrams are a way of keeping track of the terms in a particular path integral expansion known in physics as the Dyson series. See the paper by Datta *et. al.* [45] for an application of this series expansion which extends the Pirogov-Sinai theory to some quantum spin systems. To our knowledge, Ginibre [68] was the first to use path integrals in the context of quantum spin systems in a mathematically controlled fashion.

We will touch on a pair of path integral expansions in this section. Both rely on a fundamental theorem from the theory of linear operators:

Theorem 1.2.14. *[The Lie-Trotter Product Formula] For any collection of bounded operators* $\{\mathcal{A}_j\}_{j=1}^l$ *on a Hilbert space* $\mathbb{H}$*, we have*

$$e^{\sum_{j=1}^l \mathcal{A}_j} = \lim_{k\to\infty}\left(\prod_{j=1}^l e^{\frac{\mathcal{A}_j}{k}}\right)^k = \lim_{k\to\infty}\left(\prod_{j=1}^l 1+\frac{\mathcal{A}_j}{k}\right)^k.$$

This is certainly not the most general version possible, Khnemund and Wacker [93] for further results and conditions for failure. As our applications in Chapters 4 and 5 deal with finite dimensional Hilbert spaces, we are content with this formulation.

Let us briefly review the framework of equilibrium quantum spin systems. In keeping with the notation of the last two chapters of the dissertation, in this section we denote sites of a finite volume by the indexes $i, j \in \Lambda$.

In this setting, one describes each particle associated with a vertex $i \in \Lambda$ using an

n_i dimensional Hilbert space $\mathbb{C}^{n_i}$ along with a spin s_i irreducible representation of the Lie algebra $\mathfrak{su}(2)$, where $n_i = 2s_i + 1$. The representation is generated by a triplet of operators $\mathbf{S} = (S^{(x)}, S^{(y)}, S^{(z)})$. These operators satisfy the usual commutation relations

$$\left[S_i^{(\mu)}, S_j^{(\nu)}\right] = \mathrm{i}\delta_{i,j}\epsilon_{\mu\nu\eta}S^{(\eta)}$$

where $\epsilon_{xyz} = 1$ and is totally anti-commuting in the indices and $\delta_{i,j}$ is one if $i = j$ and zero otherwise.

To represent the combined system of particles, we introduce the tensor product $\mathbb{V}_\Lambda = \bigotimes_{i\in\Lambda}\mathbb{C}^{n_i}$, along with the trivial embedding of the sequence $(\mathbf{S}_i)_{i\in\Lambda}$ of representations of $\mathfrak{su}(2)$, *i.e.* $S_i^{(\mu)} \mapsto I \bigotimes \cdots \bigotimes S_i^{(\mu)} \bigotimes \ldots I$, where I is the identity of $\mathbb{C}^{n_j}$, $j \neq i$ and $\mu \in \{x, y, z\}$. As mentioned previously, the particles interact by means of a Hamiltonian $\mathcal{H}_\Lambda$ which is now an operator acting on $\mathbb{V}_\Lambda$. The weights of 'configurations' are then described by the associated Gibbs-Boltzmann operator $\mathrm{e}^{-\beta\mathcal{H}_\Lambda}$. Here, $\mathcal{H}_\Lambda$ is a self adjoint operator, typically a polynomial in the Λ-tuple of spin operators $(\mathbf{S}_i)_{i\in\Lambda}$. For example, the Hamiltonian of the simplest non-trivial quantum spin system, the mean field transverse Ising model, which shall presently be related to Example 1.2.5, is described by

$$-\mathcal{H}_N = \frac{1}{2N}\sum_{i,j=1}^{N} S_i^{(z)}S_j^{(z)} + \lambda\sum_{i=1}^{N} S_i^{(x)} \tag{1.2.14}$$

where $\lambda > 0$.

Once we specify the Hamiltonian, statistical quantities of the system may be defined: the partition function and free energy of a quantum spin system are given via the trace of the Gibbs-Boltzmann operator as

$$\begin{aligned} Z_\Lambda(\beta) =& \mathrm{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right) \\ f_\Lambda(\beta) =& \frac{-1}{|\Lambda|\beta}\log Z_\Lambda. \end{aligned}$$

Self adjoint operators on $\mathbb{V}_\Lambda$ replace functions as observables of the system and the thermal average of an observable A is defined as

$$\left\langle A \right\rangle_\beta = \frac{\mathrm{Tr}\left(A\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_\Lambda}\right)}.$$

We refer to the functional $\left\langle \cdot \right\rangle_\beta$ as the Gibbs state of the system corresponding to the Hamiltonian $\mathcal{H}_\Lambda$.

This last paragraph strongly hints at our interest in the Lie-Trotter formula. Most, if not all, physically interesting macroscopic quantities are expressed in terms of (derivatives of) the free energy. Since exact calculation of the free energy via diagonalization of the given Hamiltonian is all but impossible when the system is very large, it is hoped that introducing probabilistic representations of quantum models will allow us to attack problems of operator theory with the more familiar and intuitive tools of probability and analysis. In the remainder of this section we shall describe two ways to expand the partition function and free energy and give some indications of notable rigorous results where these expansions proved valuable.

1.2.4.1 Feynman-Kac Representations

The first and (arguably) most natural expansion we discuss is referred to as the Feynman-Kac formula [31, 68] after Feynman's extensive use of this technique in his work on quantum electrodynamics, and after M. Kac, who pointed out the connection between Feynman's expansions and Brownian path integrals. We note here that we shall rely on this expansion in Chapter 5 to prove convergence of the quenched pressure in quantum spin glasses.

Let us begin by giving a schematic and heuristic description of the Feynamn-Kac transformation. We will follow the discussion with two in depth examples. The idea is to write our Hamiltonian $\mathcal{H}_\Lambda$ as the perturbation of a classical model, *i.e.*

$$\mathcal{H}_\Lambda = \mathcal{H}_* + \mathcal{V}$$

where $\mathcal{H}_*$ has a well understood collection of eigenvalues and eigenvectors (which might even be understood on the basis of classical statistical mechanics). The hope is that $\mathcal{V}$ may be interpreted as imposing a 'dynamical evolution' on the eigenvectors.

It is convenient and instructive for us to introduce the Dirac 'bra-ket' notation from physics. Suppose $\{v_\psi\}_{\psi\in\mathfrak{I}}$, for some natural index set $\mathfrak{I}$, is a complete orthonormal set of eigenvectors corresponding to $\mathcal{H}_*$ with eigenvalues $H_*(\psi)$. In Dirac notation one makes the dependence of these vectors on their respective index more explicit by using the notation $\{|\psi\rangle\} = v_\psi$. Bra vectors $\langle\psi|$ are identified with linear functionals on the Hilbert space. Thus $\langle\psi'|\psi\rangle = (v_{\psi'}, v_\psi)$ denotes the inner product between the vectors $|\psi\rangle$ and $|\psi'\rangle$ and for any operator A, $\langle\psi'|A|\psi\rangle \triangleq (v_{\psi'}, Av_\psi) = (A^* v_{\psi'}, v_\psi)$. Finally there is the useful'resolution of the identity'

$$\mathrm{I}_\Lambda = \sum_{\psi\in\mathfrak{I}} |\psi\rangle\langle\psi| \tag{1.2.15}$$

which gives the presentation of a vector v in terms of the orthonormal basis v_ψ.

Using the Lie-Trotter formula, we expand the Gibbs-Boltzmann operator via the resolution of the identity. We have

$$\left(\mathrm{e}^{-\frac{\beta}{k}\mathcal{H}_*}\mathrm{e}^{-\frac{\beta}{k}\mathcal{V}}\right)^k = \sum_{\psi_1,\ldots,\psi_k\in\mathfrak{I}} \mathrm{e}^{-\sum_{j=1}^{k}\frac{\beta}{k}H_*(\psi_j)}|\psi_k\rangle \left[\prod_{j=1}^{k-1}\langle\psi_{j+1}|\mathrm{e}^{-\frac{\beta}{k}\mathcal{V}}|\psi_j\rangle\right]\langle\psi_1|$$

where I_Λ, in the form (1.2.15), was inserted in between each factor. Thinking of β/k as a time step, the sum in the exponent becomes a Riemann sum. In addition,

$$\langle\psi'|\mathrm{e}^{-\frac{\beta}{k}\mathcal{V}}|\psi\rangle = P_{\frac{\beta}{k}}(\psi,\psi')$$

is reminiscent of a probability transition kernel with time step β/k from the 'state' ψ to the 'state' ψ'. Therefore the full expansion looks like a weighted average of polygonal paths with values in the state space $\psi\in\mathfrak{I}$.

To make this heuristic explanation precise let us specialize to the case $n_i = 2$ for all $i \in \Lambda$. In the single particle description we denote the eigenvector for $S^{(z)}$ which corresponds to the eigenvalue $+1$ by $|+\rangle$ and the eigenvector which corresponds to the eigenvalue -1 by $|-\rangle$. On $\mathbb{V}_\Lambda$, we choose as a preferred basis the collection of all simple tensor products of eigenvectors for the operators $\{S_i^{(z)}\}_i \in \Lambda$. We naturally identify this preferred basis with classical Ising spin configurations $\sigma \in \mathfrak{I} = \{-1, +1\}^\Lambda$. For each $\sigma \in \mathfrak{I}$, we denote the corresponding basis vector by $|\sigma\rangle$.

For convenience we record the form of the spin matrices with respect to this preferred basis.

$$2S^{(z)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 2S^{(y)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad 2S^{(x)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example 1.2.15. [The $s = \frac{1}{2}$ Mean Field Transverse Ising Model]

Consider the Hamiltonian 1.2.14, where we identify the component in the z-direction with $\mathcal{H}_*$ and the component in the x-direction with $\mathcal{V}$. Then the eigenvalues of the operator $\mathrm{e}^{-\beta\mathcal{H}_*}$ are the classical Gibbs weights for the spin configurations $\{\sigma\}_{\sigma\in\{-1,1\}^N}$ at inverse temperature $\beta/4$.

For any spin configuration σ, let

$$\sigma_j^i = \begin{cases} \sigma_j \text{ if } j \neq i \\ -\sigma_i \text{ if } j = i \end{cases} \tag{1.2.16}$$

Let us define the operator $\mathcal{L}$ on $L^2\left(\{-1,1\}^N\right)$ by

$$\mathcal{L}(\sigma, \tilde{\sigma}) = \frac{\lambda}{2} \sum_{i=1}^N \delta_{\sigma^i,\tilde{\sigma}} - \frac{\lambda N}{2} \delta_{\sigma,\tilde{\sigma}}. \tag{1.2.17}$$

Then $\mathcal{L}$ generates the continuous time Markov chain on the state space $\{-1, +1\}^N$. The dynamics are determined by flipping the spin at each vertex independently according to Poisson processes with rate $\frac{\lambda}{2}$. Further, one may check that

$$\mathcal{L} = \lambda \sum_{i=1}^{N} S^{(x)} - \frac{\lambda N}{2} I = -\mathcal{V} - \frac{\lambda N}{2} I$$

Let us denote the evolution of the spin configurations by $\underline{\sigma}(t) = \big(\sigma_i(t)\big)_{i=1}^{N}$. Considering the expansion of the partition function $Z_N(\beta)$ via the Lie-Trotter formula, for all $r \in \mathbb{N}$

$$\mathrm{Tr}\left[\left(e^{\frac{\beta}{r}\mathcal{H}_*} e^{\frac{\beta}{r}\mathcal{V}}\right)^r\right] = \mathrm{e}^{\frac{\lambda \beta N}{2}} 2^N \mathbb{E}_{\lambda/2}\left[\mathrm{e}^{\frac{1}{8N}\sum_{k=1}^{r} \frac{\beta}{r}\sigma_i(\beta k/r)\sigma_j(\beta k/r)} \mathbf{1}_{\{\underline{\sigma}(0)=\underline{\sigma}(\beta)\}}\right]$$

where we start the process uniformly among all initial spin configurations and $\mathbb{E}_{\frac{\lambda}{2}}[\cdot]$ denotes the measure on piece-wise constant spin paths determined by this Markov process with Poisson arrival rate $\frac{\lambda}{2}$. We can pass to the limit as r tends to infinity using the bounded convergence theorem to conclude that

$$\mathrm{Tr}\left[e^{\beta \mathcal{H}_N}\right] = \mathrm{e}^{\frac{\lambda \beta N}{2}} 2^N \mathbb{E}_{\lambda/2}\left[\mathrm{e}^{\frac{1}{8N}\int_0^\beta \sigma_i(t)\sigma_j(t)\,\mathrm{d}t} \mathbf{1}_{\{\underline{\sigma}(0)=\underline{\sigma}(\beta)\}}\right].$$

Similar identifications can be made for the matrix elements of the Gibbs-Boltzmann operator.

Example 1.2.16. [The $s = \frac{1}{2}$ Quantum Heisenberg Ferromagnet on T_L] We consider the quantum Heisenberg ferromagnet on the torus T_L. Let us define the Hamiltonian by

$$\mathcal{H}_L = -\sum_{i,j \in T_L} J_{i,j} \sum_{\mu \in \{x,y,z\}}^{3} S_i^{(\mu)} S_j^{(\mu)} \tag{1.2.18}$$

for some fixed finite range translation invariant coupling $J_{i,j}$ such that $J_{i,j} \geq 0$, $\sum_{i \in \mathbb{Z}^d}, J_{0,i} = 1$ and $J_{i,i} = 0$. Let $S_j^{\pm} = S_j^{(x)} \pm \mathrm{i} S_j^{(y)}$. Since $[S_i^+, S_j^-] = \delta_{i,j} 2 S_i^{(z)}$, (1.2.18) can be identified with the Hamiltonian

$$\mathcal{H}_L = -\sum_{i,j \in T_L} S_i^+ \Delta_L^{\mathbf{J}}(i,j) S_j^- + \sum_{i,j \in T_L} P(i,j)\left(S_i^{(z)} + \frac{1}{2}\right)\left(S_j^{(z)} + \frac{1}{2}\right).$$

Here

$$\Delta_L^{\mathbf{J}}(i,j) = \begin{cases} -1 & \text{if } i = j \\ J_{i,j} & \text{otherwise} \end{cases}$$

and

$$P(i,j) = \begin{cases} 0 & \text{if } i = j \\ J_{i,j} & \text{otherwise} \end{cases}$$

For a spin configuration σ, we may interpret the locations of the $+1$ spins as occupied vertices and -1 spins as vacant vertices.

Let X_t^i, $i \in \mathbb{N}$ denote continuous time random walks which jump at rate one according to the probability transition kernel determined by $J_{i,j}$. For each $n \in \mathbb{N}$ let τ_n denote the first collision time for the walks X_t^i i= $1, \ldots$ n and let $\mathfrak{S}_n$ denote the symmetric group on $\{1, \ldots, n\}$. If the initial spin configuration consists of n + particles, then until τ_n, the first term of the Hamiltonian is the generator for n continuous time random walks with transition probabilities determined by $J_{i,j}$. Expanding the partition function using the Feynman-Kac transformation,

$$Z_{T_L}(\beta) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \sum_{x^1, \ldots x^n}$$
$$\mathbb{E}\left[e^{-\int_0^\beta \sum_{i,j} P(X_t^i, X_t^j)\, \mathrm{d}t} \mathbf{1}_{\{X_\beta^i = x^{\pi(i)} \text{ for } i\,=\,1,\ldots,n\}} \mathbf{1}_{\{\tau_n > \beta\}} \middle| X_0^i = x^i, \text{ for } i = 1, \ldots, n \right].$$

Specializing to the simple symmetric random walk (*i.e.* a nearest neighbor Hamiltonian) in the above example, Conlon-Solovej [41, 42] and Tòth [127] use this representation to obtain precise upper bounds on the free energy at low temperatures. In particular, further work allows the representation of this process as a symmetric exclusion process with an independent killing time and/or a random stirring process with periodic boundary conditions.

The literature using Feynman-Kac representations is quite large and we shall mention only a small selection. In 1985, Kennedy [83] used this representation to prove long range order for anisotropic ferromagnetic Heisenberg models on $\mathbb{Z}^d$, which the reader will recall from the previous section, are not reflection positive. The idea in that work is to consider cluster expansions in which the clusters evolve in the time coordinate produced by the Feynman-Kac expansion. Along these line we should mention the 1996 papers [31, 45] which develop quantum cluster expansions using alternate path integral representations. Another interesting applications appears in [84].

1.2.4.2 Quasi-state Decompositions

In contrast to the previous expansion, the following idea of Aizenmann and Nachtergaele [7, 100, 101] relates path integral expansions to dependant models of percolation, rather than the dynamical picture developed above. As a drawback, one loses the Markov process intuition. On the other hand the quasi-state decomposition can in principle be applied in wider generality.

Suppose we are given a system of particles in a finite volume Λ with a quantum Hamiltonian

$$\mathcal{H}_\Lambda = -\sum_{i,j\in\Lambda} J_{i,j}\mathcal{P}_{i,j} = -\sum_{i,j\in\Lambda} J_{i,j}\left(\mathcal{P}_{i,j}-1\right) - \sum_{i,j} J_{i,j}$$

where the $\{\mathcal{P}_{i,j}\}$ form some collection of polynomials in the spin operators for the vertices i, j and the $\{J_{i,j}\}$ are positive but otherwise arbitrary, so we do not specify an underlying graph structure. The compensated Gibbs-Boltzmann operator $\mathrm{e}^{-\beta\mathcal{H}_\Lambda+\beta\sum_{i,j\in\Lambda}J_{i,j}}$ can be viewed as follows:

To each vertex $i \in \Lambda$, we attach a copy of $[0, \beta]$ and to each pair $\{i, j\}$ we attach an independent Poisson process $N_{i,j}(t)$ with rate $J_{i,j}$. The joint measure of these Poisson processes observed to time β is denoted by $\rho^\beta_{\mathbf{J}}$. Thus arrivals of $\rho^\beta_{\mathbf{J}}$ correspond to drawing links between the corresponding intervals at the arrival times.

Let Ω be the graphical representation of the configurations of this joint process.

Realizations consist of intervals of length β associated with the vertices along with finite collections of bonds between them. For each $\omega \in \Omega$, let $\mathcal{K}(\omega) : (\mathbb{C}^{2s+1})^{\Lambda} \to (\mathbb{C}^{2s+1})^{\Lambda}$ denote the time indexed product of the $\mathcal{P}_{i,j}$'s associated to the collection of bonds of γ. The Lie-Trotter expansion allows us to write

$$\mathrm{e}^{-\beta\mathcal{H}_{\Lambda}+\beta\sum_{i,j\in\Lambda}J_{i,j}} = \int_{\Omega} \mathrm{d}\rho^{\beta}_{\mathbf{J}}(\omega)\mathcal{K}(\omega). \tag{1.2.19}$$

It is convenient when computing the partition function of the Gibbs-Boltzmann operator to consider periodic boundary conditions in the 'time' direction. Let $\mathbb{S}_{\beta}$ denote a circle with circumference β and Ω' configuration space of graphical representations of arrivals and let $\mathrm{d}\rho^{\beta}_{\mathbf{J}}(\omega')$ denote the measure induced by $\mathrm{d}\rho^{\beta}_{\mathbf{J}}(\omega)$ on $\{1, \dots, N\} \times \mathbb{S}_{\beta}$. Then linearity and cyclicity of the trace implies the partition function of the system Z_{β} satisfies

$$Z_{\beta} = \int_{\Omega'} \rho^{\beta}_{\mathbf{J}}(\mathrm{d}\omega')\mathrm{Tr}(\mathcal{K}(\omega'))$$

Aizenman and Nachtergaele discovered that, for a number of quantum systems of interest, $\mathrm{Tr}(\mathcal{K}(\omega'))$ has a geometric representation.

Irrespective of this last statement an important point to make, which shall not be pursued further here, is that whenever $\mathrm{Tr}(\mathcal{K}(\omega')) \geq 0 \ \omega' a.s.$ the quasi state decomposition can be used to represent expectations of all observables. Using (1.2.19) along with the decomposition outlined above, we have

$$\langle \mathcal{A} \rangle_{\beta} = \int \mathrm{d}\mu(\omega')E_{\omega'}(\mathcal{A})$$

where

$$\mathrm{d}\mu(\omega') = \frac{\mathrm{d}\rho^{\beta}_{\mathbf{J}}(\omega')}{Z(\beta)}\mathrm{Tr}(\mathcal{K}(\omega'))$$

is a probability measure and

$$E_{\omega'}(\mathcal{A}) = \frac{\mathrm{Tr}(\mathcal{A}\mathcal{K}(\omega'))}{\mathrm{Tr}(\mathcal{K}(\omega'))}$$

In this way, statistical quantities like correlations between spin operators, which are difficult to calculate quantum mechanically, may be represented geometrically. Also, the stochastic geometric representation sometimes allows the use of stochastic domination techniques (as in the case of quantum Heisenberg antiferromagnetic spin chains). Finally, we mention that the terminology 'quasi-states' in this subsection refers to the (random) linear functional $E_{\omega'}(\mathcal{A})$ defined on local operators. These functionals are not bounded and so do not extend to the closure of the algebra of local operators in the thermodynamic limit.

To illustrate the quasi-state decomposition outlined above, let us revisit the preceding examples:

Example 1.2.17. [The Mean Field $s = \frac{1}{2}$ Transverse Field Ising Model] There is no particular reason the above construction restricts to interactions along bonds. For example, rewriting (1.2.14) as

$$-\mathcal{H}_N = \frac{1}{N}\sum_{i,j=1}^{N}\left(\frac{1}{2}\left[1+S_i^{(z)}S_j^{(z)}\right]-1\right)+\frac{\lambda}{2}\sum_{i=1}^{N}\left(1+2S_i^{(x)}-1\right)+\lambda N \quad (1.2.20)$$

we may consider a collection of independent Poisson processes in which we have arrivals *on* the intervals $[0,\beta]$ as well as between them. The transverse field term corresponds to a Poisson process placing vertex marks on corresponding circles independently with rate $\frac{\lambda}{2}$.

Passing to periodic boundary conditions on the intervals $[0,\beta]$, let us compute Tr$(\mathcal{K}(\omega'))$ for some particular realization of this process. Note that $\langle\sigma|1+2S_i^{(x)}|\tilde{\sigma}\rangle = \delta_{\sigma,\tilde{\sigma}} + \delta_{\sigma^i,\tilde{\sigma}}$ and $\langle\sigma|\frac{1}{2}\left[1+S_i^{(z)}S_j^{(z)}\right]|\tilde{\sigma}\rangle = \delta_{\sigma\tilde{\sigma}}\mathbf{1}_{\sigma_i=\sigma_j}$. Therefore, an expansion of Tr$\,(\mathcal{K}(\omega'))$ via insertion of the resolution of the identity $I = \sum_\sigma |\sigma\rangle\langle\sigma|$ in between each factor of $K(\omega')$ gives a sum over time indexed products of constraints on spin paths.

The collection of vertex marks partitions each of the N circles into a collection of disjoint subintervals and the bond arrivals link subintervals into a collection of disjoint components. Let $c(\omega')$ be the number of components in the configuration of marks and links. There are two conditions necessary for a spin path to be allowable. The discontinuities of the spin path must be a subset of the arrival times of the processes of marks and linked subintervals of space-time must have the same sign. Thus using the graphical representation, we may express Tr $(\mathcal{K}(\omega'))$ as $2^{c(\omega')}$. The partition function reads

$$Z_{N,\lambda,\beta} = \int_{\Omega'} \rho^{\beta}_{\mathbf{J}}(\,\mathrm{d}\omega') 2^{c(\omega')}.$$

Example 1.2.18. [The Mean Field $s = \frac{1}{2}$ Quantum Heisenberg Ferromagnet] Consider the Hamiltonian

$$\mathcal{H} = -\sum_{i,j=1}^{N} \frac{1}{2N} \sum_{\mu \in \{x,y,z\}} S_i^{(\mu)} S_j^{(\mu)} \tag{1.2.21}$$

for the mean field quantum Heisenberg ferromagnet. As in Example 1.2.15, let $|\sigma\rangle$ denote the basis of eigenvectors for the $\{S_i^{(z)}\}$. Then

$$\langle \sigma | \frac{1}{2} + \mathbf{S}_i \cdot \mathbf{S}_j | \tilde{\sigma} \rangle = \delta_{\sigma^{ij}, \tilde{\sigma}}$$

where σ^{ij} denotes σ with the values of i and j interchanged. Expanding of Tr $(\mathcal{K}(\omega'))$ via insertion of the resolution of the identity $I = \sum_\sigma |\sigma\rangle\langle\sigma|$ in between each factor of $K(\omega')$ again gives a sum over time ordered products of constraints on spin paths. This time, the arrival of a particular bond process $N_{i,j}$ at time t forces

$$\sigma_i(t+) = \sigma_j(t-),\ \sigma_i(t-) = \sigma_j(t+)$$

where $\lim_{h \downarrow 0} f(t \pm h) = f(t\pm)$. All other spin paths are continuous at time t.

To count the number of allowable spin paths we place particles on space-time points $(i, 0)$ and follow their evolution in the forward time direction. Any time a particle encounters a bond from the arrivals of the Poisson process it must cross the bond and then continue in the forward time direction. The trajectories thus obtained partition $\mathbb{S}_\beta^N$ into a collection of disjoint loops. In terms of spin paths, each loop must have the same sign, $+$ or $-$. With these considerations we have

$$\text{Tr}\left(\mathcal{K}(\omega')\right) = 2^{l(\omega')}$$

where $l(\omega')$ denotes the number of loops formed by ω' and the partition function reads

$$Z_{N,\beta} = \int_{\Omega'} \rho_{\mathbf{J}}^{\beta}(\,\mathrm{d}\omega')2^{l(\omega')}.$$

The application of the quasi-states decomposition is quite general. In contrast to the Feynman-Kac expansion, we do not need restrictions like nonnegative off-diagonal matrix elements of the kernel V to make the basic expansion. In fact, most of the paper [7] is devoted to consequences of applying the quasi-states decomposition to quantum Heisenberg antiferromagnets for arbitrary spin s. The bond interaction for this model is defined by

$$\mathcal{P}_{i,j} = \frac{1}{2} - S_i \cdot S_j$$

in the $s = \frac{1}{2}$ case.

Finally we wish to point out that, curiously to the author, very few subsequent papers have appeared which employ quasi-states decompositions in any significant way. Those that have, mostly restrict their use to quantum spin chains (spin systems on the graph $\mathbb{Z}$) where a number of interesting phenomena have been predicted by physicists (particularly the Haldane gap [1, 75]).

CHAPTER 2

Mean-field driven first-order phase transitions

We consider a class of spin systems on $\mathbb{Z}^d$ with vector valued spins $(\boldsymbol{S}_x)$ that interact via the pair-potentials $J_{x,y}\,\boldsymbol{S}_x\cdot\boldsymbol{S}_y$. The interactions are generally spread-out in the sense that the $J_{x,y}$'s exhibit either exponential or power-law fall-off. Under the technical condition of reflection positivity and for sufficiently spread out interactions, we prove that the model exhibits a first-order phase transition whenever the associated mean-field theory signals such a transition. As a consequence, e.g., in dimensions $d\geq 3$, we can finally provide examples of the 3-state Potts model with spread-out, exponentially decaying interactions, which undergoes a first-order phase transition as the temperature varies. Similar transitions are established in dimensions $d=1,2$ for power-law decaying interactions and in high dimensions for next-nearest neighbor couplings. In addition, we also investigate the limit of infinitely spread-out interactions. Specifically, we show that once the mean-field theory is in a unique "state," then in any sequence of translation-invariant Gibbs states various observables converge to their mean-field values and the states themselves converge to a product measure.

2.1 Introduction

2.1.1 Motivation

The understanding of the quantitative aspects of phase transitions is one of the basic problems encountered in physical (and other) sciences. Most of the existing mathematical approaches are based on the use of contour expansions via Pirogov-Sinai theory [111, 112, 132] and/or the use of correlation inequalities [60, 120, 122]. Notwithstanding, many "practical" scientists still rely on the so-called *mean-field theory* which, in its systematic form, goes back to the work of Landau. From the perspective of mathematical physics, it is therefore desirable to shed as much light as possible on various mean-field theories and, in particular, attempt to place the subject on an entirely rigorous basis.

In a recent paper [21], two of us have established a direct connection between temperature-driven first-order phase transitions in certain ferromagnetic nearest-neighbor spin systems on $\mathbb{Z}^d$ and their mean-field counterparts. The principal result of Ref. [21] states that, once the mean-field theory signals a first-order phase transition, the actual system has a similar transition provided the dimension d is sufficiently large and/or the mean-field transition is sufficiently strong. Moreover, the transition happens for the values of parameters that are appropriately "near" the mean-field transitional values; indeed, the various error terms tend to zero as $d \to \infty$.

The principal goal of the present paper is two-fold. First, we will considerably extend the scope of systems to which the ideas of Ref. [21] apply; i.e., we will prove discontinuous phase transitions in systems which heretofore have been beyond the reach of rigorous methods. Second, we will in a general way expound on the *mean-field philosophy*. In particular, we will demonstrate that mean-field theory provides an asymptotic description of a certain class of systems regardless of the nature of their transitions.

Our approach is somewhat akin to the bulk of work on the so-called *Kac limit* of lattice [38, 32, 33, 37] as well as continuum [**?**, **?**, 94] systems. Here one considers finite-range interactions of unit total strength which are smeared out over a region of scale $1/\gamma$. As γ tends to zero, each individual site interacts with larger and larger number of other sites and so, for $\gamma \ll 1$, one is in the position to prove that the characteristics of an actual system (e.g., the magnetization) are close to those of the corresponding mean-field theory. In particular, all "approximations" (i.e., upper and lower bounds) become exact as $\gamma \downarrow 0$.

Notwithstanding, the similarity between the Kac limit and our approach ends with the above statements: Our technique involves tight bounds on the fluctuations of the effective field while the analyses of Refs. [38, 32, 33, 37] are based on coarse-graining arguments. As a consequence, we have no difficulty treating models with complicated single-spin spaces—even those exhibiting continuous internal symmetries or leading to power-law decay of correlations—or nearest-neighbor systems in large dimensions. Of course, there is a price to pay: Our technique requires the infrared bound on two-point correlation function which is presently available only for models obeying the condition of reflection positivity. Moreover, unless we assume power-law decaying interactions, the use of infrared bounds does not permit any statements in $d = 2$, while the Kac-limit approach works equally well in all $d \geq 2$.

2.1.2 Models of interest

For the duration of the paper, as in Ref. [21], we will focus on spin models with two body interactions as described by the formal Hamiltonian

$$\beta\mathscr{H} = -\beta \sum_{\langle x,y \rangle} J_{x,y}\,(\boldsymbol{S}_x, \boldsymbol{S}_y) - \sum_x (\boldsymbol{h}, \boldsymbol{S}_x). \tag{2.1.1}$$

The various objects on the right-hand side are as follows: β is the inverse temperature, $\langle x, y\rangle$ denotes an unordered pair of distinct sites, $J_{x,y}$ $(= J_{y,x})$ is the coupling constant associated with this pair, the spins $\boldsymbol{S}_x$ take values in a compact set $\Omega \subset \mathbb{R}^n$, the (reduced) external field $\boldsymbol{h}$ is a vector from $\mathbb{R}^n$ and $(\cdot,\cdot)$ denotes some inner product in $\mathbb{R}^n$. Implicit in the notation is an underlying *a priori* measure on Ω which represents the behavior of the spins in the absence of interactions. (In principle, the term which describes the coupling to the external field, namely the $(\boldsymbol{h}, \boldsymbol{S}_x)$'s, could be absorbed into the definition of the *a priori* measure. However, for æsthetic reasons, here we will often retain these terms as part of the interaction.)

Mean-field behavior is typically anticipated in situations where fluctuations are insignificant and, on general grounds, one expects this to be the case in high dimensions. These were precisely the operating conditions of Ref. [21] (as well as of Refs. [30, 88]) where, in a mathematically precise sense, the stipulation concerning the fluctuations was vindicated. However, an alternative route for ramping down fluctuations is to consider "spread out" interactions, i.e., $J_{x,y}$'s which do not go to zero too quickly. As alluded to earlier, this alternative is, in fact, the common starting point for modern mathematical studies of phase transitions based on mean-field theory, e.g., Refs. [38, 32, 33, 94, 37] and Refs. [82, 80, 81, 76, 117].

Unfortunately, we do not have complete flexibility as to how we can spread out our interactions. Indeed, our principal error estimate requires that the $(J_{x,y})$ satisfy the condition of *reflection positivity* (RP). Notwithstanding, the following three classes of interactions are available to our methods:

(1) *Nearest along with next-nearest neighbor couplings*, i.e., potentials such that $J_{x,y} = \lambda$ if x and y are nearest neighbors, $J_{x,y} = \kappa$ with $\lambda \geq 2(d-1)|\kappa|$ if x and y are next-nearest neighbors and $J_{x,y} = 0$ in the remaining cases.

(2) *Yukawa-type potentials* of the form

$$J_{x,y} = \mathrm{e}^{-\mu|x-y|_1}, \tag{2.1.2}$$

where $\mu > 0$ and $|x-y|_1$ is the ℓ^1-distance between x and y.

(3) *Power-law decaying interactions* of the specific form

$$J_{x,y} = \frac{1}{|x-y|_1^s}, \tag{2.1.3}$$

with $s > 0$.

Aside from these "pure" interactions, reflection positivity holds for

(4) any combination of the above with positive coefficients.

The derivation of the reflection-positivity property for these interactions goes back to the classic references on the subject [65, 62, 63]; for reader's convenience we will provide additional details in Sect. 2.3.1 and Sect. 2.4 (Remark 2.4.5).

We note that for all positive values of s the interactions listed in item (3) are indeed, in the technical sense, reflection positive. However, some values of s are not viable and others are not particularly useful. Specifically, if $s \leq d$, then the interaction is attractive and non-summable so there is no thermodynamics. Thus we may as well assume that $s > d$. Furthermore, if $d = 1$ and $s \geq 2$ or $d = 2$ and $s \geq 4$ then our methods break down. With some reason: In the one dimensional cases with $s > 2$, the results of Refs. [97, 128, 55, 56, 4] indicate (and in specific cases prove) that no magnetic ordering is possible. Similarly, in the above mentioned two-dimensional cases, magnetic ordering is precluded in many systems.

To summarize, we will impose the following limitations on our power-law interactions in Eq. (2.1.3):

(a) $s < 2$ in $d = 1$,

(b) $s < 4$ in $d = 2$,

(c) $s > d$ in all $d \geq 1$.

Although case (1) does not give us any real options for spreading the interaction beyond the previous recourse of taking $d \gg 1$, cases (2) and (3) offer us the possibility to do so on a *fixed* lattice. This is essentially obvious in case (2)—just take the parameter μ small. As for case (3) it is seen, after a little thought, that taking s close to d presents an additional and powerful method for smearing interactions.

2.1.3 Outline of results

Given the ability to smear interactions on a fixed lattice, much of the technology developed in Ref. [21] can be applied *without* the stipulation of "d sufficiently large." Thus it will prove possible to make statements about specific models on reasonable lattices with (more or less) reasonable interactions.

One such "specific" model will be the q-state Potts model (see Sect. 2.2.2). Here, for example, we will establish a discontinuous transition between the ordered and disordered states of a 3-state Potts model on $\mathbb{Z}^3$ with interactions decaying to zero exponentially. (And similarly for any other q-state Potts model on $\mathbb{Z}^d$ with $q \geq 3$ and $d \geq 3$.) Analogous first-order phase transitions are also proved in dimensions one and two provided we have power-law decay of the couplings as discussed above. For example, in $d = 1$, for any power-law decay exponent $s \in (1, 2)$, we produce couplings such that the 3-state Potts model has a first-order transition as the overall strength of the coupling varies.

As another illustration, we consider the low temperature behavior of the Blume-Capel model. The system will be described precisely in Sect. 2.3.4, for now it suffices to say that the spins take values in $\{-1, 0, +1\}$ with *a priori* equal weights. The zero temperature phase diagram of this model has a triple point where the three states of constant spin are degenerate in energy, however, as demonstrated in Ref. [123], this degeneracy is broken at finite temperatures in favor of the state dominated by the zeros. The previous analyses of this phenomenon required rather detailed contour estimates; here we will establish similar results by relatively painless methods.

The techniques at our disposal will allow us to put to rest some small controversies which, in recent years, have been topics of some discussion. For instance, a conjecture has been made [86, 87] which boils down to the statement that in any one-dimensional finite-state spin system with arbitrary translation-invariant, summable interaction, the set of phase-coexistence points at positive temperatures is a *subset* of the corresponding set at zero temperature. We will rule this out by our analysis of the Potts models in an external field.

In addition to predicting first-order transitions, our mean-field framework provides an explicit description of general lattice spin systems in the limit when the interactions become highly diffuse. In particular we show that, whenever the mean-field theory is in a unique "state," the magnetization and the energy density of the actual system converge to their mean-field counterparts. Moreover, *every* translation invariant Gibbs state converges to a product (i.i.d.) measure with individual-spin distribution self-consistently adjusted to produce the correct value of the magnetization. (This vindicates the assumptions typically used to "justify" mean-field theory; see Sect. 2.2.1.) Results in this direction have appeared before; cf Refs. [30, 88], but the main difference is that here we are *not* forcing $d \to \infty$ and hence it is possible to envision a limiting system towards which we are heading.

2.1.4 Organization

The organization of the remainder of this paper is as follows: In Sect. 2.2.1 we describe, in succinct terms, some general aspects of mean-field theory. In Sect. 2.2.2 we discuss the mean-field theory for the Potts model in an external field—which is the primary model studied in this work. Precise results concerning these situations are the subject of Sect. 2.2.3.

Sect. 2.3 is devoted to the statements of our main result. Specifically, in Sect. 2.3.1 we formulate a general theorem (Theorem 2.3.2) that allows us to prove first-order phase transitions in actual lattice models with interaction (2.1.1)—and RP couplings—by comparison to the associated mean-field theory. Sect. 2.3.2 provides conditions under which the mean-field theory is obtained as a limit of lattice systems when the interaction becomes infinitely spread out. Sects. 2.3.3 and 2.3.4 contain precise statements of our theorems concerning the behavior of the specific systems we study: The zero-field q-state Potts models with $q \geq 3$, the same model (with $q \geq 4$) in an external field which enhances or supresses—depending on the sign—one of the states, and the Blume-Capel model near its zero-temperature triple point. Sect. 2.3.5 mentions some recent conjectures that can be addressed using our results.

The principal subject of Sect. 2.4 is to give the proof of our general results (Theorems 2.3.2 and 2.3.3). As part of the proof, we will discuss certain interesting convexity bounds (Sect. 2.4.1), reflection positivity (Sect. 2.4.2) and infrared bounds (Sect. 2.4.3). In Sect. 2.4.5 we show how the specific interactions listed in Sect. 2.1.2 fit into our general scheme. Sect. 2.5 is devoted to the mathematical details of the mean-field theories for all the above mentioned models; in particular the proofs of all claims made in Sect. 2.2.3. Sect. 2.6 then assembles all ingredients into the proofs for actual lattice systems.

2.2 Mean-field theory and the Potts model

Here we shall recall to mind a formalism underlying (our version of) mean-field theory and provide heuristic discussion of the basic facts. The specifics will be demonstrated on an example of the q-state Potts model in an external field; first somewhat informally in Sect. 2.2.2 and then precisely in Sect. 2.2.3.

2.2.1 Mean-field heuristic

We will focus on the situations described by the Hamiltonian in Eq. (2.1.1). Of course the real models must be carefully defined on $\mathbb{Z}^d$ as limits of finite volume measures corresponding to this Hamiltonian at inverse temperature β *and* some sort of boundary conditions. We shall assume the reader is familiar with this basic theory (enough of the relevant formalism can be found in Sect. 2.3.1) and skip right to the consideration of an infinite-volume translation-invariant Gibbs state $\mu_{\beta,\boldsymbol{h}}$ corresponding to the Hamiltonian in Eq. (2.1.1) and inverse temperature β. For convience we will assume here, as in the rest of this paper,

$$J_{x,x} = 0, \quad \sum_{x \in \mathbb{Z}^d} |J_{0,x}| < \infty \quad \text{and} \quad \sum_{x \in \mathbb{Z}^d} J_{x,y} = 1. \tag{2.2.1}$$

We will let $\mathbb{E}_{\beta,\boldsymbol{h}}$ denote the expectation with respect $\mu_{\beta,\boldsymbol{h}}$ and $\mathbb{E}_0$ expectation with respect to the *a priori* (product) measure μ_0. (We will of course assume in the following that μ_0 is supported on more than one point.)

The principal idea is to study the distribution of one spin variable, e.g., the one at the origin of coordinates. Let $\boldsymbol{m}$ denote the expected value of this spin, $\boldsymbol{m} = \mathbb{E}_{\beta,\boldsymbol{h}}(\boldsymbol{S}_0)$. Then, conditioning on the configuration in the complement of the origin, we get the identity

$$\boldsymbol{m} = \mathbb{E}_{\beta,\boldsymbol{h}} \left(\frac{\mathbb{E}_0(\boldsymbol{S} \ \mathrm{e}^{(\boldsymbol{S},\beta \boldsymbol{m}_0 + \boldsymbol{h})})}{\mathbb{E}_0(\ \mathrm{e}^{\beta(\boldsymbol{S},\beta \boldsymbol{m}_0 + \boldsymbol{h})})} \right), \tag{2.2.2}$$

where $\boldsymbol{m}_0$ is the *random* variable given by the weighted average

$$\boldsymbol{m}_0 = \sum_{x\in\mathbb{Z}^d} J_{0,x}\, \boldsymbol{S}_x. \tag{2.2.3}$$

We emphasize that the expectation $\mathbb{E}_{\beta,\boldsymbol{h}}$ "acts" only on $\boldsymbol{m}_0$ while $\mathbb{E}_0$ "acts" only on the auxiliary spin variable $\boldsymbol{S}$.

When all is said and done, the underlying *assumption* behind the standard mean-field theories boils down to the statement that the quantity $\boldsymbol{m}_0$ is non-random, and therefore equal to $\boldsymbol{m}$. Postponing, momentarily, any discussion that concerns the validity of such an assumption, the immediate relevance is that in Eq. (2.2.2) we can replace $\boldsymbol{m}_0$ by $\boldsymbol{m}$ which in turn makes the outer expectation on the right-hand side redundant. We thus arrive at the self-consistency constraint

$$\boldsymbol{m} = \frac{\mathbb{E}_0\big(\boldsymbol{S}\ \mathrm{e}^{(\boldsymbol{S},\beta\boldsymbol{m}+\boldsymbol{h})}\big)}{\mathbb{E}_0\big(\ \mathrm{e}^{\beta(\boldsymbol{S},\beta\boldsymbol{m}+\boldsymbol{h})}\big)} \tag{2.2.4}$$

which is the *mean-field equation* for the magnetization. Clearly, if it can be established that the fluctuations of $\boldsymbol{m}_0$ are negligible, then the actual magnetization must be near a solution of Eq. (2.2.4).

In this light, our results are not that hard to understand: In most instances where the mean-field theory predicts a discontinuous transition this prediction is showcased by the fact that Eq. (2.2.4) simply does not admit continuous solutions. Thus if the error caused in the approximation $\boldsymbol{m}_0 \approx \boldsymbol{m}$ is much smaller than the discontinuities predicted in the mean-field approximation, jumps of the physical magnetization cannot be avoided.

As all of the above is predicated on the near constancy of the random variable $\boldsymbol{m}_0$, let us turn to a discussion of the fluctuations of this quantity. An easy calculation shows that

$$\mathrm{Var}(\boldsymbol{m}_0) = \sum_{x,y} J_{0,x}J_{0,y}\mathbb{E}_{\beta,\boldsymbol{h}}\big((\boldsymbol{S}_x,\boldsymbol{S}_y) - |\boldsymbol{m}|^2\big) \tag{2.2.5}$$

where $|\boldsymbol{m}|^2 = (\boldsymbol{m}, \boldsymbol{m})$. The quantity $\mathbb{E}_{\beta,\boldsymbol{h}}((\boldsymbol{S}_x, \boldsymbol{S}_y) - |\boldsymbol{m}|^2)$ is the thermal two-point correlation function which, on general grounds, may be presumed to tend to zero at large separations. It would thus seem that the stipulation of a "spread out interaction" along with *any* sort of decay estimate on the two-point correlations would allow us to conclude that the variance of $\boldsymbol{m}_0$ is indeed small. However, while explanations of this sort are satisfactory at a heuristic level, a second glance at Eq. (2.2.5) indicates that the task is not necessarily trivial. Indeed, of actual interest is the decay of correlations within the effective range of the interaction, which is guaranteed to be delicate. At the core of this paper is the use of *reflection positivity* to provide these sorts of estimates.

In many cases, Eq. (2.2.4) on its own is insufficient for understanding the behavior of a system—even at the level of mean-field theory. Specifically, in the case of a discontinuous transition, Eq. (2.2.4) will typically have multiple solutions the overall structure of which does not allow for a continuous solution. While this may have the advantage of signaling the existence of discontinuities, it does not provide any insight as to where the discontinuities actually occur. Thus, whenever there are multiple solutions to Eq. (2.2.4), a supplementary "rule" is needed to determine which of these solutions ought to be selected.

The supplement—or starting point of the whole theory depending on one's perspective—is the introduction of the *mean-field free-energy function* $\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m})$ defined as follows: Let $S(\boldsymbol{m})$ be the *entropy function* associated with the *a priori* measure on the spins. Formally, this quantity is defined by means of the Legendre transform

$$S(\boldsymbol{m}) = \inf_{\boldsymbol{b}\in\mathbb{R}^n} \bigl\{G(\boldsymbol{b}) - (\boldsymbol{b}, \boldsymbol{m})\bigr\} \tag{2.2.6}$$

of the cumulant generating function

$$G(\boldsymbol{b}) = \log \mathbb{E}_0\bigl(\,\mathrm{e}^{(\boldsymbol{b},\boldsymbol{S})}\bigr). \tag{2.2.7}$$

The mean-field free-energy function is then defined as the difference of the energy function, $E(\boldsymbol{m}) = -\frac{\beta}{2}|\boldsymbol{m}|^2 - (\boldsymbol{h}, \boldsymbol{m})$, and the entropy $S(\boldsymbol{m})$:

$$\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}) = -\frac{\beta}{2}|\boldsymbol{m}|^2 - (\boldsymbol{h}, \boldsymbol{m}) - S(\boldsymbol{m}). \tag{2.2.8}$$

Then, as is not hard to see, the mean-field equation is implied by the condition that $\Phi_{\beta,\boldsymbol{h}}$ be minimized. Indeed, writing $\nabla\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}) = \boldsymbol{0}$ some straightforward manipulations give us

$$\boldsymbol{m} = \nabla G(\beta\boldsymbol{m} + \boldsymbol{h}), \tag{2.2.9}$$

which is exactly Eq. (2.2.4).

Eq. (2.2.8) along with the stipulation to minimize adds a whole new dimension to the theory that was defined by Eq. (2.2.4). Foremost, in the case of multiple solutions, we now have a "rule" for the selection of the relevant solutions. Beyond this, we have a framework resembling a full thermodynamical theory: A free energy—defined by evaluating $\Phi_{\beta,\boldsymbol{h}}$ at the minimizing $\boldsymbol{m}$—along with an entropy and energy which are the corresponding functions evaluated at this magnetization. In fact, a secondary goal of this work is to demonstrate that this "more complete" mean-field theory provides an asymptotic description of the actual theories with spread out interactions.

Remark 2.2.1. We conclude this subsection with the remark that the mean-field theory for any particular Hamiltonian of the form (2.1.1) can be produced in an actual spin-system by considering the model on the *complete graph*. Explicitly, for a system with N sites, we take $J_{x,y} = \frac{1}{N}$, compute all quantities according to the standard rules of statistical mechanics and then take $N \to \infty$. The result of this procedure is the mean-field theory described in this subsection for the thermodynamics and a limiting distribution for the spins which is i.i.d. The connection between mean-field theory and complete graph models is well known and has been proved in numerous special cases (see, e.g., Ref. [50] for a recent study of ensemble equivalence for the Potts model on the complete graph). A complete proof for the general form of $\mathscr{H}$ given in Eq. (2.1.1) appears e.g. in Sect. 5 of Ref. [21].

2.2.2 Potts models in external field

The best example of a system which exhibits a rich spectrum of behaviors while remaining tractable is the Potts model in an external field. The Potts model is typically defined using discrete spin variables $\sigma_x \in \{1, \ldots, q\}$ with no apparent internal geometry. The energy of a configuration is given by the (formal) Hamiltonian

$$\beta H = \beta \sum_{x,y} J_{x,y} \delta_{\sigma_x,\sigma_y} - \sum_x h \delta_{1,\sigma_y}. \tag{2.2.10}$$

Here β is the inverse temperature, the $J_{x,y}$'s are the coupling constants for the system, and $\delta_{\sigma_x,\sigma_y}$ is the Kronecker delta. The reduced external field h is related to the physical external field $\tilde{h}$ via $\tilde{h} = h/\beta$. We have chosen only the state "1" as the state affected by the external field even though more general versions are also possible [24, 16, 18, 19].

This system is cast in the form of Eq. (2.1.1) by using the *tetrahedral representation*: We take spin variables $\boldsymbol{S}_x \in \{\hat{\mathrm{v}}_1, \ldots \hat{\mathrm{v}}_q\}$, where the $\hat{\mathrm{v}}_k$'s are the vertices of a unit tetrahedron in $\mathbb{R}^{q-1}$. Inner products (defined the usual way for vectors in $\mathbb{R}^{q-1}$) between the $\hat{\mathrm{v}}_k$'s satisfy

$$(\hat{\mathrm{v}}_k, \hat{\mathrm{v}}_l) = \begin{cases} 1, & \text{if } k = l, \\ \frac{-1}{q-1}, & \text{otherwise,} \end{cases} \tag{2.2.11}$$

and so

$$\delta_{\sigma_x,\sigma_y} - \frac{1}{q} = \frac{q-1}{q} (\boldsymbol{S}_x, \boldsymbol{S}_y). \tag{2.2.12}$$

After similar consideration of the magnetic field terms, it is seen that the Hamiltonian in Eq. (2.2.10) is manifestly of the form in Eq. (2.1.1). To stay in accord with the classic references on the subject, e.g., Ref. [130], we will keep the q-dependent prefactor suggested by Eq. (2.2.12). So, our official Hamiltonian for the Potts model will read

$$\beta \mathscr{H} = -\frac{q-1}{q} \beta \sum_{(x,y)} J_{x,y} (\boldsymbol{S}_x, \boldsymbol{S}_y) - \frac{q-1}{q} h \sum_x (\hat{\mathrm{v}}_1, \boldsymbol{S}_x) \tag{2.2.13}$$

with the J's obeying Eq. (2.2.1) and $h \in \mathbb{R}$.

The mean-field theory is best expressed in terms of the vector magnetization given by

$$\boldsymbol{m} = x_1 \, \hat{\mathrm{v}}_1 + \cdots + x_q \, \hat{\mathrm{v}}_q, \tag{2.2.14}$$

and the mean-field free-energy function is [130, 21]

$$\Phi_{\beta,h}^{(q)}(\boldsymbol{m}) = \sum_{k=1}^{q} \Bigl(-\frac{\beta}{2} x_k^2 + x_k \log x_k\Bigr) - h x_1. \tag{2.2.15}$$

Here the "barycentric" coordinates x_k are components of a probability vectors, i.e., we have $x_k \geq 0$ and $x_1 + \cdots + x_q = 1$. In the context of the Potts model on a complete graph, x_k represents the fraction of sites in the k-th spin state.

Let us start with a recapitulation of the zero-field case where the resulting theory is quite well known. For each q there is a number $\beta_{\mathrm{MF}}^{(q)}$ such that if $\beta < \beta_{\mathrm{MF}}^{(q)}$, the unique global minimizer is the "most symmetric state," $\boldsymbol{m} = \boldsymbol{0}$, while for $\beta > \beta_{\mathrm{MF}}^{(q)}$, there are exactly q (asymmetric) global minima which are permutations of one probability vector of the form $x_1 > x_2 = \cdots = x_q$. Thus we may express all quantities in terms of a *scalar* magnetization, e.g., $x_1 = \frac{1}{q} + m$ and $x_k = \frac{1}{q} - \frac{m}{q-1}$, $k = 2, \dots, q$. Then, when $\beta > \beta_{\mathrm{MF}}^{(q)}$, the mean-field magnetization is given by $m_{\,\mathrm{MF}}(\beta) = \frac{q-1}{q}\theta$, where θ is the maximal positive solution to the equation

$$\theta = \frac{\mathrm{e}^{\beta\theta} - 1}{\mathrm{e}^{\beta\theta} + q - 1}. \tag{2.2.16}$$

The crucial point—which can be gleaned form a perturbative analysis of Eq. (2.2.16)—is the division *at* $q = 2$ of two types of behavior. In particular, $m_{\,\mathrm{MF}}(\beta)$ tends to a strictly positive value as $\beta \downarrow \beta_{\mathrm{MF}}^{(q)}$ for $q > 2$, while for $q = 2$ the limit value is zero. (Indeed, for $q = 2$, there *are no* nontrivial solutions to Eq. (2.2.16) at $\beta = \beta_{\mathrm{MF}}^{(2)} = 2$.)

Remark 2.2.2. Interestingly, the values of $\beta_{\mathrm{MF}}^{(q)}$ and the limit value $m_{\,\mathrm{MF}}(\beta_{\mathrm{MF}}^{(q)})$ are explicitly computable:

$$\beta_{\mathrm{MF}}^{(q)} = 2\,\frac{q-1}{q-2}\log(q-1), \qquad m_{\,\mathrm{MF}}(\beta_{\mathrm{MF}}^{(q)}) = \frac{q-2}{q}. \tag{2.2.17}$$

This observation goes back to at least Ref. [130].

Let us now anticipate, without going to details, what happens for $h \neq 0$. (The full-blown statements and proofs will appear in Sect. 2.2.3 and Sect. 2.5, respectively.) We will capitalize on the principle that local minimizers are stable to small changes in parameters. Consider $q \geq 3$ and $h \neq 0$ such that $|h| \ll 1$. The overall situation cannot differ too drastically from the zero-field case; the only distinction is that for $h > 0$ only one of the "$h = 0$ asymmetric minimizers" is allowed while for $h < 0$ the same minimizer is suppressed in favor of the remaining $q - 1$ ones. On the other hand, for h positive and large, it is clear that the minimizer of $\Phi^{(q)}_{\beta,h}(\boldsymbol{m})$ will be unique no matter what β is. Thus, for $h > 0$ we should have a line of mean-field first-order phase transitions which terminates at a finite value of h. On general grounds, the terminal point is expected to be a critical point.

Next, let us consider $h < 0$ with $|h| \gg 1$. The situation at $h = -\infty$ is clear; this is just the $(q - 1)$-state Potts model. Thus for finite but large $|h|$, we can see a clear distinction between $q = 3$ and $q > 3$. In the former cases, the mean-field transition should be Ising like and hence continuous. In the latter case, the transition should be discontinuous. Thus, the $q = 3$ line should break at a *tricritical* point followed by a line of continuous transitions while for $q > 3$ there will be an unbroken line of discontinuous mean-field phase transitions.

Aside from general interest, the key motivation for obtaining such detailed knowledge about m_{MF} is as follows: Under specific conditions on (2.1.1), virtually all that has just been discussed pertaining to discontinuous transitions in these systems can be established with rigor in the spread out "real" systems. (On the downside is the fact that virtually nothing pertaining to the continuous transition can be proved by these methods.) To illustrate let us consider the transition at $h > 0$ when q is large. The mean-field picture is as follows: A non-convexity of $\Phi^{(q)}_{\beta,h}(\boldsymbol{m})$ develops when β is of order unity, but it does not "touch down" until β is appreciable (of order $\log q$). However, the existence of a non-convexity suggests that a strong-enough magnetic field can tilt the balance in favor of a magnetized state, even for β's of order unity. This is indeed the case for the MFT as our detailed calculations later show. As a consequence of the general techniques presented here, this result from the MFT will be processed into a theorem for actual systems.

2.2.3 Precise statements for mean-field Potts model

Our precise results for the mean-field theory of the Potts model in an external field are summarized into two theorems; one for positive fields and the other for negative fields.

Theorem 2.2.3 (Positive fields)**.** *Let* $q \geq 3$*, let* $\boldsymbol{m}$ *and the probability vector* $(x_1, \dots, x_q)$ *be related as in Eq.* (2.2.14) *and let* $\Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ *denote the function from Eq.* (2.2.15)*. Let* h_c *denote the quantity*

$$h_c = \log q - \frac{2(q-2)}{q}. \tag{2.2.18}$$

Then there is a continuous function $\beta_+^{(q)} \colon (0, h_c) \to (0, \infty)$ *such that*

(1) For all (β, h) *such that either* $h \geq h_c$ *or* $\beta \neq \beta_+^{(q)}(h)$*, there is a unique global minimizer of* $\Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ *with* $x_2 = \cdots = x_q$*. The quantity* x_1 *corresponding to this minimizer is strictly larger than the mutual value of the* x_k*'s for* $k = 2, \dots, q$*.*

(2) For all $h < h_c$*, there are two distinct global minimizers of* $\Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ *at* $(\beta_+^{(q)}(h), h)$*.*

(3) For (β, h) *such that* $h \geq h_c$ *or* $\beta \neq \beta_+^{(q)}(h)$*, let* $x_1 = x_1(\beta, h)$ *denote the first coordinate of the global minimizer of* $\Phi_{\beta,h}^{(q)}(\boldsymbol{m})$*. Then* $(\beta, h) \mapsto x_1(\beta, h)$ *is continuous with well-defined but distinct (one-sided) limits at* $(\beta, h) = (\beta_+^{(q)}(h), h)$*.*

Furthermore, writing $x_1 = \frac{1}{q} + m$*, the quantity* $\theta = \frac{q}{q-1}m$ *obeys the equation*

$$\theta = \frac{e^{\beta\theta+h} - 1}{e^{\beta\theta+h} + q - 1}. \tag{2.2.19}$$

in the region of uniqueness. At the points $(\beta_+^{(q)}(h), h)$*, both limiting values obey this equation.*

(4) *The function* $h \mapsto \beta_+^{(q)}(h)$ *is strictly decreasing on* $(0, h_c)$ *with limit values* $\beta_+^{(q)}(h) \uparrow \beta_{MF}^{(q)} = 2\frac{q-1}{q-2}\log(q-1)$ *as* $h \downarrow 0$ *and* $\beta_+^{(q)}(h) \downarrow \frac{4(q-1)}{q}$ *as* $h \uparrow h_c$*.*

In order to preserve uniformity of exposition, we will restrict the statement of negative-field results to $q \geq 4$.

Theorem 2.2.4 (Negative fields)**.** *Let* $q \geq 4$*, let* $\boldsymbol{m}$ *and the probability vector* $(x_1, \dots, x_q)$ *be related as in Eq.* (2.2.14) *and let* $\Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ *denote the function from Eq.* (2.2.15)*. Then we have:*

(1) *All global minima are permutations in the last* $q-1$ *variables of vectors with the representation*

$$x_1 < x_2 = \cdots = x_{q-1} \leq x_q. \tag{2.2.20}$$

Moreover, there exists a function $\beta_-^{(q)}\colon (-\infty, 0) \to (0, \infty)$ *such that the following hold:*

(2) *(Symmetric Minimum) For all* $\beta < \beta_-^{(q)}(h)$*, there is a unique global minimum and it has* $x_2 = \cdots = x_q$*. Moreover, if* m *is such that* $x_1 = \frac{1}{q} - m$ *and* $x_k = \frac{1}{q} + \frac{m}{q-1}$*, for all* $k = 2, \dots, q$*, then* $\theta = \frac{q}{q-1}m$ *corresponds to a global minimum when*

$$\theta = \frac{e^{\beta\theta-h} - 1}{(q-1)\, e^{\beta\theta-h} + 1}. \tag{2.2.21}$$

There is only one $\theta \in [0, \frac{1}{q-1}]$ *for which Eq.* (2.2.21) *holds.*

(3) *(Asymmetric Minima) For all $\beta > \beta_-^{(q)}(h)$, we have $q-1$ global minima. These are permutations in the last $q-1$ variables of a single minimum whose coordinate representation takes the form*

$$x_1 < x_2 = \cdots = x_{q-1} < x_q. \tag{2.2.22}$$

(4) *At $\beta = \beta_-^{(q)}(h)$ there are q global minima. One of these is of the type described in (2)—namely, the symmetric minimum—while the other $q-1$ are of the type described in (3).*

(5) *The function $h \mapsto \beta_-^{(q)}(h)$ is strictly increasing and continuous. Moreover, we have the limits*

$$\lim_{h\to-\infty} \beta_-^{(q)}(h) = \beta_{MF}^{(q-1)} \quad \textit{and} \quad \lim_{h\uparrow 0} \beta_-^{(q)}(h) = \beta_{MF}^{(q)} \tag{2.2.23}$$

Theorem 2.2.3 is proved in Sect. 2.5.3 and Theorem 2.2.4 is proved in Sect. 2.5.4. The corresponding statement for the actual lattice systems is the subject of Theorem 2.3.5.

2.3 Main results

Here we give the statements of the principal theorems which apply to any model whose interaction is of the type (2.1.1). Then we apply these to the Potts and Blume-Capel models.

2.3.1 General theory

We begin by a precise definition of the class of models we consider. Let Ω be a compact subset of $\mathbb{R}^n$, with the inner product denoted by $(\cdot,\cdot)$, and let $\operatorname{Conv}(\Omega)$ denote the convex hull of Ω. Let μ_0 be a Borel probability measure on $(\Omega, \mathscr{B})$ that describes the *a priori* distribution of the individual spins. We will consider spin configurations $(\boldsymbol{S}_x)$ from $\Omega^{\mathbb{Z}^d}$ and, abusing the notation slightly, use μ_0 to denote also the corresponding *a priori* product measure.

To define the interacting spin system, let us pick a finite set $\Lambda \subset \mathbb{Z}^d$, a spin configuration $\boldsymbol{S}_\Lambda \in \Omega^\Lambda$ in Λ and the "boundary condition" $\boldsymbol{S}_{\Lambda^{\rm c}} \in \Omega^{\Lambda^{\rm c}}$. For each $\boldsymbol{h} \in \mathbb{R}^n$ and each $\beta > 0$, we then define the finite-volume Hamiltonian $\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda, \boldsymbol{S}_{\Lambda^{\rm c}})$ by

$$\beta\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda, \boldsymbol{S}_{\Lambda^{\rm c}}) = -\beta \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda, y\in\mathbb{Z}^d}} J_{x,y}\,(\boldsymbol{S}_x, \boldsymbol{S}_y) - \sum_{x\in\Lambda}(\boldsymbol{h}, \boldsymbol{S}_x). \tag{2.3.1}$$

The first sum goes over all unordered pairs of distinct sites $\langle x, y\rangle$ at least one of which is contained in Λ.

The above Hamiltonian can now be used to define the finite-volume Gibbs measure $\nu_\Lambda^{(\boldsymbol{S}_{\Lambda^{\rm c}})}$ on spin configuration from Ω^Λ by

$$\nu_\Lambda^{(\boldsymbol{S}_{\Lambda^{\rm c}})}(\,{\rm d}\boldsymbol{S}_\Lambda) = \frac{{\rm e}^{-\beta\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda, \boldsymbol{S}_{\Lambda^{\rm c}})}}{Z_\Lambda^{(\boldsymbol{S}_{\Lambda^{\rm c}})}(\beta, \boldsymbol{h})}\mu_0(\,{\rm d}\boldsymbol{S}_\Lambda), \tag{2.3.2}$$

where the normalizing constant $Z_\Lambda^{(\boldsymbol{S}_{\Lambda^{\rm c}})}(\beta, \boldsymbol{h})$ is the partition function. Of particular interest are the (weak subsequential) limits of these measures as Λ expands to fill out the entire $\mathbb{Z}^d$. These measures obey the DLR-conditions [66] and are generally referred to as (infinite-volume) Gibbs measures. In this formalism, *phase coexistence* is said to occur for parameters β and $\boldsymbol{h}$ if there is more than one limiting Gibbs measure. Under these conditions the system is said to exhibit a *first-order phase transition*.

We proceed by formulating the precise conditions under which our results will be proved. To facilitate our next definition, for each lattice direction $\ell \in \{1, \dots, d\}$, let $\mathbb{H}_\ell$ denote the half-space

$$\mathbb{H}_\ell = \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d,\, x_\ell > 0\}. \tag{2.3.3}$$

We will use $\vartheta^{(\ell)}$ to denote the reflection $\vartheta^{(\ell)} \colon \mathbb{H}_\ell \to \mathbb{Z}^d \setminus \mathbb{H}_\ell$ defined explicitly by the formula $\vartheta^{(\ell)}(x_1, \dots, x_d) = (x_1, \dots, x_{\ell-1}, 1 - x_\ell, x_{\ell+1}, \dots, x_d)$.

Definition 2.3.1 (RP "through bonds")**.** Consider a collection of coupling constants $(J_{x,y})_{x,y\in\mathbb{Z}^d}$. We say that these are RP if the following conditions hold:

(1) (translation invariance) for any $x, y \in \mathbb{Z}^d$ we have $J_{x,y} = J_{0,y-x}$.

Moreover, for any lattice direction $\ell \in \{1, \dots, d\}$,

(2) (reflection invariance) for any $x, y \in \mathbb{H}_\ell$ we have

$$J_{x,y} = J_{\vartheta^{(\ell)}x,\vartheta^{(\ell)}y}. \tag{2.3.4}$$

(3) (reflection positivity) if $f\colon \mathbb{H}_\ell \to \mathbb{R}$ is absolutely summable with

$$\sum_{x\in\mathbb{H}_\ell} f(x) = 0, \tag{2.3.5}$$

then

$$\sum_{\substack{x\in\mathbb{H}_\ell \\ y\in\mathbb{Z}^d\smallsetminus\mathbb{H}_\ell}} J_{x,y} f(x) f(\vartheta^{(\ell)}y) \ge 0. \tag{2.3.6}$$

Given a translation-invariant Gibbs measure, we use the word *magnetization* to denote the expectation of the spin at the origin. The statement of our general result can then be viewed as a restriction on the possible values of the magnetization. However, not all magnetizations that can be physically produced are (provably) accessible to our methods. The reason is that the underlying Gibbs states for which our techniques work will have to satisfy the conditions of reflection positivity—in particular, they have to be obtained as weak limits of torus states. Our next item of business will be to define precisely the set of "allowed values" of the magnetization.

We will proceed as in Ref. [21]. Let $Z_\Lambda(\beta, \boldsymbol{h})$ be the partition function in volume Λ—the boundary condition is irrelevant—and let $F(\beta, \boldsymbol{h})$ denote the (physical) free energy defined as the limit of $-\frac{1}{|\Lambda|}\log Z_\Lambda$ as Λ increases to fill the entire $\mathbb{Z}^d$ (in the sense of van Hove [66]). The function $F(\beta, \boldsymbol{h})$ is jointly concave, so we may let $\mathscr{K}_\star(\beta, \boldsymbol{h})$ denote the set of all pairs $[e_\star, \boldsymbol{m}_\star]$ such that

$$F(\beta + \triangle\beta, \boldsymbol{h} + \triangle\boldsymbol{h}) - F(\beta, \boldsymbol{h}) \leq e_\star \triangle\beta + (\boldsymbol{m}_\star, \triangle\boldsymbol{h}) \tag{2.3.7}$$

for any $\triangle\beta \in \mathbb{R}$ and any $\triangle\boldsymbol{h} \in \mathbb{R}^n$. Now $\mathscr{K}_\star(\beta, \boldsymbol{h})$ is a convex set so we let $\mathscr{M}_\star(\beta, \boldsymbol{h})$ to denote the set of values $\boldsymbol{m}_\star$ for which there exists an $e_\star$ such that $[e_\star, \boldsymbol{m}_\star]$ is an extreme value of $\mathscr{K}_\star(\beta, \boldsymbol{h})$. Our main theorem then reads:

Theorem 2.3.2. *Consider the spin system on $\mathbb{Z}^d$ with the Hamiltonian* (2.1.1) *such that the couplings $(J_{x,y})$ are RP, the inverse temperature $\beta > 0$ and external field $\boldsymbol{h} \in \mathbb{R}^n$. For each $k \in [-\pi, \pi]^d$, let $\hat{J}(k) = \sum_{x \in \mathbb{Z}^d} J_{0,x}\, e^{\,ik\cdot x}$ and recall that $\hat{J}(0) = 1$ by Eq.* (2.2.1). *Then for any $\boldsymbol{m}_\star \in \mathscr{M}_\star(\beta, \boldsymbol{h})$,*

$$\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}_\star) \leq \inf_{\boldsymbol{m} \in \text{ Conv}\,(\Omega)} \Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}) + \beta n \frac{\kappa}{2} \mathscr{I}, \tag{2.3.8}$$

where n is the (underlying) dimension of the spin-space, $\kappa = \max_{\boldsymbol{S} \in \Omega} |\boldsymbol{S}|^2$ and

$$\mathscr{I} = \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} \frac{|\hat{J}(k)|^2}{1 - \hat{J}(k)}. \tag{2.3.9}$$

The useful aspect of Theorem 2.3.2 is that the error term $\mathcal{E} = \beta n \frac{\kappa}{2} \mathscr{I}$ can be made small by appropriate adjustment of parameters. A general statement of this sort appears in Proposition 2.4.10 but, typically, these conditions have to be verified on a case by case basis. Let us tend to the details of these adjustments later and, for the time being, simply assume that $\mathcal{E}$ is small. Then, along with the obvious supplement of Eq. (2.3.8), $\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}_\star) \geq \inf_{\boldsymbol{m} \in \text{ Conv}\,(\Omega)} \Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m})$, we have learned that the allowed values of the magnetization in the *physical* system nearly minimize the *mean-field* free energy. In this sense, the mean-field theory already provides a quantitatively accurate description of the physical system once $\mathcal{E} \ll 1$. In Sects. 2.3.3-2.3.4 we will use this fact to prove a first-order phase transitions in a few models of interest.

To demonstrate the use of Theorem 2.3.2, let us consider the "evolution" of a typical MFT phase transition, in which two local minima of $\Phi_{\beta,\boldsymbol{h}}$ exchange roles of the global minimizer as β varies. Specifically, let $\boldsymbol{m}_{\mathrm{S}}(\beta)$ and $\boldsymbol{m}_{\mathrm{A}}(\beta)$ be local minima of $\Phi_{\beta,\boldsymbol{h}}$—one of which is always global—for β near some β_{t}, and suppose that $\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}_{\mathrm{A}}) > \Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}_{\mathrm{S}})$ for $\beta > \beta_{\mathrm{t}}$ and *vice versa* for $\beta < \beta_{\mathrm{t}}$. Then Theorem 2.3.2 can be applied under the condition that, outside some small neighborhoods of $\boldsymbol{m}_{\mathrm{S}}(\beta)$ and $\boldsymbol{m}_{\mathrm{A}}(\beta)$ for $\beta \approx \beta_{\mathrm{t}}$, no magnetizations have a free energy within $\mathcal{E}$ of the absolute minimum. For $\beta \gtrapprox \beta_{\mathrm{t}}$, this stipulation applies even to the neighborhood of $\boldsymbol{m}_{\mathrm{S}}(\beta)$ and, for $\beta \lessapprox \beta_{\mathrm{t}}$, to the neighborhood of $\boldsymbol{m}_{\mathrm{A}}(\beta)$. Then, Theorem 2.3.2 tells us that in the region $\beta \lessapprox \beta_{\mathrm{t}}$, the actual magnetization is near $\boldsymbol{m}_{\mathrm{S}}(\beta)$, for $\beta \approx \beta_{\mathrm{t}}$ it could be near $\boldsymbol{m}_{\mathrm{S}}$ or $\boldsymbol{m}_{\mathrm{A}}$, and for $\beta \gtrapprox \beta_{\mathrm{t}}$ it is only near $\boldsymbol{m}_{\mathrm{A}}(\beta)$. On general grounds, as long as the difference $\boldsymbol{m}_{\mathrm{A}} - \boldsymbol{m}_{\mathrm{S}}$ is bounded uniformly away from zero, somewhere near β_{t} there has to be a point of phase coexistence.

2.3.2 Mean-field philosophy

In this section we will state some general facts about spin systems and their mean-field analogues. The stipulations that govern this section are rather mild; first we will assume that the Hamiltonian is of the form (2.1.1) with the $J_{x,y}$'s satisfying the conditions of reflection positivity. Second, we will assume that the associated mean-field free-energy function defined in Eq. (2.2.8) has a unique minimizer. Finally, we will investigate the small-$\mathscr{I}$ behavior of these models. The preferred viewpoint is a fixed dimension d with parameters μ—as defined in Eq. (2.1.2)—tending to zero or s—as defined in Eq. (2.1.3)—tending to d.

We note that special cases (usually restricted to concrete models) have been addressed elsewhere; see, in particular, Ref. [88] and references therein, but there the only mechanism to force $\mathscr{I} \to 0$ was the $d \to \infty$ limit which we find æsthetically somewhat unsatisfactory. Another possibility is to consider the aforementioned Kac limit which more or less boils down to infinite smearing out of the interaction. A contour-based analysis of this limit has been carried out, but the technical aspects have so far been overcome only for very specific models [38, 32, 33, 94, 37]. Here we provide a general result in this direction under the sole condition of reflection positivity.

Theorem 2.3.3 (Mean-field philosophy)**.** *Consider the spin system as described above and let $\Phi_{\beta,\boldsymbol{h}}$ be as in Eq.* (2.2.8)*. Suppose that the parameters $\beta > 0$ and $\boldsymbol{h} \in \mathbb{R}^n$ are such that $\Phi_{\beta,\boldsymbol{h}}$ has a unique minimizer $\boldsymbol{m}$ on Conv (Ω) in Eq.* (2.2.8)*. Let $(J_{x,y}^{(n)})$ be a sequence of coupling constants that are RP and obey Eq.* (2.2.1)*, and let $\langle - \rangle_{\beta,\boldsymbol{h}}^{(n)}$ be a sequence of translation and rotation-invariant Gibbs states corresponding to these couplings. If the sequence of integrals $\mathscr{I}_n$, obtained from $(J_{x,y}^{(n)})$ via Eq.* (2.3.9)*, satisfies*

$$\mathscr{I}_n \to 0 \quad as \quad n \to \infty, \tag{2.3.10}$$

then we have the following facts:

(1) The actual magnetization tends to $\boldsymbol{m}$, i.e.,

$$\langle \boldsymbol{S}_0 \rangle_{\beta,\boldsymbol{h}}^{(n)} \underset{n\to\infty}{\longrightarrow} \boldsymbol{m}. \tag{2.3.11}$$

(2) The energy density tends to its mean-field value, i.e.,

$$\big\langle (\boldsymbol{S}_0, \tfrac{\beta}{2}\boldsymbol{m}_0 + \boldsymbol{h}) \big\rangle_{\beta,\boldsymbol{h}}^{(n)} \underset{n\to\infty}{\longrightarrow} E(\boldsymbol{m}), \tag{2.3.12}$$

where $\boldsymbol{m}_0$ is as in Eq. (2.2.3) *and $E(\boldsymbol{m})$ is as in Sect. 2.2.1.*

In particular, in the limit $n \to \infty$, the spin variables at distinct sites become independent with distribution given by the product of the titled measures

$$e^{(\boldsymbol{S},\beta\boldsymbol{m}+\boldsymbol{h})-G(\beta\boldsymbol{m}+\boldsymbol{h})}\mu_0(\,d\boldsymbol{S}). \tag{2.3.13}$$

Here μ_0 is the a priori *measure.*

The preceding—as is the case in much of the principal results of this paper—reduces (the $\mathscr{I} \to 0$ limit of) the full problem to a detailed study of the associated mean-field theory. Two specific models will be analyzed in great detail shortly (see Sects. 2.3.3 and 2.3.4); let us mention two other well known (or well studied) examples.

First are the $O(n)$ spin systems at zero external field. Here each $\boldsymbol{S}_x$ takes values on the unit sphere in $\mathbb{R}^n$ with *a priori* uniform measure. In the mean-field theory of these models, the scalar magnetization $m(\beta)$ vanishes for β less than some β_c while for $\beta \ge \beta_\text{c}$ it is the maximal positive solution of a certain transcendental equation (see, e.g., Ref. [88]). In particular, this solution rises continuously from zero according to

$$|m(\beta)| = (\beta - \beta_\text{c})^{1/2}\left[C(n) + o(1)\right], \qquad \beta \downarrow \beta_\text{c}. \tag{2.3.14}$$

By Theorem 2.3.3, the actual magnetization converges to this function but, unfortunately, our control is not strong enough to rule out the possibility of small discontinuities (which vanish as $\mathscr{I} \to 0$).

A less well known but very interesting example is the *cubic model* where the spins point to the center of a face on an r-dimensional unit hypercube, i.e., $\boldsymbol{S}_x \in \Omega = \{\pm\,\hat{\text{e}}_1, \dots, \pm\,\hat{\text{e}}_r\}$. For $r > 3$ the transition in this model is first order (and was analyzed in Ref. [21]). The case $r = 2$ reduces to an Ising system but the borderline case, $r = 3$, while still continuous, features a somewhat anomalous (namely, tricritical) behavior. Indeed, for this system, the mean-field magnetization obeys

$$|m(\beta)| = (\beta - \beta_\text{c})^{1/4}\left[C + o(1)\right], \qquad \beta \downarrow \beta_\text{c}, \tag{2.3.15}$$

where $\beta_\text{c} = 3$. Once again, the actual magnetization converges to such a function but the control is not sufficient to rule out small discontinuities.

While these sorts of results do not establish *any* critical behavior in particular systems, they could represent a first step in proving that a variety of (mean-field) critical behaviors are possible.

2.3.3 Results for the Potts model

Our first result concerns the zero-field q-state Potts model with $q \geq 3$. Let $F(\beta, h)$ denote the free energy of the Potts model with the Hamiltonian in Eq. (2.2.10) and let $m_\star(\beta)$ be the quantity

$$m_\star(\beta) = \frac{\partial}{\partial h^+} F(\beta, h)\Big|_{h=0} - \frac{1}{q}. \tag{2.3.16}$$

(An alternative definition of $m_\star(\beta)$ would be the limiting probability that the spin at the origin is "1" in the state generated by the boundary spins all set to "1.") Let $m_{\text{MF}} = m_{\text{MF}}(\beta)$ be related to the maximal positive solution θ of Eq. (2.2.16) by $m_{\text{MF}} = \frac{q-1}{q}\theta$. Then we have:

Theorem 2.3.4. *Let $q \geq 3$ be fixed. For each $\epsilon > 0$ there exists $\delta > 0$ with the following property: For any $d \geq 1$ and any collection of coupling constants $(J_{x,y})$ on $\mathbb{Z}^d$ that are RP, obey* (2.2.1) *and for which the integral $\mathscr{I}$ in Eq.* (2.3.9) *satisfies $\mathscr{I} \leq \delta$, there exists a number $\beta_t \in (0, \infty)$ such that*

$$|\beta_t - \beta_{MF}^{(q)}| \leq \epsilon \tag{2.3.17}$$

holds and such that the physical magnetization $m_\star = m_\star(\beta)$ of the corresponding q-state Potts model obeys the bounds

$$m_\star(\beta) \leq \epsilon \quad \textit{for} \quad \beta < \beta_t \tag{2.3.18}$$

and

$$|m_\star(\beta) - m_{MF}(\beta)| \leq \epsilon \quad \textit{for} \quad \beta > \beta_t. \tag{2.3.19}$$

In particular, whenever the integral $\mathscr{I}$ is sufficiently small, $\beta \mapsto m_\star(\beta)$ undergoes a jump near the value $\beta_{MF}^{(q)}$. A similar jump occurs (at the same point) in the energy density.

This statement extends Theorem 2.1 of Ref. [21] to a class of spread-out RP interactions. (A minor technical innovation is that the bound in Eq. (2.3.19) holds uniformly.) As a consequence, we are finally able to provide examples of interactions for which the $q = 3$ state Potts models in dimension $d = 3$ can be proved to have a first-order transition. Similar conclusion holds for all $q \geq 3$ but, unfortunately, our requirements on the "smallness" of the corresponding parameters are not uniform in q.

In $d = 1$, we show that the long-range Potts models with power-law decaying interactions go first order once the exponent of the power-decay is between one and two. Models in this category have been studied in Ref. [102] in the context of percolation; the domination techniques of, e.g., Ref. [4] then imply the existence of a low temperature phase. However, the percolation-based approach alone is unable to tell whether the transition is discontinuous or not. Some additional discussion is provided in Sect. 2.3.5.

Our next item of interest will be the same system in an external field, as described by the full Hamiltonian (2.2.10). For reasons alluded to in Sect. 2.2.2, we will restrict our attention to the $q \geq 4$ cases.

Theorem 2.3.5. *Let $q \geq 4$ be fixed and let us consider the q-state Potts model with coupling constants $J_{x,y}$ that are RP and obey Eq.* (2.2.1)*. Then there exists $\delta_0 > 0$ and a function $h_0\colon (0, \delta_0] \to [0, h_c)$, where h_c is as in Eq.* (2.2.18)*, such that if* (2.3.9) *obeys $\mathscr{I} \leq \delta$ with some $\delta \leq \delta_0$, then there exists a function $\beta_t\colon (-\infty, h_0) \to (0, \infty)$ with the following properties:*

(1) A first-order transition (accompanied by a discontinuity in the energy density and the magnetization) occurs at the parameters $(h, \beta_t(h))$, for any external field $h \in (-\infty, h_0)$.

(2) Let $m_\star(\beta, h)$ be the "spin-1 density" defined by the right partial derivative $\frac{\partial}{\partial h^+} F(\beta, h)$.

Then there exists an $h_1 = h_1(\delta) < 0$ *such that* $h \mapsto m_\star(\beta, h)$ *has a discontinuity at field strength* $\tilde{h}$ *such that* $\beta = \beta_t(\tilde{h})$ *provided that* $\tilde{h} \in (h_1, h_0)$*.*

The function h_0 *is decresing while* h_1 *is increasing. Moreover,* $\lim_{\delta \downarrow 0} h_0(\delta) = h_c$ *and* $\lim_{\delta \downarrow 0} h_1(\delta) = -\infty$*.*

The second part of the theorem asserts that, even if state "1" is suppressed by the field, the order-disorder transition will be felt by the "spin-1 density" $m_\star(\beta, h)$. There is no doubt in our mind that the restriction to $h \geq h_1$ in this claim is only of technical nature. Our lack of control for h very large negative stems from the fact that the jump in the mean-field counterpart of $m_\star(\beta, h)$ decreases exponentially with $|h|$ as $h \to -\infty$. Theorems 2.3.4 and 2.3.5 are proved in Sect. 2.6.

2.3.4 Results for the Blume-Capel model

The Blume-Capel model is a system whose spins σ_x take values in the set $\Omega = \{-1, 0, 1\}$ with *a priori* equal weights. The Hamiltonian is given most naturally in the form

$$\beta\mathscr{H}(\sigma) = \beta \sum_{\langle x,y \rangle} J_{x,y}(\sigma_x - \sigma_y)^2 - \lambda \sum_x (\sigma_x)^2 - h \sum_x \sigma_x. \tag{2.3.20}$$

As is easy to see, a temporary inclusion of the terms proportional to $(\sigma_x)^2$ into the single-spin measure shows that this Hamiltonian is indeed of the general form in Eq. (2.1.1).

If we consider the situation at zero temperature ($\beta = \infty$) with λ and h finite we see that in the (λ, h)-plane there are three regions of constant spin which minimize $\beta\mathscr{H}(\sigma)$. The regions all meet at the point $h = 0$, $\lambda = 0$; tentatively we will call the origin a triple point (and the lines phase boundaries). Ostensibly one would wish to establish that this entire picture persists at finite temperature. However, we will confine attention to the line $h = 0$ which is of the greatest interest. We will show, both in the context of mean-field theory and, subsequently, realistic systems that there is indeed a finite temperature first order transition at some $\lambda_{\text{t}}(\beta)$. Of significance is the fact that this occurs at a λ_{t} which is *strictly* positive; i.e., for $1 \ll \beta < \infty$, the point $\lambda = 0$ lies inside the phase which is dominated by zeros.

We remark that results of this sort are far from new; indeed the proof of this and similar results represented one of the early triumphs of low temperature techniques Ref. [123]. The physical reason behind the shifting of the phase boundary is the enhanced ability of the "zero" phase over the plus and minus phases to harbor elementary excitations. Interestingly, in spite of the fact that our method relies on *suppression* of fluctuations, the corresponding entropic stabilization is nevertheless manifest in our derivation. In addition, while the contour-based approaches require a non-trivial amount of "low temperature labor" to ensure that the interactions between excitations are limited, our methods effortlessly incorporate whatever interactions may be present.

To simplify our discussion, from now on we will focus on the situation at zero external field, i.e., $h = 0$, and suppress h from the notation. First let us take a look at the mean-field theory. Here we find it useful to express the relevant quantities in terms of mole fractions x_1, x_0, x_{-1} of the three spin states in Ω. To within an irrelevant constant, the mean-field free-energy function is

$$\Phi_{\beta,\lambda} = 4\beta x_1 x_{-1} + \beta x_0(1 - x_0) + \lambda x_0 + \sum_{\sigma=\pm1,0} x_\sigma \log x_\sigma. \tag{2.3.21}$$

Here we have used the fact that $x_1 + x_0 + x_{-1} = 1$. Our main result concerning the mean-field theory of the Blume-Capel model is now as follows:

Theorem 2.3.6. *For all $\beta \ge 0$ and all $\lambda \in \mathbb{R}$, all local minima of $\Phi_{\beta,\lambda}$ obey the equations*

$$x_1\, e^{4\beta x_{-1}} = x_{-1}\, e^{4\beta x_1} = x_0\, e^{\beta(1-2x_0)+\lambda}. \tag{2.3.22}$$

Moreover, there exists a $\beta_0 < \infty$ such that for all $\beta \geq \beta_0$, any such (local) minimum is of the form that two components of (x_1, x_0, x_{-1}) are very near zero and the remaining one is near one. Explicitly, there exists a constant $C < \infty$ such that

(1) If x_0 is the dominant index, then $x_1 = x_{-1} = \frac{1}{2}(1 - x_0)$ and we have that $(1 - x_0) \leq C\, e^{-\beta+\lambda}$.

(2) If x_1 is the dominant index, then $x_{-1} \leq C\, e^{-4\beta}$ while $x_0 \leq C\, e^{-\beta-\lambda}$. A corresponding statement is true for the situation when x_{-1} is dominant.

Furthermore, consider two local minima at (β, λ), one dominated by x_0 and the other dominated by x_1. Let $\phi_0(\beta, \lambda)$ be the mean-field free energy corresponding to the former minimum and let $\phi_1(\beta, \lambda)$ be that corresponding to the latter minimum. Then

$$\phi_0(\beta, \lambda) - \phi_1(\beta, \lambda) = \lambda - \; e^{-\beta+\lambda} + O(\beta\, e^{-2\beta}) \tag{2.3.23}$$

where $O(\beta\, e^{-2\beta})$ denotes a quantity bounded by a constant times $\beta\, e^{-2\beta}$ for all λ in a neighborhood of the origin. In particular, for all β sufficiently large there exists $\lambda_{MF}(\beta) = \; e^{-\beta} + O(\beta\, e^{-2\beta})$ such that the global minimizes of $\Phi_{\beta,\lambda}$ have $x_{\pm 1} \ll 1$ for $\lambda < \lambda_{MF}(\beta)$ and $x_0 \ll 1$ for $\lambda > \lambda_{MF}(\beta)$.

Theorem 2.3.6 is proved in Sect. 2.5.1. Next we will draw our basic conclusions about the actual system:

Theorem 2.3.7. *Consider the Blume-Capel model in Eq.* (2.3.20)*, with zero field ($h = 0$), inverse temperature β and the coupling constants $(J_{x,y})$ that are RP and obey Eq.* (2.2.1)*. Let $\mathscr{I}$ be the integral in Eq.* (2.3.9)*. There exist constants $\beta_0 \in (0, \infty)$ and $C < \infty$ such that if $\beta \geq \beta_0$ and $\beta\mathscr{I} \ll \; e^{-\beta}$, then there is a function $\lambda_t \colon [\beta_1, \beta_2] \to \mathbb{R}$ satisfying $|\lambda_t(\beta) - \; e^{-\beta}| < \beta\mathscr{I}$ such that any translation-invariant Gibbs state $\langle - \rangle_{\beta,\lambda}$ obeys*

(1) $\langle \sigma_x^2 \rangle_{\beta,\lambda} \le C\, e^{-\beta}$ *if* $\lambda < \lambda_t(\beta)$*,*

(2) $\langle \sigma_x^2 \rangle_{\beta,\lambda} \ge 1 - C\, e^{-\beta}$ *if* $\lambda > \lambda_t(\beta)$*.*

Moreover, at $\lambda = \lambda_t(\beta)$*, there exist three distinct, translation-invariant Gibbs states* $\langle - \rangle_{\beta,\lambda}^{\sigma}$*, with* $\sigma \in \{+1, 0, -1\}$*, the typical configuration of which contains fraction at least* $1 - C\, e^{-\beta}$ *of the corresponding spin state.*

We remark that the phase transition happens at a value of λ which (at least for $\beta \gg 1$) is strictly positive. This demonstrates the phenomenon of entropic suppression (of ± 1 ground states at $\lambda = 0$) established previously in Ref. [123] by the contour-expansion techniques. The entropic nature of the above transition is also manifested by the fact that the free-energy "gap" separating the distinct states *decreases* as $\beta \to \infty$. This is the reason why, to maintain uniform level of control, we need $\mathscr{I}$ to be smaller for smaller temperatures. Theorem 2.3.7 is proved in Sect. 2.6.

2.3.5 Discussion

We close this section with a discussion of some conjectures that can be addressed via the above theorems.

Starting with the intriguing results in Ref. [85] and culminating in Refs. [86, 87], A. Kerimov formulated the following conjecture (we quote verbatim from the latter pair of references): "Any one-dimensional model with discrete (at most countable) spin space and with a unique ground state has a unique Gibbs state if the spin space of this model is finite or the potential of this model is translationally invariant." The conclusions of Theorem 2.3.4 manifestly demonstrate that this conjecture fails for the 1D Potts model in external field. Indeed, for $q \ge 3$, $h > 0$ and interactions decaying like $1/r^s$ with $s \in (1, 2)$ which are RP and satisfy the condition that the integral in Eq. (2.3.9) is sufficiently small, the Potts model has phase coexistence at some positive temperature. However, it is clear that this system enjoys a unique ground state.

In a recent paper [10], N. Berger considered random-cluster models with parameter q and interactions between sites x and y decaying as $|x-y|^{-s}$, where $d < s < 2d$. He proved, among other results, that at the percolation threshold there is no infinite cluster in the measure generated by the free boundary conditions. For ordinary percolation (i.e., $q = 1$), this implies continuity of the infinite cluster density. As to the wired boundary conditions, for $q = 2$—i.e., the Ising model—the classic results of Refs. [5, 3] show that the magnetization vanishes continuously once the model is in the "mean-field regime" $s \in (1, 3/2)$. However, for general random-cluster models with $q > 1$ and wired boundary conditions, the situation remained open.

While we cannot quite resolve the situation *at* the percolation threshold, our results prove that, for sufficiently spread out random-cluster models with RP couplings, there is a point where the free and wired densities are indeed different. To resolve the full conjecture from Ref. [10], one would need to establish that the only place such a discontinuity can occur is at the percolation threshold.

Our third application concerns the problem of partition function zeros of the Potts model in a *complex* external field with $\mathfrak{Re}\, h < 0$. Here there have been numerical results [89] claiming that no such zeros occur for the nearest-neighbor 2D Potts model with $q \leq 7$. On the basis of the classic Lee-Yang theory [131, 95], absence of such zeros would imply analyticity of the spin-1 density. The results of Refs. [14, 15, 17, 18, 19] rule this out for q very large and Theorem 2.3.5(2) also makes this impossible for reasonable values of q and sufficiently spread-out interactions (of course, for $d = 1, 2$ this requires a power-law interaction).

2.4 Proofs: General theory

The goal of this section is to prove Theorems 2.3.2 and 2.3.3. In Sect. 2.4.1 we present some general convexity results that provide the framework for the derivation of our results. However, the driving force of our proofs are the classic tools of reflection positivity and infrared bounds which are reviewed (and further developed) in Sects. 2.4.2 and 2.4.3. The principal results of this section are Theorem 2.4.1 and Lemmas 2.4.2, 2.4.8 and 2.4.9.

2.4.1 Convexity bounds

We begin with an intermediate step to Theorem 2.3.2 which gives an estimate on how far above the mean-field free energy evaluated at a *physical* magnetization is from the absolute minimum.

Theorem 2.4.1. *Suppose $(J_{x,y})$ are translation and rotation invariant couplings on $\mathbb{Z}^d$ such that Eq. (2.2.1) holds. Let $\nu_{\beta,\boldsymbol{h}}$ be a translation and rotation-invariant, infinite volume Gibbs measure corresponding to $\beta \geq 0$ and $\boldsymbol{h} \in \mathbb{R}^n$. Let $\langle - \rangle_{\beta,\boldsymbol{h}}$ denote the expectation with respect to $\nu_{\beta,\boldsymbol{h}}$ and let $\boldsymbol{m}_\star = \langle \boldsymbol{S}_0 \rangle_{\beta,\boldsymbol{h}}$. Then*

$$\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}_\star) \leq \inf_{\boldsymbol{m} \in \, Conv(\Omega)} \Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}) + \frac{\beta}{2}\bigl\{\bigl\langle(\boldsymbol{S}_0, \boldsymbol{m}_0)\bigr\rangle_{\beta,\boldsymbol{h}} - |\boldsymbol{m}_\star|^2\bigr\}, \tag{2.4.1}$$

where $\boldsymbol{m}_0 = \sum_{x \in \mathbb{Z}^d} J_{0,x} \boldsymbol{S}_x$.

Proof. The proof is very similar to that of Theorem 1.1 of Ref. [21]. Let Λ be a box of $L \times \cdots \times L$ sites in $\mathbb{Z}^d$ and let $\boldsymbol{M}_\Lambda$ be the total spin in Λ, i.e., $\boldsymbol{M}_\Lambda = \sum_{x \in \Lambda} \boldsymbol{S}_x$. Let us also recall the meaning of the mean-field quantities from (2.2.6–2.2.8). The starting point of our derivations is the formula

$$\mathrm{e}^{|\Lambda| G(\boldsymbol{b})} = \bigl\langle \ \mathrm{e}^{(\boldsymbol{b}, \boldsymbol{M}_\Lambda) + \beta \mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda | \boldsymbol{S}_{\Lambda^{\mathrm{c}}})} Z_\Lambda(\boldsymbol{S}_{\Lambda^{\mathrm{c}}}) \bigr\rangle_{\beta,\boldsymbol{h}}, \qquad \boldsymbol{b} \in \mathbb{R}^n, \tag{2.4.2}$$

which is obtained by invoking the DLR conditions for the Gibbs state $\nu_{\beta,\boldsymbol{h}}$. Here $\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda|\boldsymbol{S}_{\Lambda^{\rm c}})$ is as in Eq. (2.3.1) and $Z_\Lambda(\boldsymbol{S}_{\Lambda^{\rm c}})$ is a shorthand for the partition function in Λ given $\boldsymbol{S}_{\Lambda^{\rm c}}$.

The goal is to derive a lower bound on the right-hand side of Eq. (2.4.2). First we provide a lower bound on $Z_\Lambda(\boldsymbol{S}_{\Lambda^{\rm c}})$ which is independent of boundary conditions. To this end, let $\langle - \rangle_{0,\boldsymbol{b}}$ denote expectation with respect to the product measure

$$\mathrm{e}^{(\boldsymbol{b},\boldsymbol{M}_\Lambda)-|\Lambda|G(\boldsymbol{b})}\prod_{x\in\Lambda}\mu_0(\,\mathrm{d}\boldsymbol{S}_x) \tag{2.4.3}$$

and let $\boldsymbol{m}_{\boldsymbol{b}}$ denote the expectation of any spin in Λ with respect to this measure. Jensen's inequality then gives us

$$\begin{aligned} Z_\Lambda(\boldsymbol{S}_{\Lambda^{\rm c}}) &= e^{|\Lambda|G(\boldsymbol{b})}\big\langle\, \mathrm{e}^{-(\boldsymbol{b},\boldsymbol{M}_\Lambda)-\beta\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda|\boldsymbol{S}_{\Lambda^{\rm c}})}\big\rangle_{0,\boldsymbol{b}} \\ &\ge\ \mathrm{e}^{|\Lambda|[G(\boldsymbol{b})-(\boldsymbol{b},\boldsymbol{m}_{\boldsymbol{b}})]}\ \mathrm{e}^{-\langle\beta\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda|\boldsymbol{S}_{\Lambda^{\rm c}})\rangle_{0,\boldsymbol{b}}}. \end{aligned} \tag{2.4.4}$$

Now, (2.2.6–2.2.7) imply that $G(\boldsymbol{b})-(\boldsymbol{b},\boldsymbol{m}_{\boldsymbol{b}})=S(\boldsymbol{m}_{\boldsymbol{b}})$, while the absolute summability of $x\mapsto J_{0,x}$ implies that for all $\epsilon>0$ there is a $C_1<\infty$, depending on ϵ, the $J_{x,y}$'s and the diameter of Ω, so that

$$-\big\langle\,\beta\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda|\boldsymbol{S}_{\Lambda^{\rm c}})\big\rangle_{0,\boldsymbol{b}}\ge|\Lambda|E(\boldsymbol{m}_{\boldsymbol{b}})-\beta\epsilon|\Lambda|-\beta C_1|\partial\Lambda|, \tag{2.4.5}$$

with $E(\boldsymbol{m}_{\boldsymbol{b}})$ denoting the mean-field energy function from Sect. 2.2.1. (Note that we used also the normalization condition (2.2.1).) Invoking Eq. (2.2.8) and optimizing over all $\boldsymbol{b}\in\mathbb{R}^n$, we thus get

$$Z_\Lambda(\boldsymbol{S}_{\Lambda^{\rm c}})\ge\ \mathrm{e}^{-|\Lambda|F_{\rm MF}(\beta,\boldsymbol{h})-\beta\epsilon|\Lambda|-\beta C_1|\partial\Lambda|}, \tag{2.4.6}$$

where $F_{\rm MF}(\beta,\boldsymbol{h})$ is the absolute minimum of $\Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m})$ over all $\boldsymbol{m}\in$ Conv (Ω).

Having established the desired lower bound on the partition function, we now plug the result into Eq. (2.4.2) to get

$$\mathrm{e}^{|\Lambda|G(\boldsymbol{b})}\ge\big\langle\,\mathrm{e}^{(\boldsymbol{b},\boldsymbol{M}_\Lambda)+\beta\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda|\boldsymbol{S}_{\Lambda^{\rm c}})}\big\rangle_{\beta,\boldsymbol{h}}\ \mathrm{e}^{-|\Lambda|F_{\rm MF}(\beta,\boldsymbol{h})-\beta\epsilon|\Lambda|-\beta C_1|\partial\Lambda|}. \tag{2.4.7}$$

The expectation can again be moved to the exponent using Jensen's inequality, now taken with respect to measure $\nu_{\beta,\boldsymbol{h}}$. Invoking the translation and rotation invariance of this Gibbs state, bounds similar to Eq. (2.4.5) imply

$$\begin{aligned}\big\langle \beta\mathscr{H}_\Lambda(\boldsymbol{S}_\Lambda|\boldsymbol{S}_{\Lambda^{\rm c}})\big\rangle_{\beta,\boldsymbol{h}} \\ \ge -|\Lambda|\biggl(\sum_{x\in\mathbb{Z}^d}\frac{\beta}{2}J_{0,x}\big\langle(\boldsymbol{S}_x,\boldsymbol{S}_0)\big\rangle_{\beta,\boldsymbol{h}}+(\boldsymbol{h},\boldsymbol{m}_\star)-\epsilon\biggr)-C_2|\partial\Lambda|.\end{aligned} \tag{2.4.8}$$

Plugging this back into Eq. (2.4.7), taking logarithms, dividing by $|\Lambda|$ and letting $|\Lambda|\to\infty$ (with $|\partial\Lambda|/|\Lambda|\to 0$) followed by $\epsilon\downarrow 0$, we arrive at the bound

$$G(\boldsymbol{b})-(\boldsymbol{b},\boldsymbol{m}_\star)\ge -\frac{\beta}{2}\sum_{x\in\mathbb{Z}^d}J_{0,x}\big\langle(\boldsymbol{S}_x,\boldsymbol{S}_0)\big\rangle_{\beta,\boldsymbol{h}}-(\boldsymbol{h},\boldsymbol{m}_\star)-F_{\rm MF}(\beta,\boldsymbol{h}). \tag{2.4.9}$$

Optimizing over $\boldsymbol{b}$ gives

$$S(\boldsymbol{m}_\star)-(\boldsymbol{h},\boldsymbol{m}_\star)\le\frac{\beta}{2}\sum_{x\in\mathbb{Z}^d}J_{0,x}\big\langle(\boldsymbol{S}_x,\boldsymbol{S}_0)\big\rangle_{\beta,\boldsymbol{h}}+F_{\rm MF}(\beta,\boldsymbol{h}) \tag{2.4.10}$$

from which Eq. (2.4.1) follows by subtracting $\frac{\beta}{2}|\boldsymbol{m}_\star|^2$ on both sides. □

Similar convexity estimates allow us to establish also the following bounds between the energy density and fluctuations of the weighted magnetization $\boldsymbol{m}_0$:

Lemma 2.4.2. *Let* $\kappa=\sup_{\boldsymbol{S}\in\Omega}(\boldsymbol{S},\boldsymbol{S})$ *and let* $(J_{x,y})$ *be a collection of couplings satisfying Eq.* (2.2.1). *For each* $\beta>0$ *and* $\boldsymbol{h}\in\mathbb{R}^n$ *there exists a number* $\varkappa=\varkappa(\beta,\boldsymbol{h})$ *such that for any translation and rotation invariant Gibbs state* $\langle-\rangle_{\beta,\boldsymbol{h}}$ *we have*

$$\beta\varkappa\big\langle\,|\boldsymbol{m}_0-\boldsymbol{m}_\star|^2\big\rangle_{\beta,\boldsymbol{h}}\le\big\langle(\boldsymbol{S}_0,\boldsymbol{m}_0)\big\rangle_{\beta,\boldsymbol{h}}-|\boldsymbol{m}_\star|^2\le\beta\kappa\big\langle\,|\boldsymbol{m}_0-\boldsymbol{m}_\star|^2\big\rangle_{\beta,\boldsymbol{h}}, \tag{2.4.11}$$

where $\boldsymbol{m}_0=\sum_{x\in\mathbb{Z}^d}J_{0,x}$ *and* $\boldsymbol{m}_\star=\langle\boldsymbol{S}_0\rangle_{\beta,\boldsymbol{h}}$.

Proof. We begin with a rewrite of the correlation function in the middle of Eq. (2.4.11). First, using the DLR equations to condition on the spins in the complement of the origin, we have

$$\big\langle(\boldsymbol{m}_0,\boldsymbol{S}_0)\big\rangle_{\beta,\boldsymbol{h}}=\big\langle(\boldsymbol{m}_0,\nabla G(\beta\boldsymbol{m}_0+\boldsymbol{h})\big\rangle_{\beta,\boldsymbol{h}}. \tag{2.4.12}$$

Next, our hypotheses imply that $\boldsymbol{m}_\star = \langle \boldsymbol{m}_0 \rangle_{\beta,\boldsymbol{h}} = \langle \nabla G(\beta \boldsymbol{m}_0 + \boldsymbol{h}) \rangle_{\beta,\boldsymbol{h}}$, and so

$$\bigl\langle (\boldsymbol{m}_0, \nabla G(\beta \boldsymbol{m}_0 + \boldsymbol{h}) \bigr\rangle_{\beta,\boldsymbol{h}} - |\boldsymbol{m}_\star|^2 \\ = \bigl\langle (\boldsymbol{m}_0 - \boldsymbol{m}_\star, \nabla G(\beta \boldsymbol{m}_0 + \boldsymbol{h}) - \nabla G(\beta \boldsymbol{m}_\star + \boldsymbol{h})) \bigr\rangle_{\beta,\boldsymbol{h}}. \tag{2.4.13}$$

For the rest of this proof, let $\varXi$ abbreviate the inner product in the expectation on the right-hand side.

We will express $\varXi$ using the mean value theorem

$$\varXi = \bigl(\boldsymbol{m}_0 - \boldsymbol{m}_\star, [\nabla\nabla G(\boldsymbol{b})](\boldsymbol{m}_0 - \boldsymbol{m}_\star)\bigr), \tag{2.4.14}$$

where $\boldsymbol{b}$ is a point somewhere on the line between $\beta \boldsymbol{m}_0 + \boldsymbol{h}$ and $\beta \boldsymbol{m}_\star + \boldsymbol{h}$. The double gradient $\nabla\nabla G(\boldsymbol{b})$ is a matrix with components $(\nabla\nabla G(\boldsymbol{b}))_{i,j} = \langle S_0^{(i)} S_0^{(j)} \rangle_{0,\boldsymbol{b}} - \langle S_0^{(i)} \rangle_{0,\boldsymbol{b}} \langle S_0^{(j)} \rangle_{0,\boldsymbol{b}}$. As was shown in Ref. [21], the ℓ^2-operator norm of $\nabla\nabla G(\boldsymbol{b})$ is bounded by $\kappa = \sup_{\boldsymbol{S}\in\Omega}(\boldsymbol{S}, \boldsymbol{S})$ and so we have

$$\varXi \le \beta\kappa\, |\boldsymbol{m}_0 - \boldsymbol{m}_\star|^2. \tag{2.4.15}$$

Taking expectations on both sides, and invoking Eqs. (2.4.12–2.4.13), this proves the upper bound in Eq. (2.4.11).

To get the lower bound we note that, μ_0 almost surely, the double gradient $\nabla\nabla G(\boldsymbol{b})$ is positive definite on the linear subspace generated by vectors from Ω. (We are using that Ω *is* the support of the *a priori* measure μ_0.) Since $\beta \boldsymbol{m}_0 + \boldsymbol{h}$ takes values in a compact subset of this subspace, we have

$$\varXi \ge \beta\varkappa\, |\boldsymbol{m}_0 - \boldsymbol{m}_\star|^2 \tag{2.4.16}$$

for some (existential) constant $\varkappa > 0$. Taking expectations, the left inequality in (2.4.11) follows. □

We emphasize that in its present form, the bounds (2.4.1) and (2.4.11) are essentially of complete generality. Underlying most of the derivations in this paper is the observation that the variance term on the right-hand side of Eq. (2.4.11) is sufficiently small. Via Eq. (2.4.1), the physical magnetization $\boldsymbol{m}_\star$ is then forced to be near one of the near minima of the mean-field free energy. This reduces the problem of proving discontinuous phase transitions to:

(1) controlling the variance term in Eq. (2.4.11),

(2) a detailed analysis of the minimizers of $\Phi_{\beta,\boldsymbol{h}}$.

For (1), we will use the method of reflection positivity/infrared bounds discussed in the following subsections. As mentioned before, this does impose some restrictions on our interactions and our Gibbs states. Part (2) is model specific and, for the Potts and Blume-Capel models, is the subject of Sect. 2.5.

2.4.2 Reflection positivity

Our use of reflection positivity (RP) will require that we temporarily restrict our model to the torus $\mathbb{T}_L$ of $L \times \cdots \times L$ sites. In order to define the interaction potential on this torus, we recall that the $J_{x,y}$'s are translation invariant and define their "periodized" version by

$$J_{x,y}^{(L)} = \sum_{z \in \mathbb{Z}^d} J_{x,y+Lz}, \tag{2.4.17}$$

where Lz is the site whose coordinates are L-multiples of those of z. The torus version of the Hamiltonian (2.1.1) is then defined by

$$\beta\mathscr{H}_L(\boldsymbol{S}) = -\sum_{\substack{\langle x,y\rangle \\ x,y\in\mathbb{T}_L}} \beta J_{x,y}^{(L)}(\boldsymbol{S}_x,\boldsymbol{S}_y) - \sum_{x\in\mathbb{T}_L}(\boldsymbol{S}_x,\boldsymbol{h}). \tag{2.4.18}$$

(Here, as in Eq. (2.1.1), the first sum is over all unordered pairs of sites.) Let $\mathbb{P}_L$ denote the Gibbs measure on $\Omega^{\mathbb{T}_L}$ whose Radon-Nikodym derivative with respect to the *a priori* spin distribution $\mu_0(\,\mathrm{d}\boldsymbol{S})$ is the properly normalized $\mathrm{e}^{-\beta\mathscr{H}_L(\boldsymbol{S})}$.

Let us suppose that L is even and let us temporarily regard $\mathbb{T}_L$ as a periodized box $\{1,\dots,L\}^d$. Let $\mathbb{T}_L^+$ be those sites whose i-th coordinate ranges between 1 and $L/2$ and let $\mathbb{T}_L^-$ be the remaining sites. The two parts of the torus are related to each other by a reflection in the "hyperplane" P that separates the two halves from each other. (The geometrical image of the plane has two components.) Given such a plane P, we let $\mathscr{F}_P^+$ denote the σ-algebra of events that depend on the configuration in $\mathbb{T}_L^+$, and similarly for $\mathscr{F}_P^-$ and $\mathbb{T}_L^-$.

Let ϑ_P denote the reflection taking $\mathbb{T}_L^+$ onto $\mathbb{T}_L^-$ and *vice versa* (cf. the definition of $\vartheta^{(k)}$ in Sect. 2.3.1). In the natural way, ϑ_P induces an operator $\vartheta_P^\star$ on the set of real-valued functions on $(\Omega^{\mathbb{T}_L})$. Then we have:

Definition 2.4.3 (RP on torus)**.** We say that $\mathbb{P}_L$ is reflection positive if for every plane P as described above and any two bounded, $\mathscr{F}_P^+$-measurable random variables X and Y,

$$\mathbb{E}_L\bigl(X\vartheta_P^\star(Y)\bigr)=\mathbb{E}_L\bigl(Y\vartheta_P^\star(X)\bigr) \tag{2.4.19}$$

and

$$\mathbb{E}_L\bigl(X\vartheta_P^\star(X)\bigr)\ge 0. \tag{2.4.20}$$

Here $\mathbb{E}_L$ is the expectation with respect to $\mathbb{P}_L$.

Condition (2.4.20) in the above definition is often too complicated to be verified directly. Instead we verify a convenient sufficient condition which we will state next:

Lemma 2.4.4. *Consider a collection of coupling constants $(J_{x,y})_{x,y\in\mathbb{Z}^d}$ satisfying the properties of Definition 2.3.1 in Sect. 2.3.1. Then the measure $\mathbb{P}_L$, defined on $\mathbb{T}_L$ using the periodized coupling constants from Eq.* (2.4.17)*, is reflection positive in the sense of Definition 2.4.3.*

Proof. This is a multidimensional version of Proposition 3.4 of [62]. □

Remark 2.4.5. We note that the three classes of interactions listed in Sect. 2.1.2 are reflection positive. For the most part, interactions of this sort were discussed in Ref. [62]; however, for reader's convenience, we provide the relevant calculations below.

(1) *Nearest-neighbor/next-nearest neighbor couplings*: Consider a function $f\colon \mathbb{H}_1 \to \mathbb{C}$ which is nonzero only on the sites of $\mathbb{H}_1$ that are adjacent to $\mathbb{Z}^d \setminus \mathbb{H}_1$. (By inspection of Eq. (2.3.6), for nearest and next-nearest neighbor interactions, this is the most general function that need to be considered.) Pick $\eta \in \mathbb{R}$ and consider the function

$$g_j(x) = f(x) + \eta f(x + \hat{\mathrm{e}}_j), \qquad j = 2, \dots, d, \tag{2.4.21}$$

and define a collection of coupling constants $(J_{x,y})$ by the formula

$$\sum_{\substack{x\in\mathbb{H}_1\\ y\in\mathbb{Z}^d\smallsetminus\mathbb{H}_1}} J_{x,y}\,\overline{f(x)}f(\vartheta^{(1)}y) = \sum_{j=2,\dots,d}\,\sum_{x\in\mathbb{H}_1}\overline{g_j(x)}g_j(x) \tag{2.4.22}$$

Now the right-hand side is clearly positive and so the $J_{x,y}$'s satisfy the condition in Eq. (2.3.6).

It remains to identify the explicit form of these coupling constants. Let $x \in \mathbb{H}_1$ be a boundary site and let $x' = \vartheta^{(1)}x$ be its nearest neighbor in $\mathbb{Z}^d \setminus \mathbb{H}_1$. First we note that, for each x and j, there is an interaction of "strength" η between x and its next-nearest neighbor $x' + \hat{\mathrm{e}}_j$ and a similar interaction between x and the site $x' - \hat{\mathrm{e}}_j$. So, the next-nearest neighbors have coupling strength η. As to the nearest-neighbor terms, for a fixed x and fixed j, there is the direct interaction with x' of strength 1 and there is a term of strength η^2. Thus, upon summing, the nearest-neighbor interaction has total strength $(d-1)(1+\eta^2)$.

Since the overall strength of the interaction is irrelevant, the ratio of the strength of the next-nearest neighbor to the nearest-neighbor couplings has to be a number of the form $\frac{1}{d-1}\frac{\eta}{1+\eta^2}$ which, in particular, permits any ratio whose absolute value is bounded by $\frac{1}{2(d-1)}$.

(2) *Yukawa potentials*: Reflection positivity for the Yukawa potentials can be shown by applying the criterion from Lemma 2.4.4: Fix $\mu > 0$ and let $J_{x,y} = \mathrm{e}^{-\mu\|x-y\|_1}$. Then for any observable $f\colon \mathbb{H}_1 \to \mathbb{R}$,

$$\sum_{\substack{x\in\mathbb{H}_1\\ y\in\mathbb{Z}^d\smallsetminus\mathbb{H}_1}} J_{x,y} f(x) f(\vartheta^{(1)} y)$$

$$= \sum_{\substack{x_2,\dots,x_d\in\mathbb{Z}\\ y_2,\dots,y_d\in\mathbb{Z}}} K(x,y)\biggl(\sum_{x_1>0} \mathrm{e}^{-\mu x_1} f(x)\biggr)\biggl(\sum_{y_1>0} \mathrm{e}^{-\mu y_1} f(y)\biggr), \qquad (2.4.23)$$

where the operator kernel $K\colon \mathbb{Z}^{d-1} \to \mathbb{Z}^{d-1}$ is defined by $K(x,y) = \exp\{-\mu \sum_{j=2}^{d} |x_j - y_j|\}$. This operator is symmetric and diagonal in the Fourier basis; a direct calculation shows that K has only positive eigenvalues. This means that the right-hand side is non-negative, proving condition (3) of Definition 2.3.1. (The other conditions are readily checked as well.)

(3) *Power-laws*: We begin by noting that all conditions on $J_{x,y}$ in Definition 2.3.1 are linear in $J_{x,y}$. Therefore, any linear combination of reflection positive $J_{x,y}$'s with non-negative coefficients is also reflection positive. In particular, if we integrate a one parameter family of interactions against a positive measure, the result must also be RP. Now if we let

$$J_{x,y} = \int_0^\infty \mu^{s-1}\,\mathrm{e}^{-\mu|x-y|_1}\,\mathrm{d}\mu \qquad \text{for } s > 0, \qquad (2.4.24)$$

then $J_{x,y} = C(s)|x-y|_1^{-s}$ and so the power laws are RP as well.

We observe that in the classics, particularly, Refs. [62, 63], the above types of interactions are treated and the RP properties established with all distances expressed in ℓ_2-norms. The derivations therein all rely, to some extent, on latticization of the field-theoretic counterparts to reflection positivity which were, perhaps, better known in their heyday. Our ℓ_1 derivations, while being a more pedestrian method of extension from $d = 1$, have the advantage that they are self-contained.

2.4.3 Infrared bounds

Our principal reason for introducing reflection positivity is to establish an upper bound on the two point correlation term in Theorem 2.4.1. This will be achieved by invoking the connection between reflection positivity and infrared bounds. For spin systems this connection goes back to Ref. [65] where infrared bounds were used to provide proofs of phase coexistence in certain continuous-spin models at low temperature. Here we will follow the strategy of Ref. [21], and so we will keep our discussion brief.

In order to apply infrared bounds to the problem at hand we must first restrict consideration to those Gibbs states with the following two properties:

Property 1 (Torus state)**.** *An infinite volume Gibbs measure $\nu_{\beta,\boldsymbol{h}}$ is called a* torus state *if it can be obtained as a weak limit of finite-volume states with periodic boundary conditions. (The torus states need not correspond exactly to the values β and $\boldsymbol{h}$.)*

Property 2 (Block averages)**.** *An infinite volume Gibbs measure $\nu_{\beta,\boldsymbol{h}}$ is said to have* block average magnetization $\boldsymbol{m}_\star$ *if*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\boldsymbol{S}_x=\boldsymbol{m}_\star,\qquad \nu_{\beta,\boldsymbol{h}}\text{-almost surely.}\tag{2.4.25}$$

Similarly, the measure is said to have block average energy density $e_\star$ *if*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\sum_{\substack{\langle x,y\rangle\\ x,y\in\Lambda}}J_{x,y}\left(\boldsymbol{S}_x,\boldsymbol{S}_y\right)=e_\star,\qquad \nu_{\beta,\boldsymbol{h}}\text{-almost surely.}\tag{2.4.26}$$

Here in Eqs. (2.4.25–2.4.26) the limits are along increasing sequences of square boxes centered at the origin.

It is conceivable that not every (extremal) Gibbs state will obey these restrictions, so the reader might wonder how we are going to detect the desired phase transitions. We will use an approximation argument which goes back to Ref. [21]. Recall the definition of the set $\mathscr{M}_\star(\beta,\boldsymbol{h})$ of "extremal magnetizations" from the paragraph before Theorem 2.3.2. Then we have:

Lemma 2.4.6. *For all $\beta>0$, $\boldsymbol{h}\in\mathbb{R}^n$ and all $\boldsymbol{m}_\star\in\mathscr{M}_\star(\beta,\boldsymbol{h})$, there exists an infinite volume Gibbs state $\nu_{\beta,\boldsymbol{h}}$ for interaction* (2.1.1) *which obeys Properties 1 and 2.*

Proof. This is, more or less, Corollary 3.4 from Ref. [21] enhanced to include the block average energy density. □

Our next goal is to show that the right-hand side of Eq. (2.4.1) can be controlled for any Gibbs state satisfying Properties 1 and 2. To this end let $D^{-1}(x,y)$ denote the inverse of the (weighted) Dirichlet lattice Laplacian defined using the $J_{x,y}$'s. Explicitly, we have

$$D^{-1}(x,y)=\int_{[-\pi,\pi]^d}\frac{\mathrm{d}k}{(2\pi)^d}\frac{\mathrm{e}^{\,\mathrm{i}k\cdot(x-y)}}{1-\hat{J}(k)},\tag{2.4.27}$$

where $\hat{J}(k)=\sum_{x\in\mathbb{Z}^d}J_{0,x}\,\mathrm{e}^{\,\mathrm{i}k\cdot x}$. We will always work under the conditions for which the integral is convergent. Our principal estimate is now as follows:

Lemma 2.4.7 (Infrared bound)**.** *Assume that $k\mapsto(1-\hat{J}(k))^{-1}$ is Riemann integrable. Fix $\beta>0$, $\boldsymbol{h}\in\mathbb{R}^n$ and let $\nu_{\beta,\boldsymbol{h}}$ be an infinite-volume Gibbs measure for interaction* (2.1.1) *that satisfies Properties 1 and 2. Let $\langle-\rangle_{\beta,\boldsymbol{h}}$ denote the expectation with respect to $\nu_{\beta,\boldsymbol{h}}$ and let n be the dimension of the underlying spin space. Then the bound*

$$\sum_{x,y\in\mathbb{Z}^d}v_x\bar{v}_y\bigl\langle(\boldsymbol{S}_x-\boldsymbol{m}_\star,\boldsymbol{S}_y-\boldsymbol{m}_\star)\bigr\rangle_{\beta,\boldsymbol{h}}\le\frac{n}{\beta}\sum_{x,y\in\mathbb{Z}^d}v_x\bar{v}_yD^{-1}(x,y)\tag{2.4.28}$$

holds for all $v\colon\mathbb{Z}^d\mapsto\mathbb{C}$ such that $\sum_{x\in\mathbb{Z}^d}|v_x|<\infty$.

Proof. As this lemma and its proof are similar to Lemma 3.2 of Ref. [21] we will stay very brief. Let $J_{x,y}^{(L)}$ denote the periodized interactions corresponding to the torus $\mathbb{T}_L$ and let

$$\mathbb{T}_L^\star=\Bigl\{\Bigl(\frac{2\pi}{L}n_1,\dots,\frac{2\pi}{L}n_d\Bigr)\colon 1\le n_i\le L\Bigr\}\tag{2.4.29}$$

be the reciprocal torus. It is easy to see that the k-th Fourier component $\hat{J}^{(L)}(k)$ of the $J^{(L)}_{x,y}$'s satisfies $\hat{J}^{(L)}(k) = \hat{J}(k)$ for all $k \in \mathbb{T}^\star_L$. This means that the inverse Dirichlet Laplacian on $\mathbb{T}_L$ can be written in terms of the original coupling constants, i.e.,

$$D_L^{-1}(x,y) = \frac{1}{|\mathbb{T}^\star_L|} \sum_{k\in\mathbb{T}^\star_L \smallsetminus \{0\}} \frac{\mathrm{e}^{\,\mathrm{i}k\cdot(x-y)}}{1-\hat{J}(k)}. \tag{2.4.30}$$

The infrared bound of Ref. [62] then says that, for any Gibbs state $\langle - \rangle^{(L)}_{\beta,\boldsymbol{h}}$ on $\mathbb{T}_L$ we have

$$\sum_{x,y\in\mathbb{Z}^d} \big\langle (\boldsymbol{w}_x,\boldsymbol{S}_x)(\bar{\boldsymbol{w}}_y,\boldsymbol{S}_y)\big\rangle^{(L)}_{\beta,\boldsymbol{h}} \le \frac{1}{\beta} \sum_{x,y\in\mathbb{Z}^d} (\boldsymbol{w}_x,\bar{\boldsymbol{w}}_y) D_L^{-1}(x,y) \tag{2.4.31}$$

for any absolutely summable collection of complex vectors $(\boldsymbol{w}_x)_{x\in\mathbb{T}^L}$ with $\mathfrak{Re}\,\boldsymbol{w}_x, \mathfrak{Im}\,\boldsymbol{w}_x \in \mathbb{R}^n$ and $\sum_{x\in\mathbb{T}^L} \boldsymbol{w}_x = 0$.

Now let us consider a torus state $\nu_{\beta,\boldsymbol{h}}$ with almost-surely constant block magnetization. We will first prove that $\nu_{\beta,\boldsymbol{h}}$ satisfies the $L\to\infty$ version of Eq. (2.4.31). By the assumption on the Riemann integrability of $\frac{1}{1-\hat{J}(k)}$,

$$D_L^{-1}(x,y) \underset{L\longrightarrow\infty}{\longrightarrow} D^{-1}(x,y), \tag{2.4.32}$$

independently of x,y. Letting all $\boldsymbol{w}_x$ be parallel, i.e., $\boldsymbol{w}_x = w_x\,\hat{\mathrm{e}}$, where $\hat{\mathrm{e}}$ is a unit vector in $\mathbb{R}^n$, and passing to the limit $L\to\infty$, we thus get

$$\sum_{x,y\in\mathbb{Z}^d} w_x\bar{w}_y \big\langle (\boldsymbol{S}_x,\boldsymbol{S}_y)\big\rangle_{\beta,\boldsymbol{h}} \le \frac{n}{\beta} \sum_{x,y\in\mathbb{Z}^d} w_x\bar{w}_y D^{-1}(x,y) \tag{2.4.33}$$

whenever $w\colon \mathbb{Z}^d \to \mathbb{C}$ is absolutely summable and $\sum_{x\in\mathbb{Z}^d} w_x = 0$.

In order to make the $\boldsymbol{m}_\star$'s appear explicitly on the left-hand side, we need to relax the condition on the total sum of the w_x's. Under the condition in Property 2, this is done exactly as in Lemma 3.2 of Ref. [21]. $\square$

2.4.4 Actual proofs

A key consequence of the infrared bound is the following estimate on the variance of the quantity $\boldsymbol{m}_0 = \sum_{x\in\mathbb{Z}^d} J_{0,x}\boldsymbol{S}_x$:

Lemma 2.4.8 (Variance bound)**.** *Consider a collection* $(J_{x,y})$ *of coupling constants that are RP and obey Eq.* (2.2.1)*, and let* $\mathscr{I}$ *be the integral in Eq.* (2.3.9)*. Let* $\langle - \rangle_{\beta,\boldsymbol{h}}$ *be a translation and rotation invariant Gibbs state satisfying Properties 1 and 2 and let* $\boldsymbol{m}_\star = \langle \boldsymbol{S}_0 \rangle_{\beta,\boldsymbol{h}}$*. Then*

$$\beta \left\langle\, |\boldsymbol{m}_0 - \boldsymbol{m}_\star|^2 \right\rangle_{\beta,\boldsymbol{h}} \le n\mathscr{I}. \tag{2.4.34}$$

Proof. We have to show how the bound (2.4.28) is used to estimate the variance of $\boldsymbol{m}_0$. Let (v_x) be defined by $v_x = J_{0,x}$. Using Lemma 2.4.7 and Lemma 2.4.6, for any $\langle - \rangle_{\beta,\boldsymbol{h}}$ as above, this choice of the v_x's leads to the variance of $\boldsymbol{m}_0$ on the left-hand side of Eq. (2.4.28), while on the right-hand side the sum turns into the integral $\mathscr{I}$. □

The proof of Theorem 2.3.2 is now reduced to two lines:

Proof of Theorem 2.3.2. Combining Lemmas 2.4.6 and 2.4.8 with Eqs. (2.4.11) and (2.4.1), we obtain Eqs. (2.3.8–2.3.9). □

Armed with the conclusions of Theorem 2.3.2, we can now finish also the proof of Theorem 2.3.3:

Proof of Theorem 2.3.3. In light of the previous derivations, the claims in Theorem 2.3.3 are hardly surprising. The difficulty to be overcome is the fact that the limits in Eqs. (2.3.11–2.3.12) are claimed for sequences of *any* states, regardless of whether they obey Properties 1 and 2 above.

We begin with the proof of part (1); namely, Eq. (2.3.11). Since $\boldsymbol{m}$ is the unique minimizer of $\Phi_{\beta,\boldsymbol{h}}$, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\big\{\boldsymbol{m}' \in \text{ Conv}(\Omega)\colon \Phi_{\beta,\boldsymbol{h}}(\boldsymbol{m}') < F_{\text{MF}}(\beta,\boldsymbol{h}) + \delta\big\} \tag{2.4.35}$$

is contained in a ball $\mathcal{U}_\epsilon(\boldsymbol{m})$ of radius ϵ centered at $\boldsymbol{m}$. By Eq. (2.3.8), once $\beta n\frac{\kappa}{2}\mathscr{I} \le \delta$, all of $\mathscr{M}_\star(\beta,\boldsymbol{h})$ must be contained in this ball. But, $\mathscr{M}_\star(\beta,\boldsymbol{h})$ is the set of extremal magnetizations, and any magnetization $\boldsymbol{m}'$ that can be achieved in a translation-invariant state is thus in the convex hull of $\mathscr{M}_\star(\beta,\boldsymbol{h})$. It follows that $\boldsymbol{m}' \in \mathcal{U}_\epsilon(\boldsymbol{m})$, proving Eq. (2.3.11).

To prove Eq. (2.3.12), let $[e_\star,\boldsymbol{m}_\star]$ be an extremal pair in $\mathscr{K}_\star(\beta,\boldsymbol{h})$. (See the discussion prior to Theorem 2.3.2 for the definition of these objects.) Let $\langle-\rangle_{\beta,\boldsymbol{h}}$ be a translation and rotation invariant state for which

$$e_\star = \big\langle(\boldsymbol{S}_0, \tfrac{\beta}{2}\boldsymbol{m}_0 + \boldsymbol{h})\big\rangle_{\beta,\boldsymbol{h}} \quad \text{and} \quad \boldsymbol{m}_\star = \langle \boldsymbol{S}_0\rangle_{\beta,\boldsymbol{h}} \tag{2.4.36}$$

and suppose the state satisfies Properties 1 and 2. (The existence of such a state is guaranteed by Lemma 2.4.6.) Combining Eqs. (2.4.11) and (2.4.34), we get

$$0 \le \big\langle(\boldsymbol{S}_0,\boldsymbol{m}_0)\big\rangle_{\beta,\boldsymbol{h}} - |\boldsymbol{m}_\star|^2 \le \kappa n\mathscr{I}, \tag{2.4.37}$$

and so, invoking the result of part (1) of this theorem, $e_\star$ is close to $E(\boldsymbol{m}_\star)$ once $\mathscr{I}$ is sufficiently small. But this is true for all extremal pairs in $\mathscr{K}_\star(\beta,\boldsymbol{h})$ and so it must be true for *all* pairs in $\mathscr{K}_\star(\beta,\boldsymbol{h})$. Hence, $\mathscr{K}_\star(\beta,\boldsymbol{h})$ shrinks to a single point as $\mathscr{I} \downarrow 0$, which is what is claimed in part (2) of the theorem.

To conclude the proof of the theorem, we need to show that the spin configuration converges in distribution to a product measure. Applying the DLR conditions, the conditional distribution of $\boldsymbol{S}_0$ given a spin configuration in $\mathbb{Z}^d \setminus \{0\}$ is

$$\mathrm{e}^{(\boldsymbol{S}_0,\beta\boldsymbol{m}_0+\boldsymbol{h})-G(\beta\boldsymbol{m}_0+\boldsymbol{h})}\mu_0(\,\mathrm{d}\boldsymbol{S}_0), \tag{2.4.38}$$

i.e., the distribution of $\boldsymbol{S}_0$ depends on the rest of the spin configuration only via $\boldsymbol{m}_0 = \sum_{x\in\mathbb{Z}^d} J_{0,x}\boldsymbol{S}_x$. Hence, it clearly suffices to show that $\boldsymbol{m}_0$ converges to $\boldsymbol{m}$—the unique minimizer of $\Phi_{\beta,\boldsymbol{h}}$—in probability. But this is a direct consequence of the convexity bound on the left-hand side of Eq. (2.4.11) which tells us that, once the magnetization and energy density converge to their mean-field values, the variance of $\boldsymbol{m}_0$ tends to zero. □

While we cannot generally prove that, in systems with interaction (2.1.1) the magnetization increases with β, the estimates in the previous proof provide a bound on how bad the non-monotonicity can be:

Lemma 2.4.9 (Near monotonicity of magnetization)**.** *Let $(J_{x,y})$ be coupling constants that are RP and obey Eq.* (2.2.1)*, and let $\mathscr{I}$ be the integral in Eq.* (2.3.9)*. Let $\beta < \beta'$ and let $\boldsymbol{m}_\star \in \mathscr{M}_\star(\beta, \boldsymbol{h})$ and $\boldsymbol{m}'_\star \in \mathscr{M}_\star(\beta', \boldsymbol{h})$. Then we have:*

$$|\boldsymbol{m}_\star|^2 \le |\boldsymbol{m}'_\star|^2 + \kappa n \mathscr{I}. \tag{2.4.39}$$

Proof. Let $\langle - \rangle_{\beta,\boldsymbol{h}}$ and $\langle - \rangle_{\beta',\boldsymbol{h}}$ be (translation and rotation invariant) states satisfying Properties 1 and 2 in which the above magnetizations are achieved. (Such states exist by Lemma 2.4.6.) By Eq. (2.4.11) we have

$$\bigl\langle (\boldsymbol{S}_0, \boldsymbol{m}_0) \bigr\rangle_{\beta,\boldsymbol{h}} \ge |\boldsymbol{m}_\star|^2, \tag{2.4.40}$$

and Eqs. (2.4.11) and (2.4.37) yield

$$\bigl\langle (\boldsymbol{S}_0, \boldsymbol{m}_0) \bigr\rangle_{\beta',\boldsymbol{h}} \le |\boldsymbol{m}'_\star|^2 + \kappa n \mathscr{I}. \tag{2.4.41}$$

But the quantities on the left are, more or less, derivatives of the physical free energy with respect to β (in the parametrization introduce in Eq. (2.1.1)). Hence, standard convexity arguments give us

$$\bigl\langle (\boldsymbol{S}_0, \boldsymbol{m}_0) \bigr\rangle_{\beta',\boldsymbol{h}} \ge \bigl\langle (\boldsymbol{S}_0, \boldsymbol{m}_0) \bigr\rangle_{\beta,\boldsymbol{h}}. \tag{2.4.42}$$

Combining these inequalities the claim follows. □

2.4.5 Bounds for specific interactions

Having presented the main theorem, we now argue that by appropriately adjusting the parameters μ and s in the Yukawa and power law terms of an interaction, one can make the integral $\mathscr{I}$ as small as desired. We begin with a general criterion along these lines:

Proposition 2.4.10. *Let $(J_{x,y}^{(\lambda)})$ be a family of translation and reflection-invariant couplings depending on a parameter λ. Assume that the $J_{x,y}^{(\lambda)}$ obey Eq.* (2.2.1) *and let $\hat{J}_\lambda(k) = \sum_{x\in\mathbb{Z}^d} J_{0,x}^{(\lambda)}\, e^{\,ik\cdot x}$ be the Fourier components. Suppose that the following two conditions are true:*

(1) There exists a $\delta > 0$ and a constant $C > 0$ such that for all sufficiently small λ, we have

$$\frac{1-\hat{J}_\lambda(k)}{|k|^{d-\delta}} \geq C, \qquad k \in [-\pi,\pi]^d \setminus \{0\}. \tag{2.4.43}$$

(2) The ℓ^2-norm of $(J_{0,x}^{(\lambda)})$ tends to zero as $\lambda \to 0$, i.e.,

$$\lim_{\lambda\to 0} \sum_{x\in\mathbb{Z}^d} \bigl[J_{0,x}^{(\lambda)}\bigr]^2 = 0. \tag{2.4.44}$$

Then we have:

$$\lim_{\lambda\to 0} \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} \frac{|\hat{J}_\lambda(k)|^2}{1-\hat{J}_\lambda(k)} = 0. \tag{2.4.45}$$

Proof. Note that, by Eq. (2.2.1) and condition (1) above we have $\hat{J}_\lambda(0) = 1$ and $\hat{J}_\lambda(k) < 1$ for all $k \neq 0$. (The reflection invariance guarantees that $\hat{J}_\lambda$ is an even and real function of k.) First we will bound the part of the integral corresponding to $k \approx 0$. To that end we pick $r > 0$ and estimate

$$\int_{|k|<r} \frac{\mathrm{d}k}{(2\pi)^d} \frac{|\hat{J}_\lambda(k)|^2}{1-\hat{J}_\lambda(k)} \leq \int_{|k|<r} \frac{\mathrm{d}k}{(2\pi)^d} \frac{1}{C|k|^{d-\delta}} = C_1 r^\delta, \tag{2.4.46}$$

where $C_1 = C_1(\delta, d, C) < \infty$. Next we will attend to the rest of the integral. Let $M(r)$ be the supremum of $(1-\hat{J}_\lambda(k))^{-1}$ over all $k \in [-\pi,\pi]^d$ with $|k| \geq r$. By condition (1) above, we have that $M(r) \leq \frac{1}{C} r^{\delta - d}$. Therefore,

$$\int_{\substack{k \in [-\pi,\pi]^d \\ |k| \geq r}} \frac{\mathrm{d}k}{(2\pi)^d} \frac{|\hat{J}_\lambda(k)|^2}{1 - \hat{J}_\lambda(k)} \leq M(r) \sum_{x \in \mathbb{Z}^d} \bigl[J_{0,x}^{(\lambda)}\bigr]^2, \tag{2.4.47}$$

where we also used Parseval's identity. By condition (2) above, this vanishes as $\lambda \to 0$, while the integral in (2.4.46) can be made as small as desired by letting $r \downarrow 0$. From here the claim follows. □

Now we apply the above lemma to our specific interactions. We begin with the Yukawa potentials:

Lemma 2.4.11. *Let $(J_{x,y}^{(\mu)})$ be the Yukawa interactions with parameter μ—as described in Sect. 2.1.2—and suppose these are adjusted so that Eq.* (2.2.1) *holds. Then $(J_{x,y}^{(\mu)})$ obey conditions (1) and (2) of Proposition 2.4.10 as $\mu \downarrow 0$ with $\delta = d - 2$. Consequently, in dimensions $d \geq 3$, the corresponding integral in Eq.* (2.3.9) *tends to zero as $\mu \downarrow 0$.*

Proof. Let $(J_{x,y}^{(\mu)})$ be as above and let $\hat{J}_\mu$ denote the Fourier transform. In order to handle the overall normalization effectively, we introduce the quantity C_μ by $C_\mu \mu^d \sum_{x \neq 0} \mathrm{e}^{-\mu |x|_1} = 1$ and note that C_μ converges to a finite and positive limit as $\mu \downarrow 0$. From here we check that the ℓ^2-norm in Eq. (2.4.44) scales as μ^d and so condition (2) of Proposition 2.4.10 follows.

It remains to prove that $1 - \hat{J}_\mu(k)$ is bounded from below by a positive constant times $|k|^2$, where $|k|$ denotes the ℓ^2-norm of k. First we claim that for all $\eta > 0$ there exists a constant $A < \infty$ such that for all $k \in [-\pi,\pi]^d$,

$$\hat{J}_\mu(k) \leq 1 - \eta, \qquad |k| \geq A\mu. \tag{2.4.48}$$

Indeed, an explicit calculation gives us

$$\hat{J}_\mu(k) = \mu^d C_\mu \sum_{x \neq 0} \mathrm{e}^{-\mu |x|_1 + \mathrm{i} k \cdot x} \leq \mu^d C_\mu \prod_{j=1}^d \Bigl\{ \mathfrak{Re} \frac{1}{1 - \mathrm{e}^{-\mu + \mathrm{i} k_j}} \Bigr\}, \tag{2.4.49}$$

where we first neglected the condition $x \neq 0$, then wrote the result as the product over lattice directions and, finally, threw away some negative constants from each term in the product (the real parts are positive). Introducing the abbreviations $a = \mathrm{e}^{-\mu}$, $\epsilon = 1 - a$ and $\Delta_j = 1 - \cos(k_j)$, the ϵ-multiple of the j-th term in the product is now

$$\epsilon\, \mathfrak{Re}\, \frac{1}{1 - \mathrm{e}^{-\mu + \mathrm{i} k_j}} = \frac{\epsilon^2 + a\Delta_j \epsilon}{\epsilon^2 + 2a\Delta_j}. \tag{2.4.50}$$

Now if $\epsilon^2 \geq \Delta_j$ the right-hand side is less than $1 + a\epsilon$, while if $\epsilon^2 \leq \Delta_j$, then it is less than $\epsilon + \frac{1}{2a}\frac{\epsilon^2}{\Delta_j}$, which is $\ll 1$ once $\epsilon^2 \ll \Delta_j$. Going back to Eq. (2.4.49), if at least one component of k exceeds large constant times μ (which is itself of order ϵ), then the right-hand side of Eq. (2.4.49) is small. This proves Eq. (2.4.48) for μ small; for all other μ this holds existentially.

The condition (2.4.48) implies Eq. (2.4.44) for $|k| \geq A\mu$. As for the complementary values of k, here we pick a small number θ and write

$$1 - \hat{J}_\mu(k) \geq C_\mu \mu^d \sum_{\substack{x \neq 0 \\ |x|_1 \leq \theta/\mu}} \mathrm{e}^{-\mu|x|_1}[1 - \cos(k \cdot x)]. \tag{2.4.51}$$

By the fact that $|k| \leq A\mu$, the condition $|x|_1 \leq \theta/\mu$ (with θ sufficiently small) implies that $1 - \cos(k \cdot x) \geq c(k \cdot x)^2$ for some $c > 0$. Plugging this into Eq. (2.4.51) and using that the domain of the sum is invariant under reflection of any component of x, the result will be proportional to $|k|^2$. The constant of proportionality is of order μ^{-2} and so condition (1) is finally proved. □

Next we attend to the power laws:

Lemma 2.4.12. *Let $(J^{(s)}_{x,y})$ be the power-law interactions with exponent $s > d$—see Sect. 2.1.2—and suppose these are adjusted so that Eq.* (2.2.1) *holds. Then $(J^{(s)}_{x,y})$ obey conditions (1) and (2) of Proposition 2.4.10 as $s \downarrow d$ with any $\delta < d$. Consequently, the corresponding integral in Eq.* (2.3.9) *tends to zero as $s \downarrow d$ in all $d \geq 1$.*

Proof. Our first item of business will again be the overall normalization. Let C_s be the constant defined by

$$C_s(s-d)\sum_{x\neq 0}|x|_1^{-s}=1. \tag{2.4.52}$$

As is not hard to check, C_s tends to a positive and finite limit as $s\downarrow d$. Since $\sum_{x\neq 0}|x|_1^{-2s}$ is uniformly bounded for all $s>d$, the ℓ^2-norm in Eq. (2.4.44) is proportional to $(s-d)$. This proves condition (2) of Proposition 2.4.10.

In order to prove condition (1), we first write

$$1-\hat{J}_s(k)=C_s(s-d)\sum_{x\neq 0}|x|_1^{-s}\big(1-\cos(k\cdot x)\big), \tag{2.4.53}$$

where $\hat{J}_s$ is the Fourier transform of the $(J_{x,y}^{(s)})$. Consider the set $\mathcal{R}_k=\{x\in\mathbb{Z}^d\colon\ \cos(k\cdot x)\leq 0\}$, which we note is the union of strips of width—and separation—of the order $O(1/|k|)$ which are perpendicular to vector k. A simple bound gives us

$$\sum_{x\neq 0}|x|_1^{-s}\big(1-\cos(k\cdot x)\big)\geq\sum_{x\in\mathcal{R}_k}|x|_1^{-s}. \tag{2.4.54}$$

Next we let $\mathcal{R}_k'=\{x\in\mathbb{Z}^d\colon|x\cdot k|>\pi\}$. The fact that $|x|_1^{-s}$ decreases with distance allows us to bound the second sum in Eq. (2.4.54) by a similar sum with $x\in\mathcal{R}_k'$. Using the usual ways to bound sums by integrals, we thus get

$$1-\hat{J}_s(k)\geq C(s-d)\int_{|k\cdot x|\geq\pi}\frac{\mathrm{d}x}{|x|^s}, \tag{2.4.55}$$

where C is a positive constant (independent of s) and $|x|$ is the ℓ^2-norm of x. Extracting a factor of $|k|^{s-d}$, the resulting integral *times* $(s-d)$ is uniformly positive for all $s>d$. Hence we proved that for some $c'>0$,

$$1-\hat{J}_s(k)\geq c'|k|^{s-d} \tag{2.4.56}$$

for all $s>d$ and all $k\in[-\pi,\pi]^d$, and so condition (1) of Proposition 2.4.10 holds as stated. $\square$

2.5 Proofs: Mean-field theories

2.5.1 Blume-Capel model

We begin by giving the proof of Theorem 2.3.6 which deals with the mean-field theory of the Blume-Capel model. The core of this proof, and other proofs in this paper, are certain facts about the mean-field theory of the Ising model in an external field. In the formalism of Sect. 2.2.2, this model corresponds to the $q = 2$ Potts model. The magnetizations are parameterized by a pair of quantities (z_1, z_{-1}), where $z_1 + z_{-1} = 1$, which represent the mole-fractions of plus and minus spins. The mean-field free energy is given by

$$\Phi_{J,h} = Jz_1 z_{-1} - hz_1 + z_1 \log z_1 + z_{-1} \log z_{-1}. \tag{2.5.1}$$

The following properties are the results of straightforward calculations:

(I1) If $h = 0$ and $J \leq 2$, then the only local—and global—minimum occurs at $z_1 = z_{-1}$.

(I2) If $h = 0$ and $J > 2$, then there is only one local minimum with $z_1 \geq z_{-1}$ and it satisfies $Jz_1 > 1 > Jz_{-1}$. A corresponding local minimum with with $z_1 \geq z_{-1}$ exists and obeys $Jz_{-1} > 1 > Jz_1$.

(I3) Let now h be arbitrary. If (z_1, z_{-1}) is a local minimimum of $\Phi_{J,h}$, then $m = z_1 - z_{-1}$ satisfies $J(1 - m^2) \leq 1$.

These properties are standard; for some justification see, e.g., the proof of Lemma 4.4 in Ref. [21].

Proof of Theorem 2.3.6. Let (x_1, x_0, x_{-1}) be a triplet of positive variables which corresponds to a local minimum of the Blume-Capel free-energy function $\Phi_{\beta,\lambda}$ from Eq. (2.3.21). A simple calculations shows that the derivative of the entropy part of $\Phi_{\beta,\lambda}$

is singular in the limit when any component of (x_1, x_0, x_{-1}) tends to zero, while nothing spectacular happens to the energy. Therefore, the minimum must lie strictly inside the simplex of allowed values. Accounting for the constraint $x_1 + x_0 + x_{-1} = 1$, the condition that the gradient of $\Phi_{\beta,\lambda}$ vanish at (x_1, x_0, x_{-1}) translates into the equations (2.3.22).

Due to the symmetry between x_1 and x_{-1}, we may (and will) assume for simplicity that $x_1 \geq x_{-1}$. First we claim that, under this condition, we have $4\beta x_{-1} \leq 1$. Indeed, for a fixed x_0, the Blume-Capel mean-field free energy $\Phi_{\beta,\lambda}$ expressed in terms of (z_1, z_{-1}), where $z_{\pm 1} = x_{\pm 1}/(1 - x_0)$, is proportional to the Ising free energy (2.5.1) with $J = 4\beta(1 - x_0)$. Since the Ising pair (z_1, z_{-1}) is at its local minimum, we have $Jz_{-1} = 4\beta x_{-1} \leq 1$ by property (I2) above.

Once we know that x_{-1} is small, the question is whether x_0 and x_1 divide the amount $1 - x_{-1}$ democratically or autocratically. Here we observe that, once again, for a fixed x_{-1}, the (x_1, x_0)-portion of the Blume-Capel mean-field free energy $\Phi_{\beta,\lambda}$ is proportional to its Ising counterpart in Eq. (2.5.1) with $J = \beta(1 - x_{-1})$ and $h = 3\beta x_{-1} - \lambda$. In light of property (I3) above, the magnetization variable $m = (x_1 - x_0)/(1 - x_{-1})$ thus satisfies the bound $J(1 - m^2) \leq 1$. Using the inequality $\sqrt{1 - a} \geq 1 - a$ valid for all $a \leq 1$, we have

$$\frac{|x_1 - x_0|}{1 - x_{-1}} \geq 1 - \frac{1}{\beta(1 - x_{-1})} \tag{2.5.2}$$

once β is sufficiently large. Some simple algebra now shows that this implies

$$2\beta \min\{x_1, x_0\} \leq 1. \tag{2.5.3}$$

Using these findings in Eq. (2.3.22) and extracting appropriate inequalities we derive the bounds listed in (1) and (2) with C being a numerical constant.

To derive the asymptotics (2.3.23) on the free-energy gap for $\lambda \approx 0$, let us first evaluate the free energy at a generic local minimum. Suppose (x_1, x_0, x_{-1}) obey

Eq. (2.3.22) and let Θ denote the logarithm of the quantity in Eq. (2.3.22). A direct calculation shows that then

$$\Phi_{\beta,h} = -4\beta x_1 x_{-1} + \beta x_0^2 + \Theta. \tag{2.5.4}$$

Now let us consider a minimum with x_0 dominant. Then the inequality $\beta(1 - x_0) = \beta(x_1 + x_{-1}) \leq 3/4 < 1$ shows that the (x_1, x_{-1}) Ising pair is subcritical. By (I1) above we must have $x_1 = x_{-1} = \frac{1}{2}(1 - x_0)$ and, as is seen by a direct calculation, x_0 can be determined from the equation

$$\frac{1 - x_0}{x_0} = 2\,\mathrm{e}^{-\beta+\lambda}. \tag{2.5.5}$$

In particular, for λ bounded we have $1 - x_0 = 2\,\mathrm{e}^{-\beta+\lambda} + O(\,\mathrm{e}^{-2\beta})$. Similarly, if the minimum corresponds to a triple dominated by x_1, our bounds show that $x_0 = 1 - x_1 + O(\,\mathrm{e}^{-4\beta})$ and so we have

$$x_1 = \bigl(1 - x_1 + O(\,\mathrm{e}^{-4\beta})\bigr)\,\mathrm{e}^{\beta+\lambda+O(\beta\,\mathrm{e}^{-\beta})}. \tag{2.5.6}$$

From here we have $1 - x_1 = \,\mathrm{e}^{-\beta-\lambda} + O(\beta\,\mathrm{e}^{-2\beta})$.

Now we are ready to derive Eq. (2.3.23). First, using that $\Theta = \log x_0 + \beta(1 - 2x_0) + \lambda$ we have

$$\begin{aligned}\phi_0(\beta,\lambda) = -4\beta x_1 x_{-1} + \beta(1 - x_0)^2 + \lambda + \log x_0 \\ = \lambda - 2\,\mathrm{e}^{-\beta+\lambda} + O(\,\mathrm{e}^{-2\beta}).\end{aligned} \tag{2.5.7}$$

Next, in light of $\Theta = \log x_1 + 4\beta x_{-1}$ and the bounds proved on x_{-1} in (2) above we have

$$\begin{aligned}\phi_1(\beta,\lambda) = -4\beta x_1 x_{-1} + \beta x_0^2 + \log x_1 + 4\beta x_{-1} \\ = -\,\mathrm{e}^{-\beta-\lambda} + O(\beta\,\mathrm{e}^{-2\beta}).\end{aligned} \tag{2.5.8}$$

Combining Eqs. (2.5.7–2.5.8), the desired relation (2.3.23) is proved. $\square$

We finish this section with a computational lemma that will be useful in the proof of Theorem 2.3.7:

Lemma 2.5.1. *There exists* $\alpha > 0$ *and, for each* $C \gg 1$, *there exists* $\beta_0 < \infty$ *such that the following is true for all* $\beta \geq \beta_0$ *and all* λ *with* $|\lambda| \leq C\,e^{-\beta}$*: If* (x_1, x_0, x_{-1}) *is a triplet with*

$$\max\{x_1, x_0, x_{-1}\} = 1 - C\,e^{-\beta}, \tag{2.5.9}$$

then

$$\Phi_{\beta,\lambda}(x_1, x_0, x_{-1}) - \inf \Phi_{\beta,\lambda} \geq \alpha(C \log C)\,e^{-\beta}. \tag{2.5.10}$$

Here $\Phi_{\beta,\lambda}$ *is the function in Eq.* (2.3.21) *and* $\inf \Phi_{\beta,\lambda}$ *is its absolute minimum.*

Proof. An inspection of Eqs. (2.5.7–2.5.8) shows that, once $|\lambda| \leq C\,\mathrm{e}^{-\beta}$, we have that $|\inf \Phi_{\beta,\lambda}|$ is proportional to $C\,\mathrm{e}^{-\beta}$ and so we just have to prove that, once C is sufficiently large, $\Phi_{\beta,\lambda}(x_1, x_0, x_{-1})$ is proportional to $(C \log C)\,\mathrm{e}^{-\beta}$. We will focus on the situation when the maximum in Eq. (2.5.10) is achieved by x_1; the other cases are handled similarly.

By our assumption we have that x_0 and x_{-1} are quantities less than $C\,\mathrm{e}^{-\beta}$. Inspecting the various terms in Eq. (2.3.21), we thus have

$$\begin{aligned} \beta x_0(1 - x_0) &= \beta x_0 + O(\beta C^2\,\mathrm{e}^{-2\beta}), \\ \beta x_1 x_{-1} &= 4\beta x_{-1} + O(\beta C^2\,\mathrm{e}^{-2\beta}), \\ x_1 \log x_1 &= -C\,\mathrm{e}^{-\beta} + O(C^2\,\mathrm{e}^{-2\beta}), \end{aligned} \tag{2.5.11}$$

Plugging these back into the definition of $\Phi_{\beta,\lambda}$ we get

$$\begin{aligned} \Phi_{\beta,\lambda}(x_1, x_0, x_{-1}) = x_0[\beta + \log x_0] &+ x_{-1}[4\beta + \log x_{-1}] \\ &+ \lambda x_0 - C\,\mathrm{e}^{-\beta} + O(\beta C^2\,\mathrm{e}^{-2\beta}). \end{aligned} \tag{2.5.12}$$

Now $|\lambda x_0| \le |\lambda| \le C\,\mathrm{e}^{-\beta}$, and if β_0 is such that $\beta C\,\mathrm{e}^{-\beta} \ll 1$, the last three terms on the right-hand side are all of order $C\,\mathrm{e}^{-\beta}$. It thus suffices to to prove that the first two terms exceed a constant times $(C\log C)\,\mathrm{e}^{-\beta}$.

We first replace 4β by β in Eq. (2.5.12) and then substitute $x_0 = y_0\,\mathrm{e}^{-\beta}$ and $x_{-1} = y_{-1}\,\mathrm{e}^{-\beta}$. The relevant two terms on the right-hand side then equal $\mathrm{e}^{-\beta}[y_0 \log y_0 + y_{-1}\log y_{-1}]$. Under the condition (2.5.9)—which implies that at least one of the y's is larger than $C/2$—this is a number of order $\mathrm{e}^{-\beta} C\log C$ (for $C \gg 1$). The right-hand side of Eq. (2.5.12) is thus of order $\mathrm{e}^{-\beta} C\log C$ whenever $\beta \ge \beta_0$, which proves the desired claim. □

2.5.2 Potts model: Preliminaries

Next we turn our attention to the mean-field theory of the Potts model. In the present section we will first establish some basic properties of the (local) minimizers of the Potts mean-field free energy. The proof of Theorem 2.2.3 dealing with positive fields is then the subject of Sect. 2.5.3. The negative-field portion of our results (Theorem 2.2.4) is somewhat more involved and we defer its discussion to Sect. 2.5.4.

We invite the reader to recall the representation of magnetizations in terms of barycentric coordinates in Eq. (2.2.14), the mean-field free-energy function $\Phi_{\beta,h}$ from Eq. (2.2.15) and the transitional coupling $\beta_{\mathrm{MF}}^{(q)}$ for the q-state Potts model from Eq. (2.2.17). We begin with some general monotonicity properties of the minimizers:

Lemma 2.5.2 (Monotonicity in h)**.** *For any $\beta \ge 0$ we have:*

(1) Let $h < h'$, let x_1 be the first barycentric coordinate of a global minimum of $\Phi_{\beta,h}^{(q)}$ and let x_1' be the first barycentric coordinate of a global minimum of $\Phi_{\beta,h'}^{(q)}$. Then $x_1 \le x_1'$.

(2) Let $(x_1,\dots,x_q)$ be the probability vector corresponding to a global minimizer

of $\Phi^{(q)}_{\beta,h}$*. If* $h > 0$ *then* $x_1 > \max\{x_2, \dots, x_q\}$*. Similarly, if* $h < 0$ *then* $x_1 < \min\{x_2, \dots, x_q\}$.

(3) If $h \mapsto \boldsymbol{m}(\beta, h)$ *is a differentiable trajectory of local extrema, then*

$$\frac{d}{dh}\Phi^{(q)}_{\beta,h}(\boldsymbol{m}(\beta, h)) = -x_1(\beta, h), \tag{2.5.13}$$

where $x_1(\beta, h)$ *is the first component of* $\boldsymbol{m}(\beta, h)$ *in the decomposition into* $(\hat{v}_1, \dots, \hat{v}_q)$.

Proof. (1) Let $\boldsymbol{m} \in \operatorname{Conv}\Omega$. Then we have

$$\Phi^{(q)}_{\beta,h}(\boldsymbol{m}) - \Phi^{(q)}_{\beta,h'}(\boldsymbol{m}) = (h' - h)x_1, \tag{2.5.14}$$

where x_1 is the first component of $\boldsymbol{m}$. Let x_1 and x_1' be as above and let $\boldsymbol{m}$ and $\boldsymbol{m}'$ be the corresponding minimizers. Then Eq. (2.5.14) implies

$$x_1 \le \frac{\Phi^{(q)}_{\beta,h}(\boldsymbol{m}) - \Phi^{(q)}_{\beta,h'}(\boldsymbol{m}')}{h' - h} \tag{2.5.15}$$

Similar reasoning gives

$$x_1' \ge \frac{\Phi^{(q)}_{\beta,h}(\boldsymbol{m}) - \Phi^{(q)}_{\beta,h'}(\boldsymbol{m}')}{h' - h}. \tag{2.5.16}$$

Combining Eqs. (2.5.15) and (2.5.16) gives the result.

(2) Let $h > 0$ and let $(x_1, \dots, x_q)$ be a probability vector with $x_1 < x_2$. Interchanging x_1 and x_2 shows that, due to the interaction with the field, the q-tuple $(x_2, x_1, \dots, x_q)$ has strictly lower free energy than $(x_1, \dots, x_q)$, i.e., $(x_1, \dots, x_q)$ could not have been a global minimizer. Hence $x_1 \ge x_2$. To rule out $x_1 = x_2$ we note that $x_1, x_2 > 0$ and so the gradient of the free energy, subject to the constraint $x_1 + x_2 = \text{const}$, must vanish. Hence $x_1\,\mathrm{e}^{-\beta x_1 - h} = x_2\,\mathrm{e}^{-\beta x_2}$ which forces $x_1 \ne x_2$. The cases $h < 0$ are handled similarly.

(3) This is a consequence of the fact that the gradient $\nabla\Phi^{(q)}_{\beta,h}$ vanishes at any local extremum in the interior of $\operatorname{Conv}(\Omega)$. □

Lemma 2.5.3 (Monotonicity in β)**.** *Fix $h \in \mathbb{R}$. If $\beta \mapsto \boldsymbol{m}(\beta, h)$ is a differentiable trajectory of local extrema, then*

$$\frac{d}{d\beta}\Phi^{(q)}_{\beta,h}(\boldsymbol{m}(\beta, h)) = -\frac{1}{2}|\boldsymbol{m}(\beta, h)|^2. \tag{2.5.17}$$

Proof. The proof is analogous to that of Lemma 2.5.2(3). □

The next lemma significantly narrows the list of possible candidates for global minimizers:

Lemma 2.5.4 (Symmetries of global minimizers)**.** *Let $\Phi^{(q)}_{\beta,h}(\boldsymbol{m})$ be the mean-field free-energy function. Let $\boldsymbol{m} \in$ Conv Ω be a global minimum of $\Phi^{(q)}_{\beta,h}$ and let $(x_1, \dots, x_q)$ be the corresponding probability vector of barycentric coordinates.*

(1) If $h > 0$, then

$$x_1 > x_2 = \cdots = x_q. \tag{2.5.18}$$

(2) If $h < 0$, then $(x_1, \dots, x_q)$ is a permutation in indices $x_2, \dots, x_q$ of a vector with

$$x_1 < x_2 = \cdots = x_{q-1} \leq x_q. \tag{2.5.19}$$

Proof. The main idea of the proof is that the variables $x_2, \dots, x_q$, properly scaled, behave like a $(q-1)$-state, zero-field Potts model. Abusing the notation slighly, let us write $\Phi^{(q)}_{\beta,h}(x_1, \dots, x_q)$ instead of $\Phi^{(q)}_{\beta,h}(\boldsymbol{m})$ whenever $\boldsymbol{m}$ corresponds to the probability vector $(x_1, \dots, x_q)$. In looking for global minima, we may assume that all x_k's satisfy $x_k \in (0, 1)$. Letting

$$z_k = \frac{x_k}{1 - x_1}, \qquad k = 2, \dots, q, \tag{2.5.20}$$

this allows us to write

$$\Phi^{(q)}_{\beta,h}(x_1, \dots, x_q) = (1 - x_1)\Phi^{(q-1)}_{\beta(1-x_1),0}(z_2, \dots, z_q) + R(x_1), \tag{2.5.21}$$

where $R(x_1)$ is a function of x_1 (and β and h). The rest of the proof is based on some basic properties of the zero-field Potts free energy for which we refer the reader back to Sect. 2.2.2.

Let $(x_1, \dots, x_q)$ correspond to a global minimum. A principal conclusion coming from Eq. (2.5.21) is that the components of the vector $(x_2, \dots, x_q)$, ordered increasingly, satisfy $x_2 = \dots = x_{q-1} \le x_q$. Using part (2) of Lemma 2.5.2, this immediately implies Eq. (2.5.19). To prove Eq. (2.5.18), let $h > 0$ and let $(\tilde{x}_1, \dots, \tilde{x}_q)$ be a global minimizer at zero field with maximal value of $\tilde{x}_1$. By general facts about the zero-field problem, this forces $\beta(1 - \tilde{x}_1) < \beta_{\text{MF}}^{(q-1)}$ and, since part (2) of Lemma 2.5.2 implies that $x_1 > \tilde{x}_1$, also $\beta(1 - x_1) < \beta_{\text{MF}}^{(q-1)}$. Hence, the variables $(z_2, \dots, z_q)$ correspond to a subcritical Potts model and thus $z_2 = \dots = z_q$. Invoking again Lemma 2.5.2(2), we have Eq. (2.5.18). □

2.5.3 Potts model: Positive fields

Next we will focus on the cases with $h > 0$. Our first step is to characterize the local and global minima of $\boldsymbol{m} \mapsto \Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ for $\boldsymbol{m}$ restricted to satisfy Eq. (2.5.18). While we could appeal to the "on-axis" formalism from Ref. [21], we will keep the requisite calculations more or less self-contained.

For any probability vector satisfying Eq. (2.5.18), let us consider the parametrization $\theta = \frac{q}{q-1} m$, where m denotes the scalar magnetization defined via $x_1 = \frac{1}{q} + m$ and $x_k = \frac{1}{q} - \frac{m}{q-1}$, $k = 2, \dots, q$. (The physical values of θ are $\theta \in [0, 1]$.) Let $\phi_{\beta,h}(\theta)$ denote the value of $\Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ where $\boldsymbol{m}$ corresponds to the above $(x_1, \dots, x_q)$. Then we have:

Lemma 2.5.5 ("On-axis" minima)**.** *The local minima of* $\theta \mapsto \phi_{\beta,h}(\theta)$ *are solutions to* *the equation $\theta = f(\theta)$, where*

$$f(\theta) = \frac{e^{\beta\theta+h} - 1}{e^{\beta\theta+h} + q - 1}. \tag{2.5.22}$$

Moreover, let $\beta_0 = 4\frac{q-1}{q}$. Then

(1) For all $\beta \leq \beta_0$ and all $h \in \mathbb{R}$, the equation $\theta = f(\theta)$ has only one solution.

(2) For $\beta > \beta_0$ there exists an interval (h_-, h_+) such that $\theta = f(\theta)$ has three distinct solutions once $h \in (h_-, h_+)$ and only one solution for $h \notin [h_-, h_+]$. At $h = h_\pm$, there are two distinct solutions. Once $h \neq h_\pm$, only the extreme solutions (the largest and the smallest) correspond to local minima of $\theta \mapsto \phi_{\beta,h}(\theta)$.

Finally, for each $\beta > \beta_0$, there exists a number $h_1 = h_1(\beta) \in (h_-, h_+)$ such that the global minimizer of $\theta \mapsto \phi_{\beta,h}(\theta)$ is unique as long as $h \neq h_1$. On the other hand, for $h = h_1$ there are two distinct global minimizers (the two extreme solutions of $\theta = f(\theta)$).

Remark 2.5.6. Although the above holds as stated in complete generality, it is only useful (in the present context) for $\beta < \beta_{\text{MF}}^{(q)}$. In particular, for $\beta \geq \beta_{\text{MF}}^{(q)}$, while $h_1(\beta)$ continues on taking negative values, it does not correspond to any equilibrium commodity.

Proof of Lemma 2.5.5. Since the derivative of $\theta \mapsto \phi_{\beta,h}(\theta)$ diverges as θ tends to either zero or one, all local minima will lie in $(0, 1)$. Differentiating with respect to θ we find that these must satisfy $f(\theta) = \theta$ with f as given above.

In order to characterize the solutions to $\theta = f(\theta)$, let us calculate the first two derivatives of this function:

$$f'(\theta) = \beta \frac{\mathrm{e}^{\beta\theta+h}}{\mathrm{e}^{\beta\theta+h} + q - 1}\big(1 - f(\theta)\big) \tag{2.5.23}$$

and

$$f''(\theta) = \beta^2 \frac{e^{\beta\theta+h}}{e^{\beta\theta+h} + q - 1}\bigl(1 - f(\theta)\bigr)\left(1 - 2\frac{e^{\beta\theta+h}}{e^{\beta\theta+h} + q - 1}\right). \tag{2.5.24}$$

Since we also have $f(\theta) < 1$, we find that f is strictly increasing, strictly convex for $\theta < \theta_{\rm I}$ and strictly concave for $\theta > \theta_{\rm I}$, where $\theta_{\rm I}$ is the inflection point of f, which is given by

$$\frac{e^{\beta\theta+h}}{e^{\beta\theta+h} + q - 1} = \frac{1}{2}, \tag{2.5.25}$$

i.e., $e^{\beta\theta+h} = q - 1$. In particular, the derivative $f'(\theta)$ is maximal at $\theta = \theta_{\rm I}$, where it equals $f'(\theta_{\rm I}) = \frac{\beta}{4}\frac{q}{q-1}$.

Let us suppose that $f'(\theta_{\rm I}) \le 1$, which is equivalent to $\beta \le \beta_0$. Then there is only one solution to $\theta = f(\theta)$, proving (1) above. Let us now assume that $f'(\theta_{\rm I}) > 1$. The fact that increasing h amounts to "shifting the graph of f to the left" implies that there exists an h_0 such that $\theta_{\rm I}$ solves $\theta = f(\theta)$ for $h = h_0$. Similar arguments show that there exists a unique value $h_+ > h_0$ such that the diagonal line (at 45°) is tangent to the graph of f at some $\theta < \theta_{\rm I}$, and a similar value $h_- < h_0$ such that the diagonal line is tangent to the $\theta \ge \theta_{\rm I}$ portion of the graph of f. For $h \in [h_-, h_+]$, there are altogether three solutions, labeled $\theta_{\rm L} < \theta_{\rm M} < \theta_{\rm U}$, where $f'(\theta) \le 1$ at $\theta = \theta_{\rm L}, \theta_{\rm U}$ while $f'(\theta_{\rm M}) \ge 1$ (with the inequalities strict when $h \ne h_\pm$).

The "dynamics" of these solutions as h changes is easy to glean from the above picture. First $\theta_{\rm L}$ is defined for all $h \le h_+$ while $\theta_{\rm U}$ is defined for all $h \ge h_-$. Now, as h decreases through h_-, the middle $\theta_{\rm M}$ and upper $\theta_{\rm U}$ solutions merge and disappear; and similarly for $\theta_{\rm M}$ and $\theta_{\rm L}$ as h increases through h_+. Only the remaining solution continues to exist in the complementary part of the h-axis. Clearly, both $\theta_{\rm L}$ and $\theta_{\rm U}$ are continuous and strictly increasing on the domain of their definition with $\theta_{\rm L} \to 0$ as $h \to -\infty$ and $\theta_{\rm U} \to 1$ as $h \to \infty$. Since $\phi_{\beta,h}(\theta)$ has local maxima at $\theta = 0$ and 1, we must have that $\theta_{\rm L}$ and $\theta_{\rm U}$ are local minima and $\theta_{\rm M}$ is a local maximum of $\phi_{\beta,h}$. (These are strict except perhaps at $h \ne h_\pm$.) This finishes the proof of (2).

It remains to prove the existence of the transitional field-strength h_1. By Lemma 2.5.4, every global minimizer $\boldsymbol{m} \mapsto \Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ corresponds to either $\theta_{\rm L}$ or $\theta_{\rm U}$. Observe that, since $\theta_{\rm U}$ and $\theta_{\rm L}$ never enter the portion of the graph of f where f' exceeds one, we have $\theta_{\rm U} \geq \theta_{\rm U}(h_+) > \theta_{\rm L}(h_-) \geq \theta_{\rm L}$ and so the difference $\theta_{\rm U} - \theta_{\rm L}$ is uniformly positive. Consequently, the values $\Phi_{\beta,h}^{(q)}$ at the corresponding magnetizations change at a strictly different rate with h (see Lemma 2.5.2). In particular, there exists a unique point $h_1(\beta) \in (h_-, h_+)$, where the status of the global minimizer changes from $\theta_{\rm L}$ to $\theta_{\rm U}$. By continuity, at $h = h_1$, both one-sided limits are minimizers of $\Phi_{\beta,h}$. □

Now we are ready to finish the prove of Theorem 2.2.3.

Proof of Theorem 2.2.3. Most of the claims of the theorem have already been proved. Indeed, let h_1 be as in Lemma 2.5.5 and let $\beta \geq \beta_{\rm MF}^{(q)}$. By the properties of the zero-field Potts model, the maximal solution to $\theta = f(\theta)$ is a global minimizer of $\theta \mapsto \phi_{\beta,0}(\theta)$. It follows that $h_1(\beta) \leq 0$ for $\beta \geq \beta_{\rm MF}^{(q)}$. Invoking also Lemma 2.5.4(1), we thus conclude that for $\beta \leq \beta_0$ or $\beta \geq \beta_{\rm MF}^{(q)}$ and $h > 0$, the global minimizer of $\boldsymbol{m} \mapsto \Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ is unique, while for $\beta \in (\beta_0, \beta_{\rm MF}^{(q)})$ this is only true when $h \neq h_1(\beta)$. This establishes parts (2) and (3) of the theorem. It thus remains to prove the strict inequality between x_1 and $x_2 = \cdots = x_1$ in part (1)—the rest follows by Lemma 2.5.4(1)—and the properties of $\beta \mapsto h_1(\beta)$ in part (4).

First, it is easy to see that h_1 is continuous. Indeed, let $\beta' \in (\beta_0, \beta_{\rm MF}^{(q)}]$ and suppose that $\beta \mapsto h_1(\beta)$ has two limit points as $\beta \to \beta'$. By a simple compactness argument, there are two distinct minimizers of $\phi_{\beta',h}^{(q)}$ for h at these limit points, which contradicts the uniqueness of $h_1(\beta')$. Applying this to $\beta' = \beta_{\rm MF}^{(q)}$, we thus have that $h_1(\beta) \to 0$ as $\beta \to \beta_{\rm MF}^{(q)}$.

Second, we claim that $\beta \mapsto h_1(\beta)$ is actually strictly decreasing. To this end, let $\boldsymbol{m}_+(\beta)$ and $\boldsymbol{m}_-(\beta)$ denote the values of the two global minimizers of $\boldsymbol{m} \mapsto \Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ at $h = h_1(\beta)$ and let $x_1^+(\beta)$ and $x_1^-(\beta)$ denote the corresponding first components. From Lemmas 2.5.2 and 2.5.3 we can now extract

$$\frac{\mathrm{d}}{\mathrm{d}\beta} h_1(\beta) = -\frac{1}{2} \frac{|\boldsymbol{m}_+(\beta)|^2 - |\boldsymbol{m}_-(\beta)|^2}{x_1^+(\beta) - x_1^-(\beta)}, \tag{2.5.26}$$

which the reader will note is the Clausius-Clapeyron relation. Since both x_1 and $|\boldsymbol{m}|$ are increasing with the scalar magnetization, the right hand side is negative and so $\beta \mapsto h_1(\beta)$ is strictly decreasing.

Third, we turn our attention to the inequality $x_1 > x_2 = \cdots = x_q$ once $h > 0$. In light of Eq. (2.5.18), it suffices to show that, for $h > 0$, the state with equal barycentric coordinates is not a local minimum once $h > 0$. This is directly checked by differentiating Eq. (2.2.15) subject to appropriate constraints. Finally, we will compute the value of h at the end of the line $h \mapsto \beta_+(h)$. Let $\theta_+(h)$ and $\theta_-(h)$ denote the two distinct (extremal) solutions of $f(\theta) = \theta$, with f as in Eq. (2.5.22), for $\beta = \beta_+(h)$. As h increases, β_+ decreases to β_0 and $\theta_\pm$ converge to a single value θ_0—the *unique* solution of $f(\theta) = \theta$ at $\beta = \beta_0$. But the inflection point, θ_{I}, is always squeezed between θ_+ and θ_-, and so we must have $\theta_0 = \theta_{\mathrm{I}}$. Now the inflection point is characterized by $\mathrm{e}^{\beta\theta_{\mathrm{I}}+h} = q-1$ and the equation $\theta = f(\theta)$ gives us that $\beta_+(h) = \beta_0$ at $h = h_{\mathrm{c}}$. □

2.5.4 Potts model: Negative fields

The goal of this section is to give the proof of Theorem 2.2.4. The difficulty here is that, on the basis of Eq. (2.5.19), the full-blown optimization problem is intrinsically two-dimensional. We begin with some lemmas that encapsulate the computational parts of the proof. First we will address the symmetric minima by describing the solutions to the "on-axis" equation:

Lemma 2.5.7. *Let $\beta \geq 0$ and $h < 0$ and let $g\colon [0, \frac{1}{q-1}] \to \mathbb{R}$ be the function*

$$g(\theta) = \frac{e^{\beta\theta - h} - 1}{(q-1)\, e^{\beta\theta - h} + 1}. \tag{2.5.27}$$

Then g is increasing, concave and satisfies $g(0) > 0$ and $g(\theta) < 1$. In particular, the equation $g(\theta) = \theta$ has a unique solution on $[0, \frac{1}{q-1}]$.

Proof. This is a result of straightforward computations which are not entirely dissimilar from those in Eqs. (2.5.23–2.5.24). □

The two-parameter nature of solutions of the form (2.5.19) will be handled by fixing the first barycentric coordinate and optimizing over the remaining ones. Here the following property of the resulting "partial minimum" will turn out to be very useful:

Lemma 2.5.8. *Let $\beta > \beta_{MF}^{(q-1)}$ and let $\tilde{a}$ be the minimum of $1/q$ and the quantity a satisfying $\beta(1-a) = \beta_{MF}^{(q-1)}$. For each $x \in [0, \tilde{a}]$, let $z_2(x), \dots, z_q(x)$ denote the vector corresponding to the asymmetric minimizer of $(z_2, \dots, z_q) \mapsto \Phi_{\beta(1-x),0}^{(q-1)}(z_2, \dots, z_q)$ with $z_2 = \cdots = z_{q-1} < z_q$. Let $\psi(x)$ denote the quantity $\Phi_{\beta,h}^{(q)}(\boldsymbol{m})$ evaluated at $\boldsymbol{m} = \boldsymbol{m}(x)$ where*

$$\boldsymbol{m}(x) = x\, \hat{\mathrm{v}}_1 + (1-x) z_2(x)\, \hat{\mathrm{v}}_2 + \cdots + (1-x) z_q(x)\, \hat{\mathrm{v}}_q. \tag{2.5.28}$$

Then

$$\psi'''(x) < 0 \quad \textit{for all } x \in [0, \tilde{a}]. \tag{2.5.29}$$

Proof. Let $\psi(x)$ be as stated above. Let $t = t(x) = \beta(1-x)$ and let $\boldsymbol{z}(x) = (z_2(x), \dots, z_q(x))$ denote the asymmetric global minimum of $\Phi_{t(x),0}^{(q-1)}$. This allows us to rewrite $\psi(x)$ as

$$\psi(x) = -\frac{\beta}{2} x^2 + x \log(x) + (1-x)\log(1-x) - hx + (1-x)\Phi_{t(x),0}^{(q-1)}(\boldsymbol{z}(x)). \tag{2.5.30}$$

We will write $z_2 = \dots = z_{q-1} = \frac{1}{q-1} - \frac{m(t)}{q-2}$ and $z_q = \frac{1}{q-1} + m(t)$, where $m(t)$ is the maximal positive solution to

$$\frac{q-1}{q-2}m(t) = \frac{\exp\{t\frac{q-1}{q-2}m(t)\} - 1}{\exp\{t\frac{q-1}{q-2}m(t)\} + q - 2}. \tag{2.5.31}$$

The various steps of the proof involve two specific functions $u(t)$ and $\alpha(t)$ defined by

$$u(t) = t\frac{q-1}{q-2}m(t) \tag{2.5.32}$$

and

$$\alpha(t) = \frac{e^{u(t)}}{e^{u(t)} + q - 2}\Bigl(1 - \frac{e^{u(t)} - 1}{e^{u(t)} + q - 2}\Bigr). \tag{2.5.33}$$

We state these definitions here to facilitate later reference.

A simple argument gives that $t \mapsto m(t)$ is smooth when $t \geq \beta_{\text{MF}}^{(q-1)}$, so $\psi(x)$ is differentiable. The actual proof then commences by the calculation of the third derivative of $\psi(x)$:

$$\begin{aligned}\psi'''(x) = &-\frac{1}{x^2} + \frac{1}{(1-x)^2}\\ &+ 2\,\frac{q-2}{q-1}\Bigl(\frac{u(t)}{1-x}\Bigr)^2\Bigl(3\frac{m'(t)}{m(t)} + t\frac{m''(t)}{m(t)} + t\Bigl(\frac{m'(t)}{m(t)}\Bigr)^2\Bigr),\end{aligned} \tag{2.5.34}$$

where m' and m'' denote the first and second derivative of $t \mapsto m(t)$ and where we have used Lemma 2.5.3 to differentiate $\Phi_{t,0}^{(q-1)}$. Since we want to show $\psi'''(x) < 0$ and we know that $x \leq \tilde{a} < 1/2$, it suffices to prove the inequality

$$3\frac{m'(t)}{m(t)} + t\frac{m''(t)}{m(t)} + t\Bigl(\frac{m'(t)}{m(t)}\Bigr)^2 < 0. \tag{2.5.35}$$

Differentiating both sides of Eq. (2.5.31) and solving for $m'(t)$ yields

$$\frac{m'(t)}{m(t)} = \frac{\alpha(t)}{1 - t\alpha(t)}. \tag{2.5.36}$$

Taking another derivative with respect to t allows us to express $m''(t)/m(t)$ in terms of $\alpha(t)$ and $\alpha'(t)$. In conjunction with Eq. (2.5.36), this shows that Eq. (2.5.35) is equivalent to

$$3 + t\frac{\alpha'(t)}{\alpha(t)} < 0. \tag{2.5.37}$$

Differentiating Eq. (2.5.33) and applying Eqs. (2.5.32) and (2.5.36), we have

$$\alpha'(t) = \alpha(t)\frac{u(t)}{t[1-t\alpha(t)]}\left(1 - 2\frac{\mathrm{e}^{u(t)}}{\mathrm{e}^{u(t)}+q-2}\right). \tag{2.5.38}$$

Writing Eq. (2.5.37) back in terms of $u(t)$, we see that Eq. (2.5.35) is equivalent to the inequality

$$3\left(1 - \frac{t(q-1)\,\mathrm{e}^{u(t)}}{(\,\mathrm{e}^{u(t)}+q-2)^2}\right) < u(t)\frac{\mathrm{e}^{u(t)}-q+2}{\mathrm{e}^{u(t)}+q-2}. \tag{2.5.39}$$

The rest of the proof is spent on proving Eq. (2.5.39).

We first use that $x \le \tilde{a}$ implies $t \ge \beta_{\text{MF}}^{(q-1)} = 2\frac{q-2}{q-3}\log(q-2)$ and so the left-hand side of Eq. (2.5.39) increases if we replace t by $\beta_{\text{MF}}^{(q-1)}$. After this, there is no explicit dependence on t and so we may regard the result as an inequality for the quantity u. Clearing denominators, substituting $s = \mathrm{e}^u$, and recalling that $u(t) \ge 2\log(q-2)$ for $x \le \tilde{a}$, it suffices to show that

$$\gamma(s) = A_q s + s^2\log s - \lambda^2\log s - 3s^2 - 3\lambda^2 \tag{2.5.40}$$

is strictly positive for all $s \ge \lambda^2$ and all $q \ge 4$, where $\lambda = q-2$ and

$$A_q = 3(q-1)\beta_{\text{MF}}^{(q-1)} - 6(q-2). \tag{2.5.41}$$

Since $\beta_{\text{MF}}^{(q-1)} \ge 2.5$ for $q \ge 4$, we easily check that $A_q \ge 10$ once $q \ge 4$.

First we will observe that γ is actually increasing for all $s \ge \lambda^2$. Indeed, a simple calculation shows that, for such s, we have $\gamma'(s) \ge \omega(s)$, where

$$\omega(s) = A_q - 1 + 2s\log s - 5s. \tag{2.5.42}$$

Next we find that $\min_{s\ge0}\omega(s) = A_q - 1 - 2\,\mathrm{e}^{3/2}$. Since $\mathrm{e}^{3/2} \approx 4.48$ and $A_q \ge 10$, we have that ω—and hence γ'—are strictly positive for $s \ge \lambda^2$. Hence γ is increasing for all s of interest.

Once we know that γ is increasing, it suffices to show that $\gamma(\lambda^2)$ is positive. Here we note that

$$\gamma(\lambda^2) = \frac{q-1}{q-3}(q-2)^2\big\{(q^2-3q+6)2\log(q-2) - 3(q-1)(q-3)\big\} \qquad (2.5.43)$$

and so $\gamma(\lambda^2)$ is positive once

$$2\log(q-2) > 3\,\frac{(q-1)(q-3)}{q^2-3q+6}. \qquad (2.5.44)$$

Noting that the right-hand side is less than 3, and using that $2\log 5 > 3$, this holds trivially for $q \geq 7$. In the remaining cases $q = 4, 5, 6$, the inequality is verified by direct calculation. □

Using Lemma 2.5.8 we arrive at the following conclusion:

Corollary 2.5.9. *Let $q \geq 4$, $\beta \geq 0$ and $h < 0$. Then $\Phi_{\beta,h}^{(q)}$ has at most one (symmetric) global minimizer with $x_1 < x_2 = \cdots = x_q$ and at most one (asymmetric) global minimizer with $x_1 < x_2 = \cdots = x_{q-1} < x_q$.*

Proof. Let $(x_1, \dots, x_q)$ correspond to a minimizer of $\Phi_{\beta,h}^{(q)}$. Since $h < 0$, Lemma 2.5.4 allows us to assume that $x_1 < x_2 = \cdots = x_{q-1} \leq x_q$. If $x_2 = \cdots = x_q$, then a simple calculation shows that the quantity θ, which is related to x_1 via $x_1 = \frac{1}{q} - \frac{q-1}{q}\theta$, obeys the equation $g(\theta) = \theta$, where g is as in Eq. (2.5.27). By Lemma 2.5.7, such a solution is unique and so there is at most one symmetric minimizer.

Next let us assume that x_q exceeds the remaining components. Note that we must have that $\beta(1 - x_1) \geq \beta_{\text{MF}}^{(q-1)}$ because otherwise Eq. (2.5.21) implies that $(x_2, \dots, x_q)$, properly scaled, would correspond to the $(q-1)$-state Potts model in the high-temperature regime. Since in addition $x_1 < 1/q$, we are permitted to use Lemma 2.5.8 and conclude that x_1 is a minimizer of the function ψ from Eq. (2.5.30). As is seen from its definition and Eq. (2.5.29), ψ starts off convex (and decreasing) at $x = 0$ and, as x increases, may eventually turn concave. In particular, there could be at most two points in $[0, \tilde{a}]$ where ψ achieves its absolute minimum—one in $(0, \tilde{a})$ and the other at $\tilde{a}$.

We claim that if $\psi'(\tilde{a}) < 0$ then $\tilde{a}$ cannot be the first coordinate of an asymmetric global minimizer. Indeed, if ψ is strictly decreasing at $\tilde{a}$, then the free energy could be lowered by increasing the first component beyond $\tilde{a}$. Therefore, if $\psi'(\tilde{a}) < 0$, then ψ has at most one *relevant* minimum in $[0, \tilde{a}]$. On the other hand, the above concavity-convexity picture implies that, once $\psi'(\tilde{a}) \geq 0$, there is *only one* point in $[0, \tilde{a}]$ where ψ is minimized. Hence, in all cases, there is at most one asymmetric minimizer. □

The proof of Theorem 2.3.5 will require some comparisons between the two minimizers allowed by Corollary 2.5.9. These are stated in the following lemma.

Lemma 2.5.10. *Let $q \geq 4$, $\beta \geq 0$ and $h \in (-\infty, 0)$. Suppose that $\Phi_{\beta,h}^{(q)}$ has two minimizers, one symmetric with $x_1^{(S)} < x_2^{(S)} = \cdots = x_q^{(S)}$ and the other asymmetric with $x_1^{(A)} < x_2^{(A)} = \cdots = x_{q-1}^{(A)} < x_q^{(A)}$. Then*

$$x_1^{(A)} < x_1^{(S)} \quad \text{and} \quad x_q^{(S)} < x_q^{(A)}. \tag{2.5.45}$$

Moreover, let $e_A = [x_1^{(A)}]^2 + \cdots + [x_q^{(A)}]^2$ and $e_S = [x_1^{(S)}]^2 + \cdots + [x_q^{(S)}]^2$. Then there exists a constant $c_q > 0$ such that for any $h \in [-\infty, 0)$ and any $\beta \geq 0$ where both minimizes of $\Phi_{\beta,h}^{(q)}$ "coexist," we have

$$e_A - e_S \geq c_q. \tag{2.5.46}$$

Both parts of this lemma are based on the following fact. Let $(x_1, \ldots, x_q)$ be a minimizer of $\Phi_{\beta,h}^{(q)}$ ordered such that $x_1 < x_2 = \cdots = x_{q-1} \leq x_q$. The stationarity condition yields

$$x_1 \, \mathrm{e}^{-\beta x_1 - h} = x_2 \, \mathrm{e}^{-\beta x_2} = \cdots = x_q \, \mathrm{e}^{-\beta x_q}, \tag{2.5.47}$$

and so let Θ denote the common value of this equality. Then we have:

Lemma 2.5.11. *Let $h < 0$ and $\beta \geq 0$. If Θ and Θ' correspond to two minimizers of $\Phi^{(q)}_{\beta,h}$, and $\Theta = \Theta'$, then the minimizers are the same (up to permutations in the last $q-1$ indices).*

Proof. Suppose that both minimizers are ordered increasingly. By $h < 0$ and Lemma 2.5.2, $x_1 \leq x_2 = \cdots = x_{q-1}$. The fact that $(\frac{x_2}{1-x_1}, \dots, \frac{x_q}{1-x_1})$ is the minimizer of $\Phi^{(q-1)}_{\beta(1-x_1),h}$—see Eq. (2.5.21)—then implies $\beta x_k \leq 1$ for all $k = 1, \dots, q-1$. Since the function $r(x) = x\,\mathrm{e}^{-\beta x}$ is invertible for x with $\beta x \leq 1$, equality of the Θ's implies equality of the first $q-1$ coordinates. The constraint on the total sum implies equality of the x_q's as well. □

Proof of Lemma 2.5.10. We will first attend to the proof of Eq. (2.5.45). In light of Eq. (2.5.21), the $(q-1)$-state Potts system on $(x_2, \dots, x_q)$ is at the effective temperature $\beta^{(\mathrm{S})}_{\text{eff}} = (1 - x^{(\mathrm{S})}_1)\beta$ for the symmetric minimizer and $\beta^{(\mathrm{A})}_{\text{eff}} = (1 - x^{(\mathrm{A})}_1)\beta$ for the asymmetric minimizer. But for both symmetric and asymmetric minimizers to "coexist" we must have $\beta^{(\mathrm{S})}_{\text{eff}} \leq \beta^{(q-1)}_{\text{MF}} \leq \beta^{(\mathrm{A})}_{\text{eff}}$ and so $x^{(\mathrm{A})}_1 \leq x^{(\mathrm{S})}_1$. To rule out the equality sign, we note that if $x^{(\mathrm{A})}_1 = x^{(\mathrm{S})}_1$, then the corresponding Θ's are the same and Lemma 2.5.11 thus forces equality of all components. Once $\beta^{(\mathrm{S})}_{\text{eff}} \leq \beta^{(\mathrm{A})}_{\text{eff}}$ is known, $x^{(\mathrm{S})}_q < x^{(\mathrm{A})}_q$ follows.

In order to prove Eq. (2.5.46), let ϕ be the common value of $\Phi^{(q)}_{\beta,h}$ for the two minimizers and let Θ_{A} and Θ_{S} be the corresponding Θ's. Let us take the logarithm of every term in (2.5.47), multiply the result for the j-th term by x_j and add these all up to get

$$\phi - \frac{\beta}{2} e_{\mathrm{S}} = \log \Theta_{\mathrm{S}} \quad \text{and} \quad \phi - \frac{\beta}{2} e_{\mathrm{A}} = \log \Theta_{\mathrm{A}}. \tag{2.5.48}$$

As $x^{(\mathrm{A})}_1 < x^{(\mathrm{S})}_1 \leq 1/\beta$, we have $\Theta_{\mathrm{A}} < \Theta_{\mathrm{S}}$ for all $h \in (-\infty, 0)$; for $h = 0, -\infty$ this holds by a direct argument for the zero-field Potts model. Hence $e_{\mathrm{S}} < e_{\mathrm{A}}$ whenever the two minimizers are "coexist."

To see that the positivity of $e_{\text{A}} - e_{\text{S}}$ holds uniformly in $(h, \beta) \in [-\infty, 0] \times [0, \infty]$, we use a compactness argument. First, we only need to worry about the β's in a finite, closed interval I_q. Indeed, the effective temperature of the Potts model, $\beta_{\text{eff}} = \beta(1 - x_1)$, is a number between β and $\beta(1 - 1/q)$ and so if either $\beta < \beta_{\text{MF}}^{(q-1)}$ or $\beta(1 - 1/q) > \beta_{\text{MF}}^{(q-1)}$, then no coexistence of minimizers is possible.

Next let us consider a sequence of (h, β) in $[-\infty, 0] \times I_q$ with a topology that makes this set compact. If $e_{\text{A}} - e_{\text{S}}$ tends to zero along this sequence, the above arguments imply that the asymmetric and symmetric minimizers must coalesce as the parameters tend to a limiting point. But this is impossible because by the second half of Eq. (2.2.17), the scalar magnetization of the corresponding $(q-1)$-state Potts model, which is proportional to the ratio of $x_q^{(\text{A})} - x_2^{(\text{A})}$ and $1 - x_1^{(\text{A})}$, is always at least $\frac{q-3}{q-1}$. □

Remark 2.5.12. The previous proof kept the distinctness of e_{S} and e_{A} in the realm of the existential. A calculation actually shows that, for any $h < 0$, there are constants $e_1 < e_2$ depending only on q such that $e_{\text{S}} < e_1$ and $e_{\text{A}} > e_2$ whenever the two minimizers "coexist."

Proof of Theorem 2.2.4. Fix $\beta \geq 0$ and $h < 0$. Corollary 2.5.9 implies that, up to a permutation in all-but-the-first component, $\Phi_{\beta,h}^{(q)}$ has at most two global minimizers: one symmetric $\boldsymbol{m}_{\text{S}}$ and one asymetric $\boldsymbol{m}_{\text{A}}$. This proves part (1) of the theorem.

Among the global minima, the first barycentric coordinate $x_1 = x_1(\beta, h)$ is (strictly) increasing in h (see Lemma 2.5.2) and so the effective coupling $\beta_{\text{eff}}(h) = \beta(1 - x_1(\beta, h))$, which governs the $(q-1)$-state Potts model on $(x_2, \dots, x_q)$, is decreasing. Now if $\beta_{\text{eff}}(h) > \beta_{\text{MF}}^{(q-1)}$ then only the asymmetric minimum is relevant, while if $\beta(h) < \beta_{\text{MF}}^{(q-1)}$ then only the symmetric minimum applies. Hence, for $\beta \in (\beta_{\text{MF}}^{(q-1)}, \beta_{\text{MF}}^{(q)})$, there is a unique $h_2 = h_2(\beta)$ such that the role of minimizers changes as h increases through h_2. (For β outside $(\beta_{\text{MF}}^{(q-1)}, \beta_{\text{MF}}^{(q)})$, the minimizers are in qualitative agreement with those of $h = -\infty$ or $h = 0^-$.) In particular, the minimizer is unique for $h \neq h_2(\beta)$

and both minimizers "coexist" for $h = h_2(\beta)$.

Modulo the definition of function $\beta_-^{(q)}$, parts (2-4) of the theorem are proved. It remains to show that $\beta \mapsto h_2(\beta)$ is strictly increasing (and thus invertible), continuous and with limits $-\infty$ and 0 at the left and right endpoints of $(\beta_{\text{MF}}^{(q-1)}, \beta_{\text{MF}}^{(q)})$, respectively. By Lemma 2.5.10, the quantities e_{S} and e_{A} are separated by a "gap." A simple limiting argument (not dissimilar to that used in the proof of Theorem 2.2.3) now shows that h_2 is continuous. Moreover, by Lemma 2.5.3, the norm-squared of all minimizers increases with β, and so h_2 is strictly monotone and the limits of h_2 at the endpoints of $(\beta_{\text{MF}}^{(q-1)}, \beta_{\text{MF}}^{(q)})$ must be as stated. These facts allow us to define $\beta_-^{(q)}$ as the inverse of h_2 and verify all its properties in part (5) of the theorem. □

2.6 Proofs: Actual systems

Here we will provide the proofs of our results for actual spin systems. The main portion of the arguments has already been given in Sects. 2.4 and 2.5. We will draw freely on the notation from these sections. The proofs are fairly straightforward (and mostly existential) and so we will stay rather brief.

First we will attend to the zero-field Potts model:

Proof of Theorem 2.3.4. The proof is more or less identical to that of Theorem 2.1 of Ref. [21]; the only substantial difference is that now we are not permitted to assume that the magnetization is monotone (indeed, some of the $J_{x,y}$'s may be negative). We will base our arguments on the mean-field properties of the zero-field Potts model, as outlined in Sect. 2.2.2.

Recall the mean-field free-energy function $\Phi_{\beta,0}^{(q)}$ from Eq. (2.2.15). By the fact that the global minimizer of $\Phi_{\beta,0}^{(q)}$ changes from symmetric to asymmetric as β increases through $\beta_{\text{MF}}^{(q)}$, we can make the following conclusions: Given $\beta \approx \beta_{\text{MF}}^{(q)}$, let $\mathcal{U}_\epsilon$ be an ϵ-

neighborhood of $\boldsymbol{m} = \mathbf{0}$ and let $\mathcal{V}_\epsilon$ be the union of ϵ-neighborhoods of the asymmetric minimizers. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that for all β with $|\beta - \beta_{\text{MF}}^{(q)}| \le \epsilon$ the set

$$\mathcal{O}_\delta = \bigl\{\boldsymbol{m} \in \text{ Conv}\,(\Omega) \colon \Phi_{\beta,0}^{(q)}(\boldsymbol{m}) - F_{\text{MF}}(\beta, 0) < \delta\bigr\} \tag{2.6.1}$$

is contained in $\mathcal{U}_\epsilon \cup \mathcal{V}_\epsilon$. Moreover, if $\beta = \beta_{\text{MF}}^{(q)} - \epsilon$, then $\mathcal{O}_\delta \subset \mathcal{U}_\epsilon$ while at $\beta = \beta_{\text{MF}}^{(q)} + \epsilon$, we have $\mathcal{O}_\delta \subset \mathcal{V}_\epsilon$.

Let $\mathscr{M}_\star(\beta, 0)$ be the set of "extremal magnetizations." By Theorem 2.3.2, if the integral $\mathscr{I}$ in Eq. (2.3.9) is so small that $\beta \frac{\kappa}{2} n \mathscr{I} = \beta \frac{q-1}{2} \mathscr{I} \le \delta$ for all β with $\beta \le \beta_{\text{MF}}^{(q)} + \epsilon$, then $\mathscr{M}_\star \subset \mathcal{O}_\delta$. Now the asymmetric minimizers have norm at least $1/2$, and the near-monotonicity of the magnetization from Lemma 2.4.9 thus implies that, at some β_{t} with $|\beta_{\text{t}} - \beta_{\text{MF}}^{(q)}| \le \epsilon$, the physical magnetization jumps from some value inside $\mathcal{U}_\epsilon$ to some value inside $\mathcal{V}_\epsilon$. The jump (of this size) is unique by Lemma 2.4.9. From here the claims (2.3.17–2.3.19) follow. □

Next we dismiss the cases with non-zero field:

Proof of Theorem 2.3.5. Let h_{c} be the quantity from Theorem 2.2.3 and $\beta_{\text{MF}}^{(q)}(h)$ be the concatenation of functions β_+ and β_- from Theorems 2.2.3 and 2.2.4. An argument similar to the one used in the previous proof shows that, for each $\epsilon > 0$ there exists $\delta > 0$, such that if $\beta_{\text{MF}}^{(q)} \frac{\kappa}{2} n \mathscr{I} \le \delta$ and $h \le h_{\text{c}} - \epsilon$, a strong first-order transition occurs at some $\beta_{\text{t}}(h)$ which is within ϵ of $\beta_{\text{MF}}^{(q)}(h)$. This transition is manifested by a jump in both magnetization and energy density. This proves part (1) of the theorem.

As to part (2), by Lemma 2.5.10 we know that the first components of the two minimizers are uniformly separated whenever h is confined to a compact subset of $(-\infty, h_{\text{c}})$. Since our general bounds in Theorem 2.3.2 imply that the physical magnetizations at $(h, \beta_{\text{t}}(h))$ are very near their mean-field values provided $\mathscr{I}$ is sufficiently small, also the first components thereof must be different. Using the monotonicity of the first component of physical minimizers in h, the existence of a jump in $m_\star(\beta, h)$ on the transition line follows. $\square$

Proof of Theorem 2.3.7. The proof is based on Theorem 2.3.6 and Lemma 2.5.1. Indeed, Theorem 2.3.6 implies that all minima are characterized by the fact that one of (x_1, x_0, x_{-1}) is larger than $1 - C\,\mathrm{e}^{-\beta}$. These minima are nearly degenerate for λ of order $\mathrm{e}^{-\beta}$ with free energy difference given by $\lambda - \mathrm{e}^{-\beta} + O(\beta\,\mathrm{e}^{-\beta})$. The goal is to show that the free energy is uniformly large (on the scale of $\mathrm{e}^{-\beta}$ in the complement of the $C\,\mathrm{e}^{-\beta}$-neighborhood of these minima.

Let $C \gg 1$ be the number exceeding the corresponding constant from Theorem 2.3.6 and suppose that $|\lambda| \le C\,\mathrm{e}^{-\beta}$. Consider the set $\mathcal{O}_\beta$ of all triplets (x_1, x_0, x_{-1}) with $x_1 + x_0 + x_{-1} = 1$, such that $\max\{x_1, x_0, x_{-1}\} > 1 - C\,\mathrm{e}^{-\beta}$. We claim that for $\beta \ge \beta_0$ (with β_0 depending on C),

$$\inf_{(x_1,x_0,x_{-1})\in\mathcal{O}_\beta^{\mathrm{c}}} \Phi_{\beta,\lambda}(x_1, x_0, x_{-1}) \ge \alpha(C \log C)\,\mathrm{e}^{-\beta}, \tag{2.6.2}$$

where α is a positive number independent of C. Indeed, Theorem 2.3.6 implies that all local minima of $\Phi_{\beta,\lambda}$ lie in $\mathcal{O}_\beta$, and so the absolute minimum of $\Phi_{\beta,\lambda}$ must occur on the boundary of $\mathcal{O}_\beta^{\mathrm{c}}$. But the "outer" boundary of $\mathcal{O}_\beta^{\mathrm{c}}$ is not a possibility, and so the mimimum occurs at a point with $\max\{x_1, x_0, x_{-1}\} = 1 - C\,\mathrm{e}^{-\beta}$. The bound (2.6.2) is then a consequence of Lemma 2.5.1.

Let now the integral $\mathscr{I}$ in Eq. (2.3.9) be such that $\beta\mathscr{I} \ll (C \log C)\,\mathrm{e}^{-\beta}$. Then Theorem 2.3.2 ensures that all physical magnetizations (from $\mathscr{M}_\star$) are contained inside $\mathcal{O}_\beta$. However, by Eq. (2.3.23), for β such that $\lambda - \mathrm{e}^{-\beta} \ge O(\beta\,\mathrm{e}^{-\beta})$ the set $\mathcal{O}_\beta$ contains no triplets with dominant $x_{\pm 1}$ while for $\lambda - \mathrm{e}^{-\beta} \le O(\beta\,\mathrm{e}^{-\beta})$, there are no x_0-dominant states. The standard thermodynamic arguments imply that the amount of zero-ness decreases as λ increases. Hence, there must be a jump at some $\lambda_{\mathrm{t}} = \mathrm{e}^{-\beta} + O(\beta\,\mathrm{e}^{-\beta})$ from states dominated by 0's to those where 0's are very sparse. This finishes the proof. □

CHAPTER 3

A Fine Analysis of the Mean Field Transverse Ising Model

3.1 Introduction

One of the simplest classical systems exhibiting phase transition is the Curie-Weiss model. In this model, N Ising spins $\sigma_i = \pm 1$, $x = 1, \dots, N$, interact via the Hamiltonian

$$H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j, \tag{3.1.1}$$

where the normalization by $1/N$ makes H_N a quantity of order N. (Generalizations to multiple-spin interactions may also be considered; cf Sect. 3.2. However, for the time being, (3.1.1) will suffice.)

As is well known [58], for the spins distributed according to the measure $\mu_{N,\beta}(\{\sigma\}) \propto \mathrm{e}^{-\beta H_N(\sigma)}$, i.e., in the canonical ensemble, as $N \to \infty$ the law of the empirical mean $m_N(\sigma) = N^{-1} \sum_x \sigma_i$ converges to a mixture of point masses at $\pm m_\star$ where $m_\star = m_\star(\beta)$ is the so called spontaneous magnetization. The phase transition in this model is manifestly seen from the observation that $m_\star(\beta) \equiv 0$ for $\beta \le \beta_\mathrm{c}$ while $m_\star(\beta) > 0$ for $\beta > \beta_\mathrm{c}$. The function $\beta \mapsto m_\star(\beta)$ is in fact the maximal non-negative solution of the equation

$$m = \tanh(\beta m), \tag{3.1.2}$$

while $\beta_\mathrm{c} = 1$ is the so called critical temperature.

The main reason why the Curie-Weiss model is so approachable is the fact that the Hamiltonian is, to within an additive constant, equal to $-\frac{1}{2}Nm_N(\sigma)^2$. This permits a very explicit expression for the law of m_N whose concentration properties are then readily controlled by straightforward large-deviation arguments. However, this simple strategy breaks down once inhomogeneous terms (i.e., those not invariant under exchanges of the spins) are added to the Hamiltonian. One example where this happens is the Curie-Weiss system in random external field where the term $\sum_x h_x\sigma_i$, with h_x sampled from an i.i.d. law with zero mean, is added to H_N. While rigorous analysis is still possible in this case, the technical difficulties involved are more substantial. Significantly more complex is the Sherrington-Kirkpatrick version of (3.1.1), where the term $\sigma_i\sigma_j$ is weighed by a (fixed) random number $J_{x,y}$ that has been sampled from a symmetric distribution on $\mathbb{R}$. This model possesses a beautiful underlying structure [99, 110] which has been harnessed mathematically only very recently [72, 125, 126].

The goal of this paper is to study another natural generalization of the Curie-Weiss model, namely to the realm of *quantum mechanics*. Here the spin variables σ_i are replaced by the z-component of the triplet of Pauli matrices $(\sigma^{(x)}, \sigma^{(y)}, \sigma^{(z)})$—the generators of $\mathfrak{su}(2)$—acting on the one-particle Hilbert space $\mathscr{H}_1 =$ span$\{|+\rangle, |-\rangle\}$. The configuration space is replaced by the product space $\mathscr{H}_N = \bigotimes_{i=1}^N \mathscr{H}_1$; the spin operator $\sigma_i^{(k)}$ for the spin at i acts on a product vector $|\varphi\rangle = \bigotimes_{i=1}^N |\varphi_i\rangle \in \mathscr{H}_N$ via

$$\hat{\sigma}_i^{(k)}|\varphi\rangle = |\varphi_1\rangle \otimes \cdots \otimes (\hat{\sigma}^{(k)}|\varphi_i\rangle) \otimes \cdots \otimes |\varphi_N\rangle. \tag{3.1.3}$$

There are at least two natural ways to introduce quantum effects into the Curie-Weiss model: Either make the interaction term isotropic—this corresponds to the quantum Heisenberg model—or consider an external transverse field. Here we focus on the latter situation: The Hamiltonian is now an operator on $\mathscr{H}_N$ defined by

$$\hat{\mathcal{H}}_N = -\frac{1}{2N}\sum_{i,j=1}^{N}\sigma_i^{(z)}\sigma_j^{(z)} - \sum_{i=1}^{N}(h\sigma_i^{(z)} + \lambda\sigma_i^{(x)}). \tag{3.1.4}$$

The Gibbs probability measure $\mu_{N,\beta}$ is replaced by a *KMS state* $\langle - \rangle_{N,\beta}$ which is a positive linear functional on the $C^\star$-algebra of operators on $\mathscr{H}_N$ defined by

$$\langle A \rangle_{N,\beta} = \frac{\text{Tr}(A\ \text{e}^{-\beta\,\mathcal{H}_N})}{\text{Tr}(\ \text{e}^{-\beta\mathcal{H}_N})}. \tag{3.1.5}$$

As before, the parameter β plays the role of inverse temperature while λ, which corresponds to the strength of an external field, determines the overall strength of the quantum perturbation.

Despite its relatively clean formulation, the quantum nature makes this model very different from the classical one. Indeed, the Hamiltonian is, for $\lambda \neq 0$, not simultaneously diagonalizable with the $\sigma_i^{(z)}$'s and so the very notion of "value of spin at i" is apparently lost. (For $\lambda = 0$ the Hamiltonian is diagonalized in the basis $|\sigma\rangle$; the eigenvalues are the values of the Hamiltonian for the classical Curie-Weiss model.) Fortunately, a classical system may be recovered if one resorts to auxiliary variables. An approach along these lines leads to a graphical representation [7] whose percolation properties are often related to the existence/absence of phase transitions. A study of a model closely related to the above—namely, its quantum-percolation version—based on graphical representations has been made recently [79].

Our strategy in this paper will be a variation on this idea and will be close in spirit to the way the classical Curie-Weiss model is solved: Using the Lie-Trotter formula, we will represent the matrix element of the Gibbs-Boltzmann weight $\text{e}^{-\beta\mathcal{H}_N}$ in the basis of classical Ising states $\{|\sigma\rangle\colon \sigma\in\{-1,+1\}^N\}$—the eigenbasis of $\hat{\sigma}_i^{(z)}, x = 1,\dots,N$—as an expectation,

$$\langle\sigma|\ \text{e}^{-\beta\mathcal{H}_N}|\tilde{\sigma}\rangle = \mathbb{E}_{\lambda,\sigma}\big(\ \text{e}^{N\int_0^\beta[\frac{1}{2}m_N(t)^2 + hm_N(t)]\ \text{d}t}\big), \tag{3.1.6}$$

over a collection $\sigma(t) = (\sigma_i(t))_{1\leq x\leq N}$ of N independent Poisson point processes on circles $\mathbb{S}_\beta$ with arrival rate λ, one for each site i. The quantity $m_N(t)$ is the empirical magnetization at time t, i.e., $m_N(t) = N^{-1}\sum_{i=1}^N \sigma_i(t)$; the interaction is thus the classical interaction averaged over "time."

With this representation in place, we then apply methods of large-deviation theory to derive the leading-order $N \to \infty$ asymptotic of these expectations. As can be expected, many physical properties of the quantum system may be gleaned from the properties of the minimizer of the corresponding variational problem. The variational problem (or rather dual thereof) turns out to have intrinsic features which allow us bring methods of FK percolation to bear. Many quantitative characteristics of the system will be determined explicitly (at least in the limit $N \to \infty$). In particular, we will obtain full control of the phase diagram and stability near optimizers of the variational problem. Moreover we give an explicit characterization for the critical exponent for the decay of the optimizer as one approaches the critical curve.

3.2 The Mean Field Transverse Ising Model as a Stochastic Process

A slight generalization of our Hamiltonian of interest in this chapter is given by

$$-\mathcal{H}_N = N\psi\left(\frac{1}{N}\sum_{i=1}^N \sigma_i^{(z)}\right) + \lambda\sum_{i=1} N\sigma_i^{(x)} \tag{3.2.1}$$

where φ is some polynomial function from $\mathbb{R}$ to $\mathbb{R}$. In contrast to our previous probabilistic representation of this model 1.2.17, consider the following partial linearization in the Lie-Trotter product formula:

$$\frac{e^{-\beta\mathcal{H}_N}}{e^{\lambda N}} = \lim_{\Delta\to 0}\left(e^{\frac{\Delta}{N}\varphi(\sum_{i\leq N} {}^1/_N \sum_{i=1}^N \sigma_i^{(z)})}\prod_i\left\{(1-\Delta\lambda)\mathrm{I} + \Delta\lambda(\sigma_i^{(x)}+\mathrm{I})\right\}\right)^{\beta/\Delta}. \tag{3.2.2}$$

Since the matrices $e^{\frac{\Delta}{N}\psi(\sum_{i\leq N} 1/N \sum_{i=1}^{N}\sigma_i^{(z)})}$ are diagonal in the z-basis,

$$\left\langle \sigma | e^{\frac{\Delta}{N}\psi(\sum_{i\leq N} 1/N \sum_{i=1}^{N}\sigma_i^{(z)})} | \sigma' \right\rangle = e^{\frac{\Delta}{N}\psi(\sum_{i\leq N} 1/N \sum_{i=1}^{N}\sigma_i)}. \tag{3.2.3}$$

We begin by introducing a convenient notation for expectation values. Given a probability measure $\mathbb{P}$ on a sample space Ω and an integrable function $f : \Omega \to \mathbb{R}$, let us denote the expectation value of f by $\mathbb{P}(f)$.

Let $\mathbb{P}_\beta^\lambda$ be the distribution of the Poisson point process (of holes) on the circle $\mathbb{S}_\beta$ with arrival intensity λ. We shall use $\otimes\mathbb{P}_\beta^\lambda$ for the product distribution of N independent copies $\xi = (\xi_1, \ldots, \xi_N)$.

Given a realization of ξ let us say that a classical piece-wise constant trajectory $\underline{\sigma} : \mathbb{S}_\beta^N \mapsto \{\pm 1\}^N$ is compatible with ξ; $\underline{\sigma} \sim \xi$, if for every $i = 1, \ldots, N$ jumps of $\sigma_i(\cdot)$ occur only at arrival times of ξ_i. Passing to the limit in (3.2.2) we, in view of (3.2.3), infer

$$\frac{\mathrm{Tr}\left(e^{-\beta\mathcal{H}_N}\right)}{e^{\lambda N}} = \int \otimes\mathbb{P}_\beta^\lambda(\mathrm{d}\xi) \sum_{\underline{\sigma}\sim\xi} \exp\left\{\int_0^\beta \psi\left(1/N \sum_i \sigma_i(t)\right)\mathrm{d}t\right\}. \tag{3.2.4}$$

For every i let $\#(\xi_i)$ be the number of connected components of $\mathbb{S}_\beta \setminus \xi_i$. Evidently, the number of all compatible $\underline{\sigma} \sim \xi$ equals to $2^{\sum_i \#(\xi_i)}$. Define

$$\widetilde{\mathbb{P}}_\beta^\lambda(\mathrm{d}\xi_i) = \frac{2^{\#(\xi_i)}\mathbb{P}_\beta^\lambda(\mathrm{d}\xi_i)}{\mathbb{P}_\beta^\lambda\left(2^{\#(\xi_i)}\right)}$$

This is a one-circle FK measure with respect to the Poisson point process of arrivals. Consider probability distribution $\mathbb{Q}_\beta^\lambda$ on piece-wise constant classical one-circle spin trajectories $\sigma_i(\cdot) : \mathbb{S}_\beta \mapsto \{\pm 1\}$ which is generated by the following two step procedure: First sample ξ_i from $\widetilde{\mathbb{P}}_\beta^\lambda$, and then paint connected components of $\mathbb{S}_\beta \setminus \xi_i$ into ± 1, independently and with probability $1/2$ each. Let $\otimes\mathbb{Q}_\beta^\lambda$ be the corresponding product measure. It is straightforward to check that the righthand side of (3.2.4) equals to

$$\left[\mathbb{P}_\beta^\lambda\left(2^{\#(\xi)}\right)\right]^N \cdot \otimes\mathbb{Q}_\beta^\lambda\left(\exp\left\{\int_0^\beta \psi\left(1/N \sum_i \sigma_i(t)\right)\mathrm{d}t\right\}\right).$$

Consequently, an analysis of phase diagram of the CW model in transverse field boils down to an investigation of asymptotic properties for weighted measures

$$\otimes\widetilde{\mathbb{Q}}_\beta^\lambda(\mathrm{d}\sigma) \triangleq \frac{\otimes\mathbb{Q}_\beta^\lambda\left(\exp\left\{N\int_0^\beta \psi(m_N(t))\mathrm{d}t\right\}\,;\,\mathrm{d}\sigma\right)}{\otimes\mathbb{Q}_\beta^\lambda\left(\exp\left\{N\int_0^\beta \psi(m_N(t))\mathrm{d}t\right\}\right)}, \tag{3.2.5}$$

where,

$$m_N(t) = \frac{1}{N}\sum_i \sigma_i(t).$$

This problem belongs to the realm of theory of large deviations. Formally, the measures (3.2.5) are asymptotically concentrated around solutions of

$$\sup_m \left\{\int_0^\beta \psi(m(t))\mathrm{d}t - I(m)\right\} \triangleq \sup_m \mathfrak{G}(m), \tag{3.2.6}$$

where I is the large deviation rate function for the average m_N under the product measures $\otimes\mathbb{Q}_\beta^\lambda$. If we formulate the large deviation principle in $\mathbb{L}_2(\mathbb{S}_\beta)$, then, using $(\cdot,\cdot)_\beta$ for the corresponding scalar product,

$$I(m) = \sup_h \{(h,m)_\beta - \Lambda(h)\} \quad \text{where} \quad \Lambda(h) = \log\mathbb{Q}_\beta^\lambda\left(e^{(h,\sigma)_\beta}\right). \tag{3.2.7}$$

In general, one must proceed in the computation of the critical curve on a case by case basis. Let us specialize to the quadratic case. Let us define

$$\mathfrak{f}(\lambda,\beta) \triangleq \frac{1}{\beta}\mathbb{V}\mathrm{ar}_\beta^\lambda\left((\sigma,\mathbf{1})_\beta\right) = \frac{1}{\lambda}\tanh(\lambda\beta) \tag{3.2.8}$$

where $\mathbb{V}\mathrm{ar}_\beta^\lambda$ is the variance under the one-circle spin measure $\mathbb{Q}_\beta^\lambda$. The second equality is the result of an easy computation which we present in Section 3.6. The results of our analysis are summarized as follows:

Theorem 3.2.1. *Let* $\psi(x) = \frac{1}{2}x^2$*. The variational problem* (3.2.6) *has constant maximizers* $\pm m^*(\lambda,\beta)\cdot\mathbf{1}$*, where the spontaneous z-magnetization* m^* *satisfies:*

1. *If* $\mathfrak{f}(\lambda,\beta) \le 1$*, then* $m^* = 0$.

2. *If* $\mathfrak{f}(\lambda,\beta) > 1$*, then* $m^* > 0$*, and, consequently there are two distinct solutions to* (3.2.6).

Furthermore, away from the critical curve the solutions $\pm m^* \cdot \mathbf{1}$ *are stable in the following sense: There exists* $c = c(\lambda,\beta) > 0$ *and a strictly convex symmetric function* U *with* $U(0) = 0$ *and* $U''(0) > 0$*, such that*

$$\mathfrak{G}(\pm m^* \cdot \mathbf{1}) - \mathfrak{G}(m) \;\geq\; c \min\left\{\|m - m^* \cdot \mathbf{1}\|_\beta^2, \|m + m^* \cdot \mathbf{1}\|_\beta^2\right\} \;+\; \int_0^\beta U(m'(t))dt. \tag{3.2.9}$$

Finally we have the following expression for the decay of m^* *near the critical curve:*

$$(m^*)^2 \asymp \left|1 - \frac{1}{\beta}\mathbb{V}ar_0\left[(\sigma,\mathbf{1})_\beta\right]\right| \tag{3.2.10}$$

where the implicit constants are depend on β *and* $lambda$ *but are bounded below in compact regions of the parameter space.*

We remark that the second term of (3.2.9) is important in the super-critical regime ($\mathfrak{f}(\lambda,\beta) > 1$) since it rules out trajectories of $m_N(\cdot)$ with rapid transitions between the optimal values $\pm m^*$. The rest of this chapter develops our proof of this theorem.

3.3 Properties of the Single Spin Measure

We begin by giving a comprehensive description of the properties of the single spin measure $\mathbb{Q}_\beta^\lambda$ we need for analysis of the variational problem (3.2.6). Our method of extracting the necessary properties is not new, it has been used previously by Aizenman-Klein-Newman [6] and Campanino-Klein-Perez [35]. However, one novel contribution in this section is the derivation of quantitative bounds on the third derivative of $\Lambda(h)$ (for h constant functions) via the random current representation used by Aizenman [2] and Aizenman-Fernandez [5].

Let $\Omega = \{-1, 1\}^{\mathbb{S}_\beta}$ and let us define a natural ordering on spin paths $\sigma(t) \in \Omega$. We say that $\sigma \preceq \sigma'$ if $\sigma(t) \leq \sigma'(t)$ for all $t \in \mathbb{S}_\beta$ and that a function $f : \Omega \to \mathbb{R}$ is increasing if $f(\sigma) \leq f(\sigma')$ whenever $\sigma \preceq \sigma'$. For any positive $h \in L^2(\mathbb{S}_\beta)$ with finite exponential moment $\Lambda(h)$ let $Z(h) = e^{\Lambda(h)}$ Let us define

$$\mathrm{d}\mathbb{Q}^\lambda_{h\beta}(\sigma) = e^{(h,\sigma)_\beta} \mathbf{1}_{\sigma\sim\xi}\, \mathrm{d}\widetilde{\mathbb{P}}^\lambda_\beta(\xi) \tag{3.3.1}$$

and

$$\begin{aligned} U_h(r,s,t) = \big\langle \sigma(r)\sigma(s)\sigma(t)\big\rangle_h - \big\langle \sigma(r)\sigma(s)\big\rangle_h \big\langle \sigma(t)\big\rangle_h - \big\langle \sigma(s)\sigma(t)\big\rangle_h \big\langle \sigma(r)\big\rangle_h \\ - \big\langle \sigma(r)\sigma(t)\big\rangle_h \big\langle \sigma(s)\big\rangle_h + 2\big\langle \sigma(r)\big\rangle_h \big\langle \sigma(s)\big\rangle_h \big\langle \sigma(t)\big\rangle_h \end{aligned} \tag{3.3.2}$$

be the third Ursell function for the circle measure weighted by the external field h (in a formal sense the third derivative of $\Lambda(h)$).

Lemma 3.3.1. *[FKG and GHS inequalities] For any positive, bounded measurable function h on Ω, the weighted measure*

$$\frac{d\mathbb{Q}^\lambda_\beta(\sigma)}{Z(h)} \tag{3.3.3}$$

has the FKG property with respect to the partial order defined above. Moreover, we have

$$U_h(r,s,t) \leq 0 \text{ for all } r,s,t \in \mathbb{S}_\beta. \tag{3.3.4}$$

Corollary 3.3.2. *For any $h \in L^2(\mathbb{S}_\beta)$ we have $\Lambda(h) \leq \Lambda(|h|)$.*

The following lemma will be needed for the stability estimate (3.2.9).

Lemma 3.3.3. *[Random Current Bound] For any positive constant field h and any triplet of points $r, s, t \in \mathbb{S}_\beta$, we have the bound*

$$U_h(r,s,t) \leq -\big\langle \sigma_0 \big\rangle_h e^{-(4\lambda+2h)\beta} \frac{\lambda^2\beta^2}{2} \left[(s-r)^2 + (t-s)^2 + (\beta + r - t)^2 \right]. \tag{3.3.5}$$

Moreover as h tends to 0, we have

$$U_h(r,s,t) \asymp -\langle\sigma_0\rangle_h e^{-(4\lambda+2h)\beta}\frac{\lambda^2\beta^2}{2}$$
$$\left[(r-t)^2+(s-r)^2+2(t-s)^2+(\beta+r-s)^2)+(\beta+r-t)^2\right]. \quad (3.3.6)$$

These statements follow from the construction of a sequence of discrete Ising models on the circle $\mathbb{S}_\beta$ converging weakly to $\,\mathrm{d}\mathbb{Q}^{\lambda}_{h\beta}(\sigma)$. We shall outline this construction below. Using discrete versions of the corresponding statements, we conclude that the various inequalities hold in the limit.

Let $k \in \mathbb{N}$ be fixed, $\mathcal{D}_k$ denote the $h \in L^2(\mathbb{S}_\beta)$ constant on dyadic intervals of the form $[\frac{\beta j}{2^k}, \frac{\beta(j+1)}{2^k})$ and let $\mathcal{D} = \bigcup_k \mathcal{D}_k$.

Lemma 3.3.4. *Suppose $h \in \mathcal{D}$. Then $Z(h)^{-1}\, d\mathbb{Q}_{h,\lambda,\beta}(\sigma)$ is the weak limit of a sequence of Ising models defined on discrete subsets of $\mathbb{S}_\beta$.*

Proof. Given $h \in \mathcal{D}_k$, let $N \geq k$ be fixed and consider the Ising model on a lattice subset of the circle $\mathbb{S}_\beta$ with 2^N sites. For J, α fixed real numbers, the Hamiltonian for this model is defined by

$$-H_N(\sigma) = J\sum_{x=1}^{2^N}\sigma_x\sigma_{x+1} + \alpha\sum_{x=1}^{2^N}h_x\sigma_x. \quad (3.3.7)$$

where $h_x = h(\frac{\beta x}{2^N})$ and $\sigma_{2^N+1} = \sigma_1$ by definition. Let $\mathbb{P}_{J,\alpha,h}$ denote the Gibbs-Boltzman distribution associated to this Hamiltonian (at inverse temperature equal to one). Multiplying through by e^{-J2^N}

$$\mathbb{P}_{J,\alpha,h}(\sigma) \propto e^{-2J|I(\sigma)|+\alpha\sum_{x=1}^{2^N}h_x\sigma_x} \quad (3.3.8)$$

where

$$I(\sigma) = \bigcup_{\{x:\,\sigma_x\neq\sigma_{x+1}\}}\left[\frac{\beta x}{2^N}, \frac{\beta(x+1)}{2^N}\right) \quad (3.3.9)$$

and $|A|$ denotes the Lebesgue measure of $A \subset \mathbb{R}$.

Next we consider J, α scaling with N. Let

$$J = J_N = -\frac{1}{2}\log\frac{\lambda\beta}{2^N},\ \alpha = \alpha_N = \frac{\beta}{2^N} \tag{3.3.10}$$

and let X_t denote a Poisson point process with rate λ on the circle of length β which is independent of $\mathbb{P}_{J,\alpha,h}$. For these choices of J, α, we have

$$\mathbb{P}_{J_N,\alpha_N,h}(\sigma) \propto \left(e^{(h,\sigma)_\beta} + o(1)\right)\mathbb{P}\left(X_t \text{ only has arrivals in } I(\sigma)\right)(1+o(1)) \tag{3.3.11}$$

where the $o(1) \to 0$ as $N \to \infty$.

We identify, in the canonical way, the left hand side with a sequence of measures probability measures μ_N on the set of piecewise constant spin paths, *i.e.* μ_N are concentrated on paths which are constant on the intervals $[\frac{\beta x}{2^N}, \frac{\beta(x+1)}{2^N})$. For the choices J_N, α_N above, we have that the measures μ_N converge weakly (with respect functions continuous in the product topology on spin paths) to the measure

$$Z(h)^{-1}\,\mathrm{d}\mathbb{Q}_{h,\lambda,\beta}(\sigma) \tag{3.3.12}$$

.

It is clear that the standard notion of stochastic domination for Ising models translates directly to our stochastic domination for paths. In particular, this implies that for any positive $h \in \mathcal{D}$ the probability measure defined by (3.3.1) satisfies the FKG property. Since the coordinate functions

$$\pi_u(\sigma) = \sigma(u) \tag{3.3.13}$$

define the product topology, the GHS inequality for the discrete models carries over to the limiting measure, i.e. we have $U_h(r,s,t) \le 0$. □

Proof of Lemma 3.3.1. Any positive bounded measurable function is the limit of a uniformly bounded positive sequence $(h_n : h_n \in \mathcal{D})$. Therefore the bounded convergence theorem implies the FKG and GHS inequalities hold for all bounded measurable external fields $h \geq 0$.

□

Proof of Corollary 3.3.2. Suppose h is a bounded measurable function. Let $h^+ = h \vee 0$ and $h^- = -(h \wedge 0)$. Let us denote expectation values with respect to $Z(h)^{-1}\, \mathrm{d}\mathbb{Q}_{h,\lambda,\beta}(\sigma)$ by $\langle \cdot \rangle_h$. Then

$$\begin{aligned}\left\langle e^{(h,\cdot)}\right\rangle_0 =& \left\langle e^{\left(h^+ - h^-,\cdot\right)_\beta}\right\rangle_0 \leq \left\langle e^{\left(h^+,\cdot\right)_\beta}\right\rangle_0 \left\langle e^{\left(-h^-,\cdot\right)_\beta}\right\rangle_0 \\ =& \left\langle e^{\left(h^+,\cdot\right)_\beta}\right\rangle_0 \left\langle e^{\left(h^-,\cdot\right)_\beta}\right\rangle_0 \leq \left\langle e^{(|h|,\cdot)_\beta}\right\rangle_0\end{aligned}$$

where the two inequalities follow from FKG and the middle equality follows from spin flip symmetry.

For the general bound, we may assume $\Lambda(|h|)$ is finite, otherwise there is nothing to prove. It is then a consequence of Fatou's Lemma and the FKG property for the free measure. □

Proof of Lemma 3.3.3. We begin by recalling the random current representation of the third Ursell function for a discrete Ising model, see [5].

Following the notation of [5], consider a general finite graph Ising model on sites $i \in V$ and with bonds $b \in \mathcal{E} \subset V \times V$. Let us denote the coupling constants by J_b and local fields by h_i. Let $\underline{n} = (n_b)_{b \in \mathcal{E}}$ denote a sequence of integer valued 'fluxes' attached to the bonds of the underlying graph where the local fields h_i are interpreted as the coupling constants between the site i and a 'ghost' site $\mathfrak{g}$. We say that that a site $i \in V$ is a *source* if $\sum_{i \in b} n_b$ is odd and denote the collection of sources other than the ghost site by $\partial \underline{n}$.

We say that $x \leftrightarrow y$ if there exists a path of non-zero fluxes connecting x to y using bonds in $\mathcal{E}$ (i.e. not ghots site bonds). Moreover, $x \nrightarrow h$ means that whenever $x \leftrightarrow y$,

$n_{y,\mathfrak{g}} = 0$. For any subset of bonds $\mathcal{B} \subset \mathcal{E}$, let $\langle \cdot \rangle_{\mathcal{B}}$ denote the Gibbs state for the Ising model with the constants $(J_b)_{b \in \mathcal{B}}$ set to 0. Let

$$W(\underline{n}) = \prod_b \frac{J_b{}^{n_b}}{n_b!} \tag{3.3.14}$$

and

$$Z = \sum_{\partial \underline{n} = \varnothing} W(\underline{n}). \tag{3.3.15}$$

Suppose $r, s, t \in V$ are fixed sites in the Ising model under consideration. For a collection of fluxes $\underline{n}$, let us say that $b \in C_{\underline{n}}(r)$ if $i \leftrightarrow r$ for some $i \in b$. Then we have

$$U_h(r,s,t) = \left\{ \sum_{\partial \underline{n}_1 \{r\} \triangle \{s\},\, \partial \underline{n}_2 = \varnothing} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathbf{1}_{\underline{n}_1 + \underline{n}_2 : r \nleftrightarrow h} \times \left[\langle \sigma_t \rangle_{C^c_{\underline{n}_1 + \underline{n}_2}(r)} - \langle \sigma_t \rangle \right] \right\} + \{ s \Leftrightarrow t \} \tag{3.3.16}$$

where $\{s \Leftrightarrow t\}$ represents the first term with the roles of s and t interchanged. Here $\underline{n}_1$ and $\underline{n}_2$ are independent copies of fluxes.

Clearly, the weights $W(\underline{n})$ are proportional to the probability that a family of independent Poisson processes indexed by (generalized) bonds take a collection of values determined by $\underline{n}$. We need to differentiate between the processes associated to the bonds of the graph and the bonds with the ghost site $\mathfrak{g}$. Specifically, let $\{\mathcal{N}_b, \mathcal{M}_i\}_{b \in \mathcal{E}, i \in V}$ denote a collection of independent Poisson processes with respective parameters $\{J_b, h_i\}_{b \in \mathcal{E}, i \in V}$. Also, let $\mathbb{P}$ denote the joint probability measure associated to these processes.

The next few observations apply to Ising models on general finite graphs. It is well known that if $h \geq 0$, spin correlations are increasing with respect to coupling strengths, so each summand on the righthand side must be non-positive. Therefore, neglecting summands we obtain

$$U_h(r,s,t) \leq \sum_{\partial \underline{n}_1 \{r\} \triangle \{s\},\, \partial \underline{n}_2 = \varnothing} \frac{W(\underline{n}_1)}{Z} \frac{W(\underline{n}_2)}{Z} \mathbf{1}_{\underline{n}_1+\underline{n}_2 : r \nrightarrow h} 1_{\underline{n}_1+\underline{n}_2 : r \leftrightarrow t} \times \Big[\langle \sigma_t \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(r)} - \langle \sigma_t \rangle \Big] + \{s \Leftrightarrow t\} \tag{3.3.17}$$

where we drop analogous terms in the expression $\{s \Leftrightarrow t\}$ and $A \triangle B$ denotes the symmetric difference of two sets.

We concentrate on the first term on the righthand side of the inequality. To avoid singular cases assume that r, s, t are all distinct. We note two things. First, on the set $\{\underline{n}_1 + \underline{n}_2 : r \leftrightarrow t\} \cap \{\underline{n}_1 + \underline{n}_2 : r \nrightarrow h\}$ we have

$$\langle \sigma_t \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(r)} = 0 \tag{3.3.18}$$

by spin flip symmetry. Second, we have

$$\begin{aligned} &\{\partial \underline{n}_1 = \{r,s\}, \partial \underline{n}_2 = \varnothing\} \cap \{\underline{n}_1 + \underline{n}_2 : r \nrightarrow h\} = \\ &\qquad \{\underline{n}_1 : r \leftrightarrow s\} \cap \{\partial \underline{n}_1 = \{r,s\}, \partial \underline{n}_2 = \varnothing\} \cap \{\underline{n}_1 + \underline{n}_2 : r \nrightarrow h\}. \end{aligned} \tag{3.3.19}$$

In terms of the Poisson processes, let $E_{r,s,t}$ denote the event determined by the requirements of the first term on the righthand side of (3.3.19). After the reduction made above (3.3.17) may be expressed as

$$U_h(r,s,t) \leq \frac{\mathbb{P} \otimes \mathbb{P}(E_{r,s,t})}{\mathbb{P}(\underline{n}_1 = \varnothing)^2} + \{s \Leftrightarrow t\} \tag{3.3.20}$$

Let us now specialize to Ising models on the unit circle with 2^N vertices. We take coupling and field strengths given by (3.3.10). Without loss of generality we may assume the orientation

$$0 < r < s < t < \beta. \tag{3.3.21}$$

Continuity arguments imply that it is sufficient for us prove the bound (3.3.5) assuming each point is of the form ${}^{\beta j}/_{2^k}$ for some $j, k \in \mathbb{N}$ fixed.

We shall compute a lower bound for the probability determined by the numerator. For the readers convenience we note that

$$\mathbb{P}(\mathcal{N}_b = \text{odd}) \propto \tanh(J_b) = \frac{1 - \frac{\lambda\beta}{2^N}}{1 + \frac{\lambda\beta}{2^N}}$$
$$\mathbb{P}(\mathcal{N}_b = \text{even}) \propto 1$$
$$\mathbb{P}(\mathcal{N}_b = 0) \propto \cosh(J_b)^{-1} = \sqrt{\frac{\lambda\beta}{2^N}} + O(2^{-N})$$

and

$$\mathbb{P}(\mathcal{M}_i = \text{odd}) = \frac{h\beta}{2^N} e^{-\frac{\lambda\beta}{2^N}} + O(2^{-3N})$$
$$\mathbb{P}(\mathcal{M}_i = \text{even}) = e^{-\frac{\lambda\beta}{2^N}} + O(2^{-2N})$$
$$\mathbb{P}(\mathcal{M}_i = 0) = e^{-\frac{\lambda\beta}{2^N}}$$

where the implicit constant of proportionality is

$$(1 + o(1))e^{-\lambda\beta/2^N}. \tag{3.3.22}$$

For each $c, d \in [0, \beta]$, let $\lfloor c, d \rfloor$ denote the integers i in $[0, 2^N]$ such that ${}^{c2^N}/_\beta \leq i \leq {}^{d2^N}/_\beta$. If $c > d$ this is understood to be with respect to periodic boundary conditions. For any bond b let $b \in \lfloor c, d \rfloor$ denote the case that both vertices of b are in $\lfloor c, d \rfloor$. Finally, let

$$M_{\lfloor c,d \rfloor} = \{\mathcal{M}_i^{(1)} + \mathcal{M}_i^{(2)} > 0 \text{ for some } i \in \lfloor c, d \rfloor\}. \tag{3.3.23}$$

Based on topological considerations there are three cases to consider: $M_{\lfloor r,s \rfloor}$, $M_{\lfloor s,t \rfloor}$ and $M^c_{\lfloor r,t \rfloor}$. The requirements of (3.3.19) imply these three possibilities lead to disjoint sub-events of $E_{r,s,t}$. The cases carry a minimal but tedious amount of computation, so we shall present only one of them.

Let us consider $E \cap M^c_{\lfloor r,t \rfloor}$. For any $x < r < t < y$ in the dyadic lattice $\{\frac{\beta l}{2^N}\}$, let

$$A(x,y) = \{\mathcal{N}^{(1)}_b + \mathcal{N}^{(2)}_b > 0 \text{ for all } b \in \lfloor x, r \rfloor \cup \lfloor t, y \rfloor\} \cap \\ \{\mathcal{N}^{(1)}_b + \mathcal{N}^{(2)}_b = 0,\ b = \left\{{}^{(x-1)2^N}/_{\beta}, {}^{x2^N}/_{\beta}\right\} \text{ or } \left\{{}^{y2^N}/_{\beta}, {}^{(y+1)2^N}/_{\beta}\right\}\}. \quad (3.3.24)$$

The parity of all bonds in $\lfloor x, y \rfloor$ is completely determined on the event $E \cap M^c_{\lfloor r,t \rfloor} \cap A(x,y)$. Independence of the various Poisson processes allows us to compute:

$$\mathbb{P}\left(E \cap M^c_{\lfloor r,t \rfloor} \cap A(x,y)\right) \propto e^{-2\lambda\beta|s-r|-2h\beta|y-x|} \left(\frac{\lambda\beta}{2^N}\right)^2 + o(2^{-2N}). \quad (3.3.25)$$

A worst case lower bound for the exponential term is $e^{-(2\lambda+2h)\beta}$, so summing over x and y, we have

$$\mathbb{P}\left(E \cap M^c_{\lfloor r,t \rfloor}\right) \geq e^{-(3\lambda+2h)\beta} \frac{1}{2} [\lambda\beta\,(1 + r - t)]^2 + o(1). \quad (3.3.26)$$

Analogous estimates in the other cases lead to

$$\mathbb{P}(E) \geq \frac{\lambda^2\beta^2}{2} e^{-(3\lambda+2h)\beta} \left[(s-r)^2 + (t-s)^2 + (1+r-t)^2\right] + o(1). \quad (3.3.27)$$

As $\mathbb{P}(\partial \underline{n} = \varnothing) \leq 1$, the bound (3.3.5) follows.

The second statement follows analogously. This time we pitch out the positive contributions to (3.3.16), i.e. the terms corresponding to the less ferromagnetic systems $\langle \sigma_t \rangle_{C^c_{\underline{n}_1+\underline{n}_2}(r)}$.

□

3.4 Dual variational problem

In order to explain the implications of the properties of $\mathbb{Q}^\lambda_\beta$ listed above, it is convenient to consider the variational problem which is dual to (3.2.6). When ψ is quadratic this problem becomes

$$\sup_h \left\{ \Lambda(h) - \frac{1}{2} \int_0^\beta h^2(t) \mathrm{d}t \right\} \triangleq \sup_h \mathfrak{G}^*(h). \quad (3.4.1)$$

Any solution $\tilde{h}$ of (3.4.1) is also a solution to (3.2.6). This is a general fact from convex analysis: Let F and G be two proper lower-semicontinuous convex functionals (on say $\mathbb{L}_2(\mathbb{S}_\beta)$) and let F^* and G^* be their convex conjugates. Assume that

$$F^*(\tilde{h}) - G^*(\tilde{h}) \;=\; \max_h \left\{F^*(h) - G^*(h)\right\},$$

and assume that both F^* and G^* are Gateaux differentiable (in fact sub-differentiability would be enough) at $\tilde{h}$. Let $\tilde{m} = \nabla F^*(\tilde{h}) = \nabla G^*(\tilde{h})$. Then,

$$F^*(\tilde{h}) - G^*(\tilde{h}) = G(\tilde{m}) - F(\tilde{m}).$$

Consequently, for each couple of functions m and h,

$$\{(m,h)_\beta - G^*(h)\} - \{(m,h)_\beta - F^*(h)\} \le G(\tilde{m}) - F(\tilde{m}).$$

It follows that for every m, $G(m) - F(m) \le G(\tilde{m}) - F(\tilde{m})$.

Assume that we can quantify the stability property of the dual variational problem in the following way: There exists a non-negative functional D, such that $D = 0$ only on the solutions of the dual problem, and for any function h,

$$F^*(h) - G^*(h) + D(h) \;\le\; F^*(\tilde{h}) - G^*(\tilde{h}). \tag{3.4.2}$$

Then such a stability bound is transferable to the direct problem: Assume that $h = \nabla G(m)$. Then,

$$G(m) - F(m) + D(h) + \{F(m) + F^*(h) - (m,h)_\beta\} \;\le\; G(\tilde{m}) - F(\tilde{m}). \tag{3.4.3}$$

In particular, $G(m) - F(m) < G(\tilde{m}) - F(\tilde{m})$, whenever $\nabla G(m)$ *is not* a solution of the dual problem or whenever $h \notin \partial F(m)$.

Let us now go back to (3.2.6) and (3.4.1). In the above notation: $F(m) = I(m)$ and $G(m) = \|m\|_\beta^2/2$. Accordingly, $F^*(h) = \Lambda(h)$ and $G^*(h) = \|h\|_\beta^2/2$. In particular,

G, G^* and F^* are everywhere Gateaux differentiable. Of course, $\nabla G(m) = m$. Consequently, once we derive a stability bound of the type (3.4.2) for the dual problem, we immediately recover a stability bound

$$\frac{1}{2}\int_0^\beta m^2(t)\mathrm{d}t - I(m) + D(m) + \left\{I(m) + \Lambda(m) - \|m\|_\beta^2\right\} \le \mathfrak{G}(\tilde{m}) \tag{3.4.4}$$

for the original problem (3.2.6). In particular, any solution of (3.2.6) is a solution of (3.4.1).

We, therefore, proceed to study the dual variational problem (3.4.1).

3.5 Reduction of the Dual Problem to One Dimension

Our first step in studying (3.4.1) is to assemble facts which reduce the problem to a study of constant fields $h = c \cdot \mathbf{1}$ for $c \in \mathbb{R}$. This section does not depend on the function ψ being quadratic. Let B_1^∞ denote the collection of measurable functions on $[0, \beta]$ essentially bounded by one in absolute value. Let $\psi : \mathbb{R} \to \mathbb{R}$ be any polynomial function and let

$$\Psi(m) = \int_0^\beta \psi(m_t)\,\mathrm{d}t \tag{3.5.1}$$

for any $m \in B_1^\infty$. For $c \in \mathbb{R}$ and $h \in L^2[0, \beta]$, let

$$\psi^*(c) = \sup_{m\in[-1,1]} (c, m)_\beta - \psi(m) \tag{3.5.2}$$

and

$$\Psi^*(h) = \sup_{m\in B_1^\infty} (h, m)_\beta - \Psi(m) \tag{3.5.3}$$

be their respective Legendre transforms. Note that

$$\beta\psi^*\left(\frac{1}{\beta}\int_0^\beta h_t\,\mathrm{d}t\right) \le \Psi^*(h) \le \int_0^\beta \psi^*(h_t)\,\mathrm{d}t. \tag{3.5.4}$$

The first inequality follows from viewing $\beta\psi^*\left(\frac{1}{\beta}\int_0^\beta h_t\,\mathrm{d}t\right)$ as a restricted Legendre transform for the constant function $\frac{1}{\beta}\int_0^\beta h_t\,\mathrm{d}t\cdot\mathbf{1}$ optimizing only over constant functions $m_0 \cdot \mathbf{1}$. The second follows from subadditivity of the process of taking supremums.

We have

Lemma 3.5.1. *For any polynomial $\psi(x)$, let Ψ and Ψ^* be defined as above. Let $\Lambda(h)$ denote the cumulant generating function of the free measure $d\mathbb{Q}_{\lambda,\beta}$. Then we have*

$$\sup_{h \in L^2[0,\beta]} \Lambda(h) - \Psi^*(h) = \sup_{h \in \mathbb{R}} \Lambda(h \cdot \mathbf{1}) - \beta\psi^*(h). \tag{3.5.5}$$

In particular we point out that this lemma implies that the constant function $0 \cdot \mathbf{1}$ optimizes the left hand side if and only if 0 optimizes the right hand side.

To prove Lemma 3.5.1, we need to make a few observations. For any $h \in L^2[0,\beta]$, let h_l and h_r be the two functions obtained by reflecting h about $\frac{\beta}{2}$. That is let

$$h_l(t) = \begin{cases} h(t) & \text{if } t \in [0, \frac{\beta}{2}] \\ h(\beta - t) & \text{if } t \in [\frac{\beta}{2}, \beta] \end{cases} \tag{3.5.6}$$

and h_r be defined analogously. Let $\mathcal{S}$ denote the collection of functions in $L^2([0,\beta])$ which are symmetric about $\beta/2$.

Lemma 3.5.2. *Let $h \in L^2([0,\beta])$ and h_l, h_r be as above. Then we have*

$$\Lambda(h) \leq \frac{1}{2}\left[\Lambda(h_l) + \Lambda(h_r)\right]. \tag{3.5.7}$$

We note that (3.5.2) was originally proved in a somewhat more general (but completely analogous) context in [54].

Proof. Recall that the measure space $(\Omega,\ d\mathbb{Q}_{\lambda,\beta})$ is the continuum limit of a sequence of discrete Ising models with asymptotically singular interactions. Let $M \in \mathbb{N}$ be fixed and suppose that the field h is piecewise constant on the dyadic intervals $[\frac{k}{2^M}\beta, \frac{k+1}{2^M}\beta)$. Expressing the discrete Ising model on a circle via transfer matrices and using the

Cauchy-Schwartz inequality, one may check equation (3.5.7) directly whenever the field $h \in \mathcal{D}$. Passing to limits with respect to the discrete Ising models, we immediately have (3.5.7) for any dyadic piecewise constant function h. Standard approximation arguments extend the bound to the general case.

We should point out the ubiquity of the concept of reflection positivity. Another point of view one might take on this problem is to note that nearest neighbour circle Ising models are reflection positive. Appealing to the general theory and passing to limits gives the same conclusion.

Both of these points of view are just manifestations of the general Chessboard estimate from [62].

□

Lemma 3.5.3. *Suppose h, h_l, h_r are fixed as above. Then*

$$\Psi^*(h) = \frac{1}{2}\left[\Psi^*(h_l) + \Psi^*(h_r)\right]. \tag{3.5.8}$$

Proof. This is a simple computation with Legendre transforms. Let $m \in B_1^\infty$ and denote by m_l and m_r the symmetrizations of m about $\beta/2$. Starting from the definition of Ψ^*,

$$\begin{aligned} \Psi^*(h) &= \sup_{m_l, m_r} \frac{1}{2}\left[(h_l, m_l)_\beta - \Psi(m_l)\right] + \frac{1}{2}\left[(h_r, m_r)_\beta - \Psi(m_r)\right] \\ &= \frac{1}{2}\left(\sup_{m_l} (h_l, m_l)_\beta - \Psi(m_l) + \sup_{m_r} (h_r, m_r)_\beta - \Psi(m_r)\right) \end{aligned}$$

But for any function h_l symmetric with respect to $\beta/2$,

$$\sup_{m_l} (h_l, m_l)_\beta - \Psi(m_l) = \Psi^*(h_l) \tag{3.5.9}$$

and the lemma follows. □

Proof of Lemma 3.5.1. Equation (3.5.4) immediately gives

$$\beta\psi^*(c) = \Psi^*(c \cdot \mathbf{1}) \tag{3.5.10}$$

so the lemma is just the statement that

$$\sup_{h \in L^2[0,\beta]} \Lambda(h) - \Psi^*(h) = \sup_{c \in \mathbb{R}} \Lambda(c \cdot \mathbf{1}) - \Psi^*(c \cdot \mathbf{1}). \tag{3.5.11}$$

To prove this, observe first that both $\Lambda(h)$ and $\Psi(h)$ are invariant under rotations of the circle $\mathbb{S}_\beta$ i.e. if

$$(\tau_s \circ h)_t = h_{t+s} \tag{3.5.12}$$

then

$$\Lambda(\tau_s \circ h) = \Lambda(h) \text{ and } \Psi(\tau_s \circ h) = \Psi(h). \tag{3.5.13}$$

By continuity (l.s.c. as the case may be) properties of the functionals in 3.5.11 and density of continuous functions in $L^2(\mathbb{S}_\beta)$ the left hand side of (3.5.11) can be approximated arbitrarily well by continuous functions. Let h_0 be a fixed continuous function. Repeated applications of Lemma 3.5.2, Lemma 3.5.3 and the rotation invariance (3.5.13) imply that for each $M \in \mathbb{N}$ there exists a continuous function $h_M \in \mathcal{S}$ which is $\frac{\beta}{2^{M-1}}$ periodic, equal to h on one of the dyadic intervals $[\frac{\beta k}{2^M}, \frac{\beta(k+1)}{2^M})$ and such that

$$\Lambda(h) - \Psi^*(h) \leq \Lambda(h_M) - \Psi^*(h_M). \tag{3.5.14}$$

The family $\{h_M\}_{M \in \mathbb{N}}$ is uniformly equi-continuous by construction, so the Arzela-Ascoli theorem allows us to extract a convergent subsequence h_{M_k}. The limit, denoted by $c \cdot \mathbf{1}$, is necessarily constant. The bounded convergence theorem says $\Lambda(h_{M_k}) \to \Lambda(c \cdot 1)$ and it is easily checked that

$$\Psi^*(c \cdot 1) \leq \lim_{k \to \infty} \Psi^*(h_{M_k}). \tag{3.5.15}$$

Combining all of these facts proves the result. □

3.6 Putting Things Together

Let us assume for this section that ψ is given as a nonconstant, even polynomial with positive leading coefficient. In this case one may check that

$$\lim_{|h|\to\infty} \Lambda(h \cdot \mathbf{1}) - \beta\psi^*(h) = -\infty \tag{3.6.1}$$

so that constant optimizers always exist. Moreover a calculation with Legendre transforms shows that $\psi^*(h) = \psi^*(|h|)$ in this case. Thus we may assume $h \geq 0$ when searching for optimizers of the constant field problem.

Under this assumption, we have two tools at our disposal: The FKG property of the weighted measure

$$\mathrm{d}\nu_h(\sigma) = \frac{e^{(\sigma, h\cdot\mathbf{1})}\, \mathrm{d}\mathbb{Q}^\lambda_\beta(\sigma)}{Z(h\cdot\mathbf{1})} \tag{3.6.2}$$

and the GHS inequality which for constant fields reduces to the statement

$$\frac{\mathrm{d}^2}{\mathrm{d}h^2}\big\langle(\sigma,\mathbf{1})_\beta\big\rangle_{h\cdot\mathbf{1}} \leq 0, \tag{3.6.3}$$

i.e. the function $M(h) \triangleq \frac{1}{\beta}\big\langle(\sigma,\mathbf{1})_\beta\big\rangle_{h\cdot\mathbf{1}}$ is concave.

Proof of Theorem 3.2.1: The Critical Curve. We are look for nontrivial solutions to the equation

$$h = M(h) \triangleq \frac{1}{\beta}\big\langle(\sigma,\mathbf{1})_\beta\big\rangle_{h\mathbf{1}}. \tag{3.6.4}$$

By the FKG and GHS inequalities, the function $\frac{\mathrm{d}}{\mathrm{d}h}M(h)\big|_{h=u} = \frac{1}{\beta}\big\langle(\sigma,\mathbf{1})_\beta\,;(\sigma,\mathbf{1})_\beta\big\rangle_{u\mathbf{1}}$ is positive and decreasing in u. One easily checks that it is continuous using the bounded convergence theorem. Therefore Equation (3.6.4) has nontrivial solutions if and only if

$$\frac{\mathrm{d}}{\mathrm{d}h}M(h)\big|_{h=0} = \frac{1}{\beta}\mathbb{V}\mathrm{ar}^\lambda_\beta\left[(\sigma,\mathbf{1})_\beta\right] \geq 1 \tag{3.6.5}$$

where the variance is taken with respect to the free measure $\frac{\mathrm{d}\mathbb{Q}^\lambda\beta}{Z(0)}$. As $\big\langle(\sigma,\mathbf{1})_\beta\big\rangle_0 = 0$ by spin flip symmetry, $\mathbb{V}\mathrm{ar}^\lambda_\beta\left[(\sigma,\mathbf{1})_\beta\right] = \big\langle(\sigma,\mathbf{1})^2_\beta\big\rangle_0$.

We are reduced to computing

$$\left\langle (\sigma, \mathbf{1})_\beta^2 \right\rangle_0 = \beta \int_0^\beta \left\langle \sigma_0 \sigma_t \right\rangle_0 \mathrm{d}t, \tag{3.6.6}$$

where we have used rotational invariance in the equality. Using the ± 1 symmetry,

$$\left\langle \sigma_0 \sigma_t \right\rangle_0 = \frac{\mathbb{P}_\beta^\lambda (0 \leftrightarrow t)}{\mathbb{P}_\beta^\lambda \left[2^{\# c(\omega)} \right]} \tag{3.6.7}$$

where $\# c(\omega)$ denotes the number of connected components determined by the underlying Poisson process and $0 \leftrightarrow t$ denotes the event the 0 and t are in the same connected component of the complement of the arrival points.

There are two cases to consider. Either there is an arrival in $[t, \beta]$ or there is not. Taking this under consideration, we have we

$$\mathbb{P}_\beta^\lambda (0 \leftrightarrow t) = e^{\lambda(\beta - 2t)} + e^{-\lambda(\beta - 2t)}. \tag{3.6.8}$$

A simple computation shows

$$\mathbb{P}_\beta^\lambda \left[2^{\# c(\omega)} \right] = e^{-\lambda \beta} + e^{\lambda \beta} \tag{3.6.9}$$

Upon integrating, we find

$$\frac{1}{\beta} \mathbb{V}\mathrm{ar}_\beta^\lambda \left[(\sigma, \mathbf{1})_\beta \right] = \frac{1}{\lambda} \tanh(\lambda \beta) \tag{3.6.10}$$

□

Proof of Theorem 3.2.1: Stability. Having computed the critical curve, we turn our attention to issues of stability. As a first step we claim that whenever (λ, β) is away from the critical curve, the problem

$$\sup_{h \in \mathbb{R}} \Lambda(h \cdot \mathbf{1}) - \Psi^*(h \cdot \mathbf{1}) \tag{3.6.11}$$

is stable,

$$\left\{ \frac{1}{\beta} \Lambda \left(h \cdot \mathbf{1} \right) - \frac{1}{2} h^2 \right\} + d(h) \, \leq \, \frac{1}{\beta} \Lambda \left(\pm h^* \mathbf{1} \right) - \frac{1}{2} (h^*)^2, \tag{3.6.12}$$

where d satisfies the following bound: There exists $\alpha_1 = \alpha_1(\lambda, \beta) > 0$, such that,

$$d(h) \geq \alpha_1 e^{-2\beta|h|} \min\left\{(h-h^*)^2, (h+h^*)^2\right\}. \tag{3.6.13}$$

Indeed the bound (3.6.12) follows directly from (3.6.3).

Stability of the original variational problem. It follows that the dual variational problem (3.4.1) (recall that in our case $F^*(\cdot) = \Lambda(\cdot)$ and $G^*(\cdot) = 1/2\|\cdot\|_\beta^2$) satisfies (3.4.2) with

$$D(h) = \frac{1}{\beta}\int_0^\beta d(h(t))\mathrm{d}t.$$

Of course, the bound (3.6.12) could be improved for large values of $|h|$, however since we are primarily interested in transfering stability to the direct variational problem (3.2.6), the values of $|h| > 1$ are, in view of (3.4.4), irrelevant. In particular $D(m)$ clearly dominates (with $h^* = m^*$ and c chosen appropriately small) the first term on the right hand side of (3.2.9).

The second term $\int_0^\beta U(m'(t))\mathrm{d}t$ on the right hand side of (3.2.9) is related to a more careful analysis of $\left\{I(m) + \Lambda(m) - \|m\|_\beta^2\right\}$ term in (3.4.4), which is unfortunately not complete at the time of submission of this dissertation. We, therefore, refer (as in defer) the reader to [23]. $\square$

Proof of Theorem 3.2.1: Decay Near Critical Curve. The previous sections demonstrated that the magnetization of the system

$$m_*(\beta, \lambda) \tag{3.6.14}$$

is an order parameter for the Mean Field Transverse Ising Model which tends to 0 as we approach the critical curve $F(\beta, \lambda) = 1$. In this section we shall determine the rate of decay for the magnetization and give a probabilistic interpretation of these critical phenomena.

Let us recall that under the assumption of quadratic interaction between spins, the positive optimizer $m_*(\beta, \lambda)$ is in duality with an optimizer h^*. The relationship is

$$m_* = -\nabla G(h^*) = -\frac{1}{\beta}\big\langle(\sigma, \mathbf{1})_\beta\big\rangle_{h^*} = -h^*. \tag{3.6.15}$$

Thus we may study the decay rate of m_* via the dual variable h^*.

Let us expand h^* via the righthand side of 3.6.15.

$$\begin{aligned} h^* =& \frac{1}{\beta}\int_0^{h^*} \mathbb{V}\mathrm{ar}_h\big[(\sigma, \mathbf{1})_\beta\big]\ \mathrm{d}h \\ =& \frac{1}{\beta}\int_0^{h^*} \mathbb{V}\mathrm{ar}_h\big[(\sigma, \mathbf{1})_\beta\big] - \mathbb{V}\mathrm{ar}_0\big[(\sigma, \mathbf{1})_\beta\big]\ \mathrm{d}h + \frac{1}{\beta}\mathbb{V}\mathrm{ar}_0\big[(\sigma, \mathbf{1})_\beta\big]h^*. \end{aligned}$$

Therefore

$$\left(1 - \frac{1}{\beta}\mathbb{V}\mathrm{ar}_0\big[(\sigma, \mathbf{1})_\beta\big]\right) h^* = \frac{1}{\beta}\int_0^{h^*}\int_0^{h} \mathcal{U}_k\ \mathrm{d}k\ \mathrm{d}h. \tag{3.6.16}$$

where

$$\mathcal{U}_k = \big\langle(\sigma, \mathbf{1})_\beta^3\big\rangle_k - 3\big\langle(\sigma, \mathbf{1})_\beta^2\big\rangle_k\big\langle(\sigma, \mathbf{1})_\beta\big\rangle_k + 2\big\langle(\sigma, \mathbf{1})_\beta\big\rangle_k^3 \tag{3.6.17}$$

Up to higher order in h^*, (3.6.16) gives

$$\left(1 - \frac{1}{\beta}\mathbb{V}\mathrm{ar}_0\big[(\sigma, \mathbf{1})_\beta\big]\right) h^* = \frac{1}{\beta}\,(h^*)^2\,\mathcal{U}_{h^*}. \tag{3.6.18}$$

Recall the conclusions of Lemma 3.3.3:

$$\mathcal{U}_{h^*} \asymp \frac{1}{\beta}\big\langle(\sigma, \mathbf{1})_\beta\big\rangle_{h^*} \tag{3.6.19}$$

Therefore we conclude

$$(h^*)^2 \asymp \left|1 - \frac{1}{\beta}\mathbb{V}\mathrm{ar}_0\big[(\sigma, \mathbf{1})_\beta\big]\right| \tag{3.6.20}$$

□

CHAPTER 4

Thermodynamics and Universality for Mean Field Quantum Spin Glasses

We study aspects of the thermodynamics of quantum versions of spin glasses. By means of the Lie-Trotter formula for exponential sums of operators, we adapt methods used to analyze classical spin glass models to answer analogous questions about quantum models.

4.1 Introduction

Classical spin glass models have seen a flurry of activity over the last few years, see [8, 36, 72, 73, 74, 103, 104, 105, 118, 125, 126] to name a few; in particular [8, 72, 73, 126] all consider aspects of the (generalized) Sherrington-Kirkpatrick model, a mean-field model in which the interactions between spins are mediated by an independent collection of Gaussian random variables. In contrast, though physicists have considered both short and long range quantum spin glass models for quite some time, see [11, 67, 115, 116], there are few rigorous mathematical results; we mention [43], which provides a proof that the quenched free energy of certain short ranged quantum spin glasses exists.

Here we extend classical spin glass results to quantum models in two directions. First, using the ideas of [74], we demonstrate the existence of the quenched free energy of the Sherrington-Kirkpatrick spin glass with a transverse external field. Second, under conditions made precise below, we give a complementary result which shows that a large class of quantum spin glasses, including the transverse S-K model, satisfies universality. By this we mean that as the size of the system goes to infinity, the asymptotics of the free energy of the system do not depend on the type of disorder used to define model. This latter result is based on the work of [36].

One may view this paper as an attempt to adapt the methods of classical spin systems to the analysis of various quantum models. Guerra's interpolation scheme [72, 73, 74] and the Gaussian integration-by-parts formula are ubiquitous tools in the classical setting. A major theme of the present work is that through the systematic use of the Lie-Trotter product formula (i.e. that

$$\mathrm{e}^{A+B} = \lim_{k\to\infty} \prod_{j=1}^{k} \mathrm{e}^{\frac{A}{k}} \mathrm{e}^{\frac{B}{k}} \tag{4.1.1}$$

for any pair of matrices A, B) one may extend the interpolation scheme and integration-by-parts formula to quantum systems in useful ways. We comment here that the use of the Lie-Trotter formula in the context of quantum spin systems has a long history, going back at least to the work [68]. Other references on the application of path integral representations to quantum spin systems include [7, 54], among many others.

For concreteness, the remainder of the introduction, Section 2 and Section 4 all use the language of the spin-$1/2$ representation of $\mathfrak{su}(2)$ to describe quantum spin systems (see [46] for more background), though a careful reading shows that our methods apply with minor modifications to spin-S representations as well. In this setting, one describes each particle using a two dimensional Hilbert space $\mathbb{C}^2$ along with a representation of $\mathfrak{su}(2)$ generated by the triplet of Pauli operators $\vec{S} = (S^{(x)}, S^{(y)}, S^{(z)})$.

To represent an N-particle system, we introduce the tensor product $\mathbb{V}_N = \bigotimes_{j=1}^N \mathbb{C}^2$, one factor for each particle, along with a sequence $(\vec{S}_j)_{j=1}^N$ of N copies of the Pauli vector $\vec{S}$, where $\vec{S}_j$ acts on the j'th factor. The particles interact by means of the Hamiltonian $\mathcal{H}_N$ acting on $\mathbb{V}_N$ with weights of configurations described by the associated Gibbs-Boltzmann operator $\mathrm{e}^{-\beta\mathcal{H}_N}$. Here, $\mathcal{H}_N$ is a self adjoint operator acting on $\mathbb{V}_N$, typically a polynomial in the N-tuple of spin operators $(\vec{S}_j)_{j=1}^N$. For example, the Hamiltonian of the simplest non-trivial quantum spin system, the transverse Ising model, is described by

$$-\mathcal{H}_N = \frac{1}{N}\sum_{i,j=1}^{N} S_i^{(z)} S_j^{(z)} + \lambda \sum_{j=1}^{N} S_j^{(x)} \tag{4.1.2}$$

where $\lambda > 0$.

Once we specify the Hamiltonian, statistical quantities of the system may be defined. The partition function and free energy of a quantum spin system are defined via the trace of the Gibbs-Boltzmann operator as

$$\begin{aligned} Z_N(\beta) &= \mathrm{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_N}\right) \\ f_N(\beta) &= \frac{-1}{N\beta}\log Z_N. \end{aligned}$$

Self adjoint operators on $\mathbb{V}_N$ replace functions as observables of the system and the thermal average of an observable A is defined as

$$\langle A \rangle = \frac{\mathrm{Tr}\left(A\mathrm{e}^{-\beta\mathcal{H}_N}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_N}\right)}.$$

For future reference, we refer to the functional $\langle\cdot\rangle$ on observables as the Gibbs state of the system corresponding to the Hamiltonian $\mathcal{H}_N$.

With this formalism we present a few examples of spin glasses of particular interest. Traditionally the modeling of any spin glass necessitates the introduction of disordered interactions between spins. In general, as will be the case here, the interactions are i.i.d. The basic example, the transverse S-K model, has a Hamiltonian defined by

$$-\mathcal{H}_N = \frac{1}{2\sqrt{N}}\sum_{i,j=1}^{N} J_{i,j} S_i^{(z)} S_j^{(z)} + \lambda \sum_{j=1}^{N} S_j^{(x)}.$$

A more complicated class of models, the quantum Heisenberg spin glasses, are described by one of the Hamiltonians

$$\begin{aligned}
-\mathcal{H}_N =& \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} J_{i,j} \left[S_i^{(z)} S_j^{(z)} + \alpha S_i^{(x)} S_j^{(x)} + \gamma S_i^{(y)} S_j^{(y)} \right] \\
-\tilde{\mathcal{H}}_N =& \frac{1}{2\sqrt{N}} \sum_{\nu \in \{x,y,z\}} \sum_{i,j=1}^{N} J_{i,j}^{(\nu)} S_i^{(\nu)} S_j^{(\nu)},
\end{aligned} \tag{4.1.3}$$

where $\alpha, \gamma \in \mathbb{R}$.

There is a sharp and interesting contrast in the scope of application of our results. Unlike the quantum Heisenberg spin glasses, applying the Lie-Trotter expansion to the transverse S-K model has the added benefit of allowing a natural path integral representation, known as the Feynman-Kac representation, of statistical quantities like the free energy. More precisely, the Feynman-Kac representation allows the expression of the partition function as an integral over paths which are right continuous and have left limits (i.e. càdlàg paths) (see [12] for details on the use of such paths) with state space the classical Ising spin configurations on N sites, $\{-1, +1\}^N$. For this and other technical reasons we can only resolve the existence of the free energy for the one quantum model.

The remainder of the paper is structured as follows. Section 4.2 introduces notation and states the main theorems of the paper. Section 4.3 presents quantum generalizations of [36] which allow us to prove universality and control the fluctuations of the free energy for a class of quantum spin glasses which include the Spin-$1/2$ quantum Heisenberg spin glasses. In addition, we adapt the techniques of [74] to prove exponential decay of the fluctuations of models with Gaussian disorder. Finally, Section 4.4 details the existence of the pressure of the transverse S-K model.

4.2 Results

We begin by giving a bit of notation to be used below. We consider a collection ξ_I of random variables, with $I \in \mathfrak{I}$ for some finite index set $\mathfrak{I}$, and denote $\mathbb{E}[\cdot]$ and $\mathbb{P}(\cdot)$ integration with respect to this collection. Examples of index sets that we have in mind include the collection of all r-tuples of sites in a system of N particles. In the simplest case the index set $\mathfrak{I}$ consists of all pairs (i, j) (or more generally some subset of pairs) with $i, j \le N$. Unless otherwise specified, the variables ξ_I are assumed to be i.i.d. according to some fixed random variable ξ satisfying the conditions

$$\mathbb{E}[\xi] = 0,\ \mathbb{E}\left[\xi^2\right] = 1,\ \mathbb{E}\left[|\xi|^3\right] < \infty. \tag{4.2.1}$$

When explicitly considering Gaussian environments we denote the random variables by g_I.

Let $\underline{S}$ denote the N-tuple of Pauli vectors and $\mathfrak{I}_N$ index some collection of interactions between particles at the system size N. Consider the Hamiltonian

$$\mathcal{H}_N(\xi) = \sum_{I \in \mathfrak{I}_N} \xi_I X_I(\underline{S})$$

where each $X_I(\underline{S})$ is a self-adjoint polynomial in the components of $\underline{S}$, i.e. $X_I^*(\underline{S}) = X_I(\underline{S})$ and is regarded as an operator on $\mathbb{V}_N$ via a trivial embedding. We define the associated partition function and quenched 'pressure' by

$$Z_N(\beta, \xi) = \text{Tr}\left(e^{\beta \mathcal{H}_N}\right); \quad \alpha_N(\beta, \xi) = \mathbb{E}\left[\log Z_N(\beta, \xi)\right].$$

Note that for convenience we have omitted the minus sign from the expression for the Gibbs-Boltzman operator. For any operator A on $\mathbb{V}_N$, we denote its operator norm by $\|A\| := \sup_{v \in \mathbb{V}_N} \left[\frac{(Av,Av)}{(v,v)}\right]^{\frac{1}{2}}$. With this notation we have the following result:

Theorem 4.2.1. *Let ξ be a random variable with mean 0, variance 1, and $\mathbb{E}\left[|\xi|^3\right] < \infty$ and let g be a standard normal random variable. Consider a sequence of Hamili-*

tonians $\mathcal{H}_N$ defined as above so that

$$\sum_{I \in \mathfrak{I}_N} \|X_I(\underline{S})\|^3 = o(N).$$

Then for any $\beta \in \mathbb{R}$,

$$\left| \frac{1}{N} \alpha_N(\beta, \xi) - \frac{1}{N} \alpha_N(\beta, g) \right| = o(1)$$

as N tends to infinity. Moreover

$$\mathbb{E}\left[\left| \frac{1}{N} \log Z_N(\beta, \xi) - \frac{1}{N} \alpha_N(\beta, \xi) \right|^3 \right] \leq \frac{\sqrt{|\mathfrak{I}_N|} o(N)}{N^3}. \tag{4.2.2}$$

Remark 4.2.2. Choosing the operators X_I and index sets $\mathfrak{I}_N$ appropriately allows application to a wide variety of systems. For example, in (4.1.3) the norm of each summand is bounded by $\frac{C}{\sqrt{N}}$ and the number of pairs is of the order N^2. As a result,

$$\sum_{I \in \mathfrak{I}_N} \|X_I(\underline{S})\|^3 = O(N^{\frac{1}{2}})$$

and the theorem is immediately applicable.

The concentration estimate (4.2.2) gets better if one assumes the random variable ξ has higher moments. At the extreme end, we consider the case of fluctuations for Gaussian environments:

Proposition 4.2.3. *Let g be a standard normal random variable. Then*

$$\mathbb{P}\left(|\log Z_N(\beta, g) - \alpha_N(\beta, g)| \geq u \right) \leq 2e^{\left(- \frac{u^2}{\sum_{I \in \mathfrak{I}_N} \beta^2 \|X_I\|^2} \right)}.$$

Next we take up the more subtle question of the existence of the free energy for mean field models. In light of Theorem 4.2.1, to prove convergence of free energies as the system size goes to infinity we may assume the interactions are Gaussian without loss of generality. As mentioned above, we specialize to the transverse S-K model:

$$-\mathcal{H}_N(\lambda) = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} g_{i,j} S_i^{(z)} S_j^{(z)} + \lambda \sum_{j=1}^{N} S_j^{(x)}. \tag{4.2.3}$$

Theorem 4.2.4. *Let $\beta, \lambda > 0$ be fixed. Then*

$$\lim_{N\to\infty} \frac{-1}{\beta N} \log Tr\left(e^{-\beta \mathcal{H}_N(\lambda)}\right)$$

exists, is finite and is constant a.s. Moreover, there exists a $K > 0$ so that the following concentration property holds:

$$\mathbb{P}\left(\left|\frac{-1}{N} \log Tr\left(e^{-\beta\mathcal{H}_N(\lambda)}\right) + \frac{1}{N}\mathbb{E}\left[\log Tr\left(e^{-\beta\mathcal{H}_N(\lambda)}\right)\right]\right| \geq u\right) \leq 2e^{-N\frac{Ku^2}{\beta^2}}.$$

Remark 4.2.5. For readability, we assume here that the disordered portion of the Hamiltonian in (4.2.3) is a two body interaction. However analogous arguments allow one to treat p-spin models where we replace the disordered portion by

$$-\mathcal{H}_N^{dis}(S^{(z)}) = \sum_{r=1}^{\infty} \frac{a_r}{N^{\frac{r-1}{2}}} \sum_{i_1,\ldots i_r} g_{i_1\ldots i_r} \prod_{k=1}^{r} S_{i_k}^{(z)}$$

Here $\{g_{i_1\ldots i_r}\}$ is a collection of independent standard Gaussian random variables and $\sum_1^\infty a_r^2 q^r$ is even, convex and continuous as a function of q on $[-1, 1]$. See [8] and [74].

4.3 Universality and Fluctuations

Before proving Theorem 4.2.1, we adapt the methods of [36] so as to apply them to a wide variety of quantum spin systems. It turns out that the line of argument given there is robust enough to be followed in the quantum case, though the calculations must be adjusted to accommodate the non-commutative setting.

In the general setup, we consider a collection of self adjoint operators $\{X_i\}_{i=1}^d$ and $\mathcal{H}_0$ defined on some finite dimensional Hilbert space. Let

$$\mathcal{H}(\xi) = \sum_1^d \xi_i X_i + \frac{1}{\beta}\mathcal{H}_0$$

$$Z(\beta, \xi) = \mathrm{Tr}\left(\mathrm{e}^{\beta\mathcal{H}(\xi)}\right)$$

$$\alpha(\beta, \xi) = \mathbb{E}\left[\log Z(\beta, \xi)\right]$$

denote the Hamiltonian, partition function, and quenched 'pressure' of a system. We define the thermal average, Duhamel two point function, and the three point function for any bounded linear operators A, B, and C on $\mathbb{V}_N$ as follows:

$$\langle A \rangle = \frac{\text{Tr}\left(A e^{\beta \mathcal{H}(\xi)}\right)}{Z(\beta, \xi)}$$

$$(A, B) = \frac{\int_0^1 \text{Tr}\left(A e^{u\beta \mathcal{H}(\xi)} B e^{(1-u)\beta \mathcal{H}(\xi)}\right) \, \mathrm{d}u}{Z(\beta, \xi)}$$

$$(A, B, C) = \frac{\int_0^1 \int_0^1 u \text{Tr}\left(A e^{su\beta \mathcal{H}(\xi)} B e^{(1-s)u\beta \mathcal{H}(\xi)} C e^{(1-u)\beta \mathcal{H}(\xi)}\right) \, \mathrm{d}s \, \mathrm{d}u}{Z(\beta, \xi)}.$$

For the convenience of the reader, we recall a generalized integration-by-parts lemma proved in [36]. The left hand side of (4.3.1) is well known to be zero if the randomness is Gaussian. This fact will play a role in what follows.

Lemma 4.3.1. *Let ξ be a real valued random variable such that $\mathbb{E}\left[|\xi|^3\right] < \infty$ and $\mathbb{E}[\xi] = 0$. Let $F : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable with $\|F''\|_\infty = \sup_{x \in \mathbb{R}} |F''(x)| < \infty$. Then*

$$\left|\mathbb{E}[\xi F(\xi)] - \mathbb{E}\left[\xi^2\right] \mathbb{E}[F'(\xi)]\right| \leq \frac{3}{2}\|F''\|_\infty \mathbb{E}\left[|\xi|^3\right]. \tag{4.3.1}$$

We first express the derivative of the quenched pressure in terms of thermal averages and Duhamel two-point functions of the operators X_i and a bound on the difference between the quantities:

Lemma 4.3.2.

$$\frac{\partial \alpha(\beta, \xi)}{\partial \beta} = \beta \mathbb{E}\left[\sum_{i=1}^d (X_i, X_i) - \langle X_i \rangle^2\right] + 9O(\beta^2)\mathbb{E}\left[|\xi|^3\right] \left(\sum_{i=1}^d \|X_i\|^3\right)$$

where $|O(\beta^2)| \leq \beta^2$.

Remark 4.3.3. There is no correction in this formula when ξ is Gaussian since the integration-by-parts formula is exact in this case .

Proof. To begin the derivation, we have

$$\frac{\partial \alpha(\beta,\xi)}{\partial \beta} = \sum_{i=1}^{d} \mathbb{E}\left[\xi_i \langle X_i \rangle\right].$$

For $z \in \mathbb{R}$, let us define $\mathcal{H}_i(z) = \sum_{j \neq i} \xi_j X_j + z X_i + \frac{1}{\beta}\mathcal{H}_0$, denote the corresponding Gibbs state by $\langle \cdot \rangle_i^{(z)}$ and define the function $F_i(z) = \langle X_i \rangle_i^{(z)}$. Then

$$\mathbb{E}\left[\xi_i \langle X_i \rangle\right] = \mathbb{E}\left[\xi_i F_i(\xi_i)\right]. \tag{4.3.2}$$

To evaluate the right hand side of Equation (4.3.2), we calculate the first and second derivatives of F_i. Applying the Lie-Trotter expansion, i.e. that $\mathrm{e}^{A+B} = \lim_k \prod_1^k \mathrm{e}^{A/k}\mathrm{e}^{B/k}$, with $A = \beta z X_i$ and $B = \beta \mathcal{H}_i(z) - \beta z X_i$, we find that the first and second derivatives of F_i take the form:

$$\begin{aligned} F_i'(z) =& \beta \left[(X_i, X_i)_i^{(z)} - \left(\langle X_i \rangle^{(z)}\right)^2 \right] \\ F_i''(z) =& \beta^2 \left[2\left(X_i, X_i, X_i\right)^{(z)} - 3\left(X_i, X_i\right)^{(z)} \langle X_i \rangle^{(z)} + 2\left(\langle X_i \rangle^{(z)}\right)^3 \right]. \end{aligned}$$

In order to prove the lemma we must bound F_i''. To this end we claim that for any $a_1, \ldots, a_n > 0$ so that $a_1 + \cdots + a_n = 1$ and any self-adjoint matrices $X, \mathcal{H}$

$$\left|\mathrm{Tr}\left(X\mathrm{e}^{a_1\mathcal{H}} \cdots X\mathrm{e}^{a_n\mathcal{H}}\right)\right| \leq \|X\|^n \mathrm{Tr}\left(\mathrm{e}^{\mathcal{H}}\right) \tag{4.3.3}$$

Indeed, we may assume by continuity that $a_j = \frac{k_j}{2^m}$ for some $m > 0$ and some sequence of positive integers (k_j) summing to 2^m.

For any sequence of matrices $(B_j)_{j=1}^{2k}$,

$$\left|\mathrm{Tr}\left(\prod_{j=1}^{2k} B_j\right)\right| \leq \prod_{j=1}^{2k} \mathrm{Tr}\left(\left[B_j B_j^*\right]^k\right)^{\frac{1}{2k}}. \tag{4.3.4}$$

This can be seen as a specific case of the general method of chessboard estimates, Theorem 4.1 of [62]. Applying (4.3.4) to the left hand side of (4.3.3) with $B_j \in \{X\mathrm{e}^{\frac{\mathcal{H}}{2^m}}, \mathrm{e}^{\frac{\mathcal{H}}{2^m}}\}$ we have

$$\left|\mathrm{Tr}\left(X\mathrm{e}^{\frac{k_1}{2^m}\mathcal{H}} \cdots X\mathrm{e}^{\frac{k_n}{2^m}\mathcal{H}}\right)\right| \leq \mathrm{Tr}\left(\left[X\mathrm{e}^{\frac{\mathcal{H}}{2^{m-1}}}X\right]^{2^{m-1}}\right)^{\frac{n}{2^m}} \mathrm{Tr}\left(\mathrm{e}^{\mathcal{H}}\right)^{\frac{2^m-n}{2^m}} \tag{4.3.5}$$

Another application of (4.3.4) implies

$$\mathrm{Tr}\left(\left[X\mathrm{e}^{\frac{\mathcal{H}}{2^{m-1}}}X\right]^{2^{m-1}}\right)\leq \mathrm{Tr}\left(X^{2^{m+1}}\right)^{\frac{1}{2}}\mathrm{Tr}\left(\mathrm{e}^{2\mathcal{H}}\right)^{\frac{1}{2}}.$$

Using this bound on the right hand side of (4.3.5) and letting m pass to infinity proves the bound (4.3.3).

It follows that $\|F_i''\|_\infty \leq 6\beta^2\|X_i\|^3$. Recalling that $\mathbb{E}\left[\xi^2\right] = 1$, Lemma 4.3.1 in conjunction with the previous calculations imply

$$\left|\frac{\partial\alpha(\beta,\xi)}{\partial\beta}-\beta\mathbb{E}\left[\sum_{i=1}^{d}(X_i,X_i)-\langle X_i\rangle^2\right]\right|\leq 9\beta^2\mathbb{E}\left[|\xi|^3\right]\left(\sum_{i=1}^{d}\|X_i\|^3\right).$$

□

Next we use Lemma 4.3.2 to compare the quenched 'pressure' for ξ to that of a Gaussian environment g where g is a standard normal. For this, recall the assumptions (4.2.1) on ξ.

Lemma 4.3.4. *Let $\{\xi_i\}$ and $\{g_i\}$ be collections of i.i.d random variables distributed according to ξ and g respectively. For any $\beta\in\mathbb{R}$,*

$$|\alpha(\beta,\xi)-\alpha(\beta,g)|\leq 9|\beta|^3\mathbb{E}\left[|\xi|^3\right]\left(\sum_{i=1}^{d}\|X_i\|^3\right)$$

Proof. The similarity between the proof of this lemma and that of Proposition 7 [36] means that we will be extremely brief. Consider the interpolating partition function and corresponding quenched 'pressure' defined by

$$\begin{aligned}Z(s,t-s)=&\mathrm{e}^{\sqrt{s}\left(\sum_{i=1}^{d}\xi_iX_i\right)+\sqrt{t-s}\left(\sum_{i=1}^{d}g_iX_i\right)+\mathcal{H}_0}\\ \alpha^{(t)}(s)=&\mathbb{E}\log Z(s,t-s)\end{aligned}$$

respectively. We have

$$\alpha^{(t)}(t)=\alpha(\sqrt{t},\xi);\quad \alpha^{(t)}(0)=\alpha(\sqrt{t},g).$$

Lemma 4.3.2 along with independence between the two environments implies that for all $s \in [0,t]$ we have

$$\left|\frac{\partial \alpha^{(t)}(s)}{\partial s}\right| \leq 9\sqrt{t}\mathbb{E}\left[|\xi|^3\right]\left(\sum_{i=1}^{d}\|X_i\|^3\right).$$

For $\beta \geq 0$, if we let $t = \beta^2$ and integrate this inequality the result follows. For $\beta < 0$ we instead consider the environments $-\xi, -g$. □

Next we attend to the fluctuations of the 'pressure' determined by the random environment (ξ_i):

Lemma 4.3.5. *There exists some universal constant $c > 0$ so that*

$$\mathbb{E}\left[|\log Z(\beta,\xi) - \alpha(\beta,\xi)|^3\right] \leq c\,\mathbb{E}\left[|\xi|^3\right]\beta^3\sqrt{d}\left(\sum_{i=1}^{d}\|X_i\|^3\right).$$

Proof. Consider the filtration $\mathcal{F}_k = \sigma\{\xi_1 \ldots \xi_k\}, k \geq 1$ determined by the sequence of independent random variables (ξ_k). Let

$$\Delta_i := \mathbb{E}\left[\log Z(\beta,\xi)\middle|\mathcal{F}_i\right] - \mathbb{E}\left[\log Z(\beta,\xi)\middle|\mathcal{F}_{i-1}\right]$$

We have

$$\log Z(\beta,\xi) - \alpha(\beta,\xi) = \sum_{i=1}^{d}\Delta_i.$$

Burkholder's martingale inequality (see [34] for a statement) implies the existence of a universal constant c' so that

$$\mathbb{E}\left|\sum_{i=1}^{d}\Delta_i\right|^3 \leq c'\mathbb{E}\left(\sum_{i=1}^{d}\Delta_i^2\right)^{\frac{3}{2}}.$$

To bound the increment Δ_i, consider the partition function

$$Z_i(\beta,\xi) = \mathrm{Tr}\left(\mathrm{e}^{\beta\psi_i}\right)$$

where $\psi_i = \psi_i(\xi) := \mathcal{H}(\xi) - \xi_i X_i$. Since $Z_i(\beta,\xi)$ is independent of ξ_i,

$$\Delta_i = \mathbb{E}\left[\log \frac{Z(\beta,\xi)}{Z_i(\beta,\xi)} \big| \mathcal{F}_i\right] - \mathbb{E}\left[\log \frac{Z(\beta,\xi)}{Z_i(\beta,\xi)} \big| \mathcal{F}_{i-1}\right].$$

We use this identity to estimate Δ_i.

We claim that

$$\frac{Z(\beta,\xi)}{Z_i(\beta,\xi)} \leq \mathrm{e}^{\beta|\xi_i|\cdot\|X_i\|}.$$

Indeed, the Lie-Trotter formula implies

$$Z(\beta,\xi) = \lim_{k\to\infty} \mathrm{Tr}\left(\prod_{j=1}^{2^k} \mathrm{e}^{\frac{\beta\psi_i}{2^k}} \mathrm{e}^{\frac{\beta\xi_i X_i}{2^k}}\right) = \lim_{k\to\infty} \mathrm{Tr}\left(\prod_{j=1}^{2^k} \mathrm{e}^{\frac{\beta\xi_i X_i}{2^{k+1}}} \mathrm{e}^{\frac{\beta\psi_i}{2^k}} \mathrm{e}^{\frac{\beta\xi_i X_i}{2^{k+1}}}\right).$$

Since

$$\left\|\mathrm{e}^{\frac{\beta\xi_i X_i}{2^k}}\right\| \leq \mathrm{e}^{\frac{\beta|\xi_i|\cdot\|X_i\|}{2^k}},$$

Inequality (4.3.3) applied to the righthand side for k finite gives

$$\mathrm{Tr}\left(\prod_{j=1}^{2^k} \mathrm{e}^{\frac{\beta\psi_i}{2^k}} \mathrm{e}^{\frac{\beta\xi_i X_i}{2^k}}\right) \leq \mathrm{e}^{\beta|\xi_i|\cdot\|X_i\|}\mathrm{Tr}\left(\mathrm{e}^{\beta\psi_i}\right)$$

from which our claim follows.

From this we estimate:

$$|\Delta_i| \leq \beta\|X_i\|(|\xi_i| + \mathbb{E}|\xi_i|).$$

Therefore,

$$\mathbb{E}\left[|\log Z(\beta,\xi) - \alpha(\beta,\xi)|^3\right] \leq c\mathbb{E}\left[\left(\sum_{i=1}^d \Delta_i^2\right)^{\frac{3}{2}}\right] \leq c\beta^3\left(\sum_{i=1}^d \|X_i\|^2\mathbb{E}\left[(|\xi_i| + \mathbb{E}|\xi_i|)^2\right]\right)^{\frac{3}{2}}$$
$$\leq 8c\beta^3\mathbb{E}\left[|\xi|^3\right]\sqrt{d}\left(\sum_{i=1}^d \|X_i\|^3\right)$$

where the constants c is independant of d by Burkholder's inequality and we have used the Hölder's inequality in the last bound. □

Proof of Theorem 4.2.1. Our first theorem now follows as an application of the above machinery: the first statement is an application of Lemma 4.3.4, while the second follows from Lemma 4.3.5. □

Finally, we consider fluctuations in Gaussian environments:

Proof of Proposition 4.2.3. Let $Z_\beta(t)$ be defined as the auxiliary partition function given by two independent collections of Gaussian disorder $g^{(1)}$ and $g^{(2)}$,

$$Z_\beta(t) = \mathrm{Tr}\left(\mathrm{e}^{\beta\sqrt{t}\sum_{i=1}^d g_i^{(1)}X_i + \beta\sqrt{1-t}\sum_{i=1}^d g_i^{(2)}X_i + \mathcal{H}_0}\right)$$

with t an interpolation parameter varying between 0 and 1. Let $\mathbb{E}_j$ denote the average with respect the random variables $g^{(j)}$ for $j = 1, 2$. Given any $s \in \mathbb{R}$, let

$$Y(t) = \exp(s\mathbb{E}_2 \log Z_\beta(t)); \quad \phi(t) = \log \mathbb{E}_1 [Y(t)] . \tag{4.3.6}$$

We note that

$$\phi_N(1) - \phi_N(0) = \log \mathbb{E}\left[\exp(s(\log Z(\beta, g) - \alpha(\beta, g)))\right]. \tag{4.3.7}$$

In order to estimate this difference, consider

$$\phi'(t) = \frac{s\beta}{2\mathbb{E}_1[Y(t)]}\mathbb{E}_1\left[Y(t)\mathbb{E}_2\left[\frac{1}{\sqrt{t}}\left\langle\sum_{i=1}^d g_i^{(1)}X_i\right\rangle_t - \frac{1}{\sqrt{1-t}}\left\langle\sum_{i=1}^d g_i^{(2)}X_i\right\rangle_t\right]\right]$$

where the notation $\langle\cdot\rangle_t$ represents the Gibbs state induced by the interpolating Hamiltonian. A calculation involving the Lie-Trotter expansion and the Gaussian version of the integration-by-parts formula implies

$$\phi'(t) = \frac{s^2\beta^2}{2\mathbb{E}_1[Y(t)]}\mathbb{E}_1\mathbb{E}_2\left[Y(t)\sum_{i=1}^d \langle X_i\rangle_t^2\right]$$

so that

$$|\phi'(t)| \leq \frac{s^2\beta^2\sum_{i=1}^d \|X_i\|^2}{2}. \tag{4.3.8}$$

Using Equation (4.3.8) and the fact that $\mathrm{e}^{|x|} \leq \mathrm{e}^{x} + \mathrm{e}^{-x}$ we have

$$\exp(|s||\log Z(\beta, g) - \alpha(\beta, g)|) \leq 2\exp\left(\frac{s^2\beta^2\sum_{i=1}^{d}\|X_i\|^2}{2}\right).$$

Finally, applying Markov's inequality we have

$$\mathbb{P}\left(|\log Z(\beta, g) - \alpha(\beta, g)| \geq u\right) \leq 2\exp\left(\frac{s^2\beta^2\sum_{i=1}^{d}\|X_i\|^2}{2} - su\right)$$

for any $s \in \mathbb{R}$. Optimizing over s concludes the proof. □

4.4 Existence of the Pressure: the Transverse Ising Spin Glass

Recalling the $\mathfrak{su}(2)$ formalism from the introduction, via the trivial embedding of N one particle spaces into $\mathbb{V}_N$ we choose as a preferred basis the collection of all simple tensor products of eigenvectors for the operators $\{S_j^{(z)}\}$. If we denote the eigenvector for $S^{(z)}$ which corresponds to the eigenvalue $+1$ by $|+\rangle$ and the eigenvector for which corresponds to the eigenvalue -1 by $|-\rangle$, we may identify this preferred basis with classical Ising spin configurations $\sigma \in \{-1, +1\}^N$. For each σ, we denote the corresponding basis vector by $|\sigma\rangle$.

The proof of Theorem 4.2.4 proceeds in two steps. The first step consists of a concentration estimate following essentially the same argument as that of Proposition 4.2.3. Widening the scope beyond the transverse Ising spin glass, for this first step we consider quantum Hamiltonians of the form

$$\mathcal{H}_N := \mathcal{H}_N^{\text{dis}}(S^{(z)}) + \mathcal{H}_N^{\text{det}} \tag{4.4.1}$$

We assume that only $\mathcal{H}_N^{\text{dis}}$ involves Gaussian disorder and that the deterministic operator $\mathcal{H}_N^{\text{det}}$ takes a sufficiently nice form so as to admit a Feynman-Kac representation in terms of the basis of eigenvectors for the $S^{(z)}$ operators. By this we mean that

$$\langle\sigma|\exp(-u\mathcal{H}_N^{\text{det}})|\tilde{\sigma}\rangle \geq 0$$

for all $u \geq 0$ and all spin configurations $\sigma, \tilde{\sigma}$. Assuming that all off-diagonal matrix elements of $\mathcal{H}_N^{\text{det}}$ are non-positive gives a necessary and sufficient condition which guarantees the existence of a Feynman-Kac representation. As mentioned in Section 4.2, we assume for our treatment that $\mathcal{H}_N^{\text{dis}}$ takes the form

$$-\mathcal{H}_N^{\text{dis}}(S^{(z)}) = \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} S_i^{(z)} S_j^{(z)}.$$

where the collection $\{g_{i,j}\}$ is assumed to be i.i.d. standard normal, though more general interactions involving the $\{S_j^{(z)}\}$ may be considered.

To illustrate the significance of the Feynman-Kac representation, note that if we are willing to modify $\mathcal{H}_N^{\text{det}}$ by adding a uniquely determined matrix D_N, diagonal with respect to the preferred basis, we can force the rows of the matrix representation of $\mathcal{H}_N^{\text{det}}$ to sum to zero. This implies that the modified Hamiltonian $\bar{\mathcal{H}}_N^{\text{det}}$ is the generator of a continuous time Markov chain with state space $\{-1,+1\}^N$. Let Ω_N be the measurable space of $\{-1,+1\}^N$ valued càdlàg paths. Let $\underline{\sigma}$ denote the spin path $\{\sigma(u) : u \in [0,\beta]\}$ and let us denote the induced Markov chain path measure started from initial configuration σ by $\mathrm{d}\nu_\sigma(\underline{\sigma})$.

Now

$$\mathcal{H}_N = \mathcal{H}_N^{dis}(S^{(z)}) - D_N + \bar{\mathcal{H}}_N^{\text{det}}$$

and the term $\mathcal{H}_N^{dis}(S^{(z)}) - D_N$ is diagonal with respect to the preferred basis. Consider any matrix element $\langle\sigma|\mathrm{e}^{-u\mathcal{H}_N}|\tilde{\sigma}\rangle$. Expanding the exponential for finite k using the Lie-Trotter formula (4.1.1) with $A = -u(\mathcal{H}_N^{\text{dis}}(S^{(z)}) - D_N)$ and $B = -u\bar{\mathcal{H}}_N^{\text{det}}$ and inserting the complete orthogonal set $\{|\sigma'\rangle\}$ in between each factor $\mathrm{e}^{\frac{A}{k}}\mathrm{e}^{\frac{B}{k}}$ we have

$$\langle\sigma|\left(\mathrm{e}^{\frac{A}{k}}\mathrm{e}^{\frac{B}{k}}\right)^k|\tilde{\sigma}\rangle = \sum_{\sigma(0),\ldots,\sigma(k)} \prod_{i=0}^{k-1} \langle\sigma(i)|\mathrm{e}^{-\frac{u}{k}\left(\mathcal{H}_N^{dis}(S^{(z)}-D_N)\right)}\mathrm{e}^{-\frac{u}{k}\bar{\mathcal{H}}_N^{det}}|\sigma(i+1)\rangle \quad (4.4.2)$$

where $\sigma(0) = \sigma$ and $\sigma(k) = \tilde{\sigma}$. Let $\mu(\sigma)$ denote the eigenvalue for $-\mathcal{H}_N^{\text{dis}}(S^{(z)}) + D_N$ corresponding to the eigenvector $|\sigma\rangle$. Contracting the operator $\mathrm{e}^{-\frac{u}{k}\left(\mathcal{H}_N^{dis}(S^{(z)}-D_N)\right)}$

against $\langle\sigma(i)|$ in (4.4.2), we have

$$\langle\sigma|\left(\mathrm{e}^{\frac{A}{k}}\mathrm{e}^{\frac{B}{k}}\right)^k|\tilde{\sigma}\rangle=\sum_{\sigma(0),\ldots,\sigma(k)}\mathrm{e}^{\sum_{i=0}^{k-1}\frac{u}{k}\mu(\sigma(i))}\prod_{i=0}^{k-1}\langle\sigma(i)|\mathrm{e}^{-\frac{u}{k}\bar{\mathcal{H}}_N^{det}}|\sigma(i+1)\rangle \tag{4.4.3}$$

Interpreting the last expression in terms of the path measure $\mathrm{d}\nu_\sigma(\underline{\sigma})$, $\sigma(i)$ is the value of the process at time $\frac{ui}{k}$. Therefore

$$\langle\sigma|\left(\mathrm{e}^{\frac{A}{k}}\mathrm{e}^{\frac{B}{k}}\right)^k|\tilde{\sigma}\rangle=\int_{\Omega_N}\mathrm{e}^{\sum_{i=0}^{k-1}\frac{u}{k}\mu(\sigma(\frac{iu}{k}))}\,\mathbf{1}_{\{\sigma(u)=\tilde{\sigma}\}}\,\mathrm{d}\nu_\sigma(\underline{\sigma}) \tag{4.4.4}$$

Passing to the limit as k tends to infinity and using the bounded convergence theorem on the right hand side, we have

$$\langle\sigma|\mathrm{e}^{-u\mathcal{H}_N}|\tilde{\sigma}\rangle=\int_{\Omega_N}\mathrm{e}^{\int_0^u\mu(\sigma(s))\,\mathrm{d}s}\,\mathbf{1}_{\{\sigma(u)=\tilde{\sigma}\}}\,\mathrm{d}\nu_\sigma(\underline{\sigma}).$$

Remark 4.4.1. In the case of most interest at present, the exact form of the matrix D_N and the dynamics of the induced Markov chain are not difficult to determine: for the transverse S-K model, the diagonal matrix D_N is $\lambda N\cdot\mathrm{I}$, and the induced measure is defined by starting from an initial configuration and evolving in time via spin flips which occur independently at each site according to the arrivals of a Poisson process of rate λ. Thus for (4.4.1)

$$Z_N=\mathrm{Tr}\left(\mathrm{e}^{-\beta\mathcal{H}_N}\right)=\mathrm{e}^{\lambda\beta N}\sum_{\sigma\in\{-1,1\}^N}\int_{\Omega_N}\mathrm{e}^{\int_0^\beta\frac{1}{2\sqrt{N}}\sum_{i,j=1}^N g_{ij}\sigma_i(s)\sigma_j(s)\,\mathrm{d}s}\,\mathbf{1}_{\{\sigma(\beta)=\sigma(0)\}}\,\mathrm{d}\nu_\sigma(\underline{\sigma}).$$

We use this representation throughout Section 4.4.

Returning to the general set up let us introduce definitions and notation to be used below. In order to prove convergence of the quenched pressure we need to consider truncated partition functions Z_N^F defined via the Feynman-Kac transformation by

$$\frac{1}{N}\log Z_N^F=\frac{1}{N}\log\left(\sum_{\sigma\in\{-1,1\}^N}\int_F\mathrm{e}^{\int_0^\beta\mu_N(\sigma(s))\,\mathrm{d}s}\,\mathbf{1}_{\{\sigma(\beta)=\sigma(0)\}}\,\mathrm{d}\nu_\sigma(\underline{\sigma})\right)$$

where F is some deterministic (i.e. not depending on Gaussian disorder) subset of Ω_N. For any pair of spin configurations $\sigma, \tilde{\sigma} \in \{-1,1\}^N$, let

$$R(\sigma, \tilde{\sigma}) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i \tilde{\sigma}_i.$$

Let

$$\zeta_N(\underline{\sigma}) = \int_0^\beta \frac{1}{2\sqrt{N}} \sum_{i,j=1}^{N} g_{ij} \sigma_i(u) \sigma_j(u) \, \mathrm{d}u$$

denote the Gaussian random variable defined by the classical Gibbs weight of the spin path $\underline{\sigma}$. An easy calculation shows that

$$\mathbb{E}\left[\zeta_N(\underline{\sigma})\zeta_N(\underline{\tilde{\sigma}})\right] = \frac{N}{4} \int_0^\beta \int_0^\beta R^2(\sigma(u), \tilde{\sigma}(s)) \, \mathrm{d}u \, \mathrm{d}s$$

where the function R is the classical overlap for Ising spin configurations. Finally, let $\gamma_N(\sigma)$ denote the eigenvalue of D_N corresponding to the eigenvector $|\sigma\rangle$

Lemma 4.4.2. *Let $\beta > 0$ be given. Fix any Feynman-Kac representable deterministic Hamiltonian $\mathcal{H}_N^{det}$. Let $F \subset \Omega_N$. Then*

$$\mathbb{P}\left(\left| \frac{1}{N} \log Z_N^F - \frac{1}{N} \mathbb{E}\left[\log Z_N^F\right] \right| \geq u \right) \leq 2 \exp\left(-\frac{2u^2}{\beta^2} N \right). \tag{4.4.5}$$

Proof. With the appropriate modifications, the method of proof of Proposition 4.2.3 may be followed here: Let $Z_N^F(t)$ be defined as the auxiliary partition function given by two independent collections of Gaussian disorder $g^{(1)}$ and $g^{(2)}$,

$$Z_N^F(t) = \sum_{\sigma \in \{-1,1\}^N} \int_F \mathrm{e}^{\sqrt{t}\zeta_N^{(1)}(\underline{\sigma}) + \sqrt{1-t}\zeta_N^{(2)}(\underline{\sigma}) + \int_0^\beta \gamma_N(\sigma(s))\,\mathrm{d}s} \, \mathbf{1}_{\{\sigma(\beta)=\sigma(0)\}} \, \mathrm{d}\nu_\sigma(\underline{\sigma})$$

with t an interpolation parameter varying between 0 and 1 and $\zeta^{(i)}$ the Gaussian corresponding to $g^{(i)}$. Let $Y_N(t)$ and $\phi_N(t)$ be defined in terms of $Z_N^F(t)$ as in (4.3.6).

Replacing the corresponding quantities appearing in the proof of Proposition 4.2.3

we have, using the Feyman-Kac representation,

$$\phi_N'(t) = \frac{s^2}{2\mathbb{E}_1\left[Y_N(t)\right]}$$
$$\mathbb{E}_1\left[Y_N(t) \sum_{\sigma,\tilde\sigma\in\{-1,1\}^N} \int_{F\times F} \left(\frac{N}{4}\int_0^\beta\int_0^\beta R^2(\sigma(u),\tilde\sigma(s))\,\mathrm{d}u\,\mathrm{d}s\right) w_N(t,\underline{\sigma}) w_N(t,\underline{\tilde\sigma})\,\mathrm{d}\nu_\sigma(\underline{\sigma})\,\mathrm{d}\nu_{\tilde\sigma}(\underline{\tilde\sigma})\right]$$

where

$$w_N(t,\underline{\sigma}) = \frac{\mathrm{e}^{\sqrt{t}\zeta_N^{(1)}(\underline{\sigma})+\sqrt{1-t}\zeta_N^{(2)}(\underline{\sigma})+\int_0^\beta \gamma_N(\sigma(s))\,\mathrm{d}s}}{Z_N^F(t)}\mathbf{1}_{\{\sigma(\beta)=\sigma(0)\}}$$

is the truncated 'Gibbs weight' corresponding to the event F. Thus

$$|\phi_N'(t)| \le \frac{s^2\beta^2 N}{8}.$$

The bound now follows as in the proof of Proposition 4.2.3.

□

Remark 4.4.3. More generally Lemma 4.4.2 applies to the p-spin models defined in Remark 4.2.5, though the bound stated in the lemma must be modified slightly.

Unfortunately the use of the Feynman-Kac transformation alone does not allow our method to go through. In particular, we were unable to treat the quantum system with deterministic quadratic couplings in the x and y directions: the ferromagnetic version does permit a Feynman-Kac representation but the interaction is convex, which turns out to have exactly the wrong sign in the expression for the derivative of the interpolating 'pressure'.

The only natural example that we found amenable to our method is the transverse field Ising model. Notice that the deterministic portion of this particular Hamiltonian is *linear*, which simplifies the interpolation scheme that we employ to compare the thermodynamics at different system sizes. For inverse temperature β and transverse field strength $\lambda > 0$, we refer to the partition function of this model by $Z_N(\beta,\lambda)$

and denote $p_N(\beta,\lambda) = -\frac{1}{N}\log Z_N(\beta,\lambda)$. As mentioned in Section 2, in light of the preceding results the proof of Theorem 4.2.4 reduces to the following lemma:

Lemma 4.4.4. *Let $\beta, \lambda > 0$ be fixed. Then*

$$\lim_{N\to\infty} \mathbb{E}\left[p_N(\beta,\lambda)\right] \equiv p(\beta,\lambda)$$

exists.

Proof. For notational convenience let

$$\mathrm{d}\nu\left(\underline{\sigma}\right) = \sum_{\sigma\in\{-1,1\}^N} \mathbf{1}_{\{\sigma(\beta)=\sigma(0)\}}\, \mathrm{d}\nu_\sigma\left(\underline{\sigma}\right)$$

This proof, like that of the previous lemma, relies on the proof of an analogous statement in [74]. The main idea is to partition the space of paths into subsets on which we may control the time correlated self overlap $R(\sigma(u),\sigma(s))$.

To this end, let $\epsilon,\ \delta > 0$ be fixed, where for convenience we assume $\frac{\beta}{\delta} \in \mathbb{N}$. Let N be fixed. For any function $g : [0,\beta]\times[0,\beta] \to [-1,1]$ we define the event $A(\delta,\epsilon,g) \subset \Omega_N$ by

$$\begin{aligned} A(\delta,\epsilon,g) = \{\sigma : g(i\delta,j\delta) \le R(\sigma(i\delta),\sigma(j\delta)) < g(i\delta,j\delta) + \epsilon\ \forall\ i,j \le \frac{\beta}{\delta} - 1, \\ |g(u,s) - R(\sigma(u),\sigma(s))| \le 2\epsilon\ \forall\ (u,s)\ \in\ [0,\beta)\times[0,\beta)\}. \end{aligned}$$

When there is no possibility of confusion we suppress the dependance on δ,ϵ and denote this event by A_g. Let $S_\delta(\epsilon)$ be the set of functions which are constant on the squares $[j\delta,(j+1)\delta)\times[k\delta,(k+1)\delta)$ for $j,k \in \{0,\ldots\frac{\beta}{\delta}-1\}$ and take values in $\{i\epsilon : i \in [-\frac{1}{\epsilon},\frac{1}{\epsilon}]\cap\mathbb{N}\}$. We define the event

$$A = A(\delta,\epsilon) = \cup_{g\in S_\delta(\epsilon)} A(\delta,\epsilon,g).$$

Though A definitely does not cover the full sample space Ω_N, we claim it is enough to prove convergence of the truncated pressure

$$p_N^A(\beta,\lambda) = -\frac{1}{N}\log Z_N^A.$$

More precisely suppose $\epsilon = \epsilon_N, \delta = \delta_N$. If $-\epsilon_N \log \delta_N$ is sufficiently large as N tends to infinity, then

$$\lim_{N\to\infty} \mathbb{E}\left[p_N(\beta,\lambda) - p_N^{A(\delta_N,\epsilon_N)}(\beta,\lambda)\right] = 0. \tag{4.4.6}$$

Indeed, let $\Delta\mathcal{N}_{i\delta}(\underline{\sigma})$ denote the total number of jumps made by the spin path $\underline{\sigma}$ in the time interval $[i\delta,(i+1)\delta]$. To determine how large to take $-\epsilon\log\delta$ with N, let $A_* \subset \Omega_N$ be defined by

$$A_* = \{\underline{\sigma} : \max_{i\le\frac{\beta}{\delta}-1} \Delta\mathcal{N}_{i\delta}(\underline{\sigma}) \le \frac{\epsilon N}{4}\}.$$

Then $A^c \subset A_*^c$ so that

$$\frac{Z_N^{A^c}(\beta,\lambda)}{Z_N^{A}(\beta,\lambda)} \le \frac{Z_N^{A_*^c}(\beta,\lambda)}{Z_N^{A_*}(\beta,\lambda)}$$

By Jensen's inequality,

$$\mathbb{E}\left[\log\left(1+\frac{Z_N^{A_*^c}(\beta,\lambda)}{Z_N^{A_*}(\beta,\lambda)}\right)\right] \le \log\left(1+\int_{A_*^c}\mathbb{E}\left[\frac{\exp\left(\zeta_N(\underline{\sigma})\right)}{Z_N^{A_*}(\beta,\lambda)}\right]\,\mathrm{d}\nu(\underline{\sigma})\right). \tag{4.4.7}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\left[\frac{\exp\left(\zeta_N(\underline{\sigma})\right)}{Z_N^{A_*}(\beta,\lambda)}\right] &\le \mathbb{E}\left[\exp\left(2\zeta_N(\underline{\sigma})\right)\right]^{\frac12}\mathbb{E}\left[\left(Z_N^{A_*}(\beta,\lambda)\right)^{-2}\right]^{\frac12}\\ &\le \mathbb{E}\left[\exp\left(2\zeta_N(\underline{\sigma})\right)\right]^{\frac12}\mathbb{E}\left[\exp\left(\frac{2}{\nu(A_*)}\int_{A_*}\zeta_N(\underline{\sigma})\;\mathrm{d}\nu(\underline{\sigma})\right)\right]^{\frac12}\nu(A_*)^{-1}.\end{aligned}$$

The last line follows from Jensen's inequality applied with respect to the path measure $\frac{\mathbf{1}_{A_*}\,\mathrm{d}\nu}{\nu(A_*)}$. Since $2\zeta_N(\underline{\sigma})$ is a Gaussian random variable with variance $N\int_0^\beta\int_0^\beta R^2(\sigma(u),\sigma(s))\,\mathrm{d}u\,\mathrm{d}s$,

$$\mathbb{E}\left[\exp\left(2\zeta_N(\underline{\sigma})\right)\right] = \exp\left(\frac{N}{2}\int_0^\beta\int_0^\beta R^2(\sigma(u),\sigma(s))\,\mathrm{d}u\,\mathrm{d}s\right).$$

Similarly, after a short calculation we have

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\frac{2}{\nu(A_*)}\int_{A_*}\zeta_N(\underline{\sigma})\;\mathrm{d}\nu(\underline{\sigma})\right)\right] =\\ &\qquad\exp\left(\frac{N}{2}\int_{A_*\times A_*}\int_0^\beta\int_0^\beta R^2(\sigma(u),\tilde{\sigma}(s))\,\mathrm{d}u\,\mathrm{d}s\frac{\mathrm{d}\nu(\underline{\sigma})\,\mathrm{d}\nu(\underline{\tilde{\sigma}})}{\nu(A_*)^2}\right)\end{aligned}$$

As a result of the preceding estimates

$$\int_{A_*^c} \mathbb{E}\left[\frac{\exp\left(\zeta_N(\underline{\sigma})\right)\ \mathrm{d}\nu(\underline{\sigma})}{\int_{A_*}\exp\left(\zeta_N(\underline{\sigma})\right)\ \mathrm{d}\nu(\underline{\sigma})}\right]\ \mathrm{d}\nu(\underline{\sigma}) \leq \mathrm{e}^{\frac{N\beta^2}{2}}\frac{\nu(A_*^c)}{\nu(A_*)}.$$

Standard calculations imply that for all ϵ small enough and $\delta < \epsilon^2$,

$$\frac{\nu(A_*^c)}{\nu(A_*)} \leq \frac{1}{\delta}\mathrm{e}^{CN\epsilon\log\delta} \tag{4.4.8}$$

for some constant $C > 0$ depending only on the transverse field strength λ. If we require δ and ϵ to be chosen so that $C\epsilon\log\frac{1}{\delta} \geq \beta^2 + \frac{1}{N}\log\frac{1}{\delta}$, then putting estimates together and taking the appropriate limits proves our claim. Let us note that these conditions can be arranged by letting

$$\epsilon_N = N^{-1/4},\ \ \delta_N = \mathrm{e}^{-\frac{N^{1/2}}{2\beta^2}} \tag{4.4.9}$$

and taking N large.

Thus we are reduced to showing that the mean of the truncated quenched pressure $\mathbb{E}\left[p_N^{A(\delta_N,\epsilon_N)}(\beta,\lambda)\right]$ converges. For convenience of exposition we first consider subsequences of the form $N_k = N_0 n^k$ for some $N_0, n \in \mathbb{N}$. For any k, we may view $\underline{\sigma} \in \Omega_{N_k}$ as an n-tuple of spin paths $(\underline{\sigma}^{(1)}, \ldots, \underline{\sigma}^{(n)})$ so that $\underline{\sigma}^{(l)} \in \Omega_{N_{k-1}}$. In order to compare thermodynamics at consecutive system sizes, let us define the interpolating Hamiltonian

$$\zeta_{N_k}(t,\underline{\sigma}) = \sqrt{t}\zeta_{N_k}(\underline{\sigma}) + \sqrt{1-t}\sum_{l=1}^{n}\zeta_{N_{k-1}}^{(l)}(\underline{\sigma}^{(l)})$$

where $\zeta_{N_{k-1}}^{(l)}(\underline{\sigma}^{(l)})$ involve disorder couplings which are mutually independent and independent from the couplings in $\zeta_{N_k}(\underline{\sigma})$. In addition, for any $F \subset \Omega_N$, we introduce the partition function

$$Z_N^F(t) = \int_F \exp\left(\zeta_N(t,\underline{\sigma})\right)\ \mathrm{d}\nu(\underline{\sigma})$$

Let $\epsilon, \delta > 0$ be fixed for the moment, not necessarily as ϵ_N, δ_N . Let

$$\tilde{A}_g = \tilde{A}_{N_k}(\delta,\epsilon,g) = \{\underline{\sigma} = (\underline{\sigma}^{(1)},\ldots,\underline{\sigma}^{(n)})\ :\ \underline{\sigma}^{(l)} \in A_{N_{k-1}}(\delta,\epsilon,g)\}$$

Obviously, for all k, $\tilde{A}_g \subset A_g$. Therefore

$$-\log Z_{N_k}^{A_g}(t) \leq -\log Z_{N_k}^{\tilde{A}_g}(t) \tag{4.4.10}$$

For any $g \in S_\delta(\epsilon)$ consider

$$\phi_{N_k}^{(g)}(t) = -\frac{1}{N_k}\mathbb{E}\log Z_{N_k}^{\tilde{A}_g}(t).$$

After bit of work employing the integration-by-parts formula we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\phi_{N_k}^{(g)}(t) = \\
&-\frac{1}{8}\mathbb{E}\left[\left\langle \int_0^\beta \int_0^\beta R^2(\sigma(u),\sigma(s))\,\mathrm{d}u\,\mathrm{d}s - \frac{1}{n}\sum_1^n \int_0^\beta \int_0^\beta R^2(\sigma^{(l)}(u),\sigma^{(l)}(s))\,\mathrm{d}u\,\mathrm{d}s \right\rangle_t^{\tilde{A}_g,(1)}\right] \\
&+\frac{1}{8}\mathbb{E}\left[\left\langle \int_0^\beta \int_0^\beta R^2(\sigma(u),\tilde{\sigma}(s))\,\mathrm{d}u\,\mathrm{d}s - \frac{1}{n}\sum_1^n \int_0^\beta \int_0^\beta R^2(\sigma^{(l)}(u),\tilde{\sigma}^{(l)}(s)) \right\rangle_t^{\tilde{A}_g,(2)}\right]
\end{aligned} \tag{4.4.11}$$

where $\langle\cdot\rangle_t^{\tilde{A}_g,(1)}$ corresponds to the truncated Gibbs weight determined by the Hamiltonian $\zeta_{N_k}(t,\underline{\sigma})$ and $\langle\cdot\rangle_t^{\tilde{A}_g,(2)}$ corresponds to the product Gibbs weight determined by an independent pair of spin paths $\underline{\sigma},\underline{\tilde{\sigma}}$ and Hamiltonian $\zeta_{N_k}(t,\underline{\sigma}) + \zeta_{N_k}(t,\underline{\tilde{\sigma}})$. We stress that corresponding terms in this pair Hamiltonian involve the *same* realizations of disorder.

As the function $f(x) = x^2$ is Lipschitz continuous on $[-1,1]$ with constant 2, from the definition of $\tilde{A}_g$ the first term in (4.4.11) can be bounded by $\beta^2\epsilon$ in absolute value. Since f is convex, the second term is less than or equal to zero. Thus, by evaluating $\phi_{N_k}^{(g)}$ at zero and one and applying (4.4.10) we have

$$\frac{-1}{N_k}\mathbb{E}\left[\log Z_{N_k}^{A_g}\right] \leq \frac{-1}{N_{k-1}}\mathbb{E}\left[\log Z_{N_{k-1}}^{A_g}\right] + \beta^2\epsilon$$

for any $g \in S_\delta(\epsilon)$.

Next, by Lemma 4.4.2, we have

$$\mathbb{P}\left(\left|\frac{1}{N_k}\log Z_{N_k}^{A_g}(\beta,\lambda)-\frac{1}{N_k}\mathbb{E}\left[\log Z_{N_k}^{A_g}(\beta,\lambda)\right]\right|\geq u\right)\leq 2\exp\left(-\frac{2u^2}{\beta^2}N_k\right).$$

Setting $u=\epsilon$, there exists a set $\mathcal{S}$ so that

$$\mathbb{P}(\mathcal{S})\geq 1-\frac{4}{\delta^2\epsilon}\exp\left(-\frac{2\epsilon^2}{\beta^2}N_{k-1}\right)$$

and so that on $\mathcal{S}$

$$\frac{-1}{N_k}\log Z_{N_k}^{A_g}(\beta,\lambda)\leq\frac{-1}{N_{k-1}}\log Z_{N_{k-1}}^{A_g}(\beta,\lambda)+(2+\beta^2)\epsilon$$

for all $g\in S_\delta(\epsilon)$.

Now let us specialize so that $\delta=\exp(-\frac{N^{\frac{1}{2}}}{2\beta^2})$ and $\epsilon=N_k^{-1/4}$, i.e. $\delta_{N_k},\epsilon_{N_k}$ as in (4.4.9). Let

$$\begin{aligned}F_k=\{&\frac{-1}{N_k}\log Z_{N_k}^{A(\delta_{N_k},\epsilon_{N_k},g)}(\beta,\lambda)\geq\\&\frac{-1}{N_{k-1}}\log Z_{N_{k-1}}^{A(\delta_{N_k},\epsilon_{N_k},g)}(\beta,\lambda)+(2+\beta^2)N_k^{-1/4}\text{ for some }g\in S_{\delta(N_k)}(N_k^{-1/4})\}.\end{aligned}$$

A standard application of the Borel-Cantelli lemma implies that

$$\mathbb{P}\left(F_k\ \ i.o.\right)=0.$$

Therefore, for N_k large enough and $\delta=\delta_{N_k},\epsilon=\epsilon_{N_k}$ we have

$$\begin{aligned}Z_{N_k}^{A(\delta,\epsilon)}(\beta,\lambda)&=\sum_{g\in S_\delta(\epsilon)}Z_{N_k}^{A(\delta,\epsilon,g)}(\beta,\lambda)\\&\geq\sum_{g\in S_\delta(\epsilon)}\mathrm{e}^{-(2+\beta^2)N_k^{3/4}}\left[Z_{N_{k-1}}^{A(\delta,\epsilon,g)}(\beta,\lambda)\right]^n\\&\geq\mathrm{e}^{-(2+\beta^2)N_k^{3/4}}\mathrm{e}^{-(n-1)\beta^{-2}N_k^{1/2}}N_k^{\frac{1-n}{4}}\left[Z_{N_{k-1}}^{A(\delta,\epsilon)}(\beta,\lambda)\right]^n\end{aligned}$$

where we used the inequality $\sum_1^k x_i^n\geq k^{1-n}(\sum_1^k x_i)^n$ for $n\geq 1$ and $x_i\geq 0$.

By (4.4.6) we have

$$\frac{1}{N_{k-1}}\mathbb{E}\left[\left|\log Z_{N_{k-1}}^{A(\delta_{N_k},\epsilon_{N_k})}(\beta,\lambda)-\log Z_{N_{k-1}}^{A(\delta_{N_{k-1}},\epsilon_{N_{k-1}})}(\beta,\lambda)\right|\right]\underset{k\to\infty}{\longrightarrow}0$$

Therefore the sequence of truncated pressures $-\frac{1}{N_k}\log Z_{N_k}^{A(\delta_{N_k},\epsilon_{N_k})}(\beta,\lambda)$ is nearly decreasing and converges a.s., though the limit may be $-\infty$.

Let $X_k = -\frac{1}{N_k}\log Z_{N_k}^{A(\delta_{N_k},\epsilon_{N_k})}(\beta,\lambda)$. Then $(\mathbb{E}[X_k])_{k=1}^{\infty}$ is a uniformly bounded sequence, so the concentration inequality (4.4.5) along with the Borel-Cantelli lemma implies that $(X_k)_{k=1}^{\infty}$ is a bounded sequence a.s. and the convergence is to a finite non-random constant $f(\beta,\lambda)$. It is now a small matter to prove convergence of the truncated quenched averages along these subsequences: The bound (4.4.5) implies that X_k are uniformly integrable, i.e.

$$\lim_{\kappa\to\infty}\sup_k \mathbb{E}\left[|X_k|\,\mathbf{1}_{|X_k|\geq\kappa}\right]=0,$$

thus $\mathbb{E}[X_k]\to f(\beta,\lambda)$ as well. Finally, as $-\epsilon_{N_k}\log\delta_{N_k}=\frac{N_k^{1/4}}{2\beta^2}$, the arguments given after (4.4.6) imply that $\lim_k \mathbb{E}[p_{N_k}(\beta,\lambda)]=f(\beta,\lambda)$. Another application of the concentration inequality then implies that $p_{N_k}(\beta,\lambda)\to f(\beta,\lambda)$ a.s.

It remains to prove uniqueness of this limit and then convergence along arbitrary subsequences. We shall briefly sketch a standard argument from which these two statements follow.

Fix any (not necessarily geometric) sequence $(N_k)_{k=1}^{\infty}$ and some integer $N\in\mathbb{N}$. For k large we compare the system at size N with the system at size N_k. Now $N_k = Q_kN+R_k$ where $R_k\leq N$. Dividing the spin path $\underline{\sigma}_{N_k}$ into Q_k subpaths of size N and one subpath of size R_k, we may proceed analogously to the above to show that there exists $k_0\in\mathbb{N}$ and a set $\mathcal{S}'$ with

$$\mathbb{P}(\mathcal{S}')\geq 1-\frac{4}{\delta^2\epsilon}\exp\left(-\frac{2\epsilon^2}{\beta^2}N\right)$$

on which we have

$$\frac{-1}{N_k}\log Z_{N_k}^{A(\delta,\epsilon,g)}(\beta,\lambda) \leq \frac{Q_k N}{N_k}\left(\frac{-1}{N}\log Z_N^{A(\delta,\epsilon,g)}(\beta,\lambda)\right) + O(1)\frac{1}{N_k} + (2+\beta^2)\epsilon$$

for all $g \in S_\delta(\epsilon)$ and for all $k > k_0$. Here, the term $O(1)$ accounts for the system of size R_k (among other errors) and is maximized so that it depends on N, δ and ϵ but not on N_k.

Thus on S',

$$\begin{aligned} Z_{N_k}^{A(\delta,\epsilon)}(\beta,\lambda) &= \sum_{g\in S_\delta(\epsilon)} Z_{N_k}^{A(\delta,\epsilon,g)}(\beta,\lambda) \\ &\geq \sum_{g\in S_\delta(\epsilon)} \mathrm{e}^{-(2+\beta^2)\epsilon N_k + O(1)} \left[Z_N^{A(\delta,\epsilon,g)}(\beta,\lambda)\right]^{Q_k} \\ &\geq \mathrm{e}^{-(2+\beta^2)\epsilon N_k + O(1)} \left(\delta^2\epsilon\right)^{Q_k - 1} \left[Z_N^{A(\delta,\epsilon)}(\beta,\lambda)\right]^{Q_k}. \end{aligned}$$

If we choose $-C\epsilon\log\delta > \beta^2$ then the argument in support of (4.4.6) along with (4.4.5) implies

$$\lim_{k\to\infty}\left(\frac{-1}{N_k}\log Z_{N_k}^{A(\delta,\epsilon)}(\beta,\lambda) + \frac{1}{N_k}\log Z_{N_k}(\beta,\lambda)\right) = 0 \text{ a.s.}$$

Therefore, combining this observation with the previous paragraph

$$\limsup_{k\to\infty}\frac{-1}{N_k}\log Z_{N_k}(\beta,\lambda) \leq (2+\beta^2)\epsilon - \frac{1}{N}\log(\delta^2\epsilon) + \frac{-1}{N}\log Z_N^{A(\delta,\epsilon)}(\beta,\lambda)$$

on S'.

Choosing $\delta = \delta_N, \epsilon = \epsilon_N$ as in (4.4.9) and applying the Borel-Cantelli lemma once more

$$\limsup_{k\to\infty}\frac{-1}{N_k}\log Z_{N_k}(\beta,\lambda) \leq \liminf_{N\to\infty}\frac{-1}{N}\log Z_N(\beta,\lambda) \text{ a.s.}$$

As the sequence $(N_k)_{k=1}^\infty$ was arbitrary, we may conclude convergence of the pressures occurs a.s. To prove L^1 convergence, we proceed as above. □

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