

Introduction to solvable lattice models in statistical and mathematical physics *

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Abstract

Some features of integrable lattice models are reviewed for the case of the six-vertex model. By the Bethe ansatz method we derive the free energy of the six-vertex model. Then, from the expression of the free energy we show analytically the critical singularity near the phase transition in the anti-ferroelectric regime, where the essential singularity similar to the Kosterlitz-Thouless transition appears. We discuss the connection of the six-vertex model to the conformal field theory with $c = 1$. We also introduce various exactly solvable models defined on two-dimensional lattices such as the chiral Potts model and the IRF models. We show that the six-vertex model has rich mathematical structures such as the quantum groups and the braid group.

The graphical approach is emphasized in this review. We explain the meaning of the Yang-Baxter equation by its diagram. Furthermore, we can understand the defining relation of the algebraic Bethe ansatz by the graphical representation. We can thus easily translate formulas of the algebraic Bethe ansatz into those of the statistical models. As an illustration, we show explicitly how we can derive Baxter's expressions from those of the algebraic Bethe ansatz.

Keywords: exactly solvable models, the Yang-Baxter equation, statistical mechanics, phase transitions, the Bethe ansatz, the six-vertex model, integrable lattice models, conformal field theory, two-dimensional lattice

1 Introduction

We introduce the six-vertex model defined on a two-dimensional square lattice. We describe the model in detail, since it gives an important prototype of many solvable lattice models defined on two-dimensional lattices [1]. The transfer matrix of the six-vertex model generalizes the XXZ quantum spin chain which plays a central role among integrable quantum spin chains [2, 3]. The eight-vertex model, which generalizes the six-vertex model directly, may be considered as the most important exactly solvable model in statistical mechanics [4]. Moreover, many mathematical theories such as the algebraic Bethe ansatz [7] and quantum groups [5, 6] are closely related to the six-vertex model. Starting from the six-vertex model, one may have a wide viewpoint on various physical and mathematical topics related to solvable models. There are quite a large number of topics related to exactly solvable models in physics and mathematics [1, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

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We explain in §2 some features of the six-vertex model defined on a square lattice. We introduce the Boltzmann weights and the transfer matrix for the six-vertex model. We review a method for diagonalizing the transfer matrix, which is called the coordinate Bethe ansatz, and we show the expressions of the free energy per site in the ferroelectric, the anti-ferroelectric, and the disordered phases, respectively [30, 31]. The disordered phase is gapless, while the ferroelectric and the anti-ferroelectric phases have gaps [32, 33]. We derive a critical singularity appearing at the phase transition from the anti-ferroelectric to the disordered phases. We review the calculation of the singular part of the free energy through the analytic continuation, as shown in Ref. [1]. The critical singularity is very weak and has the essential singularity similar to the Kosterlitz-Thouless transition. We have thus derived the KT-like singularity through exact calculation. After reviewing the finite-size analysis of conformal invariance [34, 35, 36, 37], we discuss that the massless phase of the six-vertex model is related to the conformal field theory of $c = 1$ which has $U(1)$ symmetry. The $c = 1$ CFT has the critical line where critical exponents change continuously with respect to some parameter of the model [38, 39, 40]. There are quite a few papers on the finite-size corrections of the integrable models [41, 42, 43, 44, 45]. (For a review, see [46, 16, 47].) The critical line is also characteristic to the Tomonaga-Luttinger liquid [48]. The existence of a critical line was first discovered by R.J. Baxter through the exact solution of the eight-vertex model [49].

In §3, we review various integrable models in statistical mechanics [1]. We briefly introduce the Ising model [8, 50, 51, 52], the Potts models [53, 54, 55, 56], and the chiral Potts model [57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68] and then the eight-vertex model [49, 4, 69, 70] and IRF models [71, 72, 73, 74, 75, 76, 77, 78]. In §4, we solve explicitly the Yang-Baxter equations for the six-vertex model. We introduce the algebraic Bethe ansatz [7, 79, 80, 81, 16]. Here we show that the Yang-Baxter equation of the algebraic Bethe ansatz can be expressed by graphs. In §5, we discuss some mathematical theories associated with integrable models such as the braid group [82, 83] and the quantum groups [84].

There have been novel developments in the mathematical physics associated with the six-vertex model [84, 20]. The integrable vertex models associated with various Lie algebras have been obtained, which generalize the six-vertex model [85, 86]. The crystal basis of the quantum groups is derived from the mathematical analysis of the corner transfer matrix which is fundamental for calculating the one-point functions of the vertex models and IRF models [87, 88]. Through the q -vertex operators, correlation functions of the XXZ spin chain or the six vertex model are obtained [20]. The dynamical Yang-Baxter equation [89] and the elliptic quantum groups [90, 91, 92] have also been extensively discussed. In fact, we can derive the R matrix of the eight-vertex model systematically from the elliptic quantum group through the twists [92]. Furthermore, the correlation functions of the XXZ model calculated with the q -vertex operators have been rederived for large but finite chains through the algebraic Bethe ansatz with the Drinfeld twists [93, 94]. Here we note that the q -vertex operator can be defined only on the infinite chain, while the algebraic Bethe ansatz with the Drinfeld twists can be applied to any finite chain. By taking the thermodynamic limit, it has been shown that the two approaches indeed give the same results. These papers indeed illustrate nontrivial physical applications of the Drinfeld twists. It is recently found that the symmetry of the six-vertex model is enhanced at some particular coupling constants: the transfer matrix commutes with the generators of the sl_2 loop algebra for the six-vertex model at roots of unity [95].

Let us discuss some physical motivations for the six-vertex model. The exact solution of the six-vertex model was originally introduced for studying the statistical mechanics of ferro-electrics such as the residual entropy and the ferroelectric transitions [30, 31]. However, it seems that the physical motivation of the six-vertex model for ferro-electricity has decreased. On the other hand, there are many different physical applications of the six-vertex model. Here we consider a few examples, the domain wall theory [96, 97], crystal growth [98, 99, 100, 101, 102, 103] and the thermodynamics of the XXZ spin chain through the quantum transfer matrix [104, 105, 106, 107, 108, 109]. The crystal growth on surfaces has been discussed by applying exact solutions of the six-vertex model

[98, 99, 100, 101, 102] and some extensions [103]. The free energy of the six vertex model gives the equilibrium crystal shapes [98, 99]. The finite-temperature thermodynamics of the XXZ spin chain has been studied extensively through the quantum transfer matrix, which is a version of the inhomogeneous six-vertex transfer matrix [105, 106, 107, 108, 109]. The quantum transfer matrix is obtained by regarding one direction of the square lattice as the imaginary time or the inverse temperature [104]. There have been considerable efforts to evaluate thermal quantities analytically or numerically. Several functional equations on the eigenvalues of the transfer matrix have been devised [108, 109]. Finally, we remark that a universal relation between the dispersion curve and the ground-state correlation length in quantum spin chains are discussed by using the exact solutions of the vertex models [110].

In §2, we employ mainly the notation of Baxter’s textbook [1] except for the transfer matrix. In §4, however, we briefly show how the notation in statistical mechanics is related to the notation of the algebraic Bethe ansatz or the Quantum Inverse Scattering Method. The graphical illustration should be useful. Finally in §5 we discuss many connections of the six-vertex model to several mathematical developments such as the quantum groups.

2 Solvable vertex models

2.1 The six-vertex model

2.1.1 Ice rule

Let us consider a square lattice as a model of two-dimensional ferroelectric crystal. Molecules are placed on the vertices of the lattice. Arrows are placed on the edges of the lattice, which correspond to directions of dipole moments of hydrogen bonds. As a crystal of hydrogen bonding, we may consider ice, i.e., the crystal of water molecules. In this review, however, we simplify the molecular background of the model (For instance, see [9]). We assume that the dipole moments defined on edges take only two values ± 1 .

At a vertex in the lattice, there are four edges. There are possible 16 configurations of the four edges around the vertex since each of the edges takes two values ± 1 .

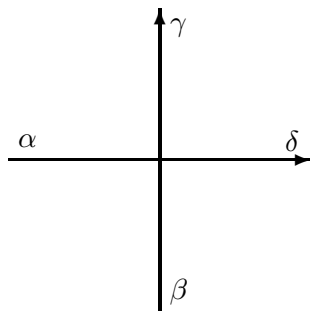


Figure 2.1: Configuration of polarizations around a vertex: $\alpha, \beta, \gamma, \delta$. The Boltzmann weight is expressed by $w(\alpha, \beta | \gamma, \delta)$. Positive directions of dipole moments are given by upward or rightward.

Let symbols $\alpha, \beta, \gamma, \delta$ denote the values of dipole moments around the vertex. Due to charge neutrality, they should satisfy the following condition

$$\alpha + \beta = \gamma + \delta \tag{2.1}$$

For an illustration, let us consider the case when α, β, γ and δ are given by $+1$. Then, α and β give

+ 2 to the vertex, while γ and δ deprive + 2 from it, so that the net charge around the vertex is kept neutral: $\alpha + \beta - \gamma - \delta = 0$.

There are only 6 configurations satisfying the condition. The other configurations that do not satisfy the condition are not allowed in thermal equilibrium. Thus, we call the model the six-vertex model. The condition (2.1) is sometimes called *ice rule*, since ice as a crystal consists of water molecules connected by hydrogen bonding.

We denote by 1 and 2 the values of polarization 1 and -1 , respectively. The symbols 1 and 2 are useful for matrix notation. Let p denote the notation of ± 1 and k 1 and 2. Then, they are related by the relation: $k = 1 + (1 - p)/2$.

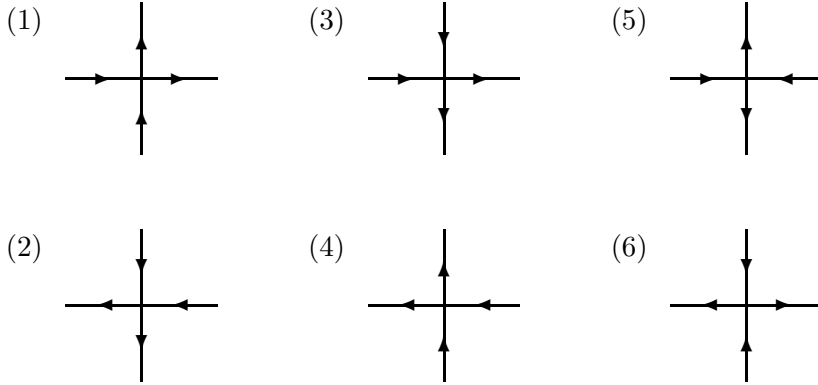


Figure 2.2: Vertex configurations satisfying the ice rule. They have the Boltzmann weights $w(\alpha, \beta|\gamma, \delta)$ as follows: (1) $w(1, 1|1, 1)$; (2) $w(2, 2|2, 2)$; (3) $w(1, 2|2, 1)$; (4) $w(2, 1|1, 2)$; (5) $w(1, 2|1, 2)$; (6) $w(2, 1|2, 1)$. The configurations (1) and (2) are for the weight a , (3) and (4) for b , and (5) and (6) for c .

2.1.2 Boltzmann weights

It is a key idea in exactly solvable models that we define the model by the Boltzmann weights not by the energies of configurations. Let us introduce the Boltzmann weights for configurations around a vertex. For a vertex configuration $\alpha, \beta, \gamma, \delta$, we denote by $\epsilon(\alpha, \beta|\gamma, \delta)$ the energy at the vertex. Then, the Boltzmann weight for a temperature T is given by

$$w(\alpha, \beta|\gamma, \delta) = \exp(-\epsilon(\alpha, \beta|\gamma, \delta)/k_B T) \quad (2.2)$$

Under the ice rule, there are only six configurations allowed round a vertex. Here, it is assumed that the energy of a configuration violating the ice rule should be infinite. We denote by ϵ_j the energy of the j th vertex configuration shown in Fig. 2.2.

Under no external field, the Boltzmann weights must be invariant when reversing all the polarizations simultaneously. Thus, we have $\epsilon_1 = \epsilon_2$, $\epsilon_3 = \epsilon_4$ and $\epsilon_5 = \epsilon_6$, when there is no external field. We denote the Boltzmann weights as follows.

$$\begin{aligned} w(1, 1|1, 1) &= w(2, 2|2, 2) = w_1 = a \\ w(1, 2|2, 1) &= w(2, 1|1, 2) = w_2 = b \\ w(1, 2|1, 2) &= w(2, 1|2, 1) = w_3 = c \end{aligned} \quad (2.3)$$

The Boltzmann weights of the zero-field six-vertex model have essentially only two parameters. For instance, we may choose a/c and b/c . Note that the probability for the vertex configuration of a is given by $a/(a + b + c)$, which does not change by replacing a, b and c with $\rho a, \rho b$ and ρc .

Let us consider $\pi/2$ rotation of the square lattice. If we rotate vertex configuration (1) of Fig. 2.2 by the angle $\pi/2$ in the counterclockwise direction, then, it becomes vertex configuration (4). Under the $\pi/2$ rotation, the weight a is exchanged with the weight b , while the weight c does not change.

2.2 Partition function and the transfer matrix

Let us discuss the partition function of the system. We now set the boundary conditions. Here, we consider the periodic boundary conditions for the two-dimensional lattice. We take a product of the Boltzmann weights over all the vertices of the lattice, and sum up the product over all the allowed configurations of arrows on the lattice.

$$Z = \sum_{\text{config}} \prod_{j:\text{vertices}} w(a_j, b_j | c_j, d_j) \quad (2.4)$$

The partition function of the square lattice can be formulated as the trace of the products of the transfer matrices. Let us define the transfer matrix τ of the six-vertex model. The matrix elements of the transfer matrix τ acting on N lattice sites are given by

$$\tau_{b_1, \dots, b_N}^{a_1, \dots, a_N} = \sum_{c_1, \dots, c_N} w(c_1, b_1 | a_1, c_2) w(c_2, b_2 | a_2, c_3) \cdots w(c_N, b_N | a_N, c_1) \quad (2.5)$$

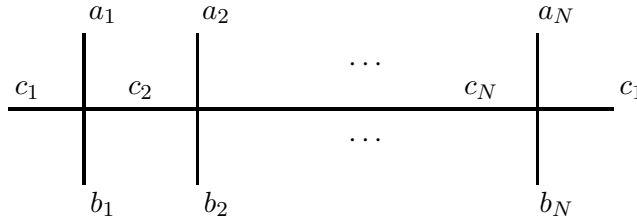


Figure 2.3: Matrix elements of the transfer matrix $\tau_{b_1, \dots, b_N}^{a_1, \dots, a_N}$

Under the periodic boundary conditions, the partition function $Z_{NN'}$ of $N \times N'$ lattice is given by the trace of the N' th power of the transfer matrices:

$$Z_{NN'} = \text{Tr} \left(\tau^{N'} \right) = \sum_{a_1, \dots, a_N} \left(\tau^{N'} \right)_{a_1, \dots, a_N}^{a_1, \dots, a_N} = \Lambda_1^{N'} + \Lambda_2^{N'} + \cdots + \Lambda_{2N}^{N'} \quad (2.6)$$

Here Λ_j denotes the eigenvalue of the transfer matrix τ .

The free energy per site f is given by

$$f = -k_B T \log Z_{NN'} / (NN'). \quad (2.7)$$

In the thermodynamic limit: $N, N' \rightarrow \infty$, the free energy per site is given by the largest eigenvalue Λ_{max} of the transfer matrix τ .

2.3 Diagonalization of the transfer matrix

2.3.1 The Yang-Baxter relations for six-vertex model

Let us consider three sets of the Boltzmann weights: $(w_1, w_2, w_3) = (a, b, c)$, (a', b', c') , and (a'', b'', c'') . We denote by τ' and τ'' the transfer matrices constructed from the sets of the Boltzmann weights

(a', b', c') and (a'', b'', c'') , respectively. If the three sets of the Boltzmann weights satisfy the Yang-Baxter equation

$$\begin{aligned} & \sum_{\alpha, \beta, \gamma} w(\alpha, \gamma | a_1, a_2) w'(\beta, b_3 | \gamma, a_3) w''(b_1, b_2 | \alpha, \beta) \\ = & \sum_{\alpha, \beta, \gamma} w''(\beta, \alpha | a_2, a_3) w'(b_1, \gamma | a_1, \beta) w(b_2, b_3 | \gamma, \alpha), \end{aligned} \quad (2.8)$$

then the transfer matrices τ' and τ'' commute. The derivation of the commutation relation is given in Appendix A. We note that a graphical presentation of the Yang-Baxter equation (2.8) will be shown in Fig. 4.1.

Let us define the parameter Δ as follows

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}. \quad (2.9)$$

For the zero-field six-vertex model, we can show that if the two sets of the Boltzmann weights have the same value of the parameter Δ , then their transfer matrices commute. We shall explicitly discuss in §4 that it is indeed derived from the Yang-Baxter equations (2.8).

2.3.2 The coordinate Bethe ansatz

Let us consider the matrix element $\tau_{b_1, \dots, b_N}^{a_1, \dots, a_N}$ of the transfer matrix τ . Due to the ice rule, we may express the suffix a_1, \dots, a_N by the positions of the value 2, as follows. Suppose that there are n suffices given by the value 2 among the N suffices a_1, \dots, a_N . The n suffices are expressed as $a_{x_1}, a_{x_2}, \dots, a_{x_n}$ where x_j 's are in increasing order: $x_1 < x_2 < \dots < x_n$. Then, the entry a_1, \dots, a_N is equivalent to the set of the x_j 's: x_1, \dots, x_n . For an illustration, let us consider the case $N = 5$ and $n = 3$. Then, $(x_1, x_2, x_3) = (1, 3, 4)$ corresponds to $(a_1, a_2, a_3, a_4, a_5) = (2, 1, 2, 2, 1)$. Thus, the matrix element $\tau_{b_1, \dots, b_N}^{a_1, \dots, a_N}$ can be denoted briefly as $\tau_{y_1, \dots, y_n}^{x_1, \dots, x_n}$.

Let us now discuss how to solve the secular equation: $\tau g = \Lambda g$. Here, the transfer matrix τ is a $2^N \times 2^N$ matrix, g is a 2^N -dimensional eigenvector with eigenvalue Λ . In terms of matrix elements, the secular equation is written as

$$\sum_{y_1, \dots, y_n} \tau_{y_1, \dots, y_n}^{x_1, \dots, x_n} g(y_1, \dots, y_n) = \Lambda g(x_1, \dots, x_n) \quad (2.10)$$

Here, $g(x_1, \dots, x_n)$ denotes the matrix element of the vector g for the entry (x_1, \dots, x_n) . In the coordinate Bethe ansatz, we assume the following form for the matrix element of the possible eigenvector g

$$g(x_1, \dots, x_n) = \sum_{P \in S_n} A_P \exp(k_{P_1} x_1 + \dots + k_{P_n} x_n) \quad (2.11)$$

Here, S_n denotes the symmetric group of order n , and P is a permutation of n letters, $1, 2, \dots, n$, where P maps j into Pj . The expression (2.11) is called the Bethe ansatz wave-function. If the vector g whose elements are of the form (2.11) is an eigenvector of the transfer matrix, then we call it a Bethe ansatz eigenvector.

For general n , the vector (2.11) becomes an eigenvector of the transfer matrix, if the wave-numbers k_j 's satisfy the Bethe ansatz equations. They are given by the following

$$\exp(iNk_j) = (-1)^{n-1} \prod_{\ell=1}^n \exp(-i\Theta(k_j, k_\ell)), \quad \text{for } j = 1, \dots, n, \quad (2.12)$$

where $\Theta(p, q)$ is defined by

$$\exp(-i\Theta(p, q)) = \frac{1 - 2\Delta e^{ip} + e^{i(p+q)}}{1 - 2\Delta e^{iq} + e^{i(p+q)}} \quad (2.13)$$

For the solutions k_j 's to the Bethe ansatz equations, the eigenvalue Λ of the transfer matrix is given by

$$\Lambda(k_1, \dots, k_n) = a^N L(z_1) \cdots L(z_n) + b^N M(z_1) \cdots M(z_n), \quad (2.14)$$

where $z_j = \exp(ik_j)$ for $j = 1, \dots, n$ and the functions $L(z)$ and $M(z)$ are defined by

$$L(z) = \frac{ab + (c^2 - b^2)z}{a(a - bz)}, \quad M(z) = \frac{a^2 - c^2 - abz}{b(a - bz)}. \quad (2.15)$$

When we discuss the spectrum of an integrable model through the coordinate Bethe ansatz, we often assume that all the eigenvectors of the transfer matrix are characterized by the Bethe ansatz wave-function (2.11). However, it is not certain whether the assumption is valid or not. Thus, we have to check it by other methods. In fact, there are several numerical studies on the validity of the completeness of the Bethe ansatz for some integrable models.

On the other hand, there is no doubt on the mathematical structure of the Bethe ansatz wave-function. We can derive the expression (2.11) by the algebraic Bethe ansatz through the 'two-site' model [79, 16]. (There is an instructive notice in [111].) It was shown that the matrix elements of the product of B operators acting on the vacuum are given by the Bethe ansatz wave-function (2.11) with k_j 's being generic.

The Yang-Baxter relation leads to not only the integrability of the six-vertex model but also the systematic construction of the eigenvectors. In fact, we shall see in §4 that the algebraic Bethe ansatz is solely based on the Yang-Baxter equation.

2.3.3 An example of the eigenvector

For an illustration, let us consider the eigenvector g for the case of $n = 1$: $g(x) = A \exp(ikx)$. Through a direct calculation, we have

$$\begin{aligned} \sum_{y=1}^N \tau_y^x g(y) &= \sum_{y=1}^{x-1} a^{x-y-1} b^{N-x+y-1} c^2 g(y) + \sum_{y=x+1}^N a^{N+x-y-1} b^{y-x-1} c^2 g(y) \\ &= \left(a^N L(z) + b^N M(z) \right) g(x) + \frac{a^{N-1} b^{N-x} c^2 z}{a - bz} (1 - z^N) \end{aligned} \quad (2.16)$$

Therefore, $g(x)$ becomes an eigenvector if the Bethe ansatz equation $z^N = 1$ is satisfied.

2.4 The free energy of the six-vertex model

2.4.1 Three phases of the six-vertex model

There are three phases for the zero-field six-vertex model. They are given by the regions of the parameter Δ : ferroelectric phase when $\Delta > 1$; anti-ferroelectric phase when $\Delta < -1$; disordered phase when $-1 < \Delta < 1$. It is found that the disordered phase ($-1 < \Delta < 1$) is gapless (massless), while the ferroelectric phase ($\Delta > 1$) and the anti-ferroelectric phase ($\Delta < -1$) are gapful (massive).

Let us consider the phase diagram of the six vertex model shown in Fig. 2.4. We recall that the ratios a/c and b/c determine the model. Here we note that the number of independent parameters is given by two, since the overall normalization factor is arbitrary. The regimes I and II give the ferroelectric phase, the regimes III and IV are anti-ferroelectric and disordered, respectively. In terms of the Boltzmann weights, the regime I is given by $a > b + c$, the regime II by $b > a + c$, and the regime IV by $a + b < c$. Then, the regime III is given by $a < b + c$, $b < c + a$ and $c < a + b$.

Let us derive the three phases through an intuitive argument. For the ferroelectric regime of $\Delta > 1$, we have $a^2 + b^2 - c^2 > 2ab$, which leads to the inequality: $|a - b| > c$. When $a > b$, we have $a > b + c$. Thus, the configuration where all the Boltzmann weights are given by the weight a should

be the largest contribution to the partition function Z . In fact, when $n = 0$ in eq. (2.14), we have $\Lambda = a^N + b^N$. When $a > b$, the free energy per site, f , is given by ϵ_1 : $f = -k_B T \log a$. When $b > a$, we have $f = \epsilon_3$, similarly.

For the anti-ferroelectric regime of $\Delta < -1$, we have the inequality $a + b < c$. Thus, the vertex configurations for c should be more favorable than those of a or b . In fact, it is shown that the transfer matrix has the largest eigenvalue when $n = N/2$. Furthermore, if we send $|\Delta|$ to infinity, all the vertex configurations should be given by those of c . The phase is thus called anti-ferroelectric.

We may explain the reason why it is called anti-ferroelectric. Let us consider the configuration (5) of Fig. 2.2. We see that the arrow coming from the left goes upward, while the arrow coming from the right goes downward. If all the vertex configurations on the square lattice are given by (5), then the lines coming from South West to North East and the lines coming from North East to South West occupy the lattice alternatively. This gives the anti-ferroelectric order.

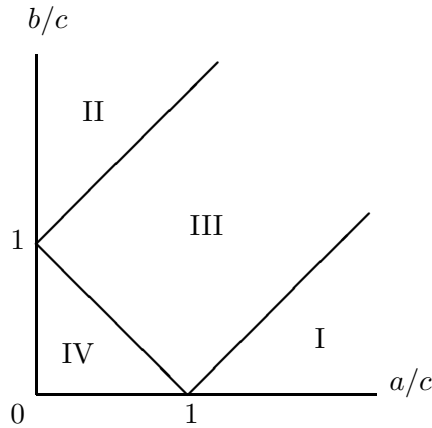


Figure 2.4: Phase diagram of the six-vertex model: regimes I and II are ferroelectric, regime III is disordered, and regime IV is anti-ferroelectric.

2.4.2 Parametrization of the Boltzmann weights

It is nontrivial how to parametrize the Boltzmann weights. Recall that there are two independent parameters for the six-vertex model. Thus, if we consider Δ as a parameter, there is only another one. As we shall see later, it is related to the spectral parameter.

We recall that the phases of the zero-field six-vertex model are classified into the following: $\Delta < -1$, $-1 < \Delta < 1$ and $1 < \Delta$.

(1) *Anti-ferroelectric phase*

For $\Delta < -1$, we define a real parameter λ by

$$\Delta = -\cosh \lambda \quad (0 < \lambda) \quad (2.17)$$

and we parametrize the Boltzmann weights as follows

$$a = \rho \sinh\left(\frac{\lambda - v}{2}\right), \quad b = \rho \sinh\left(\frac{\lambda + v}{2}\right), \quad c = \rho \sinh \lambda, \quad (-\lambda < v < \lambda) \quad (2.18)$$

where ρ is the normalization factor. Let us define the rapidity α for the wavenumber k as

$$\exp(ik) = \frac{e^\lambda - e^{-i\alpha}}{e^{\lambda - i\alpha} - 1} = -\frac{\sin \frac{1}{2}(\alpha - i\lambda)}{\sin \frac{1}{2}(\alpha + i\lambda)} \quad (2.19)$$

By replacing the wave numbers p and q with the rapidities α and β in eq. (2.13), the phase factor $\Theta(p, q)$ is written as follows

$$\exp(-i\Theta(p, q)) = \frac{e^{2\lambda} - e^{-i(\alpha-\beta)}}{e^{2\lambda-i(\alpha-\beta)} - 1} = -\frac{\sin \frac{1}{2}((\alpha - \beta) - 2i\lambda)}{\sin \frac{1}{2}((\alpha - \beta) + 2i\lambda)} \quad (2.20)$$

(2) *Disordered phase*

For $-1 < \Delta < 1$, we define a positive real parameter μ by

$$\Delta = -\cos \mu \quad (0 < \mu < \pi) \quad (2.21)$$

and we parameterize the Boltzmann weights as follows

$$a = \rho \sin\left(\frac{\mu - w}{2}\right), \quad b = \rho \sin\left(\frac{\mu + w}{2}\right), \quad c = \rho \sin \mu, \quad (-\mu < w < \mu). \quad (2.22)$$

Here ρ is the normalization factor. We define the rapidity α for the wavenumber k by

$$\exp(ik) = \frac{e^{i\mu} - e^\alpha}{e^{i\mu+\alpha} - 1} = -\frac{\sinh \frac{1}{2}(\alpha - i\mu)}{\sinh \frac{1}{2}(\alpha + i\mu)} \quad (2.23)$$

In terms of rapidities α and β , the phase factor $\Theta(p, q)$ is expressed as

$$\exp(-i\Theta(p, q)) = \frac{e^{2i\mu} - e^{\alpha-\beta}}{e^{2i\mu+\alpha-\beta} - 1} = -\frac{\sinh \frac{1}{2}((\alpha - \beta) - 2i\mu)}{\sinh \frac{1}{2}((\alpha - \beta) + 2i\mu)} \quad (2.24)$$

(3) *Ferroelectric phase*

For $\Delta > 1$, we define the real parameter λ by

$$\Delta = \cosh \lambda \quad (0 < \lambda) \quad (2.25)$$

and we may parameterize the Boltzmann weights as follows: When $a > b$ (the regime I),

$$a = \rho \sinh\left(\frac{\lambda - v}{2}\right), \quad b = -\rho \sinh\left(\frac{\lambda + v}{2}\right), \quad c = \rho \sinh(\lambda), \quad (v < -\lambda). \quad (2.26)$$

When $a < b$ (the regime II),

$$a = -\rho \sinh\left(\frac{\lambda - v}{2}\right), \quad b = \rho \sinh\left(\frac{\lambda + v}{2}\right), \quad c = \rho \sinh(\lambda), \quad (v > \lambda). \quad (2.27)$$

Here we recall that ρ is the normalization factor. The wavenumber k is related to the rapidity α by

$$\exp(ik) = -\frac{e^\lambda - e^{-i\alpha}}{e^{\lambda-i\alpha} - 1} = \frac{\sin \frac{1}{2}(\alpha - i\lambda)}{\sin \frac{1}{2}(\alpha + i\lambda)} \quad (2.28)$$

The phase factor $\Theta(p, q)$ is written as

$$\exp(-i\Theta(p, q)) = \frac{e^{2\lambda} - e^{-i(\alpha-\beta)}}{e^{2\lambda-i(\alpha-\beta)} - 1} = -\frac{\sin \frac{1}{2}((\alpha - \beta) - 2i\lambda)}{\sin \frac{1}{2}((\alpha - \beta) + 2i\lambda)} \quad (2.29)$$

Let us consider curved lines given by changing the spectral parameter w or v continuously. All the curves pass through the points $(a/c, b/c) = (1, 0)$ and $(0, 1)$. Except for the two points, however, any points on the horizontal axis: $b/c = 0$ or the vertical axis: $a/c = 0$ are never reached by the above parametrizations with finite values.

2.4.3 Expressions of the free energy

(1) *Anti-ferroelectric phase*

When $\Delta < -1$, the system is in the anti-ferroelectric phase. The free energy per site f is given by

$$f = -k_B T \log a - k_B T \left(\frac{\lambda + v}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\lambda} \sinh m(\lambda + v)}{m \cosh m\lambda} \right) \quad (2.30)$$

$$= -k_B T \log b - k_B T \left(\frac{\lambda - v}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\lambda} \sinh m(\lambda - v)}{m \cosh m\lambda} \right) \quad (2.31)$$

for $-\lambda < v < \lambda$.

(2) *Disordered phase*

When $-1 < \Delta < 1$, we have

$$f = -k_B T \log a - k_B T \int_{-\infty}^{\infty} \frac{\sinh(\mu + w)x \sinh(\pi - \mu)x}{2x \sinh \pi x \cosh \mu x} dx \quad (2.32)$$

$$= -k_B T \log b - k_B T \int_{-\infty}^{\infty} \frac{\sinh(\mu - w)x \sinh(\pi - \mu)x}{2x \sinh \pi x \cosh \mu x} dx \quad (2.33)$$

for $-\mu < w < \mu$.

(3) *Ferroelectric phase*

When $a > b$ (the regime I), we have $f = -k_B T \log a$, while when $b > a$ (the regime II), we have $f = -k_B T \log b$.

2.4.4 Correlation length

Correlation length has been calculated for the six-vertex model [32, 1]. In fact, it is calculated for the eight-vertex model. In the ferroelectric and the anti-ferroelectric regimes of the six-vertex model, the correlation length is finite. It becomes very large near their phase boundaries to the disordered phase. In the disordered phase of the six-vertex model, the correlation length diverges throughout the regime. Thus, the disordered phase is critical.

2.5 Critical singularity in the anti-ferro regime near the phase boundary

We now discuss a singular behavior of the free energy shown at the phase transition from the anti-ferroelectric to the disordered phases [1]. Here we note that the former one is gapful, while the latter is gapless. Recall that the phase boundary between the regimes III and IV is given by $a + b = c$. When $T > T_c$, the system should be in the disordered regime ($-1 < \Delta < 1$), while when $T < T_c$ it is in the anti-ferroelectric regime ($\Delta < -1$). In the lower temperature, the system should be ordered.

Let us calculate the analytic continuation of the high-temperature free energy (2.32) into the low-temperature phase, so that we can single out the singularity of the free energy near T_c and for $T < T_c$. First, we reformulate the integral of eq. (2.32) as follows.

$$\int_{-\infty}^{\infty} \frac{\sinh(\mu + w)x \sinh(\pi - \mu)x}{2x \sinh \pi x \cosh \mu x} dx = \mathcal{P} \int_{-\infty}^{\infty} \frac{\sinh(\mu + w)x \exp(\pi - \mu)x}{2x \sinh \pi x \cosh \mu x} dx \quad (2.34)$$

Here \mathcal{P} denote the principal value integral. Second, we set λ to be a very small positive real number, and take a value v satisfying $-\lambda < v < \lambda$. We consider the path in the complex μ -plane:

$$\mu = \lambda \exp(-i\theta), \quad \text{for } 0 \leq \theta \leq \pi/2. \quad (2.35)$$

Along the path, we calculate the analytic continuation of the high-temperature free energy (2.32). Here, we also consider the path of w : $w = v \exp(-i\theta)$ for $0 \leq \theta \leq \pi/2$. Then, we have

$$\begin{aligned} & \mathcal{P} \int_{-\infty}^{\infty} \frac{-i \sin(\lambda + v)x \exp(\pi + i\lambda)x}{2x \sinh \pi x \cos \lambda x} dx \\ &= \frac{\lambda + v}{2} + \sum_{m=1}^{\infty} \frac{e^{-m\lambda} \sinh m(\lambda + v)}{m \cosh m\lambda} - i \sum_{m=1}^{\infty} \frac{(-1)^m \cos((m-1/2)\pi v/\lambda) e^{-(m-1/2)\pi^2/\lambda}}{(m-1/2) \sinh((m-1/2)\pi^2/\lambda)} \end{aligned} \quad (2.36)$$

The real part of the analytic continuation corresponds to the expression of the anti-ferroelectric free energy. Therefore, we obtain the singular part of the free energy

$$f_{sing} = ik_B T \sum_{m=1}^{\infty} \frac{(-1)^m \cos((m-1/2)\pi v/\lambda) \exp(-(m-1/2)\pi^2/\lambda)}{(m-1/2) \sinh((m-1/2)\pi^2/\lambda)} \quad (2.37)$$

We define the reduced temperature t by

$$t = (a + b - c)/c. \quad (2.38)$$

In the low-temperature phase ($T < T_c$) and near T_c , t is given by

$$t \approx -\frac{1}{8}(\lambda^2 - v^2) \quad (2.39)$$

Approximately, t is given by $t \approx -\lambda^2/8$. Near T_c , we have

$$f_{sing} \approx -4ik_B T e^{-\pi^2/\lambda} \cos\left(\frac{\pi v}{2\lambda}\right) \quad (2.40)$$

Thus, we have

$$f_{sing} \propto \exp\left(-\frac{\text{constant}}{\sqrt{-t}}\right) \quad (2.41)$$

Near T_c , the free energy has an essential singularity.

The singularity of the free energy is very close to that of the Kosterlitz-Thouless transition. In fact, calculating exactly, we can show that the correlation length ξ diverges at T_c as $\xi \propto \exp(\text{constant}/\sqrt{-t})$, when T approaches T_c in the anti-ferroelectric phase.

2.6 XXZ spin chain and the transfer matrix

The logarithmic derivative of the transfer matrix of the six-vertex model gives the Hamiltonian of the XXZ spin chain.

$$\frac{d}{dv} \log \tau|_{v=-\lambda} = \tau^{-1} \frac{d}{dv} \tau \propto H_{XXZ} + \text{constant} \quad (2.42)$$

where H_{XXZ} is given by

$$H_{XXZ} = J \sum_{j=1}^L \left(\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right). \quad (2.43)$$

Intuitively, we may express it by $\tau_{6V}(v) \approx \exp(-vH_{XXZ})$.

The XXZ spin chain and the six-vertex transfer matrix have the same eigenvectors in common thanks to eq. (2.42). Taking logarithm of the Bethe ansatz equations, we have

$$Nk_j = 2\pi I_j - \sum_{\ell=1}^M \Theta(k_j, k_\ell), \quad \text{for } j = 1, \dots, M. \quad (2.44)$$

where M is the number of down spins. (We assume $2M \leq N$.) Here I_j is an integer if M is odd and half an integer if M is even.

The ground state of the XXZ spin chain for $\Delta < 1$ was obtained by Yang and Yang [3]. The ground state is specified by the integers $I_j = j - (M + 1)/2$ for $j = 1, \dots, M$. When N is very large, the distribution of k_j 's becomes continuous. The number of k_j 's between k and $k + dk$ can be approximated by $N\rho(k)dk$. Thus, we have the integral equation of $\rho(k)$

$$2\pi\rho(k) = 1 + \int_{-Q}^Q \frac{\partial\Theta(k, k')}{\partial k} \rho(k') dk' \quad (2.45)$$

where Q is determined by the normalization condition

$$\int_{-Q}^Q \rho(k) dk = M/N. \quad (2.46)$$

The integral equation (2.45) can be solved by changing the variable k to rapidity α , and then by taking the Fourier transform for the half-filling case $M/N = 1/2$. When M/N is close to $1/2$, the integral equation can be solved by the Wiener-Hopf method [3].

2.7 Low-lying excited spectrum of the transfer matrix and conformal field theory

In this section, we assume that the low excited spectra of the transfer matrix of gapless models should be characterized by the conformal invariance, if the system size is large enough. The assumption is not rigorous, however, there are many studies which confirm it numerically for integrable models. We review the finite-size corrections for the XXZ spin chain and the six-vertex model.

2.7.1 Finite-size corrections

Let us consider a conformally invariant field theory defined in the two-dimensional Euclidean space with coordinates r_1 and r_2 . The energy momentum tensor $T_{\mu\nu}$ for $\mu, \nu = 1, 2$ should be symmetric and traceless due to the conformal symmetry. Introducing the complex coordinates $z = r_1 + ir_2$, $\bar{z} = r_1 - ir_2$, we define the chiral operator: $T = (T_{11} - T_{22} - 2iT_{12})/4$, and the anti-chiral operator: $\bar{T} = (T_{11} - T_{22} + 2iT_{12})/4$. The operator T (or \bar{T}) depends only on the variable z (or \bar{z}).

The energy momentum tensor has the operator product expansion

$$T(z_1)T(z_2) = \frac{c/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{z_1 - z_2} + \dots \quad (2.47)$$

Here c is called the central charge. We define the operators L_n by the expansion: $T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$. The operator product expansion (2.47) corresponds to the Virasoro algebra

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad (2.48)$$

Under a conformal transformation $z \rightarrow w$, the energy momentum tensor is transformed as

$$T(z) = \left(\frac{dw}{dz}\right)^2 \tilde{T}(w) + \frac{c}{12}\{w, z\} \quad (2.49)$$

where the symbol $\{w, z\}$ denotes the Schwarzian derivative: $(d^3w/dz^3)/(dw/dz) - (3/2)(d^2w/dz^2)^2/(dw/dz)^2$.

Let us consider the conformal mapping from the z -plane to a cylinder of circumference L :

$$z \rightarrow w = \frac{L}{2\pi} \log z \quad (2.50)$$

Here $w = \tau - ix$ with imaginary time $\tau = it$. The Hamiltonian \hat{H} on the cylinder is given by the space integral of the (1,1) component of the energy momentum tensor $(T_{cyl})_{\mu\nu}$

$$\begin{aligned}\hat{H} &= \frac{1}{2\pi} \int_0^L dx (T_{cyl}(w) + \bar{T}_{cyl}(w)) \\ &= \frac{2\pi}{L} (L_0 + \bar{L}_0) - \frac{\pi c}{6L}\end{aligned}\quad (2.51)$$

Here we have used (2.49). For the momentum operator on the cylinder, we have

$$\begin{aligned}\hat{P} &= \frac{1}{2\pi} \int_0^L dx (T_{cyl}(w) - \bar{T}_{cyl}(w)) \\ &= \frac{2\pi}{L} (L_0 - \bar{L}_0)\end{aligned}\quad (2.52)$$

Let us now discuss the application of the formulas (2.51) and (2.52) to the quantum spin chains. We assume that the low excited energies should be gapless and conformally invariant. In other words, we assume that the excitations near the ground state have a linear dispersion relation. Let v denote the velocity of the linear dispersion. Then, for the ground-state energy E_0 , we have

$$E_0 = L e_\infty - \frac{\pi v c}{6L} \quad (2.53)$$

and for the excited energy E_{ex} and the momentum P_{ex} we have

$$\begin{aligned}E_{ex} - E_0 &= \frac{2\pi v}{L} (h + \bar{h} + N + \bar{N}) \\ P_{ex} - P_0 &= \frac{2\pi}{L} (h - \bar{h} + N - \bar{N})\end{aligned}\quad (2.54)$$

where h and \bar{h} are the conformal weights related to the zero modes of the field, and the eigenvalues of N and \bar{N} are given by non-negative integers.

There is another viewpoint on the finite-size scaling. Let us consider the t axis as the space axis for an infinitely long quantum spin chain, and x axis as the imaginary time axis. Here we assume that $L = v\beta = v/T$. Thus, our system now becomes the quantum spin chain in the finite temperature T . Replacing E_0 with $v\beta f$, where f denotes the finite-temperature free energy of the spin chain, we have $f = -\pi c T^2 / 6v$. Thus, we may calculate the specific heat C by the formula $C = -T \partial^2 f / \partial T^2$, and we have

$$C = \frac{\pi c}{3v} T \quad (2.55)$$

2.7.2 The free Boson: CFT with $c = 1$

Let us consider a free Bose field $\varphi(x, t)$ defined on a cylinder of circumference L . The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} g \int dx \left\{ (\partial_t \varphi)^2 - (\partial_x \varphi)^2 \right\} \quad (2.56)$$

We define the mode φ_n by the Fourier expansion $\varphi(x, t) = \sum \varphi_n(t) \exp(-2\pi i n x / L)$. From the canonical quantization, we have the conjugate momentum $\pi_n = g L \dot{\varphi}_{-n}$, and the commutation relation $[\varphi_n, \pi_m] = i \delta_{nm}$. With the operators a_n and \bar{a}_n for $n \neq 0$ satisfying

$$[a_n, a_m] = n \delta_{n+m} \quad [a_n, \bar{a}_m] = 0 \quad [\bar{a}_n, \bar{a}_m] = n \delta_{n+m}, \quad (2.57)$$

the Fourier mode is expressed as $\varphi_n = i (a_n - \bar{a}_{-n}) / (n \sqrt{4\pi g})$, for $n \neq 0$. The Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2gL} \pi_0^2 + \frac{2\pi}{2L} \sum_{n \neq 0} (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n) \quad (2.58)$$

Hereafter, we assume $g = 1/4\pi$. The convention is consistent with the conformally invariant partition functions.

We now discuss the compactification of the Boson with radius R . Suppose that the field operator φ takes its value only on the circle of radius R . In other words, we may identify φ with $\varphi + 2\pi R$. Then, the eigenvalue of the momentum π_0 conjugate to φ_0 is given by n/R for an integer n . Here we recall that the wavenumber of a one-dimensional system of size L is given by $2\pi n/L$ ($n \in \mathbf{Z}$), and also that the range of φ_0 is given by $2\pi R$, which corresponds to L . Furthermore, we may assign on the operator φ the boundary condition for an integer m

$$\varphi(x + L, t) = \varphi(x, t) + 2\pi mR \quad (2.59)$$

Here, the integer m is called the *winding number*. The mode expansion of φ is given by

$$\varphi(x, t) = \varphi_0 + \frac{4\pi}{L}\pi_0 t + \frac{2\pi Rm}{L}x + i \sum_{n \neq 0} \frac{1}{n} \left(a_n e^{2\pi i n(x-t)/L} - \bar{a}_{-n} e^{2\pi i n(x+t)/L} \right) \quad (2.60)$$

In terms of the coordinates $z = \exp(2\pi(\tau - ix)/L)$ and $\bar{z} = \exp(2\pi(\tau + ix)/L)$, $\varphi(x, t)$ is given by the sum of the holomorphic and anti-holomorphic parts: $\varphi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$. Here, they are given by

$$\phi(z) = \frac{\varphi_0}{2} - ia_0 \log(z) + i \sum_{k \neq 0} \frac{1}{k} a_k z^{-k} \quad \bar{\phi}(\bar{z}) = \frac{\varphi_0}{2} - i\bar{a}_0 \log(\bar{z}) + i \sum_{k \neq 0} \frac{1}{k} \bar{a}_k \bar{z}^{-k} \quad (2.61)$$

with $a_0 = n/R + mR/2$ and $\bar{a}_0 = n/R - mR/2$. Here we can show that the operator $J(z) = i\partial\phi(z)/\partial z$ is the $U(1)$ current operator.

Making use of Noether's theorem, we have

$$T(z) = -\frac{1}{2} : \left(\frac{\partial\phi(z)}{\partial z} \right)^2 :, \quad \bar{T}(\bar{z}) = -\frac{1}{2} : (\partial\bar{\phi}(\bar{z}))^2 : \quad (2.62)$$

Here $::$ denotes a proper normal ordering. Then, through the Laurent expansion of powers of z , we have

$$L_0 = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_{-n}a_n \quad (2.63)$$

Thus, the conformal weights h_{nm} and \bar{h}_{nm} are given by

$$h_{n,m} = \frac{1}{2} \left(\frac{n}{R} + \frac{1}{2} mR \right)^2, \quad \bar{h}_{n,m} = \frac{1}{2} \left(\frac{n}{R} - \frac{1}{2} mR \right)^2 \quad (2.64)$$

2.7.3 The XXZ spin chain and CFT with $c = 1$

We discuss the finite-size corrections to the XXZ spin chain. The finite-size corrections to the ground state energy is calculated in [42, 43, 44, 45] based on the method with the Euler-MacLaurin formula [41]. (For a review, see [16, 46, 47].) The result is

$$E_{ex} = Le_{\infty} - \frac{\pi v}{6L} + \frac{2\pi v}{L} \left((\Delta D)^2 \xi^2 + \frac{(\Delta M)^2}{4\xi^2} + N + \bar{N} \right) \quad (2.65)$$

$$P_{ex} - P_0 = 2k_F \Delta D + \frac{2\pi}{L} (\Delta D \Delta M + N - \bar{N}) \quad (2.66)$$

Here the term e_{∞} denotes the ground-state energy per site. The Fermi velocity is obtained by the derivative of the dressed energy [16] with respect to the rapidity at the Fermi level.

The central charge c is given by 1. ΔD and ΔM are integers. ΔM denotes the change in the number of down spins, and ΔD the number of particles jumping over the Fermi sea through the backscattering.

We note that the difference of the conformal weights (2.64) is given by $h_{nm} - \bar{h}_{nm} = nm$. Thus, ΔM and ΔD correspond to n and m of the $c = 1$ CFT, respectively. N and \bar{N} are derived from the particle-hole excitations near the Fermi surface. The Fermi wavenumber k_F is given by $k_F = \pi M/L$ where M is the number of down spins. If the dispersion is linear, k_F is consistent with the number of particles M .

The parameter ξ is given by the dressed charge, which is defined by an integral equation. We note that the sum of the conformal weights (2.64) is given by $h_{n,m} + \bar{h}_{n,m} = n^2/R^2 + m^2R^2/4$. Thus, the dressed charge ξ corresponds to the radius R of the $c = 1$ CFT, $\xi = R/2$. Under zero magnetic field, the dressed charge ξ or the radius R is given by [33]

$$R = \left(\frac{2\pi}{\pi - \mu} \right)^{1/2} \quad (0 \leq \mu \leq \pi). \quad (2.67)$$

3 Various integrable models on two-dimensional lattices

3.1 Ising model and Potts model

3.1.1 Ising model

Let us consider the Ising model defined on a square lattice. Each lattice site has spin variable which takes the two values ± 1 . We denote by σ_j the spin variable of lattice site j . The Hamiltonian of the Ising model is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \quad (3.1)$$

where the symbol $\langle i, j \rangle$ denote that sites i and j are nearest neighbors, and we take the sum over all the pairs of adjacent sites on the lattice.

There have been many papers written on the two-dimensional Ising model [8]. However, it should be remarked that correlation functions are calculated exactly for the two-dimensional Ising model in the scaling limit [50, 51].

We remark that exact solutions are discussed for the Ising model defined on various two-dimensional lattices such as the Kagome lattice (For a review, see Ref. [52]).

3.1.2 Self-dual Potts model

The Potts model generalizes the Ising model into a p -state model with $p > 2$ [1, 53, 54]. Let us consider the three-state Potts model defined on a square lattice [53]. Each lattice site has spin variable which takes three values 1, 2, 3. We denote by σ_j the spin variable of lattice site j . The Hamiltonian of the Potts model is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j) \quad (3.2)$$

Here the symbol $\delta(a, b)$ denote the Kronecker delta.

In general, the Potts model is not solvable. However, at its criticality, it is equivalent to some variant of the six-vertex model, and is solvable. Thus, the self-dual Potts model is integrable [55, 56].

3.1.3 Ashkin-Teller model

With each site i we associate two spins: s_i and t_i . They take values ± 1 . The Hamiltonian of the model is given by

$$\mathcal{H} = - \sum_{\langle i,j \rangle} [K_2(s_i s_j + t_i t_j) + K_4 s_i s_j t_i t_j] \quad (3.3)$$

It is known that the Ashkin-Teller model and the eight-vertex model are in the same universality class [38]. The universality class is described by the $c = 1$ CFT with the twisted boson [40].

3.2 Chiral Potts model

3.2.1 General case

There is another version of the Potts model [1, 53]. We may assign the chiral symmetry, or $\mathbf{Z}/p\mathbf{Z}$ symmetry on the Potts model, which we call the chiral Potts model.

We now explain the most general chiral Potts model defined on a square lattice [64]. Let a and b denote the spin variables defined on two nearest-neighboring sites. The interaction energy between the spins depends on the difference $n = a - b \pmod{N}$ as

$$\mathcal{E}(n) = \sum_{j=1}^{N-1} E_j \omega^{jn} \quad \omega = e^{2\pi/N} \quad (3.4)$$

We note that $\mathcal{E}(N+n) = \mathcal{E}(n)$. The parameter E_j can be written as

$$\frac{E_j}{k_B T} = -K_j \omega^{\Delta_j}, \quad \text{for } j = 1, \dots, \left[\frac{N}{2}\right] \quad (3.5)$$

where K_j and Δ_j constitute $N-1$ independent variables, and the symbol $[\cdot]$ denotes the Gaussian symbol. For a real number x , $[[x]]$ denotes the biggest integer not larger than x . We also assume that E_{N-j} is complex conjugate to E_j : $E_{N-j} = E_j^*$. When N is odd, we have

$$-\frac{\mathcal{E}(n)}{k_B T} = \sum_{j=1}^{[[\frac{N-1}{2}]]} 2K_j \cos\left(\frac{2\pi}{N}(jn + \Delta_j)\right) + K_{N/2} (-1)^n \frac{((-1)^N + 1)}{2} \quad (3.6)$$

3.2.2 Integrable chiral Potts model

Let us discuss the integrable restriction of the most general chiral Potts model defined on the square lattice. It has horizontal and vertical couplings. Suppose that spin variables a and b are located on two neighboring sites connected by a horizontal line. When the line goes rightward from a to b , the horizontal coupling has the energy $\mathcal{E}_{pq}(a-b)$ and the Boltzmann weight $W_{pq}(n)$. Here p and q are ‘rapidity’ parameters. For spin variables c and d located on two neighboring sites connected by a vertical line, the vertical coupling has the energy $\mathcal{E}_{pq}(c-d)$ and the Boltzmann weight $\bar{W}_{pq}(c-d)$ when the vertical line goes upward from c to d .

The model is called solvable if the Boltzmann weights W ’s and \bar{W} ’s satisfy the star-triangle equation

$$\sum_{d=1}^N \bar{W}_{qr}(b-d) W_{pr}(a-d) \bar{W}_{pq}(d-c) = R_{pqr} W_{pq}(a-b) \bar{W}_{pr}(b-c) W_{qr}(a-c) \quad (3.7)$$

The solution to the above equation is given by

$$\begin{aligned} W_{pq}(n) &= W_{pq}(0) \prod_{j=1}^n \left(\frac{\mu_p}{\mu_q} \cdot \frac{y_q - x_p \omega^j}{y_p - x_q \omega^j} \right) \\ \bar{W}_{pq}(n) &= \bar{W}_{pq}(0) \prod_{j=1}^n \left(\mu_p \mu_q \cdot \frac{\omega x_p - x_q \omega^j}{y_q - y_p \omega^j} \right) \end{aligned} \quad (3.8)$$

with a constant R_{pqr} depending on the three rapidity variables. The constraint that the Boltzmann weight should have periodicity modulo N : $W_{pq}(n+N) = W_{pq}(n)$ gives for all rapidity pairs p and q

$$\left(\frac{\mu_p}{\mu_q} \right)^N = \frac{y_p^N - x_q^N}{y_q^N - x_p^N}, \quad (\mu_p \mu_q)^N = \frac{y_q^N - y_p^N}{x_p^N - x_q^N} \quad (3.9)$$

We can define k and k' such that

$$\mu_p = \frac{k'}{1 - kx_p^N} = \frac{1 - ky_p^N}{k'} \quad (3.10)$$

$$x_p^N + y_p^N = k(1 + x_p^N y_p^N) \quad (3.11)$$

where $k^2 + (k')^2 = 1$. Thus, the rapidities are placed on a curve of genus $g > 1$ of Fermat type.

We make some comments. Multiplying the two equations of (3.9) and noting that p and q are independent, we can show eq. (3.10) and then eq. (3.11). The star-triangle equation is proven illustratively in the appendix of Ref. [60]. We note that the vertex-type formulation of the chiral Potts model is related to the tetrahedron equation [63]. The integrable chiral Potts model generalizes the self-dual Z_N model given by Fateev and Zamolodchikov [65]. An elliptic extension of the self-dual Z_N model is introduced in Ref. [66], and the Yang-Baxter equation is proven for the model in Ref. [67]. Some nontrivial connections among higher-rank chiral Potts models, elliptic IRF models and Belavin's Z_N symmetric model are explicitly discussed in Ref. [68].

3.3 The eight-vertex model

Let us explain the eight-vertex model which was solved by Baxter in 1972 [4]. The Boltzmann weights $w(\alpha, \beta | \gamma, \delta; u)$ are nonzero if the charge is conserved modulo 2: $\alpha + \beta = \gamma + \delta \pmod{2}$. We have the six configurations around a vertex shown in Fig. 2.2 and the two in Fig. 3.1. We assume that there is no external field. The weights therefore become symmetric: $\epsilon_1 = \epsilon_2$, $\epsilon_3 = \epsilon_4$, $\epsilon_5 = \epsilon_6$, and $\epsilon_7 = \epsilon_8$.

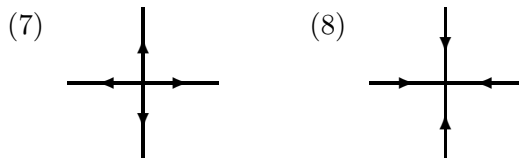


Figure 3.1: Vertex configurations for the eight-vertex model. (7) $w(2, 2 | 1, 1)$; (8) $w(1, 1 | 2, 2)$.

$$\begin{aligned} w(1, 1 | 1, 1) &= w(2, 2 | 2, 2) = w_1 = a_{8V} \\ w(1, 2 | 2, 1) &= w(2, 1 | 1, 2) = w_2 = b_{8V} \\ w(1, 2 | 1, 2) &= w(2, 1 | 2, 1) = w_3 = c_{8V} \\ w(1, 1 | 2, 2) &= w(2, 2 | 1, 1) = w_4 = d_{8V} \end{aligned} \quad (3.12)$$

We give a parametrization of the Boltzmann weights. We define the theta function

$$\theta(z; \tau) = 2p^{1/4} \sin \pi z \prod_{n=1}^{\infty} (1 - p^{2n})(1 - p^{2n} \exp(2\pi iz))(1 - p^{2n} \exp(-2\pi iz)), \quad (3.13)$$

where the nome p is related to the parameter τ by $p = \exp(\pi i \tau)$ with $\text{Im } \tau > 0$. We also define the theta functions $\theta_0(z)$ and $\theta_1(z)$ satisfying $\theta_\alpha(z + 1) = (-1)^\alpha \theta_\alpha(z)$ and $\theta_\alpha(z + \tau) =$

$ie^{-\pi i(z+\tau/2)}\theta_{1-\alpha}(z)$ for $\alpha = 0, 1$, where we define $\theta_1(z)$ by $\theta_1(z; \tau) = \theta(z; 2\tau)$. The Boltzmann weights $a_{8V}(z), b_{8V}(z), c_{8V}(z)$ and $d_{8V}(z)$ are expressed as

$$\begin{aligned} a_{8V}(z) &= \frac{\theta_0(z)\theta_0(2\eta)}{\theta_0(z-2\eta)\theta_0(0)}, & b_{8V}(z) &= \frac{\theta_1(z)\theta_0(2\eta)}{\theta_1(z-2\eta)\theta_0(0)}, \\ c_{8V}(z) &= -\frac{\theta_0(z)\theta_1(2\eta)}{\theta_1(z-2\eta)\theta_0(0)}, & d_{8V}(z) &= -\frac{\theta_1(z)\theta_1(2\eta)}{\theta_0(z-2\eta)\theta_0(0)}. \end{aligned} \quad (3.14)$$

3.4 IRF models

3.4.1 Unrestricted 8V SOS model

We introduce unrestricted Solid-on-Solid models. They are also called Interaction-Round-a-Face models or IRF models, briefly [1].

To each site i of a two-dimensional square lattice, a spin a_i is associated. Let i, j, k , and ℓ be the lattice sites surrounding a face (or a square), where i, j, k, ℓ are placed counterclockwise from the southwest corner. We assume that an elementary configuration is given by that of the four spin variables around the face, and the probability of having a_i, a_j, a_k, a_ℓ is denoted by the Boltzmann weight $w(a_i, a_j, a_k, a_\ell; z)$. Here the variable z is called the spectral parameter.

For unrestricted 8V SOS model, a_i can take any integer. When sites i and j are nearest neighboring, then the states a_i and a_j are said to be admissible if and only if $|a_i - a_j| = 1$ [69, 1]. The Yang-Baxter equations are given by

$$\begin{aligned} &\sum_g w(a, b, g, f; z-w)w(f, g, d, e; z)w(g, b, c, d; w) \\ &= \sum_g w(f, a, g, e; w)w(a, b, c, g; z)w(g, c, d, e; z-w), \end{aligned} \quad (3.15)$$

where the summation of the variable g is taken over all the admissible states.

The Boltzmann weights are given by

$$\begin{aligned} w(d+1, d+2, d+1, d; z, w_0) &= w(d, d-1, d-2, d-1; z) = \frac{\theta(2\eta-z)}{\theta(2\eta)} \\ w(d-1, d, d+1, d; z, w_0) &= w(d+1, d, d-1, d; z, w_0) \\ &= \frac{\theta(z)}{\theta(2\eta)} \frac{\sqrt{\theta(2\eta(d+1)+w_0)\theta(2\eta(d-1)+w_0)}}{\theta(2\eta d+w_0)} \\ w(d+1, d, d+1, d; z, w_0) &= \frac{\theta(z+2\eta d+w_0)}{\theta(2\eta d+w_0)} \\ w(d-1, d, d-1, d; z, w_0) &= \frac{\theta(z-2\eta d-w_0)}{\theta(2\eta d+w_0)} \end{aligned} \quad (3.16)$$

3.4.2 RSOS models

Let us explain restricted Solid-on-Solid models (RSOS models) [71]. Let s denote the number of elements in S . Consider a $s \times s$ matrix C satisfying the following conditions [76, 77]:

- (i) $C_{ab} = C_{ba} = 0$ or 1
- (ii) $C_{aa} = 0$
- (iii) For each $a \in S$, there should exist $b \in S$ such that $C_{ab} = 1$

For such choice of C , we impose a restriction that two states a and b can occupy the neighboring lattice sites if and only if $C_{ab} = 1$. We call such a pair of the states (a, b) admissible. For the case of unrestricted models, the infinite matrix C satisfies the conditions (i), (ii), and (iii) with an infinite set S .

For an illustration, let us consider the restricted eight-vertex Solid-on-Solid model (the restricted 8V SOS model), which we also call the ABF model [71]. For the N -state case, we have $S = \{1, 2, \dots, N\}$. The nonzero matrix elements of C are given by $C_{j,j+1} = C_{j+1,j} = 1$ for $j = 1, 2, \dots, N-1$; other matrix elements such as $C_{1,N}$ and $C_{N,1}$ are given by zero. Setting $w_0 = 0$, then we have the Boltzmann weights of the ABF model. The Boltzmann weights satisfy the Yang-Baxter relations (3.15) with the finite set: $S = \{1, 2, \dots, N\}$.

Let us explain CSOS models, another type of RSOS models [75, 76, 77]. Here we assume that $2N\eta = m_1$, where integer m_1 has no common divisor with N . If we set $w_0 \neq 0$, then we have the Boltzmann weights of the cyclic SOS model (CSOS model). We can show that the Boltzmann weights satisfy the Yang-Baxter relations with the finite set: $S = \{1, 2, \dots, N\}$ and the cyclic admissible conditions: $C_{1,N} = C_{N,1} = 1$.

We remark that the connection between the 6j symbols and the Boltzmann weights of IRF models is first discussed in Ref. [78].

3.4.3 Fusion IRF models and ABCD IRF models

The 8VSOS model was generalized into the fusion IRF models. [72] IRF models associated with $A_n^{(1)}$ Lie algebra are constructed. [73] The IRF models associated with $B^{(1)}$ $C^{(1)}$ $D^{(1)}$ typed Lie algebra are also obtained [74].

In the IRF models, the one-point function, which is the magnetization per site, can be calculated by the corner transfer matrix method invented by Baxter [1].

3.4.4 Gauge transformations

It is sometimes convenient to employ a gauge transformation

$$w(a, b, c, d; z) \rightarrow w(a, b, c, d; z) \frac{g_c}{g_a}. \quad (3.17)$$

The transformed Boltzmann weights also satisfy the Yang-Baxter relations (3.15). For instance, we may set $g_a = \exp(\pi i a / 2) \sqrt{\theta(2\eta a + w_0)}$ ($a \in \mathbf{Z}$).

4 Yang-Baxter equation and the algebraic Bethe Ansatz

4.1 Solutions to the Yang-Baxter equation

4.1.1 Derivation of a solution for the six-vertex model

Let us solve the Yang-Baxter equation for the six-vertex model. We recall that it is given by

$$\begin{aligned} & \sum_{\alpha, \beta, \gamma} w(\alpha, \gamma | a_1, a_2) w'(\beta, b_3 | \gamma, a_3) w''(b_1, b_2 | \alpha, \beta) \\ &= \sum_{\alpha, \beta, \gamma} w''(\beta, \alpha | a_2, a_3) w'(b_1, \gamma | a_1, \beta) w(b_2, b_3 | \gamma, \alpha). \end{aligned} \quad (4.1)$$

The Yang-Baxter equation is illustrated in Fig. 4.1.

There are $2^3 \times 2^3 = 64$ cases for the entries (a_1, a_2, a_3) and (b_1, b_2, b_3) . Due to the ice rule, however, the Yang-Baxter equation is trivial unless $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. We have thus only 20 entries: $({}_3C_0)^2 + ({}_3C_1)^2 + ({}_3C_2)^2 + ({}_3C_3)^2 = 1 + 9 + 9 + 1 = 20$.

Let us consider symmetries of the Boltzmann weights given in (2.3). If we exchange 1 and 2, the Boltzmann weights do not change. Thus, we have reduced 20 cases into 10 cases.

$$(a_1, a_2, a_3; b_1, b_2, b_3) = (1, 1, 1; 1, 1, 1),$$

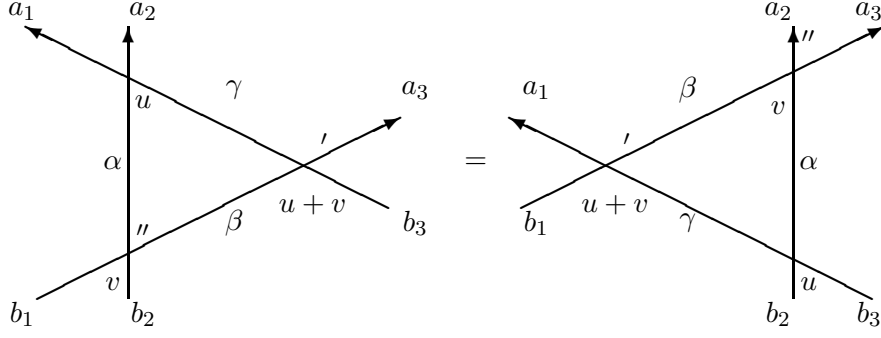


Figure 4.1: The Yang-Baxter equations for vertex models. The spectral parameters are shown by the angles between pairs of straight lines.

$$\begin{aligned}
& (1, 1, 2; 1, 1, 2), (1, 1, 2; 1, 2, 1), (1, 1, 2; 2, 1, 1), \\
& (1, 2, 1; 1, 1, 2), (1, 2, 1; 1, 2, 1), (1, 2, 1; 2, 1, 1), \\
& (2, 1, 1; 1, 1, 2), (2, 1, 1; 1, 2, 1), (2, 1, 1; 2, 1, 1)
\end{aligned} \tag{4.2}$$

We now recall that the Boltzmann weights (2.3) have the symmetry

$$w(\alpha, \beta | \gamma, \delta) = w(\gamma, \delta | \alpha, \beta) = w(\beta, \alpha | \delta, \gamma) \tag{4.3}$$

Combining these symmetries we can show that the Yang-Baxter equation for the two entries (1) and (2) are equivalent: (1) $(a_1, a_2, a_3; b_1, b_2, b_3)$; (2) $(b_3, b_2, b_1; a_3, a_2, a_1)$. Precisely, the L.H.S. (or R.H.S) of the Yang-Baxter equation of the case (1) corresponds to the R.H.S. (or L.H.S) of eq. (4.1). Thus, we have the three cases as follows.

$$(1, 1, 2; 1, 1, 2), (1, 1, 2; 1, 2, 1), (1, 2, 1; 1, 1, 2) \tag{4.4}$$

For the three cases, the Yang-Baxter equations are given by

$$\begin{aligned}
ac'a'' &= bc'b'' + ca'c'' \\
ab'c'' &= ba'c'' + cc'b'' \\
cb'a'' &= ca'b'' + bc'c''
\end{aligned} \tag{4.5}$$

A nontrivial solution (a'', b'', c'') exists only if the determinant vanishes

$$\begin{vmatrix} ac' & -bc' & -ca' \\ 0 & cc' & ba' - ab' \\ cb' & -ca' & -bc' \end{vmatrix} = abca'b'c' \left(\frac{(a')^2 + (b')^2 - (c')^2}{a'b'} - \frac{a^2 + b^2 - c^2}{ab} \right) \tag{4.6}$$

We define the parameter Δ by

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} \tag{4.7}$$

The condition that the determinant vanishes is given by

$$\Delta = \Delta' \tag{4.8}$$

Thus, the transfer matrices τ and τ' commute, if the two sets of weights (a, b, c) and (a', b', c') have the same Δ .

In terms of the spectral parameter u , we can parametrize the three Boltzmann weights a , b and c . We express the weight as $w(\alpha, \beta | \gamma, \delta; u)$. Let u and v be arbitrary. We denote $w(\alpha, \beta | \gamma, \delta)$, $w'(\alpha, \beta | \gamma, \delta)$, and $w''(\alpha, \beta | \gamma, \delta)$ as $w(\alpha, \beta | \gamma, \delta; u)$, $w(\alpha, \beta | \gamma, \delta; u + v)$, and $w(\alpha, \beta | \gamma, \delta; v)$, respectively. The Yang-Baxter equations are depicted in Fig. 4.2. As a solution, we may have $(a, b, c) = (\rho \sinh(u + 2\eta), \rho \sinh u, \rho \sinh 2\eta)$. Here we set $\Delta = \cosh 2\eta$. The transfer matrices $\tau(u)$ and $\tau(v)$ commute: $\tau(u)\tau(v) = \tau(v)\tau(u)$.

4.1.2 Gauge transformations for vertex models

Let us suppose that w 's satisfy the Yang-Baxter equation. Then we can show that transformed weights \tilde{w} 's defined by

$$\tilde{w}(\alpha, \beta | \gamma, \delta; u) = (\epsilon)^{\alpha+\gamma} \exp(\kappa(\alpha + \gamma - \beta - \delta)u) w(\alpha, \beta | \gamma, \delta; u) \quad (4.9)$$

also satisfy the Yang-Baxter equations [82, 83]. Here $\epsilon = \pm 1$, and the number κ is arbitrary.

The gauge transformation is important in the derivation of the Jones polynomial from the symmetric Boltzmann weights of the six-vertex model under zero field [82]. It is also quite useful when we discuss the relation of the six-vertex model to the quantum group, as we shall see in §5 (see also the appendix of Ref. [95]).

4.2 Algebraic Bethe ansatz

4.2.1 R matrix and the L operator

Let us diagonalize the transfer matrix by the method of the algebraic Bethe ansatz. [80, 81, 16]

Let us introduce the notation of the matrix tensor product. We define the direct product $A \otimes B$ of matrices A and B . Let A_k^j denote the matrix element for the entry of column j and row k of the matrix A . Then, the matrix element of column (j_1, j_2) and (k_1, k_2) is defined by

$$(A \otimes B)_{k_1, k_2}^{j_1, j_2} = A_{k_1}^{j_1} B_{k_2}^{j_2} \quad (4.10)$$

We now define the R matrix of the XXZ spin chain. The element R_{cd}^{ab} corresponds to the entry of column (a, b) and row (c, d) .

$$R(z) = \begin{pmatrix} R(z)_{11}^{11} & R(z)_{12}^{11} & R(z)_{21}^{11} & R(z)_{22}^{11} \\ R(z)_{11}^{12} & R(z)_{12}^{12} & R(z)_{21}^{12} & R(z)_{22}^{12} \\ R(z)_{11}^{21} & R(z)_{12}^{21} & R(z)_{21}^{21} & R(z)_{22}^{21} \\ R(z)_{11}^{22} & R(z)_{12}^{22} & R(z)_{21}^{22} & R(z)_{22}^{22} \end{pmatrix} = \begin{pmatrix} a(z) & 0 & 0 & 0 \\ 0 & c(z) & b(z) & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & 0 & 0 & a(z) \end{pmatrix} \quad (4.11)$$

Here $a(z)$, $b(z)$ and $c(z)$ are given by

$$a(z) = \sinh(z + 2\eta), \quad b(z) = \sinh z, \quad c(z) = \sinh 2\eta. \quad (4.12)$$

Here, the functions $a(z)$, $b(z)$ and $c(z)$ are equivalent to the Boltzmann weights a , b and c in §2

We now introduce L operators. We write the matrix element the L operator with entry (j, k) as $(L_n(z))_{jk}$ or $L_n(z)_k^j$. The L operator for the XXZ spin chain is given by

$$L_n(z) = \begin{pmatrix} L_n(z)_1^1 & L_n(z)_2^1 \\ L_n(z)_1^2 & L_n(z)_2^2 \end{pmatrix} = \begin{pmatrix} \sinh(zI_n + \eta\sigma_n^z) & \sinh 2\eta \sigma_n^- \\ \sinh 2\eta \sigma_n^+ & \sinh(zI_n - \eta\sigma_n^z) \end{pmatrix} \quad (4.13)$$

Here I_n and σ_n^a ($n = 1, \dots, L$) are acting on the n th vector space V_n . The L operator is an operator-valued matrix which acts on the auxiliary vector space V_0 . The symbols σ^\pm denote $\sigma^+ = E_{12}$ and $\sigma^- = E_{21}$, and $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices

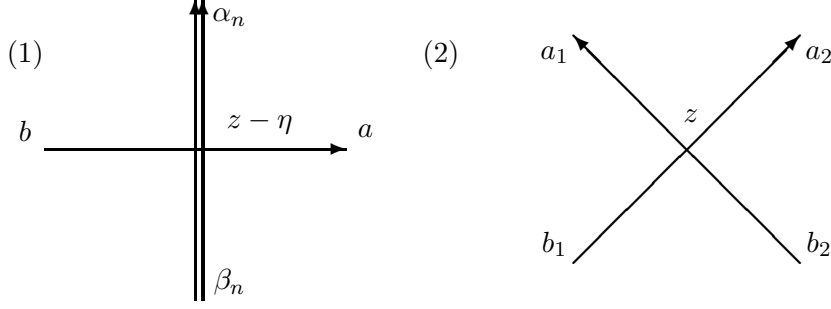


Figure 4.2: (1) $L_n(z)_b^a |_{\alpha_n, \beta_n}$; (2) $R(z)_{b_1, b_2}^{a_1, a_2}$

In terms of the R matrix and L operators, the Yang-Baxter equation is expressed as

$$R(z-t)(L_n(z) \otimes L_n(t)) = (L_n(t) \otimes L_n(z))R(z-t) \quad (4.14)$$

Here the tensor symbol in $L_n(z) \otimes L_n(t)$ denotes the tensor product of auxiliary spaces.

The Yang-Baxter equation (4.14) gives the relation between the two products of 4×4 matrices. For an illustration, we consider the L.H.S. of (4.14).

$$\begin{aligned} \left[R(z-t) \cdot L_n(z) \otimes L_n(t) \right]_{b_1, b_2}^{a_1, a_2} &= \sum_{c_1, c_2} R(z-t)_{c_1, c_2}^{a_1, a_2} (L_n(z) \otimes L_n(t))_{b_1, b_2}^{c_1, c_2} \\ &= \sum_{c_1, c_2} R(z-t)_{c_1, c_2}^{a_1, a_2} L_n(z)_{b_1}^{c_1} \times L_n(t)_{b_2}^{c_2} \end{aligned} \quad (4.15)$$

Here the symbol \times denotes the product of matrices acting on the n th space V_n . Expressing the

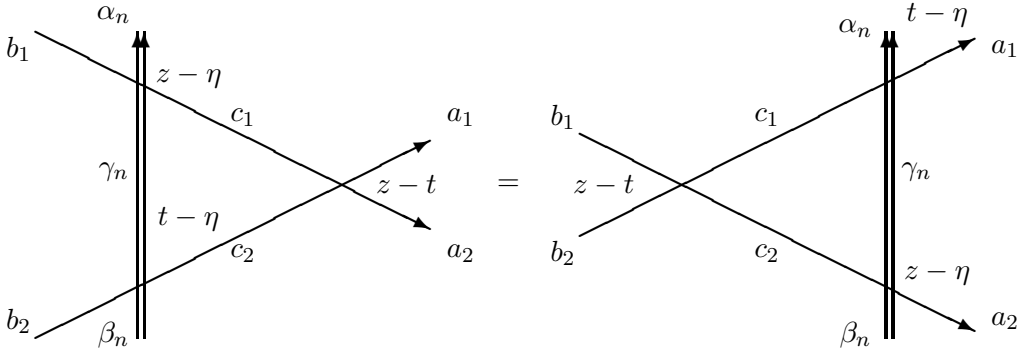


Figure 4.3: The Yang-Baxter equation: $R(z-t)(L(z) \otimes L(t)) = (L(t) \otimes L(z))R(z-t)$, eq. (4.14).

operator products in the n th space V_n , the L.H.S. of (4.14) is written as follows

$$\begin{aligned} \left[R(z-t) \cdot L_n(z) \otimes L_n(t) \right]_{b_1, b_2}^{a_1, a_2} |_{\alpha_n, \beta_n} &= \sum_{c_1, c_2} R(z-t)_{c_1, c_2}^{a_1, a_2} (L_n(z) \otimes L_n(t))_{b_1, b_2}^{c_1, c_2} |_{\alpha_n, \beta_n} \\ &= \sum_{c_1, c_2} \sum_{\gamma_n} R(z-t)_{c_1, c_2}^{a_1, a_2} L_n(z)_{b_1}^{c_1} |_{\alpha_n, \gamma_n} L_n(t)_{b_2}^{c_2} |_{\gamma_n, \beta_n} \end{aligned} \quad (4.16)$$

4.2.2 Monodromy matrix and the construction of the eigenvector

We define the monodromy matrix by the product of the L operators (see also Fig. A.1)

$$T(z) = L_N(z) \cdots L_2(z) L_1(z). \quad (4.17)$$

The transfer matrix $\tau_{6V}(z)$ of the six vertex model is given by the trace of $T(z)$

$$\tau_{6V}(z) = \text{tr}T(z) = A(z) + D(z), \quad \text{where} \quad T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \quad (4.18)$$

The Yang-Baxter equation (4.14) leads to the commutation relation: $R(z-t)(T(z) \otimes T(t)) = (T(t) \otimes T(z))R(z-t)$, from which we have many relations among the operators A , B , C , and D . For instance, we have $B(z)B(t) = B(t)B(z)$. Furthermore, we have

$$A(z)B(t) = \frac{a(t-z)}{b(t-z)}B(t)A(z) - \frac{c(t-z)}{b(t-z)}B(z)A(t) \quad (4.19)$$

$$D(z)B(t) = \frac{a(z-t)}{b(z-t)}B(t)D(z) - \frac{c(z-t)}{b(z-t)}B(z)D(t) \quad (4.20)$$

We define the ‘vacuum’ by

$$|0\rangle = \overbrace{|\uparrow\rangle_1 |\uparrow\rangle_2 \cdots |\uparrow\rangle_N}^N \quad (4.21)$$

Multiplying $A(z)$ and $D(z)$ on the vacuum, we have

$$A(z)|0\rangle = a(z-\eta)^N|0\rangle, \quad D(z)|0\rangle = b(z-\eta)^N|0\rangle. \quad (4.22)$$

Let us consider the vector generated by the product of B operators.

$$|M\rangle = B(t_1) \cdots B(t_M)|0\rangle \quad (4.23)$$

Then, through the commutation relations such as (4.19) and (4.20) we can show [7, 16] that the vector $|M\rangle$ gives an eigenvector of the transfer matrix $\tau_{6V}(z)$ if rapidities t_1, t_2, \dots, t_M satisfy the set of equations

$$\left(\frac{a(t_j - \eta)}{b(t_j - \eta)} \right)^N = - \frac{c(t_k - t_j)}{b(t_k - t_j)} \frac{b(t_j - t_k)}{c(t_j - t_k)} \prod_{k=1; k \neq j}^M \left(\frac{a(t_j - t_k)}{b(t_j - t_k)} \frac{b(t_k - t_j)}{a(t_k - t_j)} \right), \quad \text{for } j = 1, \dots, M. \quad (4.24)$$

They are the Bethe ansatz equations (2.12) with different parametrization. For a set of solutions, t_1, t_2, \dots, t_M to eqs. (4.24), the eigenvalue of the transfer matrix $\tau_{6V}(z)$ is given by

$$\begin{aligned} \Lambda(z; t_1, t_2, \dots, t_M) &= a(z-\eta)^N \prod_{j=1}^M \frac{a(t_j - z)}{b(t_j - z)} + b(z-\eta)^N \prod_{j=1}^M \frac{a(z - t_j)}{b(z - t_j)} \\ &= \sinh^N(z + \eta) \prod_{j=1}^M \frac{\sinh(t_j - z + 2\eta)}{\sinh(t_j - z)} + \sinh^N(z - \eta) \prod_{j=1}^M \frac{\sinh(t_j - z - 2\eta)}{\sinh(t_j - z)}. \end{aligned} \quad (4.25)$$

4.2.3 Connection to the coordinate Bethe ansatz result

Let us compare the result of the coordinate Bethe ansatz in §2. We consider the disordered phase: $-1 < \Delta < 1$. First, we change the variables w and α defined in §2 into u and ζ by $w = 2u - \mu$ and $\alpha = 2\zeta - i\mu$, respectively. Thus, from the expressions (2.22) we have $(a, b, c) = (\sin(\mu - u), \sin u, \sin \mu) = i(\sinh(-i\mu + iu), \sinh(-iu), \sinh(-i\mu))$ and

$$L^{Baxter}(z_j) = \frac{ab + (c^2 - b^2)z_j}{a(a - bz_j)} = - \frac{\sinh((\alpha_j - iw - 2i\mu)/2)}{\sinh((\alpha_j - iw)/2)} = \frac{\sinh(-(\zeta_j - iu) + i\mu)}{\sinh(\zeta_j - iu)} \quad (4.26)$$

Here we recall that the symbol $L^{Baxter}(z)$ has been defined in eq. (2.15) in order to denote the eigenvalue of the transfer matrix of the six-vertex model. The expression (4.26) can be derived from the formula (4.25) of the algebraic Bethe ansatz as follows. First, we take the gauge transformation: $b(z) \rightarrow -b(z)$, which corresponds to the case $\epsilon = -1$ and $\kappa = 0$ in eq. (4.9). Then, we replace the variables z , t_j 's, and 2η in (4.25) by $z - \eta \rightarrow -iu$, $t_j - \eta \rightarrow -\zeta_j$, and $2\eta \rightarrow i\mu$, respectively. We have the following:

$$\frac{a(t_j - z)}{b(t_j - z)} \rightarrow -\frac{a(t_j - z)}{b(t_j - z)} = -\frac{\sinh(t_j - z + 2\eta)}{\sinh(t_j - z)} \rightarrow \frac{\sinh(-(\zeta_j - iu) + i\mu)}{\sinh(\zeta_j - iu)} \quad (4.27)$$

and $\sinh^N(z + \eta) \rightarrow \sinh^N(-iu + i\mu)$. Thus, the formula (4.25) reproduces the expression (2.14) for the eigenvalues of the transfer matrix except for the normalization factor i^N .

5 Mathematical structures of integrable lattice models

5.1 Braid group

5.1.1 The Yang-Baxter equation in operator formalism

The Yang-Baxter equation in §2 gives a sufficient condition for the existence of commuting transfer matrices. However, there are other viewpoints on the Yang-Baxter equation.

Let E_k^j denote the matrix given by

$$\left(E_k^j\right)_b^a = \delta_{a,j}\delta_{b,k} \quad a, b = 1, 2. \quad (5.1)$$

We define operators $X_j(u)$'s by

$$X_j(z) = \sum_{a,b,c,d} w(a, b|c, d; z) \overbrace{I \otimes \cdots \otimes I}^{\otimes(j-1)} \otimes E_a^c \otimes E_b^d \otimes \overbrace{I \otimes \cdots \otimes I}^{\otimes(N-j-1)} \quad \text{for } j = 1, \dots, N-1. \quad (5.2)$$

Here, the symbol $w(a, b|c, d; z)$ corresponds to $R_{a,b}^{c,d}(z)$, and the essential part of $X_j(z)$ is given by

$$X(z) = \sum_{a,b,c,d} w(a, b|c, d; z) E_a^c \otimes E_b^d \quad (5.3)$$

which is equivalent to $R(z)$. In terms of the $X_j(z)$'s, the Yang-Baxter equation is expressed by the following

$$X_j(z)X_{j+1}(z+t)X_j(t) = X_{j+1}(t)X_j(z+t)X_{j+1}(z), \quad \text{for } j = 1, \dots, N-1. \quad (5.4)$$

5.1.2 The braid group

The braid group B_N for N strings is an infinite group which is generated by the generators b_1, \dots, b_{N-1} satisfying the defining relations

$$\begin{aligned} b_j b_{j+1} b_j &= b_{j+1} b_j b_{j+1} \\ b_i b_j &= b_j b_i \quad \text{for } |i-j| > 1 \end{aligned} \quad (5.5)$$

Let us assume that the limit: $\lim_{z \rightarrow \infty} X_j(z)$ exists. Then, eq. (5.4) becomes

$$X_j(\infty)X_{j+1}(\infty)X_j(\infty) = X_{j+1}(\infty)X_j(\infty)X_{j+1}(\infty), \quad \text{for } j = 1, \dots, N-1. \quad (5.6)$$

This is nothing but the defining relations of the braid group. Thus, the Boltzmann weights of solvable models expressed in terms of the spectral parameter lead to representations of the braid group.

We now show that from a given exactly solvable model, one can derive two different representations of the braid group [82]. Here, the gauge transformation (4.9) plays a central role. This technical point is quite fundamental when we make connections of exactly solvable models to quantum groups and the Temperley-Lieb algebra (for instance, see the appendix of Ref. [95]).

We first consider the Boltzmann weights of the zero-field six-vertex model given by eqs. (4.12). Taking the infinite limit to them, we have the following

$$\lim_{z \rightarrow \infty} X(z)/\sinh(z+2\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \exp(-2\eta) & 0 \\ 0 & \exp(-2\eta) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.7)$$

The representation of the braid group (5.7) leads to a link polynomial equivalent to the linking number.

Let us now apply the gauge transformation (4.9) with $\epsilon = 1$ and $\kappa = 1/2$ to the weights given by eqs. (4.12) [82]. Then, from the transformed Boltzmann weights we have

$$\lim_{z \rightarrow \infty} \tilde{X}(z)/\sinh(z+2\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \exp(-2\eta) & 0 \\ 0 & \exp(-2\eta) & 1 - \exp(-4\eta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (5.8)$$

This matrix representation of the braid group leads to the Jones polynomial with $q = \exp(2\eta)$ [82].

We remark that based on the representations of the braid group which are derived from the Boltzmann weights of exactly solvable models, we can construct various invariants of knots and links (see, for reviews, Refs. [83, 112, 113]).

5.2 Quantum groups (Hopf algebras)

From the quantum groups (the Hopf algebras) we can systematically construct representations of the braid group such as derived from the six-vertex model. Almost all the solutions of the Yang-Baxter equations can be constructed in some framework of quantum groups. Furthermore, the connection of solvable models to the quantum groups is useful for investigating nontrivial properties of integrable models.

Let us introduce the quantum group $U_q(sl_2)$, which is a q -analog of the universal enveloping algebra of sl_2 . Generators X^\pm, H satisfy

$$KX^\pm K^{-1} = q^{\pm 2}X^\pm, \quad [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}} \quad (5.9)$$

We may express K as $K = q^H$. Taking the limit of q to unity, the relations are reduced into the commutation relations of sl_2 .

The tensor product is defined by the following

$$\begin{aligned} \Delta(K) &= K \otimes K \\ \Delta(X^+) &= X^+ \otimes I + K \otimes X^+ \\ \Delta(X^-) &= X^- \otimes K^{-1} + I \otimes X^- \end{aligned} \quad (5.10)$$

The operation $\Delta(\cdot)$ is called the comultiplication. In the quantum group, the comultiplication does not commute with the exchange operator σ defined by $\sigma a \otimes b = b \otimes a$. However, there is an operator R which satisfy the following

$$R\Delta(x) = \sigma\Delta(x)R, \quad \text{for } x \in U_q(sl_2) \quad (5.11)$$

Thus, the tensor product $V_1 \otimes V_2$ can be related to $V_2 \otimes V_1$ through the R matrix of the quantum group.

In the $U_q(sl_2)$, the R matrix can be constructed in the operator formalism. If operators X^\pm and H satisfy the defining relations of the $U_q(sl_2)$, then the operator \mathcal{R} defined by the following satisfy the intertwining relation (5.11)

$$\mathcal{R} = q^{-H \otimes H/2} \exp_q \left(-(q - q^{-1})K^{-1}X^+ \otimes X^-K \right) \quad (5.12)$$

where $\exp_q x$ denotes the infinite series

$$\exp_q x = \sum_{n=0}^{\infty} q^{-n(n-1)/2} \frac{x^n}{[n]!} \quad (5.13)$$

The operator \mathcal{R} is called the universal R matrix.

The representation (5.8) of the braid group corresponds to the representation of the universal R matrix on the tensor product of two fundamental representations.

We note that the universal R matrix can be constructed canonically through Drinfeld's quantum double construction (for instance, see [24]). This is similar to the Sugawara construction which derives the energy momentum tensor from the current operator.

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A Commuting transfer matrices and the Yang-Baxter equations

We show that if given two sets of the Boltzmann weights of the six vertex model satisfy the Yang-Baxter relation, then their transfer matrices commute. We consider three sets of the Boltzmann weights: $(w_1, w_2, w_3) = (a, b, c)$, (a', b', c') , and (a'', b'', c'') . Let us denote by τ' and τ'' the transfer matrices constructed from the sets of the Boltzmann weights (a', b', c') and (a'', b'', c'') , respectively. Then, we can show that if the three sets of the Boltzmann weights satisfy the Yang-Baxter equations given in §2.3, then the transfer matrices τ' and τ'' commute.

Let us now explicitly discuss the commutation relation. We first introduce the monodromy matrix. It is an N ranked tensor, whose (α, β) elements are defined as follows

$$(T_{\alpha, \beta})_{b_1, \dots, b_N}^{a_1, \dots, a_N} = \sum_{c_2, \dots, c_N} w(\beta, b_1 | a_1, c_2) w(c_2, b_2 | a_2, c_3) \cdots w(c_N, b_N | a_N, \alpha) \quad (\text{A.1})$$

The transfer matrix is given by the trace of the monodromy matrix

$$\tau = \text{tr}(T) = \sum_{\alpha=1,2} T_{\alpha, \alpha} \quad (\text{A.2})$$

In terms of the matrix elements, we have

$$(\tau)_{b_1, \dots, b_N}^{a_1, \dots, a_N} = \sum_{\alpha=1,2} (T_{\alpha, \alpha})_{b_1, \dots, b_N}^{a_1, \dots, a_N} \quad (\text{A.3})$$

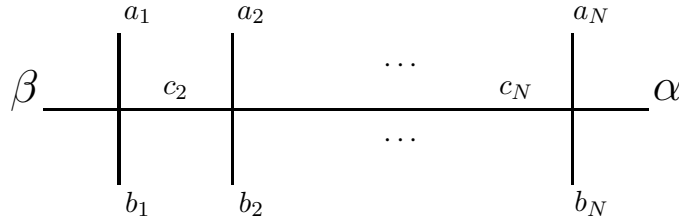


Figure A.1: The matrix element (α, β) of the monodromy matrix $(T_{\alpha, \beta})_{b_1, \dots, b_N}^{a_1, \dots, a_N}$

Let us denote by T' and T'' the monodromy matrices for the sets of the Boltzmann weights (a', b', c') and (a'', b'', c'') , respectively. We consider the product of the matrix elements of the two monodromy matrices: $T'_{\alpha, \beta} T''_{\gamma, \delta}$. The entry of (a_1, \dots, a_N) and (b_1, \dots, b_N) of the product $T'_{\alpha, \beta} T''_{\gamma, \delta}$ is given by

$$\begin{aligned} & (T'_{\gamma_1, \gamma_{N+1}} T''_{\delta_1, \delta_{N+1}})_{b_1, \dots, b_N}^{a_1, \dots, a_N} = \sum_{e_1, \dots, e_N} (T'_{\gamma_1, \gamma_{N+1}})_{e_1, \dots, e_N}^{a_1, \dots, a_N} (T''_{\delta_1, \delta_{N+1}})_{b_1, \dots, b_N}^{e_1, \dots, e_N} \\ &= \sum_{e_1, \dots, e_N} \sum_{c_2, \dots, c_N} w'(\gamma_1, e_1 | a_1, c_2) w'(c_2, e_2 | a_2, c_3) \cdots w'(c_N, e_N | a_N, \gamma_{N+1}) \\ & \quad \times \sum_{d_2, \dots, d_N} w''(\delta_1, b_1 | e_1, d_2) w''(d_2, b_2 | e_2, d_3) \cdots w''(d_N, b_N | e_N, \delta_{N+1}) \\ &= \sum_{c_2, \dots, c_N} \sum_{d_2, \dots, d_N} \sum_{e_1, \dots, e_N} \left(w'(\gamma_1, e_1 | a_1, c_2) w''(\delta_1, b_1 | e_1, d_2) \right) \cdot \\ & \quad \left(w'(c_2, e_2 | a_2, c_3) w''(d_2, b_2 | e_2, d_3) \right) \cdots \left(w'(c_N, e_N | a_N, \gamma_{N+1}) w''(d_N, b_N | e_N, \delta_{N+1}) \right) \\ &= \sum_{c_2, \dots, c_N} \sum_{d_2, \dots, d_N} S(a_1, b_1)_{\delta_1, d_2}^{\gamma_1, c_2} \cdot S(a_2, b_2)_{d_2, d_3}^{c_2, c_3} \cdots S(a_N, b_N)_{d_N, \delta_{N+1}}^{c_N, \gamma_{N+1}} \end{aligned} \quad (\text{A.4})$$

Here $S(a_j, b_j)_{d_j, d_{j+1}}^{c_j, c_{j+1}}$ has been defined by

$$S(a_j, b_j)_{d_j, d_{j+1}}^{c_j, c_{j+1}} = \sum_{e_j} w'(c_j, e_j | a_j, c_{j+1}) w''(d_j, b_j | e_j, d_{j+1}) \quad (\text{A.5})$$

We define the matrix element $M_{d_0, d_1}^{c_0, c_1}$ as follows

$$M_{d_0, d_1}^{c_0, c_1} = w(d_0, d_1 | c_0, c_1) \quad (\text{A.6})$$

Here we assume that $M_{d_0, d_1}^{c_0, c_1}$ denotes the matrix element for column (c_0, d_0) and row (c_1, d_1) of the matrix M . Multiplying the matrix M to the product $T'_{\alpha, \beta} T''_{\gamma, \delta}$ and applying the Yang-Baxter relation N times, we can derive the following

$$\sum_{c_1, d_1} M_{\delta_0, d_1}^{\gamma_0, c_1} T'_{c_1, \gamma_{N+1}} T''_{d_1, \delta_{N+1}} = \sum_{c_N, d_N} T''_{c_1, c_{N+1}} T'_{d_1, d_{N+1}} M_{d_N, \delta_{N+1}}^{c_N, \gamma_{N+1}} \quad (\text{A.7})$$

Figure A.2: Pictorial proof of the commutation relation: $MT'T'' = T''T'M$. Open and closed circles denote the Boltzmann weights w' and w'' , respectively. The summation over variables c_1, \dots, c_N and d_1, \dots, d_N is assumed.

Let us briefly discuss the derivation of relation (A.7). It is depicted in Fig. A.2. In the first equality of Fig. 2.6, we have applied the Yang-Baxter relation formulated as follows

$$\sum_{c_1, d_1} M_{\delta_0, d_1}^{\gamma_0, c_1} S(a_1, b_1)_{d_1, d_2}^{c_1, c_2} = \sum_{c_1, d_1} S'(a_1, b_1)_{\delta_0, d_1}^{\gamma_0, c_1} M_{d_1, d_2}^{c_1, c_2} \quad (\text{A.8})$$

Here the symbol $S'(a_j, b_j)_{d_j, d_{j+1}}^{c_j, c_{j+1}}$ has been defined by

$$S'(a_j, b_j)_{d_j, d_{j+1}}^{c_j, c_{j+1}} = \sum_{e_j} w''(c_j, e_j | a_j, c_{j+1}) w'(d_j, b_j | e_j, d_{j+1}) \quad (\text{A.9})$$

We also note that the LHS of (A.7) corresponds to the sum

$$\sum_{c_1, c_2, \dots, c_N} \sum_{d_1, d_2, \dots, d_N} M_{\delta_0, d_1}^{\gamma_0, c_1} S(a_1, b_1)_{d_1, d_2}^{c_1, c_2} \cdot S(a_2, b_2)_{d_2, d_3}^{c_2, c_3} \cdots S(a_N, b_N)_{d_N, \delta_{N+1}}^{c_N, \gamma_{N+1}} \quad (\text{A.10})$$

Let us consider the inverse of the matrix M

$$\left(M^{-1} M \right)_{d_1, d_2}^{c_1, c_2} = \left(M M^{-1} \right)_{d_1, d_2}^{c_1, c_2} = \delta_{c_1, c_2} \delta_{d_1, d_2} \quad (\text{A.11})$$

Multiplying the inverse M^{-1} to both hand sides of (A.7) we have

$$MT'T''M^{-1} = T''T' \quad (\text{A.12})$$

Noting $tr(MT'T''M^{-1}) = tr(T'T'')$, we obtain the commutation relation of the transfer matrices

$$\tau' \tau'' = \tau'' \tau' \quad (\text{A.13})$$