The Optimum Formula for the Gain of a Flow Graph or a Simple Derivation of Coates' Formula*

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Summary—**Starting from the definition of a determinant and using a few of its elementary properties, this paper gives an independent derivation of the optimum formula for the gain of a flow graph. Thus, a simpler path is shown to Coates' important result. This paper is self-contained, so that no previous knowledge of flow graphs is required. For motivation, the reader is referred to some well known papers and books.**

I. Introduction

MASON'S signal flow graphs[1,2] constitute a very useful tool for analysis of engineering problems, linear problems especially. The reason for their popularity and usefulness is that they display in a very intuitive manner the causal relationships between the several variables of the system under study. Many people have successfully used flow graph ideas in various fields.[1–9] Therefore, the publication of Mason's second paper[2] giving a systematic method for writing down almost by inspection the gain of a linear system was an important addition to the flow graph literature. Presently the state has been reached where a great many engineering schools include signal flow graphs in one of their senior courses.[5]

More recently, C. L. Coates[3] has shown that Mason's general gain formula is not the simplest expansion and has given a rigorous and lucid derivation of a new gain formula that is optimum in the sense that, *in general*, 1) no cancellations can occur between common factors of the numerator and denominator, and, 2) no cancellation can occur among the terms of the algebraic sums of the numerator and of the denominator.

The purpose of this paper is to present an independent derivation of Coates' formula. The author's hope is that this derivation is so simple that even seniors can grasp it![10] The only required background is the definition of a determinant. For the reader's convenience, this definition and the properties of determinants used in the derivation are listed in the Appendix.

For further background, motivation and examples, the reader is referred to the literature.[1–3,5] The present paper is self-contained in that it can be read independently of Mason's and Coates' papers.[1–3]

A word about the organization of the paper: There is a difference between Mason's and Coates' procedure for drawing the flow graph of a linear system. The difference, however, is very slight. The first five sections constitute an independent derivation of Coates' formula; they use exclusively Coates' procedure for associating a flow graph to a system of equations. Section VII discusses the difference between Mason's signal-flow graphs and Coates' flow graphs; it shows how, given one of them, the other may easily be obtained.

II. The Set of Equations and its Associated Flow Graph

Our purpose is to solve by topological methods the set of linear algebraic equations

$$\sum_{j=1}^{n} a_{kj}x_j - b_k = 0 \qquad (k = 1, 2, \cdots, n). \tag{1}$$

To this set of equations we shall associate a *flow graph* which is defined as follows:

> Definition 1—A *flow graph* is a set of weighted oriented branches which connect at nodes. That is, each branch has a positive direction and a weight, the branch gain.

The flow graph associated with (1) has n nodes and one source node. The case $n=2$ is illustrated in Fig. 1. The process of associating a flow graph to a set of equations is as follows:

1) To the source node is associated an input variable that is taken to be unity.
2) To each of the other nodes is associated one of the variables $x_1, x_2, \cdots, x_n$ of the set.

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[10] Some school will insist that the word "seniors" be replaced by "sophomores." More power to them!

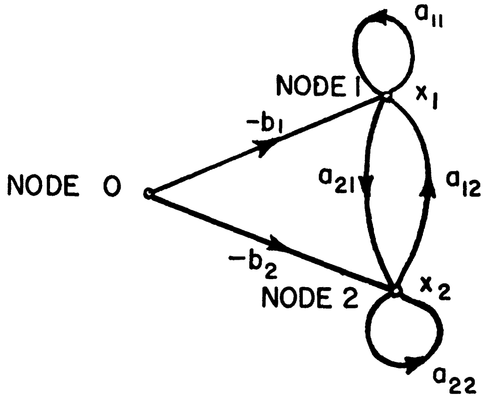


Fig. 1—Flow graph for the case of 2 equations with 2 unknowns.

3) Each node is labelled by one of the integers from 1 to n such that the node labelled k is associated with x_k. The source node is labelled 0.
4) If $a_{jk} \neq 0$, there is a branch directed from node k to node j with gain a_{jk}.
5) If $b_k \neq 0$, there is a branch connecting the source node 0 to node k. This branch is directed from 0 to k and its branch gain is $-b_k$.

This process describes how one goes from the equations to the graph. The reverse process, to obtain the equations from the graph, is very simple.

Consider Fig. 1 and concentrate on node 1. In order to write the equation associated with node 1, first consider all the branches coming into node 1; they are a_{11}, a_{12} and $-b_1$. The equation is obtained by equating to zero the sum of the products of their branch gains times the variables these branches originate from, viz:

$$a_{11} x_1 + a_{12} x_2 - b_1 \cdot 1 = 0.$$

Physically, we may think of these nodes as high gain operational amplifiers whose feedback loop is open. Because, in that case, if the output voltage is in the linear range of the amplifier, the sum of the currents into the input node must be very nearly zero.

At this stage an important point should be brought up. It is clear that to every set of equations such as (1) corresponds a flow graph and conversely. If, however, the order in which the equations appear in (1) is changed, the corresponding flow graph changes in a nontrivial manner. For example, to the set of equations

$$\begin{cases} \alpha x_2 + \beta x_3 = 0 \\ \gamma x_1 + \delta x_3 = b_1 \\ \epsilon x_1 + \eta x_2 = b_2 \end{cases}$$

corresponds the flow graph of Fig. 2.

Rearranging them as follows, we get a new set of equations

$$\begin{cases} \gamma x_1 + \delta x_3 = b_1 \\ \epsilon x_1 + \eta x_2 = b_2 \\ \alpha x_2 + \beta x_3 = 0 \end{cases}$$

and the flow graph of Fig. 3.

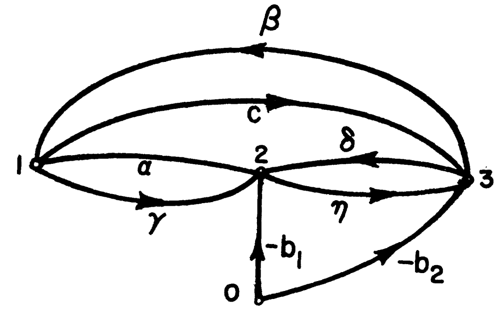


Fig. 2—Flow graph of the system (2).

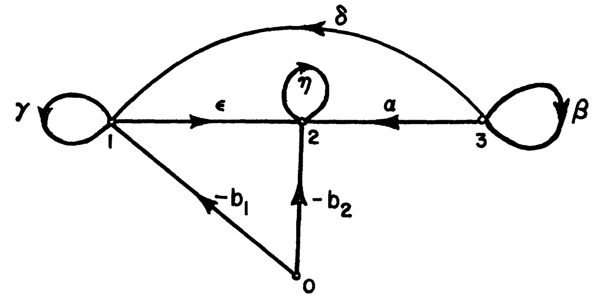


Fig. 3—Flow graph of the system (3).

Thus, once a system of linear homogeneous algebraic equations, such as (1), has been ordered in a fixed way, there is a one-to-one correspondence between the equations and the flow graph.

III. Algebraic Solution of (1)

If the matrix of the coefficients of the x_j's in (1) is nonsingular, the solution is given by

$$x_l = \frac{\sum_{k=1}^{n} \Delta_{kl} b_k}{\Delta} \qquad (l = 1, 2, \cdots, n) \tag{2}$$

where

Δ is the determinant of the coefficient matrix (a_{ij}).
Δ_{kl} is the cofactor of the element of the kth row and lth column.

In the following we shall devise a topological method for evaluating Δ and products of the form $b_k \Delta_{kl}$.

IV. Topological Evaluation of Δ

In order to evaluate Δ by topological means, we require a few topological ideas; hence we define:

G to be the flow graph of the system (1), where the equations are taken in the order in which they appear;
G_0 to be the flow graph obtained from G by deleting the source node 0.

Definition 2—*A connection* of the flow graph G is a subgraph of G such that

1) each node of G is included,

2) each node has only one branch terminating to it and one branch originating from it.

Definition 3—*A directed loop* is a connected subgraph whose branches $b_1, b_2, \cdots, b_l$ can be ordered in such a way that

1) The tip of b_k is the origin of b_{k+1} ($k=1, 2, \cdots, l-1$).
2) The origin of b_1 is the tip of b_l.
3) Each node along the directed loop is encountered only once.

Thus, a directed loop is precisely what is meant by a loop in the ordinary language. Fig. 4 illustrates the concept. Fig. 5(a) shows a flow graph G and Fig. 5(b) its five connections. It is clear that a connection is either a directed loop or a collection of nontouching directed loops (they are nontouching because of 2) in definition 2).

In addition, we shall need this definition:

Definition 4—*The connection-gain* of a connection of G is the product of the branch gains of the branches of that connection. It is denoted by $C(G)$.

The first link between the determinant Δ and the flow graph G is obtained by referring to the definition of a determinant:[4]

$$\Delta = \sum_P (\text{sgn } P) a_{1i_1} a_{2i_2} \cdots a_{ni_n} \tag{3}$$

where the summation is taken over all the $n!$ permutations $P=(i_1, i_2, \cdots, i_n)$ of the integers, $1, 2, \cdots, n$ and (sgn P) is $+1$ or -1 depending on whether the permutation P is even or odd.[11]

Lemma 1: A product appears in (3) if and only if it is a connection-gain $C(G_0)$ of the flow graph G_0.

Proof: Recall that G_0 is the graph G with the source node 0 deleted. Consider a particular product $\prod$ in the sum (3). Let $\prod'$ be the set of all branches whose gain appear in the product $\prod$. Since $\prod$ is a product of factors a_{ki_k}, with k running from 1 to n, there is one and only one branch of $\prod'$ that terminates at each of the n nodes of G_0. Since $i_1, i_2, \cdots, i_n$ is a permutation of $1, 2, \cdots, n$, there is one and only one branch of $\prod'$ originating from each node of G_0. Hence $\prod'$ is a connection of G_0.

Conversely, given an arbitrary connection of G_0, since it contains by definition one branch terminating in each of the nodes of G_0 and one branch leaving each one of the same nodes, then its connection-gain can be written as

$$a_{1j_1} a_{2i_2} \cdots a_{ni_n}$$

where $i_1, i_2 \cdots, i_n$ is a permutation of the numbers $1, 2, \cdots, n$. Hence, it will appear as one of the products of the expansion (3).

[11] The Appendix lists the three properties of determinants that will be used later.

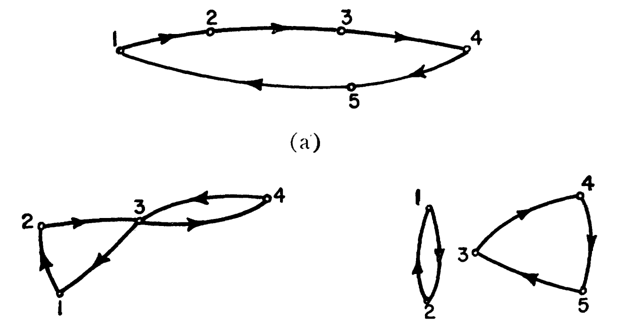


Fig. 4—(a) Example of a directed loop. (b) This is not a directed loop; when traversing the loop in the positive direction, node 3 is encountered more than once. (c) This is not a directed loop because it is not a connected subgraph.

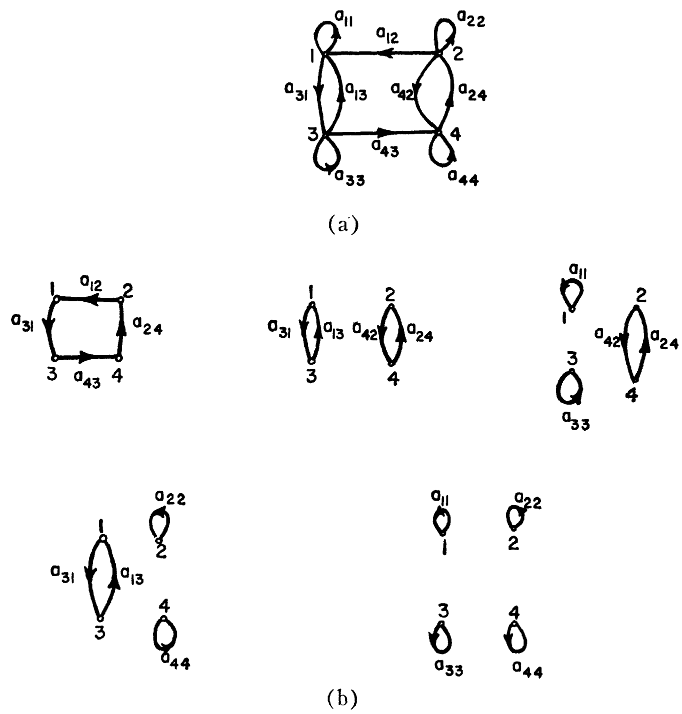


Fig. 5—(a) Flow graph associated with the matrix A. (b) The five connections of the flow graph shown in Fig. 5(a).

Thus by simply listing all the connections of G_0, as is done on Fig. 5, one obtains all the terms of the sum (3). The question of the signs remains. First let us make two rather obvious statements:

Statement 1—If, in a determinant, such as the one of the matrix A, any two rows are interchanged *and* the corresponding columns are also interchanged, the value of the determinant is not affected. (See Appendix, Property 3.)

Statement 2—Consider the system of equations (1) and its associated flow graph G. Suppose any two equations, say the ith and the kth, are interchanged and also the two variables located in the corresponding columns (*i.e.*, x_i and x_k): then to the resulting set of equations (1′) corresponds a new flow graph G'. A little thought will show that G and G' are identical except for an interchange of the labels of the ith and kth node.

The method required for specifying the sign of each term of (3) is obtained by the following reasoning:

Consider again a particular product $\prod$ of the sum (3) and its associated connection $\prod'$ of the graph G_0. $\prod'$ is a collection of directed loops; for simplicity let us assume that $\prod'$ consists of three nontouching directed loops having respectively n_1, n_2, n_3 nodes. Clearly, $n_1+n_2+n_3=n$, since all nodes of G_0 are included.

As a consequence of Statements 1 and 2 we can, without affecting any of the terms of the determinant expansion (3), relabel the nodes of $\prod'$ so that along the first directed loop of $\prod'$ as it is traversed in the positive direction one traverses the nodes $1, 2, \cdots, n_1$ in that order; and similarly for the other two directed loops. The branch gain product of the first directed loop is then

$$a_{1n_1}a_{21}a_{32}\cdots a_{n_1,n_1-1}.$$

Note that the factors of this product are ordered so that their row subscripts occur in their natural order as required by (3). Hence, the sign assigned to $\prod$ is that assigned to the permutation, defined by the column subscripts:

$$n_1, 1, 2, 3, \cdots, n_1-1; n_1+n_2, n_1+1, \cdots (n_1+n_2-1);$$
$$(n_1+n_2+n_3), (n_1+n_2+1), \cdots (n_1+n_2+n_3-1).$$

To rearrange this permutation in the natural order, $(n_1-1)+(n_2-1)+(n_3-1)=n-3$ interchanges between adjacent symbols are required. Hence, the sign of the permutation is $(-1)^{n-3}=(-1)^n(-1)^{+3}$. Note that there are three directed loops in the product $\prod$. It is clear that if the connection $\prod'$ had L directed loops the sign would have been $(-1)^{n+L}$.

Thus, we obtain the general

Theorem 1: The determinant Δ of the system (1) can be evaluated from its flow graph G by the formula

$$\Delta = (-1)^n \sum_p (-1)^{L_p} C(G_0)_p \tag{4}$$

where

L_p is the number of directed loops in the pth connection.
$C(G_0)_p$ is the connection gain of the pth connection.
G_0 is the flow graph G with the source node 0 deleted.
The summation of the connection gains $C(G_0)$ is taken over all connections of G_0.

Example: The determinant of the matrix associated with the graph of Fig. 5. From the graph G of Fig. 5, we obtain the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

From the five connections of G shown on the figure,

$$\det A = -a_{12}a_{24}a_{43}a_{31} + a_{13}a_{31}a_{42}a_{24} - a_{11}a_{33}a_{42}a_{24}$$
$$- a_{22}a_{44}a_{13}a_{31} + a_{11}a_{22}a_{33}a_{44}.$$

This expansion can also be obtained by Mason's method, but his general formula will give 25 terms which will eventually reduce to the five listed above.

V. Evaluation of the Numerator of (2)

The numerator of (2) is a sum of terms all having the same form. Consider one of them in particular; say, $\Delta_{kl}b_k$. In other words, we are going to evaluate the numerator of (2) assuming that there is only one branch, with gain $-b_k$, connecting the source node 0 to the rest of the graph, as shown on Fig. 6(a). Since we know how to evaluate an $n\times n$ determinant by topological means, let us note that $\Delta_{kl}b_k$ is equal to the determinant obtained by replacing the lth column of Δ by a column of zero except for the element of the kth row, which is b_k.

$$\begin{vmatrix} & 0 & \\ & 0 & \\ & b_k & \\ & 0 & \\ & \vdots & \\ & 0 & \\ & 0 & \end{vmatrix} \leftarrow k\text{th row}$$

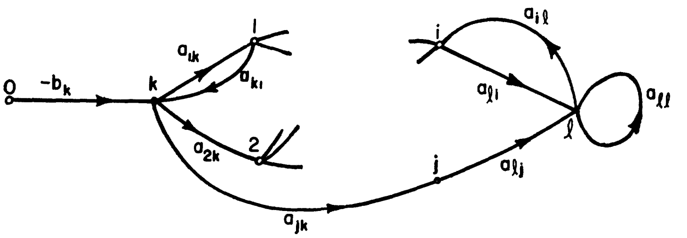
$\uparrow$ lth column

This determinant will not change if all the elements of the kth row, with the single exception of b_k, are replaced by zero. The result is the determinant Δ'.

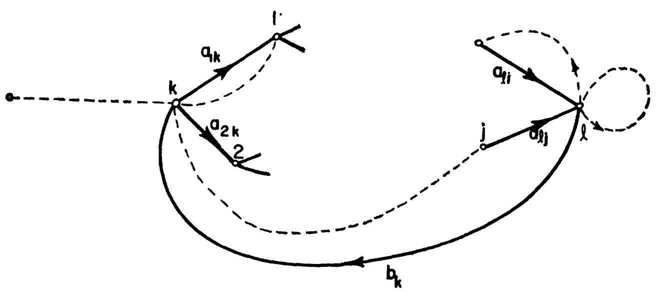
$$\Delta' = \begin{vmatrix} & & 0 & & \\ & & \vdots & & \\ & & 0 & & \\ 0\ 0\ \cdots & & 0\ b_k & 0 \cdots & 0 \\ & & \vdots & & \\ & & 0 & & \\ & & 0 & & \end{vmatrix} \leftarrow k\text{th row}$$

$\uparrow$ lth column

In order to evaluate Δ' by topological means, let us note that the flow graph G' associated with Δ' is obtained from G_0 by: 1) deleting all branches leaving node l (the node with which the variable x_l, being sought, is associated); 2) deleting all branches coming into node k; and 3) adding the branch b_k oriented from node l to node k. This operation is illustrated on Figs. 6(a) and



(a)



(b)

Fig. 6—(a) Flow graph G. (b) Flow graph G' obtained from G by deleting all branches leaving node l, deleting all branches coming into node k and adding branch b_k oriented from node l to node k.

6(b). From (4),

$$b_k\Delta_{kl} = \Delta' = (-1)^n \sum_{\tau} (-1)^{L_\tau} C(G')_\tau$$

where

$C(G')_\tau$ is the connection gain of the τth connection of G'.

L_τ is the number of directed loops of the τth connection.

The summation is taken over all connections of G'.

In order to interpret this result in terms of the graph G, let us define as follows:

Definition 5—*A one-connection* from p to q of a flow graph G is a subgraph of G which includes all the nodes of G, and such that

1) no branch terminates at p and only one branch of the subgraph originates from p, thus:

o———→——
p

2) no branch originates from q and only one branch of the subgraph terminates at q, thus:

——→——o
q

3) all other included nodes have exactly one incoming and one outgoing branch.

An example of a set of one-connections is shown on Figs. 7 and 8. It is apparent from Fig. 8 that, in general, a one-connection is a forward path together with some directed loops.

Consider Fig. 6(b). Each one of the connections of G'

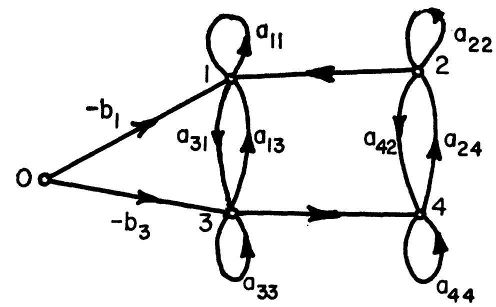


Fig. 7—Flow graph associated with the set of equations (6).

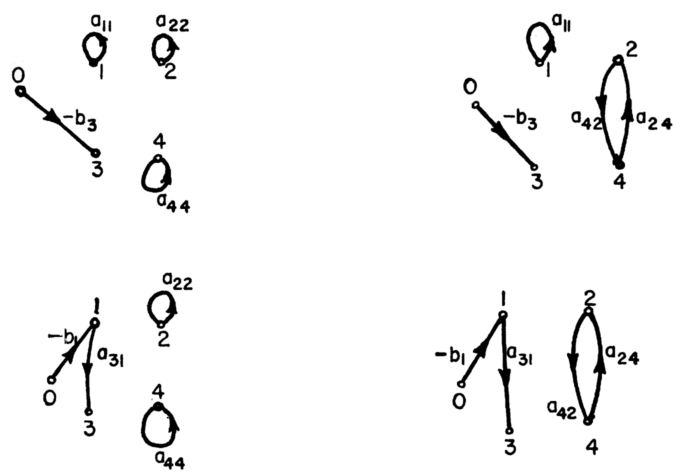


Fig. 8—The one-connections from 0 to 3 of the flow graph of Fig. 7.

includes the branch b_k because it is the only branch that leaves the lth node. In each one of these connections let us remove the origin of branch b_k from node l and place it at the source node 0 where it was originally in graph G. Finally, let us change the branch gain from b_k to $-b_k$. In each case the resulting configuration is a *one-connection* from 0 to l of flow graph G. Since by displacing the branch b_k one directed loop has been "opened," the number of directed loops in the one-connections of G is one less than that of the original connections of G'; this will result in a change of sign which will cancel the one caused by the change of sign of the branch gain b_k. Consequently,

$$b_k\Delta_{kl} = \Delta' = (-1)^n \sum_{\sigma} (-1)^{L_\sigma} C(G; 0-l)_\sigma$$

where

$C(G; 0-l)_\sigma$ is the one-connection gain (*i.e.*, the product of the branch gains) of the σth one-connection from 0 to l of the flow graph G.

L_σ is the number of directed loops in the σth one-connection.

The summation is taken over all one-connections from 0 to l of G.

VI. General Formula

In general, there is more than one branch connecting the source node to the rest of the graph; obviously, in such cases the individual contributions of each such branch must be summed and the result takes this form:

Theorem 2: In order to solve for the variable x_l defined by the set of equations

$$\sum_{j=1}^{n} a_{kj}x_j = b_k \qquad (k = 1, 2, \cdots, n) \tag{1}$$

1) Set up the associated flow graph G as specified in Section II.
2) Draw all the connections of the flow graph G_0 (=graph G with source node 0 deleted) and list their connection gains: $C(G_0)_1$, $C(G_0)_2$, $\cdots$
3) Draw all the one-connections from the source node 0 to the node l of the graph G and list their one-connection gains: $C(G; 0-l)_1$, $C(G; 0-l)_2$, $\cdots$
4)

$$x_l = \frac{\sum_\sigma (-1)^{L_\sigma} C(G; 0 - l)_\sigma}{\sum_\rho (-1)^{L_\rho} C(G_0)_\rho} \tag{5}$$

where

L_σ = number of directed loops in the σth one-connection from 0 to l of the flow graph G.
L_ρ = number of directed loops in the ρth connection of G_0.
the summations are taken over all connections and one-connections from 0 to l of graphs G_0 and G respectively.

Example: Consider the system of equations,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & = b_1 \\ \qquad a_{22}x_2 \qquad + a_{24}x_4 & = 0 \\ a_{31}x_1 \qquad + a_{33}x_3 & = b_3 \\ \qquad a_{42}x_2 = a_{43}x_3 + a_{44}x_4 & = 0. \end{cases} \tag{6}$$

Let us solve for x_3. The graph G is shown on Fig. 7. All the connections of G are listed in Fig. 5(b). The one-connections from 0 to 3 of G are shown on Fig. 8 Then, by (5), we have

$$x_3 = \frac{b_3a_{11}a_{22}a_{44} - b_3a_{11}a_{42}a_{24} - b_1a_{31}a_{22}a_{44} + b_1a_{31}a_{24}a_{42}}{\Delta}$$

where

$$\Delta = a_{11}a_{22}a_{33}a_{44} - a_{12}a_{24}a_{43}a_{31} + a_{13}a_{31}a_{42}a_{24} - a_{11}a_{33}a_{42}a_{24} - a_{22}a_{44}a_{13}a_{51}.$$

The general expression (5) calls for two important comments.

Comment 1—*In general*, there can be no cancellation of terms in the algebraic sums of either the numerator or the denominator of (5).

The reason for this is quite obvious: Since each term is a connection gain (a one-connection gain) of a connection (one-connection) distinct from all the other connections (one-connections) there can, in general, be no cancellations; for it would imply the existence of special relationships between the gains of various branches.

There is, however, one simplification that any engineer worth his salt would instinctively take advantage of. Suppose x_1 is to be computed and suppose there are some node variables x_{k_1}, x_{k_2}, $\cdots$, x_{k_m} that have no effect on x_1, then these nodes may be deleted from the graph when x_1 is computed.

This idea can be expressed precisely if the following definition is introduced:

Definition 6—*A forward path* from p to q of the flow graph G is a connected subgraph whose branches b_1, b_2, $\cdots$, b_l can be ordered such that
1) the tip of b_k is the origin of b_{k+1} ($k=1, 2, \cdots, l-1$)
2) each node of the forward path has only one branch terminating to it and one branch originating from it, with the exception of p and q which, respectively, have only one branch originating and terminating to them.

A forward path from p to q can be obtained from each one-connection from p to q by deleting from the one-connections all the directed loops.

The second comment takes the form

Comment 2—When solving for x_1, delete from the graph G all the nodes x_{k_1}, x_{k_2}, $\cdots$, x_{k_m} which have the property that there is no forward path that connects each one of them to x_1.

Example: Consider the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 \qquad\qquad &= 0 \\ a_{21}x_1 + a_{22}x_2 \qquad\qquad &= + b_2 \\ a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= 0 \\ a_{43}x_3 + a_{44}x_4 &= 0. \end{aligned}$$

The corresponding flow graph is shown in Fig. 9. It is obvious in this simple example, from the graph and from the equations, that the variables x_3 and x_4 may be disregarded in solving for x_1; also, there is no forward path from 3 to 1 and from 4 to 1.

A straightforward analysis shows that if the nodes $x_{k_1} \cdots x_{k_m}$ are not deleted, all the connection gains and the one-connection gains of (5) will have a common factor which will cancel from numerator and denominator. This leads to the very important conclusion that, provided the precaution of Comment 2 is taken into ac-

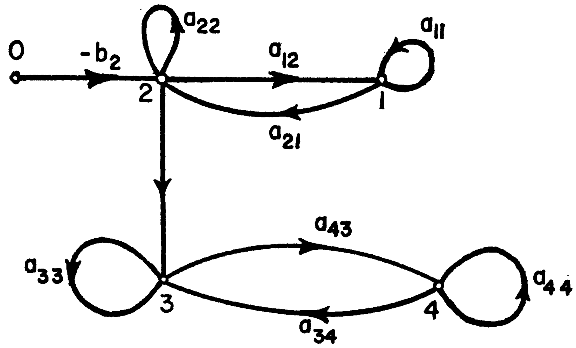


Fig. 9—Illustration of a flow graph that has no forward path from nodes 3 and 4 to node 1.

count, Theorem 2 gives the optimum gain formula for a flow graph.

Thus, the following important conclusion is reached: Given the problem of solving (1) by topological methods then, provided the Comment 2 is taken into account, the expression given in Theorem 2 is the simplest expression for the solution in terms of the b_k's and a_{ij}'s.

VII. The Relationship Between Signal-Flow Graphs (Mason) and Flow Graphs (Coates)

The process of association of a flow graph (Coates) to a set of equations has been described in detail in Section II. For reference let us note that the equations are written

$$\sum_{j=1}^{n} a_{kj}x_j = b_k \qquad (k = 1, 2, \cdots, n) \tag{1}$$

where

a_{kj} is the gain of the branch, directed from j to k, connecting node j to node k, and

$-b_k$ is the gain of the branch connecting the source node to node k.

Mason,[2] on the other hand, writes his equations thus:

$$\sum g_{kj}x_j = x_k + b_k \qquad (k = 1, 2, \cdots, n) \tag{7}$$

where

g_{kj} is the gain of the branch, directed from j to k, connecting node j to node k, and

$-b_k$ is the gain of the branch connecting the source node to node k.

Simply by looking at the equations we can see clearly that (1) and (7) will be identical if and only if

$$g_{kj} = a_{kj} \text{ if } k \neq j \quad \text{and} \quad g_{kk} - 1 = a_{kk}.$$

This gives the following rules:

1) To obtain a flow graph (Coates) from a given signal-flow graph (Mason), simply subtract one from the gain of each existing self loop and to each node of the signal-flow graph devoid of self loop, insert one with gain -1.

2) To obtain a signal-flow graph (Mason) from a flow graph (Coates), add unity to the gain of each existing self loop and to each node of the flow graph devoid of self loop, insert a self loop of gain $+1$.

Physically, we can interpret both graphs in terms of analog computer concepts:

A. Signal-Flow Graphs

Node variables x_j: potential of node j with respect to ground.

Gain g_{kj}: admittance of the branch connecting j to k, thus $g_{kj}x_j$ is a current entering node k.

Node: electronic summing amplifier: its output voltage is equal to the sum of the input currents:

$$x_k = \sum_{j=1}^{n} g_{kj}x_j - b_k \cdot 1.$$

Note that this summing amplifier does not invert the sign as is usually the case with analog computer amplifiers.

B. Flow Graphs

Node variables x_j: potential of node j with respect to ground.

Gain a_{kj}: admittance of the branch connecting j to k.

Node: operational amplifier with its feedback loop open; thus if the output voltage x_k is in the linear range, the sum of the input currents is negligibly small in view of the high gain of the amplifier; hence

$$\sum a_{kj}x_j - b_k \cdot 1 = 0.$$

Appendix

By definition, the determinant[4] of a matrix A is

$$\det A = \Delta = \sum_P (\text{sgn } P) a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where Σ/P denotes that the summation is taken only over the $n!$ permutations $i_1, i_2, \cdots, i_n$ of $1, 2, \cdots, n$ and (sgn P) is $+1$ or -1 depending on whether the permutation $i_1, i_2, \cdots, i_n$ is even or odd.

The key properties that are used in the paper are the following:

1) The interchange of any two adjacent symbols of a permutation changes the permutation into one of the opposite parity.

2) Exactly one element from each row and one element from each column appears in each term of the expansion of Δ.

3) If any two parallel lines (rows or columns) of A are interchanged, the determinant of the resulting matrix is $-\det A$.