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Parametrisation of unitary matrices

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Abstract. An algorithm for parametrisation of arbitrary $n \times n$ unitary matrices is presented. It is given with a minimum number of real parameters out of which a part take values in the positive unit cube, the others being arbitrary phases. The case $n = 4$ is worked out in detail.

1. Introduction

In many domains of theoretical physics where one is dealing with unitary matrices it is useful to have convenient parametrisations of them. Of course there is no unique way to parametrise them, the product of two unitary matrices being again unitary, the choice of parametrisation depending upon the problems to be solved. Those in which the relevant physical quantities are expressed in simple forms are preferred.

The physicists working in phase shift analyses and multichannel scattering were looking for a parametrisation which generalises to an arbitrary dimension n that given by Watson (1954) for 2×2 unitary matrices, i.e. a parametrisation with a minimum set of real parameters consisting of a number of parameters which are non-negative and less than or equal to unity and a set of unconstrained phases.

A parametrisation which satisfies these requirements can be obtained from that given in a different context by Murnaghan (1962), but it seems that people working in circuit theory and elementary particle physics were not aware of it and carried out much work to treat particular cases (Butterweck 1966, Eftimiu 1971, Waldenstrom 1974, Mennessier and Nuyts 1974, Babelon *et al* 1976, Waldenstrom 1981).

Attempting to develop the work of these authors, I discovered, independently of Murnaghan's work, a new algorithm for constructing a parametrisation of arbitrary $n \times n$ unitary matrices which is a straightforward generalisation of Watson's parametrisation.

The method uses operator techniques and is based on recent work on matrix contractions (Arsene and Gheondea 1981, Shmulyan and Yanovskaia 1981).

The algorithm is recursive, allowing the parametrisation of matrices of dimension n through the parametrisation of matrices of dimension $n-2$, the parametrisation of $(n-1)$-dimensional matrices being directly obtained from it without other computations.

Having in view the considerable interest of such parametrisations for people working in circuit theory, phase shift analyses and multichannel scattering, the existence of several explicit parametrisations can be useful, the potential users having the possibility to choose the most suitable.

The structure of the paper is as follows. The algorithm is presented in § 2. In § 3 the case $n = 4$ is worked out in detail and the paper ends with a few concluding remarks.

2. General case

Let S be an $n \times n$ unitary matrix written in the form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the partition blocks are arbitrary. For definiteness we shall suppose that A is an $m \times m$ matrix ($m \leqslant n/2$). The unitarity condition implies

$$AA^* + BB^* = I, \qquad A^*A + C^*C = I, \qquad CC^* + DD^* = I. \tag{2.1a, b, c}$$

The relations (2.1) are necessary conditions for solving the problem.

As the matrices AA^*, BB^*, CC^* and DD^* are all non-negative operators, the relations (2.1) imply that A, B, C and D are contractions, i.e. operators with norm less than or equal to unity.

Let us suppose that we know a parametrisation of contraction A. The problem then reduces to that of finding B, C and D blocks such that the matrix S be unitary. Thus, apparently, the first problem to be solved is the parametrisation of a contraction. This can be done using recent results on matrix contractions (Arsene and Gheondea 1981, Shmulyan and Yanovskaia 1981), but the method we propose here avoids it; more precisely, it requires only the parametrisation of the simplest contraction, namely of a complex number of modulus less than unity.

If T is a contraction let D_T and D_{T^*} be the defect operators defined as

$$D_T = (I - T^*T)^{1/2}, \qquad D_{T^*} = (I - TT^*)^{1/2},$$

which have the property

$$TD_T = D_{T^*}T, \qquad T^*D_{T^*} = D_TT^*. \tag{2.2}$$

The matrix blocks B and C are easily constructed using the following result by Douglas (1966).

Lemma 1. Let A and B be bounded operators on a Hilbert space H. The following conditions are equivalent.

(1) Range $(A) \subset$ Range (B).

(2) $AA^* \leqslant \lambda^2 BB^*$ for some constant $\lambda > 0$.

(3) $A = BC$ for some bounded operator C on H.

The unitarity relations (2.1a) and (2.1b) can be written as

$$BB^* = D_{A^*}^2, \qquad C^*C = D_A^2.$$

According to Douglas's lemma there exist two contractions U and V such that

$$B = D_{A^*}U, \qquad C = VD_A. \tag{2.3}$$

In fact U^* and V are isometric operators, i.e. they satisfy the relations

$$UU^* = I, \qquad V^*V = I. \tag{2.4a}$$

These relations imply

$$D_V = 0, \qquad D_{U^*} = 0. \tag{2.4b}$$

If B and C are $k \times k$ matrices, U and V are unitary operators. Thus the B and C matrices are determined by the defect operators D_A and D_{A^*} up to some isometries U and V which are much simpler operators.

In order to find the last matrix block we shall use the following result.

Lemma 2. The formula

$$D = -VA^*U + D_{V^*}KD_U \tag{2.5}$$

establishes a one-to-one correspondence between all the bounded operators D such that

$$S = \begin{pmatrix} A & D_{A^*}U \\ VD_A & D \end{pmatrix}$$

is a contraction, and all bounded contractions K.

The proof of this beautiful result can be found in Arsene and Gheondea (1981) and Shmulyan and Yanovskaia (1981).

A unitary operator being a special case of a contraction, the D block will be of the form (2.5), but has to satisfy the unitarity relation (2.1c). The substitution of (2.3) and (2.5) in the relation (2.1c) provides us with the relation

$$D_{V^*}KD_U^2K^*D_{V^*} = D_{V^*}^2 \tag{2.6}$$

which is the condition which will allow us to find K.

It is easily seen, owing to relations (2.4), that the non-negative operators D_U and D_{V^*} are orthogonal projections, their eigenvalues being equal to zero and unity.

Let X and Y be those unitary matrices which bring these operators to a diagonal form, i.e.

$$X^*D_{V^*}X = P, \qquad Y^*D_UY = P, \tag{2.7}$$

where

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Since we supposed that A is an $m \times m$ matrix, the identity operator entering P will act on the space of $(n-2m)\times(n-2m)$ matrices. This is due to the fact that the multiplicity of zero as eigenvalue of D_U is equal to m.

Multiplying the relation (2.6) to the left by X^* and to the right by X, one gets the formula

$$PX^*KYPY^*K^*XP = P. \tag{2.8}$$

We shall denote by M the matrix

$$M = X^*KY$$

and we shall write it in the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

where M_{11} is an $(n-2m)\times(n-2m)$ matrix. With this notation the relation (2.8) is equivalent to

$$M_{11}M_{11}^* = I. \tag{2.9}$$

The last relation tells us that M_{11} is an arbitrary unitary matrix. Thus K is given by

$$K = XMY^*$$

and with it one gets from (2.5)

$$D = -VA^*U + D_{V^*}XMY^*D_U.$$

The relations (2.7) can be written also as

$$D_{V^*}X = XP, \qquad Y^*D_U = PY^*,$$

so that

$$D = -VA^*U + XPMPY^*.$$

But

$$PMP = \begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

In this way we obtain the final form for D

$$D = -VA^*U + X\begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix}Y^* \tag{2.10}$$

which shows that D is completely specified by the matrices A, U, V and M_{11}.

The formulae (2.3) and (2.10) give the required parametrisation. At this stage it is evident that we did not use the off-diagonal conditions like $AC^* + BD^* = 0$. The reason is that they are redundant, and this originates in lemmas 1 and 2 which give necessary and sufficient conditions for solving the problem. One easily sees that the off-diagonal conditions are identically satisfied. For example, we have

$$\begin{aligned} 0 = AC^* + BD^* &= AD_A V^* - D_{A^*}UU^*AV^* + D_{A^*}UD_UK^*D_{V^*} \\ &= AD_AV^* - D_{A^*}AV^* + D_{A^*}D_{U^*}UK^*D_{V^*} \\ &= (AD_A - D_{A^*}A)V^* = 0 \end{aligned}$$

where we have used the properties (2.2) and (2.4).

Therefore the parametrisation of an $n\times n$ unitary matrix is equivalent to the parametrisation of a contraction A, of a unitary matrix M_{11} and two isometries U^* and V. As all these objects have lesser dimensions than those of the original matrix S, our task is considerably simplified. In fact, the above procedure is a recursive one as can be easily seen.

We suggest taking the contraction A as simple as possible, namely a complex scalar such that its parametrisation is $A = a \exp(\mathrm{i}\varphi)$ with $0 \leqslant a \leqslant 1$ and φ an arbitrary phase. Thus the defect operators D_A and D_{A^*} are scalar functions and are given by $D_A = D_{A^*} = (1-a^2)^{1/2}$.

In our choice U and V are row and column vectors respectively,

$$(U)_j = u_j, \qquad (V)_j = v_j, \qquad j = 1, 2, \ldots, n-1.$$

The conditions (2.4a) imply

$$\sum_{i=1}^{n-1} |u_i|^2 = \sum_{i=1}^{n-1} |v_i|^2 = 1. \tag{2.11}$$

The parametrisation of a vector U whose components u_i satisfy the relation (2.11) is direct. The main practical problem is the finding of X and Y matrices which diagonalise the defect operators D_U and D_{V^*}. As is well known, they are given by the orthonormal eigenvectors of the corresponding operators arrayed columnwise.

One easily sees that $\det(D_U^2 - \lambda I) = (\lambda - 1)^{n-2}(\lambda - 1 - \Sigma_{i=1}^{n-1}|u_i|^2)$ and this shows indeed that D_U is an orthogonal projection as we said before. Since $\Sigma_{i=1}^{n-1}|u_i|^2 = 1$, its eigenvalues are $\lambda = 0$ and $\lambda = 1$, the multiplicity of the latter being equal to $n-2$. The eigenvectors with $\lambda = 1$ span an $(n-2)$-dimensional vector space that is given by the relation

$$\sum_{i=1}^{n-1} a_i u_i = 0 \tag{2.12}$$

where by a_i we have denoted the components of a generic vector a. For $\lambda = 0$ the eigenvector is

$$(a^{n-1})_j = \bar{u}_j, \qquad j = 1, 2, \ldots, n-1 \tag{2.13}$$

where a bar means complex conjugation.

It is easy to find $n-2$ eigenvectors that satisfy the relation (2.12) but, in general, they will not be orthogonal. For practical purposes we suggest using the following base of eigenvectors which for high values of n leads to many zeros among the matrix elements of X and Y:

$$(a^k)_j = \frac{|u_{n-k-1}|}{(|u_k|^2 + |u_{n-k-1}|^2)^{1/2}} \left(\delta_{jk} - \frac{u_k}{u_{n-k-1}} \delta_{n-j-1,n-k-1} \right), \tag{2.14}$$

$$j = 1, 2, \ldots, n-1, \qquad k = 1, 2, \ldots, [n/2] - 1.$$

If n is odd we may add to (2.14) the vector

$$(a^{(n-1)/2})_j = \frac{|u_{n-1}|}{(|u_{n-1}|^2 + |u_{(n-1)/2}|^2)^{1/2}} \left(\delta_{j,(n-1)/2-1} - \frac{u_{(n-1)/2}}{u_{n-1}} \delta_{n-1,j} \right),$$

$$j = 1, 2, \ldots, n-1. \tag{2.15}$$

The eigenvectors (2.13), (2.14) and (2.15) are orthogonal by construction. For the remaining ones we take

$$(a^{[n/2]+k})_j = \alpha(k) \left(\delta_{j,k} + \frac{\bar{u}_{n-k-1}}{\bar{u}_k} \delta_{n-j-1,n-k-1} - \frac{|u_k|^2 + |u_{n-k-1}|^2}{\bar{u}_k u_{n-1}} \delta_{j,n-1} \right),$$

$$j = 1, 2, \ldots, n-1, \qquad k = 1, 2, \ldots, [n/2] - 1, \tag{2.16}$$

where

$$\alpha(k) = \frac{|u_k u_{n-1}|}{[(|u_k|^2 + |u_{n-k-1}|^2)(|u_k|^2 + |u_{n-k-1}|^2 + |u_{n-1}|^2)]^{1/2}}.$$

These last vectors are not orthogonal to each other, although they are orthogonal on the system (2.13)–(2.15). We can orthogonalise them by the usual Gram–Schmidt procedure.

Before, we supposed all $u_i \neq 0$. If one or more u_i are equal to zero, a number of eigenvectors will be of the form

$$(a^i)_j = \delta_{ij} \tag{2.17}$$

and it is evident how we have to modify our system of eigenvectors (2.14)–(2.16) in order to be orthogonal on the vectors (2.17). In this way the construction of X and Y matrices is finished.

With our choice of A, M_{11} is an $(n-2)\times(n-2)$ matrix, so if we succeeded in parametrising a unitary matrix this one can be used as input in the parametrisation of another one of higher dimension. This recursive feature of our approach makes it very appealing for practical calculations.

The U and V vectors whose components satisfy (2.11) are easily parametrised by a number of real and independent parameters a_i, $0 \leqslant a_i \leqslant 1$ and a number of arbitrary phases. Since a 2×2 unitary matrix is parametrised in the same manner, the above mentioned recursive feature allows us to state that our parametrisation will preserve all the characteristic features of the two-dimensional one.

3. Parametrisation of 4×4 unitary matrices

The relations (2.11) have the form

$$|u_1|^2+|u_2|^2+|u_3|^2=|v_1|^2+|v_2|^2+|v_3|^2=1$$

and a parametrisation of U and V is

$$\begin{aligned}
&u_1 = b\exp(\mathrm{i}\varphi_{11}), \qquad u_2 = c(1-b^2)^{1/2}\exp(\mathrm{i}\varphi_{13}),\\
&u_3 = [(1-b^2)(1-c^2)]^{1/2}\exp(\mathrm{i}\varphi_{14}), \qquad v_1 = d\exp(\mathrm{i}\varphi_{21}),\\
&v_2 = f(1-d^2)^{1/2}\exp(\mathrm{i}\varphi_{31}), \qquad v_3 = [(1-d^2)(1-f^2)]^{1/2}\exp(\mathrm{i}\varphi_{41}).
\end{aligned} \tag{3.1}$$

X and Y matrices are

$$X = \begin{pmatrix} \alpha f(1-d^2)^{1/2} & \alpha d[(1-d^2)(1-f^2)]^{1/2} & d\exp(\mathrm{i}\varphi_{21}) \\ -\alpha d\exp\mathrm{i}(\varphi_{31}-\varphi_{21}) & \alpha f(1-d^2)(1-f^2)^{1/2}\exp\mathrm{i}(\varphi_{31}-\varphi_{21}) & f(1-d^2)^{1/2}\exp(\mathrm{i}\varphi_{31}) \\ 0 & -\alpha^{-1}\exp\mathrm{i}(\varphi_{41}-\varphi_{21}) & [(1-d^2)(1-f^2)]^{1/2}\exp(\mathrm{i}\varphi_{41}) \end{pmatrix}$$

where $\alpha = (f^2+d^2-f^2d^2)^{-1/2}$,

$$Y = \begin{pmatrix} \beta c(1-b^2)^{1/2} & \beta b[(1-b^2)(1-c^2)]^{1/2} & b\exp(-\mathrm{i}\varphi_{12}) \\ -\beta b\exp\mathrm{i}(\varphi_{12}-\varphi_{13}) & \beta c(1-b^2)(1-c^2)^{1/2}\exp\mathrm{i}(\varphi_{12}-\varphi_{13}) & c(1-b^2)^{1/2}\exp(-\mathrm{i}\varphi_{13}) \\ 0 & -\beta^{-1}\exp\mathrm{i}(\varphi_{12}-\varphi_{14}) & [(1-b^2)(1-c^2)]^{1/2}\exp(-\mathrm{i}\varphi_{14}) \end{pmatrix} \tag{3.2}$$

where $\beta = (b^2+c^2-b^2c^2)^{-1/2}$. M_{11} is an arbitrary 2×2 unitary matrix that we take in the form

$$M_{11} = \begin{pmatrix} h\exp(\mathrm{i}\varphi_{22}) & (1-h^2)^{1/2}\exp(\mathrm{i}\varphi_{23}) \\ (1-h^2)^{1/2}\exp(\mathrm{i}\varphi_{32}) & -h\exp\mathrm{i}(\varphi_{23}+\varphi_{32}-\varphi_{22}) \end{pmatrix}. \tag{3.3}$$

We have the following notation:

$$A = S_{11} = a\exp(\mathrm{i}\varphi_{11}), \qquad D_A = D_{A^*} = (1-a^2)^{1/2},$$

$$B = (S_{12}, S_{13}, S_{14}), \qquad C^{\mathrm{T}} = (S_{21}, S_{31}, S_{41}),$$

where T means transpose,

$$D=\begin{pmatrix} S_{22} & S_{23} & S_{24} \\ S_{32} & S_{33} & S_{34} \\ S_{42} & S_{43} & S_{44} \end{pmatrix}.$$

Using the above notation in the relations (2.3) and (2.10) we get the following parametrisation of 4×4 unitary matrices:

$$\begin{aligned}
&S_{11}=a\exp(\mathrm{i}\varphi_{11}), \qquad S_{12}=b(1-a^2)^{1/2}\exp(\mathrm{i}\varphi_{22}),\\
&S_{13}=c[(1-a^2)(1-b^2)]^{1/2}\exp(\mathrm{i}\varphi_{13}), \qquad S_{14}=[(1-a^2)(1-b^2)(1-c^2)]^{1/2}\exp(\mathrm{i}\varphi_{14}),\\
&S_{21}=d(1-a^2)^{1/2}\exp(\mathrm{i}\varphi_{21}),\\
&S_{22}=-abd\exp[\mathrm{i}(\varphi_{12}+\varphi_{21}-\varphi_{11})]+\alpha\beta[(1-b^2)(1-d^2)]^{1/2}\\
&\qquad\times[\![cfh\exp(\mathrm{i}\varphi_{22})+cd[(1-f^2)(1-h^2)]^{1/2}\exp(\mathrm{i}\varphi_{32})\\
&\qquad+b(1-c^2)^{1/2}\exp(\mathrm{i}\varphi_{23})\{f(1-h^2)^{1/2}-dh(1-f^2)^{1/2}\exp[\mathrm{i}(\varphi_{32}-\varphi_{22})]\}]\!],\\
&S_{23}=-acd(1-b^2)^{1/2}\exp[\mathrm{i}(\varphi_{21}+\varphi_{13}-\varphi_{11})]\\
&\qquad-\alpha\beta(1-d^2)^{1/2}\exp[\mathrm{i}(\varphi_{13}-\varphi_{12})][\![b\{fh\exp(\mathrm{i}\varphi_{22})\\
&\qquad+d[(1-f^2)(1-h^2)]^{1/2}\exp(\mathrm{i}\varphi_{32})\}-c(1-b^2)(1-c^2)^{1/2}\\
&\qquad\times\exp(\mathrm{i}\varphi_{23})\{f(1-h^2)^{1/2}-dh(1-f^2)^{1/2}\exp[\mathrm{i}(\varphi_{32}-\varphi_{22})]\}]\!],\\
&S_{24}=-ad[(1-b^2)(1-c^2)]^{1/2}\exp[\mathrm{i}(\varphi_{21}+\varphi_{14}-\varphi_{11})]\\
&\qquad-(\alpha/\beta)(1-d^2)^{1/2}\exp[\mathrm{i}(\varphi_{14}+\varphi_{23}-\varphi_{12})]\\
&\qquad\times\{f(1-h^2)^{1/2}-dh(1-f^2)\exp[\mathrm{i}(\varphi_{32}-\varphi_{22})]\},\\
&S_{31}=f[(1-a^2)(1-d^2)]^{1/2}\exp(\mathrm{i}\varphi_{31}),\\
&S_{32}=-abf(1-d^2)^{1/2}\exp[\mathrm{i}(\varphi_{12}+\varphi_{31}-\varphi_{11})]\\
&\qquad+\alpha\beta(1-b^2)^{1/2}\exp(-\mathrm{i}\varphi_{21})[\![c\{-dh\exp[\mathrm{i}(\varphi_{22}+\varphi_{31})]\\
&\qquad+f(1-d^2)[(1-f^2)(1-h^2)]^{1/2}\exp[\mathrm{i}(\varphi_{31}+\varphi_{32})]\}\\
&\qquad-b(1-c^2)^{1/2}\exp[\mathrm{i}(\varphi_{23}+\varphi_{31})]\{d(1-h^2)^{1/2}\\
&\qquad+hf(1-d^2)(1-f^2)^{1/2}\exp[\mathrm{i}(\varphi_{32}-\varphi_{22})]\}]\!],\\
&S_{33}=-acf[(1-b^2)(1-d^2)]^{1/2}\exp[\mathrm{i}(\varphi_{13}+\varphi_{31}-\varphi_{11})]\\
&\qquad-\alpha\beta\exp[\mathrm{i}(\varphi_{13}-\varphi_{12}-\varphi_{21})][\![-bdh\exp[\mathrm{i}(\varphi_{31}+\varphi_{22})]\\
&\qquad+bf(1-d^2)[(1-f^2)(1-h^2)]^{1/2}\exp[\mathrm{i}(\varphi_{31}+\varphi_{32})]\\
&\qquad+c(1-b^2)(1-c^2)^{1/2}\exp[\mathrm{i}(\varphi_{23}+\varphi_{31})]\\
&\qquad\times\{d(1-h^2)^{1/2}+hf(1-d^2)(1-f^2)^{1/2}\exp[\mathrm{i}(\varphi_{32}-\varphi_{22})]\}]\!],\\
&S_{34}=-af[(1-b^2)(1-c^2)(1-d^2)]^{1/2}\exp[\mathrm{i}(\varphi_{31}+\varphi_{14}-\varphi_{11})]\\
&\qquad+\alpha^{-1}\beta\exp[\mathrm{i}(\varphi_{14}+\varphi_{31}+\varphi_{23}-\varphi_{12}-\varphi_{21})]\\
&\qquad\times\{d(1-h^2)^{1/2}+hf(1-d^2)(1-f^2)^{1/2}\exp[\mathrm{i}(\varphi_{32}-\varphi_{22})]\},
\end{aligned}\tag{3.4}$$

$$S_{41}=[(1-a^2)(1-d^2)(1-f^2)]^{1/2}\exp(i\varphi_{41}),$$

$$\begin{aligned}S_{42}=&-ab[(1-d^2)(1-f^2)]^{1/2}\exp[i(\varphi_{12}+\varphi_{41}-\varphi_{11})]\\&-\alpha\beta^{-1}(1-b)^{1/2}\exp[i(\varphi_{32}+\varphi_{41}-\varphi_{21})]\\&\times\{c(1-h^2)^{1/2}-bh(1-c^2)^{1/2}\exp[i(\varphi_{23}-\varphi_{22})]\},\end{aligned}$$

$$\begin{aligned}S_{43}=&-ac[(1-b^2)(1-d^2)(1-f^2)]^{1/2}\exp[i(\varphi_{13}+\varphi_{41}-\varphi_{11})]\\&+\alpha\beta^{-1}\exp[i(\varphi_{32}+\varphi_{13}+\varphi_{41}-\varphi_{12}-\varphi_{21})]\\&\times\{b(1-h^2)^{1/2}+ch(1-b^2)(1-c^2)^{1/2}\exp[i(\varphi_{23}-\varphi_{22})]\},\end{aligned}$$

$$\begin{aligned}S_{44}=&-a[(1-b^2)(1-c^2)(1-d^2)(1-f^2)]^{1/2}\exp[i(\varphi_{14}+\varphi_{41}-\varphi_{11})]\\&-\alpha\beta h\exp[i(\varphi_{14}+\varphi_{41}+\varphi_{23}+\varphi_{32}-\varphi_{12}-\varphi_{21}-\varphi_{22})].\end{aligned}$$

For symmetric matrices $S_{ij}=S_{ji}$ the parameters satisfy the supplementary conditions

$$b=d,\qquad c=f,\qquad \varphi_{12}=\varphi_{21},\qquad \varphi_{13}=\varphi_{31},\qquad \varphi_{14}=\varphi_{41},\qquad \varphi_{23}=\varphi_{32}.$$

From a parametrisation of an $n\times n$ unitary matrix we can easily obtain the parametrisation of an $(n-1)\times(n-1)$ matrix by setting $S_{ni}=S_{in}=0, i=1,2,\ldots,n-1$, $S_{nn}=\exp(i\psi)$. Thus the above parametrisation provides us with a parametrisation of 3×3 unitary matrices and it is obtained from (3.4) by setting $c=f=h=1$. The particular case $\varphi_{11}=\varphi_{13}=\varphi_{21}=\varphi_{31}=0$ and $\varphi_{12}=\pi$ of this last parametrisation gives the fermion mass matrix in the form introduced by Kobayashi and Maskawa (1973).

4. Concluding remarks

The recursive feature of the approach allows us to find the numbers $p(n)$ and $\varphi(n)$ of inelasticity parameters and phases respectively which enter the parametrisation for arbitrary n. For the non-symmetric case ($S_{ij}\neq S_{ji}$) it is easily seen from the relations (2.10) and (2.11) that $p(n)$ and $\varphi(n)$ satisfy the equations

$$p(n)=2n-3+p(n-2),\qquad \varphi(n)=2n-1+\varphi(n-2),$$

with the initial conditions

$$p(1)=0,\qquad p(2)=1,\qquad \varphi(1)=1,\qquad \varphi(2)=3.$$

The solutions of the above equations are

$$p(n)=n(n-1)/2,\qquad \varphi(n)=n(n+1)/2,\qquad n=1,2,\ldots.$$

For symmetric matrices the result is

$$p(n)=[n/2][(n+1)/2],\qquad \varphi(n)=[(n+1)/2][(n+2)/2],\qquad n=1,2,\ldots,$$

where $[r]$ denotes the integral part of r.

A parametrisation of a unitary matrix is unique modulo a multiplication by another unitary matrix. This arbitrariness might be useful in some cases. Until now all the matrix elements S_{ij} have been considered as pure complex numbers, but in many problems from particle physics they are more, they are boundary values of analytic functions. Thus naturally a problem arises: given a contraction A which is the boundary value of an analytic function, find the analytic matrix blocks B, C and D such that matrix S be unitary. No simple solution of this problem seems to exist, though its solution would be of much practical interest for elementary particle applications and here the above-mentioned arbitrariness might be useful.

Applications of our parametrisation in particle physics will be given elsewhere.

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