Two Lectures on Max-plus Algebra

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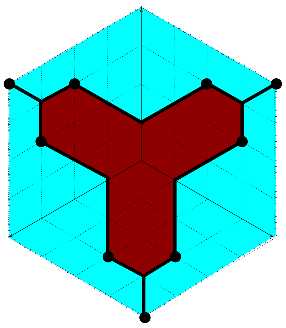
March 1998



These lecture notes have been prepared for the 26th Spring School on Theoretical Computer Science, Noirmoutier, 1998, on "Max-plus algebras and applications to Theoretical Computer Science and Automatic Control".

These notes consist in three parts.

1. A recent survey on max-plus linear algebra, which presents and motivates basic algebraic results and techniques.

2. A (rather informal) lecture on semirings, written specially for the school, with a selective synthesis of results on semirings, taken from various sources. These results are in general classical or folkloric, except the presentation of minimal generating families in terms of abstract convexity, some closure properties of finitely generated semimodules, and a few observations on max-plus Green classes, which are recent.

3. An appendix with a brief description of the max-plus policy iteration algorithm.

Some of these points are closely related to the presentations of G. Cohen (residuation), P. Butkovič (permanents), J. Gunawardena (non-expansive maps) in this school. We left voluntarily these topics untouched in the chapter on semirings, up to a few remarks and pointers, just to show connections. Hence, the reader should consult these three contributions in line with these notes to have a reasonably complete picture. Additional illustrations and applications of max-plus spectral theory can be found in the contribution of J. Mairesse.

These notes are in general strongly inspired by the work of the author within the Max Plus group, particularly with G. Cohen, J.P. Quadrat, M. Akian, and M. Viot. Some points have become more mature thanks to discussions with J. Gunawardena, R. Bapat, and P. Butkovič. The author is glad to acknowledge these debts.

The picture of the cover page represents the projection à la Mairesse of a subsemimodule of $(\mathbb{R}_{\max})^3$ with 9 generators. See Chap. 2,§7.

Contents

CHAPTER 1

Motivations and Survey of Results

This introductive chapter is taken verbatim from the survey "Methods and Applications of (max,+) Linear Algebra", S. Gaubert and M. Plus[1], which appeared in the Proceedings of STACS'1997, Lübeck, R. Reischuk and M. Morvan Eds., Springer LNCS number 1200 (also INRIA report number 3088, Jan. 1997).

Some points and some proofs are detailed in the next chapter.

Abstract

Exotic semirings such as the "(max, +) semiring" $(\mathbb{R}\cup\{-\infty\}, \max, +)$, or the "tropical semiring" $(\mathbb{N}\cup\{+\infty\}, \min, +)$, have been invented and reinvented many times since the late fifties, in relation with various fields: performance evaluation of manufacturing systems and discrete event system theory; graph theory (path algebra) and Markov decision processes, Hamilton-Jacobi theory; asymptotic analysis (low temperature asymptotics in statistical physics, large deviations, WKB method); language theory (automata with multiplicities).

Despite this apparent profusion, there is a small set of common, non-naive, basic results and problems, in general not known outside the (max, +) community, which seem to be useful in most applications. The aim of this short survey paper is to present what we believe to be the minimal core of (max, +) results, and to illustrate these results by typical applications, at the frontier of language theory, control, and operations research (performance evaluation of discrete event systems, analysis of Markov decision processes with average cost).

Basic techniques include: solving all kinds of systems of linear equations, sometimes with exotic symmetrization and determinant techniques; using the (max, +) Perron-Frobenius theory to study the dynamics of (max, +) linear maps. We point out some open problems and current developments.

1. Introduction: the (max, +) and tropical semirings

The "max-algebra" or "(max, +) semiring" $\mathbb{R}_{\max}$, is the set $\mathbb{R}\cup\{-\infty\}$, equipped with max as addition, and $+$ as multiplication. It is traditional to use the notation $\oplus$ for max ($2\oplus 3=3$), and $\otimes$ for $+$ ($1\otimes 1=2$). We denote[2] by $\mathbb{0}$ the *zero* element for $\oplus$ (such that $\mathbb{0}\oplus a=a$, here $\mathbb{0}=-\infty$) and by $\mathbb{1}$ the *unit* element for $\otimes$ (such that $\mathbb{1}\otimes a=a\otimes\mathbb{1}=a$, here $\mathbb{1}=0$). This structure satisfies all the semiring axioms, i.e. $\oplus$ is associative, commutative, with zero element, $\otimes$ is associative, has a unit, distributes over $\oplus$, and zero is absorbing (all the ring axioms are satisfied, except that $\oplus$ need not be a group law). This semiring is *commutative* ($a\otimes b=b\otimes a$), *idempotent* ($a\oplus a=a$), and non zero elements have an inverse for $\otimes$ (we call *semifields* the semirings that satisfy this property). The term *dioid* is sometimes used for an *idempotent* semiring.

[1] Max Plus is a collective name for a working group on (max, +) algebra, at INRIA Rocquencourt, comprising currently: Marianne Akian, Guy Cohen, S.G., Jean-Pierre Quadrat and Michel Viot.

[2] The notation for the zero and unit is one of the disputed questions of the community. The symbols ε for zero, and e for the unit, often used in the literature, are very distinctive and well suited to handwritten computations. But it is difficult to renounce to the traditional use of ε in Analysis. The notation $\mathbb{0}$, $\mathbb{1}$ used by the Idempotent Analysis school has the advantage of making formulæ closer to their usual analogues.

$\mathbb{R}_{\max}$	$(\mathbb{R}\cup\{-\infty\}, \max, +)$	$(\max, +)$ semiring max algebra	idempotent semifield
$\overline{\mathbb{R}}_{\max}$	$(\mathbb{R}\cup\{\pm\infty\}, \max, +)$	completed $(\max, +)$ semiring	$-\infty + (+\infty) = -\infty$, for $\mathbb{0}\otimes a = \mathbb{0}$
$\mathbb{R}_{\max,\times}$	$(\mathbb{R}^+, \max, \times)$	$(\max, \times)$ semiring	isomorphic to $\mathbb{R}_{\max}$ $(x \mapsto \log x)$
$\mathbb{R}_{\min}$	$(\mathbb{R}\cup\{+\infty\}, \min, +)$	$(\min, +)$ semiring	isomorphic to $\mathbb{R}_{\max}$ $(x \mapsto -x)$
$\mathbb{N}_{\min}$	$(\mathbb{N}\cup\{+\infty\}, \min, +)$	tropical semiring	(famous in Language Theory)
$\mathbb{R}_{\max,\min}$	$(\mathbb{R}\cup\{\pm\infty\}, \max, \min)$	bottleneck algebra	not dealt with here
$\mathbb{B}$	({false, true}, or, and)	Boolean semiring	isomorphic to $(\{\mathbb{0}, \mathbb{1}\}, \oplus, \otimes)$, for any of the above semirings
$\mathbb{R}_h$	$(\mathbb{R}\cup\{-\infty\}, \oplus_h, +)$ $a \oplus_h b = h\log(e^{a/h} + e^{b/h})$	Maslov semirings	isomorphic to $(\mathbb{R}^+, +, \times)$ $\lim_{h\to 0^+} \mathbb{R}_h = \mathbb{R}_0 = \mathbb{R}_{\max}$

TABLE 1. The family of $(\max, +)$ and tropical semirings ...

Using the new symbols $\oplus$ and $\otimes$ instead of the familiar max and $+$ notation is the price to pay to easily handle all the familiar algebraic constructions. For instance, we will write, in the $(\max, +)$ semiring:

$$ab = a\otimes b,\quad a^n = a\otimes\cdots\otimes a \ \ (n \text{ times}),\qquad 2^3 = 6\ ,\qquad \sqrt{3} = 1.5\ ,$$

$$\begin{bmatrix}2 & 0\\ 4 & \mathbb{0}\end{bmatrix}\begin{bmatrix}10\\ 103\end{bmatrix} = \begin{bmatrix}2\otimes 10\oplus 0\otimes 103\\ 4\otimes 10\oplus \mathbb{0}\otimes 103\end{bmatrix} = \begin{bmatrix}103\\ 14\end{bmatrix},$$

$$(3\oplus x)^2 = (3\oplus x)(3\oplus x) = 6\oplus 3x\oplus x^2 = 6\oplus x^2 \ \ (= \max(6, 2\times x))\ .$$

We will systematically use the standard algebraic notions (matrices, vectors, linear operators, semimodules — i.e. modules over a semiring—, formal polynomials and polynomial functions, formal series) in the context of the $(\max, +)$ semiring, often without explicit mention. Essentially all the standard notions of algebra have obvious semiring analogues, provided they do not appeal to the invertibility of addition.

There are several useful variants of the $(\max, +)$ semiring, displayed in Table 1. In the sequel, we will have to consider various semirings, and will universally use the notation $\oplus, \otimes, \mathbb{0}, \mathbb{1}$ with a context dependent meaning (e.g. $\oplus = \max$ in $\mathbb{R}_{\max}$ but $\oplus = \min$ in $\mathbb{R}_{\min}$, $\mathbb{0} = -\infty$ in $\mathbb{R}_{\max}$ but $\mathbb{0} = +\infty$ in $\mathbb{R}_{\min}$).

The fact that $\oplus$ is idempotent instead of being invertible ($\mathbb{R}_h$ is an exception, for $h \neq 0$), is the main original feature of these "exotic" algebras, which makes them so different from the more familiar ring and field structures. In fact the idempotence and cancellativity axioms are exclusive: if for all a, b, c, $(a\oplus b = a\oplus c \Rightarrow b = c)$ and $a\oplus a = a$, we get $a = \mathbb{0}$, for all a (simplify $a\oplus a = a\oplus \mathbb{0}$).

This paper is not a survey in the usual sense. There exist several comprehensive books and excellent survey articles on the subject, each one having its own bias and motivations. Applications of $(\max, +)$ algebras are too vast (they range from asymptotic methods to decidability problems), techniques are too various (from graph theory to measure theory and large deviations) to be surveyed in a paper of this format. But there is a small common set of useful basic results, applications and problems, that we try to spotlight here. We aim neither at completeness, nor at originality. But we wish to give an honest idea of the services that one should expect from $(\max, +)$ techniques. The interested reader is referred to the books [**23, 84, 13, 3, 60**], to the survey papers listed in the bibliography, and to the recent collection of articles [**41**] for an up-to-date account of the maxplusian results and motivations. Bibliographical and historical comments are at the end of the paper.

2. Seven good reasons to use the $(\max, +)$ semiring

2.1. An Algebra for Optimal Control. A standard problem of calculus of variations, which appears in Mechanics (least action principle) and Optimal Control, is the following. Given a Lagrangian L and suitable boundary conditions (e.g. $q(0), q(T)$ fixed), compute

$$\inf_{q(\cdot)} \int_0^T L(q, \dot{q})dt \ . \tag{1}$$

This problem is intrinsically (min, +) linear. To see this, consider the (slightly more general) discrete variant, with sup rather than inf,

$$\xi(n) = x,\ \xi(k) = f(\xi(k-1), u(k)),\ k = n+1, \dots, N, \tag{2a}$$

$$J_n^N(x, u) = \sum_{k=n+1}^{N} c(\xi(k-1), u(k)) + \Phi(\xi(N)) \ , \tag{2b}$$

$$V_n^N(x) = \sup_u J_n^N(x, u) \ , \tag{2c}$$

where the sup is taken over all sequences of *controls* $u(k), k = n+1, \cdots, N$, selected in a finite *set of controls* U, $\xi(k)$, for $k = n, \dots, N$, belongs to a finite set X of *states*, x is a distinguished *initial state*, $f : X \times U \to X$ is the *dynamics*, $c : X \times U \to \mathbb{R} \cup \{-\infty\}$ is the *instantaneous reward*, and $\Phi : X \to \mathbb{R} \cup \{-\infty\}$ is the *final reward* (the $-\infty$ value can be used to code forbidden final states or transitions). These data form a deterministic *Markov Decision Process* (MDP) with additive reward.

The function $V_n^N(\cdot)$, which represents the optimal reward from time n to time N, as a function of the starting point, is called the *value* function. It satisfies the backward dynamic programming equation

$$V_N^N = \Phi, \qquad V_k^N(x) = \max_{u \in U} \left\{ c(x, u) + V_{k+1}^N(f(x, u)) \right\} \ . \tag{3}$$

Introducing the *transition matrix* $A \in (\mathbb{R}_{\max})^{X \times X}$,

$$A_{x,y} = \sup_{u \in U,\ f(x,u) = y} c(x, u), \tag{4}$$

(the supremum over an empty set is $-\infty$), we obtain:

FACT 1 (DETERMINISTIC MDP = (max, +)-LINEAR DYNAMICS). *The value function V_k^N of a finite deterministic Markov decision process with additive reward is given by the* (max, +) *linear dynamics:*

$$V_N^N = \Phi, \qquad V_k^N = A V_{k+1}^N. \tag{5}$$

The interpretation in terms of paths is elementary. If we must end at node j, we take $\Phi = \mathbb{1}_j$ (the vector with all entries $\mathbb{0}$ except the j-th equal to $\mathbb{1}$), Then, the value function $V_0^N(i) = (A^N)_{ij}$ is the maximal (additive) weight of a path of length N, from i to j, in the graph canonically associated[3] with A.

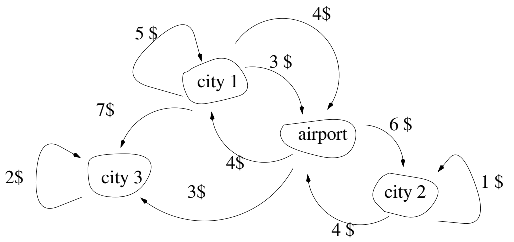
Example 1 (Taxicab). Consider a taxicab which operates between 3 cities and one airport, as shown in Fig. 1. At each state, the taxi driver has to choose his next destination, with deterministic fares shown on the graph (for simplicity, we assume that the demand is deterministic, and that the driver can choose the destination). The taxi driver considers maximizing his reward over N journeys. The (max, +) matrix associated with this MDP is displayed in Fig. 1.

Let us consider the optimization of the average reward:

$$\chi(x) = \sup_u \limsup_{N \to \infty} \frac{1}{N} J_0^N(x, u) \ . \tag{6}$$

Here, the sup is taken over infinite sequences of controls $u(1), u(2), \dots$ and the trajectory (2a) is defined for $k = 0, 1, \dots$. We expect J_0^N to grow (or to decrease) linearly, as a function of the horizon N. Thus, $\chi(x)$

[3] With a $X \times X$ matrix A we associate the weighted (directed) graph, with set of nodes X, and an arc (x, y) with weight $A_{x,y}$ whenever $A_{x,y} \neq \mathbb{0}$.



$$A = \begin{array}{c} \\ c_1 \\ a \\ c_2 \\ c_3 \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & a & c_2 & c_3 \end{array} \\ \begin{pmatrix} 5 & 4 & \mathbb{0} & 7 \\ 4 & \mathbb{0} & 6 & 3 \\ \mathbb{0} & 4 & 1 & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 2 \end{pmatrix} \end{array}$$

FIGURE 1. Taxicab Deterministic MDP and its matrix

represents the optimal average reward (per time unit), starting from x. Assuming that the sup and lim sup commute in (6), we get:

$$\chi(x) = \limsup_{N\to\infty} \frac{1}{N} \times (A^N\Phi)_x \tag{7}$$

(this is an hybrid formula, $A^N\Phi$ is in the (max, +) semiring , $1/N \times (\cdot)$ is in the conventional algebra). To evaluate (7), let us assume that the matrix A admits an *eigenvector* v in the (max, +) semiring:

$$Av = \lambda v, \text{ i.e. } \quad \max_j\{A_{ij} + v_j\} = \lambda_i + v_i \tag{8}$$

(the eigenvector v must be nonidentically $\mathbb{0}$, $\lambda \in \mathbb{R}_{\max}$ is the eigenvalue). Let us assume that v and Φ have only finite entries. Then, there exist two finite constants μ, ν such that $\nu + v \le \Phi \le \mu + v$. In (max, +) notation, $\nu v \le \Phi \le \mu v$. Then $\nu\lambda^N v = \nu A^N v \le A^N\Phi \le \mu A^N v = \mu\lambda^N v$, or with the conventional notation:

$$\nu + N\lambda + v \le A^N\Phi \le \mu + N\lambda + v. \tag{9}$$

We easily deduce from (9) the following.

FACT 2 ("EIGENELEMENTS = OPTIMAL REWARD AND POLICY"). *If the final reward Φ is* finite*, and if A has a finite eigenvector with eigenvalue λ, the optimal average reward $\chi(x)$ is a constant (independent of the starting point x), equal to the eigenvalue λ. An optimal control is obtained by playing in state i any u such that $c(i,u) = A_{ij}$ and $f(i,u) = j$, where j is in the* arg max *of* (8) *at state i.*

The existence of a finite eigenvector is characterized in Theorems 11 and 15 below.

We will not discuss here the extension of these results to the infinite dimensional case (e.g. (1)), which is one of the major themes of Idempotent Analysis [**60**]. Let us just mention that all the results presented here admit or should admit infinite dimensional generalizations, presumably up to important technical difficulties.

There is another much simpler extension, to the (discrete) semi-Markov case, which is worth being mentioned. Let us equip the above MDP with an additional map $\tau : X \times U \to \mathbb{R}^+ \setminus \{0\}$; $\tau(x(k-1), u(k))$ represents the physical time elapsed between decision k and decision $k+1$, when control $u(k)$ is chosen. This is very natural in most applications (for the taxicab example, the times of the different possible journeys in general differ). The optimal average reward per time unit now writes:

$$\chi(x) = \sup_u \limsup_{N\to\infty} \frac{\sum_{k=1}^N c(x(k-1), u(k)) + \Phi(x(N))}{\sum_{k=1}^N \tau(x(k-1), u(k))} \ . \tag{10}$$

Of course, the specialization $\tau \equiv 1$ gives the original problem (6). Let us define $\mathcal{T}_{ij} = \{\tau(i,u) \mid f(i,u) = j\}$, and for $t \in \mathcal{T}_{ij}$,

$$A_{t,i,j} = \sup_{u\in U, f(i,u)=j, \tau(i,u)=t} c(i,u) \ . \tag{11}$$

Arguing as in the Markov case, it is not too difficult to show the following.

FACT 3 (GENERALIZED SPECTRAL PROBLEM FOR SEMI-MARKOV PROCESSES). *If the generalized spectral problem*

$$\max_j \max_{t\in\mathcal{T}_{ij}}\{A_{t,i,j} - \lambda t + v_j\} = v_i \tag{12}$$

has a finite solution v, and if Φ is finite, then the optimal average reward is $\chi(x) = \lambda$, for all x. An optimal control is obtained by playing any u in the arg max *of* (11)*, with j, t in the* arg max *of* (12)*, when in state i.*

Algebraically, (12) is nothing but a generalized spectral problem. Indeed, with an obvious definition of the matrices A_t, we can write:

$$\bigoplus_{t\in\overline{\mathcal{T}}} \lambda^{-t} A_t v = v \enspace , \qquad \text{where } \overline{\mathcal{T}} = \bigcup_{i,j} \mathcal{T}_{ij} \enspace . \tag{13}$$

2.2. An Algebra for Asymptotics. In Statistical Physics, one looks at the asymptotics when the temperature h tends to zero of the spectrum of *transfer matrices*, which have the form

$$\mathcal{A}_h = (\exp(h^{-1}A_{ij}))_{1\le i,j\le n} \enspace .$$

The real parameters A_{ij} represent potential terms plus interaction energy terms (when two adjacent sites are in states i and j, respectively). The Perron eigenvalue[4] $\rho(\mathcal{A}_h)$ determines the free energy per site $\lambda_h = h\log\rho(\mathcal{A}_h)$. Clearly, λ_h is an eigenvalue of A in the semiring $\mathbb{R}_h$, defined in Table 1. Let $\rho_{\max}(A)$ denote the maximal (max, +) eigenvalue of A. Since $\lim_{h\to 0^+} \mathbb{R}_h = \mathbb{R}_0 = \mathbb{R}_{\max}$, the following result is natural.

FACT 4 (PERRON FROBENIUS ASYMPTOTICS). *The asymptotic growth rate of the Perron eigenvalue of $\mathcal{A}_h$ is equal to the maximal* (max, +) *eigenvalue of the matrix A:*

$$\lim_{h\to 0^+} h\log\rho(\mathcal{A}_h) = \rho_{\max}(A) \enspace . \tag{14}$$

This follows easily from the (max, +) spectral inequalities (24),(25) below. The normalized Perron eigenvector v_h of $\mathcal{A}_h$ also satisfies

$$\lim_{h\to 0^+} h\log(v_h)_i = u_i \enspace ,$$

where u is a special (max, +) eigenvector of A which has been characterized recently by Akian, Bapat, and Gaubert [**2**]. Precise asymptotic expansions of $\rho(\mathcal{A}_h)$ as sum of exponentials have been given, some of the terms having combinatorial interpretations.

More generally, (max, +) algebra arises almost everywhere in asymptotic phenomena. Often, the (max, +) algebra is involved in an elementary way (e.g. when computing exponents of Puiseux expansions using the Newton Polygon). Less elementary applications are WKB type asymptotics (see [**60**]), which are related to Large Deviations (see e.g. [**27**]).

2.3. An Algebra for Discrete Event Systems. The (max, +) algebra is popular in the Discrete Event Systems community, since (max, +) linear dynamics correspond to a well identified subclass of Discrete Event Systems, with only synchronization phenomena, called Timed Event Graphs. Indeed, consider a system with n repetitive tasks. We assume that the k-th execution of task i (firing of transition i) has to wait τ_{ij} time units for the $(k-\nu_{ij})$-th execution of task j. E.g. tasks represent the processing of parts in a manufacturing system, ν_{ij} represents an initially available stock, and τ_{ij} represents a production or transportation time.

FACT 5 (TIMED EVENT GRAPHS ARE (max, +) LINEAR SYSTEMS). *The earliest date of occurrence of an event i in a Timed Event Graph, $x_i(k)$, satisfies*

$$x_i(k) = \max_j \left[\tau_{ij} + x_j(k-\nu_{ij})\right] \enspace . \tag{15}$$

[4]The Perron eigenvalue $\rho(B)$ of a matrix B with nonnegative entries is the maximal eigenvalue associated with a nonnegative eigenvector, which is equal to the spectral radius of B.

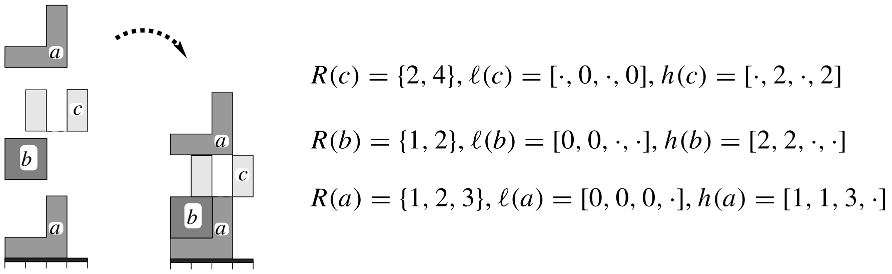


FIGURE 2. Heap of Pieces

Eqn 15 coincides with the value iteration of the deterministic semi-Markov Decision Process in § 2.1, that we only wrote in the Markov version (3). Therefore, the asymptotic behavior of (15) can be dealt with as in § 2.1, using (max, +) spectral theory. In particular, if the generalized spectral problem $v_i = \max_j[\tau_{ij} - \lambda \nu_{ij} + v_j]$ has a finite solution (λ, v), then $\lambda = \lim_{k\to\infty} k^{-1} \times x_i(k)$, for all i (λ is the *cycle time*, or inverse of the *asymptotic throughput*). The study of the dynamics (15), and of its stochastic [**3**], and non-linear extensions [**16, 40**] (fluid Petri Nets, minmax functions), is the major theme of (max, +) discrete event systems theory.

Another linear model is that of *heaps of pieces*. Let $\mathcal{R}$ denote a set of *positions* or *resources* (say $\mathcal{R} = \{1, \dots, n\}$). A *piece* (or *task*) a is a rigid (possibly non connected) block, represented geometrically by a set of occupied positions (or requested resources) $R(a) \subset \mathcal{R}$, a lower contour (starting time) $\ell(a) : R(a) \to \mathbb{R}$, an upper contour (release time) $h(a) : R(a) \to \mathbb{R}$, such that $\forall a \in R(a), h(a) \geq \ell(a)$. The piece corresponds to the region of the $\mathcal{R} \times \mathbb{R}$ plane: $P_a = \{(r, y) \in R(a) \times \mathbb{R} \mid \ \ell(a)_r \leq y \leq h(a)_r\}$, which means that task a requires the set of resources (machines, processors, operators) $R(a)$, and that resource $r \in R(a)$ is used from time $\ell(a)_r$ to time $h(a)_r$. A piece P_a can be translated vertically of any λ, which gives the new region defined by $\ell'(a) = \lambda + \ell(a),\ h'(a) = \lambda + h(a)$. We can execute a task earlier or later, but we cannot change the differences $h(a)_r - \ell(a)_s$ which are invariants of the task. A *ground* or *initial condition* is a row vector $g \in (\mathbb{R}_{\max})^{\mathcal{R}}$. Resource r becomes initially available at time g_r. If we drop k pieces $a_1 \dots a_k$, in this order, on the ground g (letting the pieces fall down according to the gravity, forbidding horizontal translations, and rotations, as in the famous Tetris game, see Fig 2), we obtain what we call a *heap of pieces*. The upper contour $x(w)$ of the heap $w = a_1 \dots a_k$ is the row vector in $(\mathbb{R}_{\max})^{\mathcal{R}}$, whose r-th component is equal to the position of the top of the highest piece occupying resource r. The *height* of the heap is by definition $y(w) = \max_{r\in\mathcal{R}} x(w)_r$. Physically, $y(w)$ gives the *makespan* (= completion time) of the sequence of tasks w, and $x(w)_r$ is the release time of resource r.

With each piece a within a set of pieces $\mathcal{T}$, we associate the matrix $M(a) \in (\mathbb{R}_{\max})^{\mathcal{R}\times\mathcal{R}}$, $M(a)_{r,s} = h(a)_s - \ell(a)_r$ if $r, s \in R(a)$, and $M(a)_{r,r} = \mathbb{1}$ for diagonal entries not in $R(a)$ (other entries are $\mathbb{0}$). The following result was found independently by Gaubert and Mairesse (in [**41**]), and Brilman and Vincent [**8**].

FACT 6 (TETRIS GAME IS (max, +) LINEAR). *The upper contour $x(w)$ and the height $y(w)$ of the heap of pieces $w = a_1 \dots a_k$, piled up on the ground g, are given by the* (max, +) *products:*

$$x(w) = gM(a_1) \dots M(a_k), \qquad y(w) = x(w)\mathbb{1}_{\mathcal{R}},$$

($\mathbb{1}_X$ denotes the column vector indexed by X with entries $\mathbb{1}$).

In algebraic terms, the height generating series $\bigoplus_{w\in\mathcal{T}^*} y(w)w$ is *rational* over the (max,+) semiring ($\mathcal{T}^*$ is the free monoid on $\mathcal{T}$, basic properties of rational series can be found e.g. in [**71**]).

Let us mention an open problem. If an infinite sequence of pieces $a_1a_2 \dots a_k \dots$ is taken at random, say in an independent identically distributed way with the uniform distribution on $\mathcal{T}$, it is known [**21, 3**] that there exists an asymptotic growth rate $\lambda \in \mathbb{R}^+$:

$$\lambda = \lim_{k\to\infty} \frac{1}{k} y(a_1 \dots a_k) \quad \text{a.s.} \tag{16}$$

The effective computation of the constant λ (Lyapunov exponent) is one of the main open problems in (max,+) algebra. The Lyapunov exponent problem is interesting for general random matrices (not only for special matrices associated with pieces), but the heap case (even with unit height, $h(a) = 1+\ell(a)$) is typical and difficult enough to begin with. Existing results on Lyapunov exponents can be found in [**3**]. See also the paper of Gaujal and Jean-Marie in [**41**], and [**8**].

2.4. An Algebra for Decision. The "tropical" semiring $\mathbb{N}_{\min} = (\mathbb{N}\cup\{+\infty\}, \min, +)$, has been invented by Simon [**74**] to solve the following classical problem posed by Brzozowski: *is it decidable whether a rational language L has the Finite Power Property (FPP): $\exists m \in \mathbb{N},\ L^* = L^0 \cup L \cup \cdots \cup L^m$*. The problem was solved independently by Simon and Hashiguchi.

FACT 7 (SIMON). *The FPP problem for rational languages reduces to the finiteness problem for finitely generated semigroups of matrices with entries in $\mathbb{N}_{\min}$, which is decidable.*

Other (more difficult) decidable properties (with applications to the polynomial closure and star height problems) are the *finite section* problem, which asks, given a finitely generated semigroup of matrices S over the tropical semiring, whether the set of entries in position i, j, $\{s_{ij} \mid s \in S\}$ is finite; and the more general *limitation* problem, which asks whether the set of coefficients of a rational series in $\mathbb{N}_{\min}$, with noncommuting indeterminates, is finite. These decidability results due to Hashiguchi [**43**], Leung [**57**] and Simon [**76**] use structural properties of long optimal words in $\mathbb{N}_{\min}$-automata (involving multiplicative rational expressions), and combinatorial arguments. By comparison with basic Discrete Event System and Markov Decision applications, which essentially involve semigroups with a single generator ($S = \{A^k \mid k \geq 1\}$), these typically noncommutative problems represent a major jump in difficulty. We refer the reader to the survey of Pin in [**41**], to [**76, 43, 57**], and to the references therein. However, essential in the understanding of the noncommutative case is the one generator case, covered by the (max, +) Perron-Frobenius theory detailed below.

Let us point out an open problem. The semigroup of *linear projective maps* $\mathbb{P}\mathbb{Z}_{\max}^{n\times n}$ is the quotient of the semigroup of matrices $\mathbb{Z}_{\max}^{n\times n}$ by the proportionality relation: $A \sim B \Leftrightarrow \exists\lambda \in \mathbb{Z},\ A = \lambda B$ (i.e. $A_{ij} = \lambda + B_{ij}$). We ask: *can we decide whether a finitely generated semigroup of linear projective maps is finite ?* The motivation is the following. If the image of a finitely generated semigroup with generators $M(a) \in \mathbb{Z}_{\max}^{n\times n}, a \in \Sigma$ by the canonical morphism $\mathbb{Z}_{\max}^{n\times n} \to \mathbb{P}\mathbb{Z}_{\max}^{n\times n}$ is finite, then the Lyapunov exponent $\lambda = \text{a.s.} \lim_{k\to\infty} k^{-1} \times \|M(a_1)\ldots M(a_k)\|$ (same probabilistic assumptions as for (16), $\|A\| = \sup_{ij} A_{ij}$, by definition) can be computed from a finite Markov Chain on the associated projective linear semigroup [**32, 33**].

3. Solving Linear Equations in the (max, +) Semiring

3.1. A hopeless algebra? The general system of n (max, +)-linear equations with p unknowns $x_1, \ldots, x_p$ writes:

$$Ax \oplus b = Cx \oplus d, \qquad A, C \in (\mathbb{R}_{\max})^{n\times p},\ b, d \in (\mathbb{R}_{\max})^n \ . \tag{17}$$

Unlike in conventional algebra, a square linear system ($n = p$) is not generically solvable (consider $3x\oplus 2 = x \oplus 0$, which has no solution, since for all $x \in \mathbb{R}_{\max}$, $\max(3 + x, 2) > \max(x, 0)$).

There are several ways to make this hard reality more bearable. One is to give general structural results. Another is to deal with natural subclasses of equations, whose solutions can be obtained by efficient methods. The *inverse* problem $Ax = b$ can be dealt with using *residuation*. The *spectral* problem $Ax = \lambda x$ (λ scalar) is solved using the (max, +) analogue of Perron-Frobenius theory. The *fixed point* problem $x = Ax \oplus b$ can be solved via rational methods familiar in language theory (introducing the "star" operation $A^* = A^0 \oplus A \oplus A^2 \oplus \cdots$). A last way, which has the seduction of forbidden things, is to say: "certainly, the solution of $3x\oplus 2 = x\oplus 0$ is $x = \ominus -1$. For if this equation has no ordinary solution, the symmetrized equation (obtained by putting each occurrence of the unknown in the other side of the equality) $x' \oplus 2 = 3x' \oplus 0$

has the unique solution $x' = -1$. Thus, $x = \ominus - 1$ is the requested solution." Whether or not this argument is valid is the object of *symmetrization* theory.

All these approaches rely, in one way or another, on the *order* structure of idempotent semirings that we next introduce.

3.2. Natural Order Structure of Idempotent Semirings. An idempotent semiring $\mathcal{S}$ can be equipped with the following *natural* order relation

$$a \preceq b \iff a \oplus b = b. \tag{18}$$

We will write $a \prec b$ when $a \preceq b$ and $a \neq b$. The natural order endows $\mathcal{S}$ with a sup-semilattice structure, for which $a \oplus b = a \vee b = \sup\{a, b\}$ (this is the least upper bound of the set $\{a, b\}$), and $\mathbb{0} \preceq a, \forall a, b \in \mathcal{S}$ ($\mathbb{0}$ is the *bottom* element). The semiring laws preserve this order, i.e. $\forall a, b, c \in \mathcal{S},\ a \preceq b \implies a \oplus c \preceq b \oplus c,\ ac \preceq bc$. For the $(\max, +)$ semiring $\mathbb{R}_{\max}$, the natural order $\preceq$ coincides with the usual one. For the $(\min, +)$ semiring $\mathbb{R}_{\min}$, the natural order is the opposite of the usual one.

Since addition coincides with the sup for the natural order, there is a simple way to define infinite sums, in an idempotent semiring, setting $\bigoplus_{i \in I} x_i = \sup\{x_i \mid i \in I\}$, for any possibly infinite (even non denumerable) family $\{x_i\}_{i \in I}$ of elements of $\mathcal{S}$, when the sup exists. We say that the idempotent semiring $\mathcal{S}$ is *complete* if any family has a supremum, and if the product distributes over infinite sums. When $\mathcal{S}$ is complete, $(\mathcal{S}, \preceq)$ becomes automatically a complete lattice, the greatest lower bound being equal to $\bigwedge_{i \in I} x_i = \sup\{y \in \mathcal{S} \mid y \leq x_i,\ \forall i \in I\}$. The $(\max, +)$ semiring $\mathbb{R}_{\max}$ is not complete (a complete idempotent semiring must have a maximal element), but it can be embedded in the complete semiring $\overline{\mathbb{R}}_{\max}$.

3.3. Solving $Ax = b$ using Residuation. In general, $Ax = b$ has no solution[5], but $Ax \preceq b$ always does (take $x = \mathbb{0}$). Thus, a natural way of attacking $Ax = b$ is to relax the equality and study the set of its subsolutions. This can be formalized in terms of *residuation* [**7**], a notion borrowed from ordered sets theory. We say that a monotone map f from an ordered set E to an ordered set F is *residuated* if for all $y \in F$, the set $\{x \in E \mid f(x) \leq y\}$ has a maximal element, denoted $f^\sharp(y)$. The monotone map $f^\sharp$, called *residual* or *residuated map* of f, is characterized alternatively by $f \circ f^\sharp \leq \mathrm{Id}$, $f^\sharp \circ f \geq \mathrm{Id}$. An idempotent semiring $\mathcal{S}$ is *residuated* if the right and left multiplication maps $\lambda_a : x \mapsto ax$, $\rho_a : x \mapsto xa$, $\mathcal{S} \to \mathcal{S}$, are residuated, for all $a \in \mathcal{S}$. A *complete* idempotent semiring is automatically residuated. We set

$$a \backslash b \stackrel{\text{def}}{=} \lambda_a^\sharp(b) = \max\{x \mid ax \preceq b\} \ , \quad b/a \stackrel{\text{def}}{=} \rho_a^\sharp(b) = \max\{x \mid xa \preceq b\} \ .$$

In the completed $(\max, +)$ semiring $\overline{\mathbb{R}}_{\max}$, $a \backslash b = b/a$ is equal to $b - a$ when $a \neq \mathbb{0} (= -\infty)$, and is equal to $+\infty$ if $a = \mathbb{0}$. The residuated character is transfered from scalars to matrices as follows.

PROPOSITION 2 (MATRIX RESIDUATION). *Let $\mathcal{S}$ be a complete idempotent semiring. Let $A \in \mathcal{S}^{n \times p}$. The map $\lambda_A : x \mapsto Ax, \mathcal{S}^p \to \mathcal{S}^n$, is residuated. For any $y \in \mathcal{S}^n$, $A \backslash y \stackrel{\text{def}}{=} \lambda_A^\sharp(y)$ is given by $(A \backslash y)_i = \bigwedge_{1 \leq j \leq n} A_{ji} \backslash y_j$.*

In the case of $\overline{\mathbb{R}}_{\max}$, this reads:

$$(A \backslash y)_i = \min_{1 \leq j \leq n} (-A_{ji} + y_j) \ , \tag{19}$$

with the convention dual to that of $\overline{\mathbb{R}}_{\max}$, $(+\infty) + x = +\infty$, for any $x \in \mathbb{R} \cup \{\pm\infty\}$. We recognize in (19) a matrix product in the semiring $\overline{\mathbb{R}}_{\min} = (\mathbb{R} \cup \{\pm\infty\}, \min, +)$, involving the transpose of the opposite of A.

COROLLARY 3 (SOLVING $Ax = y$). *Let $\mathcal{S}$ denote a complete idempotent semiring, and let $A \in \mathcal{S}^{n \times p}$, $y \in \mathcal{S}^n$. The equation $Ax = y$ has a solution iff $A(A \backslash y) = y$.*

[5] It is an elementary exercise to check that the map $x \mapsto Ax$, $(\mathbb{R}_{\max})^p \to (\mathbb{R}_{\max})^n$, is surjective (resp. injective) iff the matrix A contains a monomial submatrix of size n (resp. p), a very unlikely event — recall that a square matrix B is *monomial* if there is exactly one non zero element in each row, and in each column, or (equivalently) if it is a product of a permutation matrix and a diagonal matrix with non zero diagonal elements. This implies that a matrix has a left or a right inverse iff it has a monomial submatrix of maximal size, which is the analogue of a well known result for nonnegative matrices [**6**, Lemma 4.3].

Corollary 3 allows us to check the existence of a solution x of $Ax = y$ in time $O(np)$ (scalar operations are counted for one time unit). In the (max, +) case, a refinement (due to the total order) allows us to decide the existence of a solution by inspection of the minimizing sets in (19), see [**23, 84**].

3.4. Basis Theorem for Finitely Generated Semimodules over $\mathbb{R}_{\max}$. A finitely generated semimodule $\mathcal{V} \subset (\mathbb{R}_{\max})^n$ is the set of linear combinations of a finite family $\{u_1, \dots , u_p\}$ of vectors of $(\mathbb{R}_{\max})^n$:

$$\mathcal{V} = \Big\{ \bigoplus_{i=1}^{p} \lambda_i u_i \;\big|\lambda_1, \dots , \lambda_p \in \mathbb{R}_{\max} \Big\} \enspace .$$

In matrix terms, $\mathcal{V}$ can be identified to the *column space* or *image* of the $n \times p$ matrix $A = [u_1, \dots , u_p]$, $\mathcal{V} = \mathrm{Im}A \stackrel{\text{def}}{=} \{Ax \mid x \in (\mathbb{R}_{\max})^p\}$. The *row space* of A is the column space of A^T (the transpose of A). The family $\{u_i\}$ is a *weak basis* of $\mathcal{V}$ if it is a generating family, minimal for inclusion. The following result, due to Moller [**63**] and Wagneur [**79**] (with variants) states that finitely generated subsemimodules of $(\mathbb{R}_{\max})^n$ have (essentially) a unique weak basis.

THEOREM 4 (BASIS THEOREM). *A finitely generated semimodule $\mathcal{V} \subset (\mathbb{R}_{\max})^n$ has a weak basis. Any two weak bases have the same number of generators. For any two weak bases $\{u_1, \dots , u_p\}$, $\{v_1, \dots , v_p\}$, there exist invertible scalars $\lambda_1, \dots , \lambda_p$ and a permutation σ of $\{1, \dots , p\}$ such that $u_i = \lambda_i v_{\sigma(i)}$.*

The cardinality of a weak basis is called the *weak rank* of the semimodule, denoted $\mathrm{rk}_w\mathcal{V}$. The *weak column rank* (resp. weak row rank) of the matrix A is the weak rank of its column (resp. row) space. Unlike in usual algebra, the weak row rank in general differs from the weak column rank (this is already the case for Boolean matrices). Theorem 4 holds more generally in any idempotent semiring $\mathcal{S}$ satisfying the following axioms: ($a \succeq \alpha a$ and $a \neq \mathbb{0}$) $\implies \mathbb{1} \succeq \alpha$, ($a = \alpha a \oplus b$ and $\alpha \prec \mathbb{1}$) $\implies a = b$. The axioms needed to set up a general rank theory in idempotent semirings are not currently understood. Unlike in vector spaces, there exist finitely generated semimodules $\mathcal{V} \subset (\mathbb{R}_{\max})^n$ of arbitrarily large weak rank, if the dimension of the ambient space n is at least 3; and not all subsemimodules of $(\mathbb{R}_{\max})^n$ are finitely generated, even with $n = 2$.

Example 5 (Cuninghame-Green [**23**]*,Th. 16.4).* The weak column rank of the $3 \times (i + 1)$ matrix

$$A_i = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & i \\ 0 & -1 & \dots & -i \end{bmatrix}$$

is equal to $i + 1$ for all $i \in \mathbb{N}$. This can be understood geometrically using a representation due to Mairesse. We visualize the set of vectors with finite entries of a semimodule $\mathcal{V} \subset (\mathbb{R}_{\max})^3$ by the subset of $\mathbb{R}^2$, obtained by projecting $\mathcal{V}$ orthogonally, on any plane orthogonal to $(1, 1, 1)$. Since $\mathcal{V}$ is invariant by multiplication by any scalar λ, i.e. by the usual addition of the vector $(\lambda, \lambda, \lambda)$, the semimodule $\mathcal{V}$ is well determined by its projection. We only loose the points with $\mathbb{0}$ entries which are sent to some infinite end of the $\mathbb{R}^2$ plane. The semimodules $\mathrm{Im}A_1, \mathrm{Im}A_2, \mathrm{Im}A_3$ are shown on Fig 3. The generators are represented by bold points, and the semimodules by gray regions. The broken *line* between any two generators u, v represents $\mathrm{Im}[u, v]$. This picture should make it clear that a weak basis of a subsemimodule of $(\mathbb{R}_{\max})^3$ may have as many generators as a convex set of $\mathbb{R}^2$ may have extremal points. The notion of weak rank is therefore a very coarse one.

Let $A \in (\mathbb{R}_{\max})^{n \times p}$. A weak basis of the semimodule $\mathrm{Im}A$ can be computed by a greedy algorithm. Let $A[i]$ denote the i-th column of A, and let $A(i)$ denote the $n \times (p - 1)$ matrix obtained by deleting column i. We say that column i of A is *redundant* if $A[i] \in \mathrm{Im}A(i)$, which can be checked by Corollary 3. Replacing A by $A(i)$ when $A[i]$ is redundant, we do not change the semimodule $\mathrm{Im}A$. Continuing this process, we terminate in $O(np^2)$ time with a weak basis.

Application 6 (Controllability). The fact that ascending chains of semimodules need not stationnarize yields pathological features in terms of Control. Consider the controlled dynamical system:

$$x(0) = \mathbb{0}, \quad x(k) = Ax(k-1) \oplus Bu(k), \; k = 1, 2, \dots \tag{20}$$

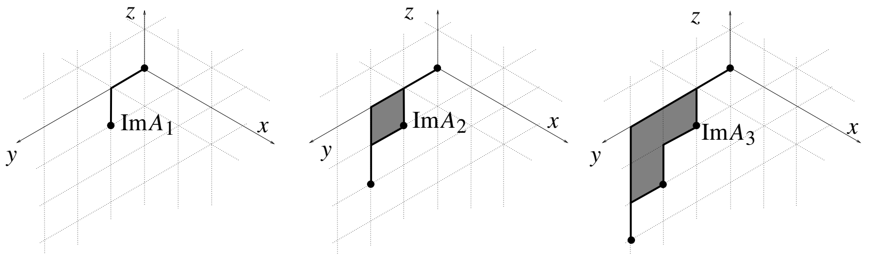


FIGURE 3. An infinite ascending chain of semimodules of $(\mathbb{R}_{\max})^3$ (see Ex. 5).

where $A \in (\mathbb{R}_{\max})^{n\times n}$, $B \in (\mathbb{R}_{\max})^{n\times q}$, and $u(k) \in (\mathbb{R}_{\max})^q$, $k = 1, 2, \dots$ is a sequence of control vectors. Given a state $\xi \in (\mathbb{R}_{\max})^n$, the *accessibility* problem (in time N) asks whether there is a control sequence u such that $x(N) = \xi$. Clearly, ξ is accessible in time N iff it belongs to the image of the *controllability* matrix $\mathcal{C}_N = [B, AB, \dots, A^{N-1}B]$. Corollary 3 allows us to decide the accessibility of ξ. However, unlike in conventional algebra (in which $\mathrm{Im}\mathcal{C}_N = \mathrm{Im}\mathcal{C}_n$, for any $N \geq n$, thanks to Cayley-Hamilton theorem), the semimodule of accessible states $\mathrm{Im}\mathcal{C}_N$ may grow indefinitely as $N \to \infty$.

3.5. Solving $Ax = Bx$ by Elimination. The following theorem is due to Butkovič and Hegedüs [**11**]. It was rediscovered in [**31**, Chap. III].

THEOREM 7 (FINITENESS THEOREM). *Let $A, B \in (\mathbb{R}_{\max})^{n\times p}$. The set $\mathcal{V}$ of solutions of the homogeneous system $Ax = Bx$ is a finitely generated semimodule.*

This is a consequence of the following universal elimination result.

THEOREM 8 (ELIMINATION OF EQUALITIES IN SEMIRINGS). *Let $\mathcal{S}$ denote an arbitrary semiring. Let $A, B \in \mathcal{S}^{n\times p}$. If for any $q \geq 1$ and any row vectors $a, b \in \mathcal{S}^q$, the* hyperplane *$\{x \in \mathcal{S}^q \mid ax = bx\}$ is a finitely generated semimodule, then $\mathcal{V} = \{x \in \mathcal{S}^p \mid Ax = Bx\}$ is a finitely generated semimodule.*

The fact that hyperplanes of $(\mathbb{R}_{\max})^q$ are finitely generated can be checked by elementary means (but the number of generators can be of order q^2). Theorem 8 can be easily proved by induction on the number of equations (see [**11, 31**]). In the $\mathbb{R}_{\max}$ case, the resulting naive algorithm has a doubly exponential complexity. But it is possible to incorporate the construction of weak bases in the algorithm, which much reduces the execution time. The making (and complexity analysis) of efficient algorithms for $Ax = Bx$ is a major open problem. When only a single solution is needed, the algorithm of Walkup and Borriello (in [**41**]) seems faster, in practice.

There is a more geometrical way to understand the finiteness theorem. Consider the following correspondence between semimodules of $((\mathbb{R}_{\max})^{1\times n})^2$ (couples of row vectors) and $(\mathbb{R}_{\max})^{n\times 1}$ (column vectors), respectively:

$$
\begin{array}{rcl}
\mathcal{W} \subset ((\mathbb{R}_{\max})^{1\times n})^2 & \longrightarrow & \mathcal{W}^\top = \{x \in (\mathbb{R}_{\max})^{n\times 1} \mid ax = bx,\ \forall (a,b) \in \mathcal{W}\}\ , \\
\mathcal{V}^\perp = \{(a,b) \in ((\mathbb{R}_{\max})^{1\times n})^2 \mid ax = bx,\ \forall x \in \mathcal{V}\} & \longleftarrow & \mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1}\ .
\end{array}
\tag{21}
$$

Theorem 7 states that if $\mathcal{W}$ is a finitely generated semimodule (i.e. if all the row vectors $[a, b]$ belong to the row space of a matrix $[A, B]$) then, its orthogonal $\mathcal{W}^\top$ is finitely generated. Conversely, if $\mathcal{V}$ is finitely generated, so does $\mathcal{V}^\perp$ (since the elements (a, b) of $\mathcal{V}^\perp$ are the solutions of a finite system of linear equations). The orthogonal semimodule $\mathcal{V}^\perp$ is exactly the set of *linear equations* $(a, b) : ax = bx$ satisfied by all the $x \in \mathcal{V}$. Is a finitely generated subsemimodule $\mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1}$ defined by its equations ? The answer is positive [**31**, Chap. IV,1.2.2]:

THEOREM 9 (DUALITY THEOREM). *For all finitely generated semimodules $\mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1}$, $(\mathcal{V}^\perp)^\top = \mathcal{V}$.*

In general, $(\mathcal{W}^\top)^\perp \supsetneq \mathcal{W}$. The duality theorem is based on the following analogue of the Hahn-Banach theorem, stated in [**31**]: *if $\mathcal{V} \subset (\mathbb{R}_{\max})^{n\times 1}$ is a finitely generated semimodule, and $y \notin \mathcal{V}$, there exist $(a, b) \in ((\mathbb{R}_{\max})^{1\times n})^2$ such that $ay \neq by$ and $ax = bx,\ \forall x \in \mathcal{V}$.*

The *kernel* of a linear operator C should be defined as $\mathrm{Ker}C = \{(x, y) \mid Cx = Cy\}$. When is the projector on the image of a linear operator B, parallel to $\mathrm{Ker}C$, defined? The answer is given in [**17**].

3.6. Solving $x = Ax \oplus b$ using Rational Calculus. Let $\mathcal{S}$ denote a complete idempotent semiring, and let $A \in \mathcal{S}^{n\times n}$, $b \in \mathcal{S}^n$. The least solution of $x \succeq Ax \oplus b$ is A^*b, where the star operation is given by:

$$A^* \stackrel{\text{def}}{=} \bigoplus_{n\in\mathbb{N}} A^n \enspace . \tag{22}$$

Moreover, $x = A^*b$ satisfies the equation $x = Ax \oplus b$. All this is most well known (see e.g. [**71**]), and we will only insist on the features special to the (max, +) case. We can interpret A^*_{ij} as the *maximal weight* of a path from i to j of any length, in the graph[3] associated with A. We next characterize the convergence of A^* in $(\mathbb{R}_{\max})^{n\times n}$ (A^* is a priori defined in $(\overline{\mathbb{R}}_{\max})^{n\times n}$, but the $+\infty$ value which breaks the semifield character of $\mathbb{R}_{\max}$ is undesired in most applications). The following fact is standard (see e.g. [**3**, Theorem 3.20]).

PROPOSITION 10. *Let $A \in (\mathbb{R}_{\max})^{n\times n}$. The entries of A^* belong to $\mathbb{R}_{\max}$ iff there are no circuits with positive weight in the graph[3] of A. Then, $A^* = A^0 \oplus A \oplus \cdots \oplus A^{n-1}$.*

The matrix A^* can be computed in time $O(n^3)$ using classical universal Gauss algorithms (see e.g. [**38**]). Special algorithms exist for the (max, +) semiring. For instance, the sequence $x(k) = Ax(k-1) \oplus b$, $x(0) = \mathbb{0}$ stationarizes before step n (with $x(n) = x(n+1) = A^*b$) iff A^*b is finite. This allows us to compute A^*b very simply. A complete account of existing algorithms can be found in [**38**].

3.7. The (max, +) Perron-Frobenius Theory. The most ancient, most typical, and probably most useful (max, +) results are relative to the spectral problem $Ax = \lambda x$. One might argue that 90% of current applications of (max, +) algebra are based on a complete understanding of the spectral problem. The theory is extremely similar to the well known Perron-Frobenius theory (see e.g. [**6**]). The (max, +) case turns out to be very appealing, and slightly more complex than the conventional one (which is not surprising, since the (max, +) spectral problem is a somehow degenerate limit of the conventional one, see §2.2). The main discrepancy is the existence of two graphs which rule the spectral elements of A, the weighted graph canonically[3] associated with a matrix A, and one of its subgraphs, called *critical* graph.

First, let us import the notion of *irreducibility* from the conventional Perron-Frobenius theory. We say that i *has access* to j if there is a path from i to j in the graph of A, and we write $i \stackrel{*}{\to} j$. The *classes* of A are the equivalence classes for the relation $i\mathcal{R}j \Leftrightarrow (i \stackrel{*}{\to} j \text{ and } j \stackrel{*}{\to} i)$. A matrix with a single class is *irreducible*. A class $\mathcal{C}$ is upstream $\mathcal{C}'$ (equivalently $\mathcal{C}'$ is downstream $\mathcal{C}$) if a node of $\mathcal{C}$ has access to a node of $\mathcal{C}'$. Classes with no other downstream classes are *final*, classes with no other upstream classes are *initial*.

The following famous (max, +) result has been proved again and again, with various degrees of generality and precision, see [**69, 78, 23, 84, 39, 3, 60**].

THEOREM 11 ("(max, +) PERRON-FROBENIUS THEOREM"). *An irreducible matrix $A \in (\mathbb{R}_{\max})^{n\times n}$ has a unique eigenvalue, equal to the maximal circuit mean of A:*

$$\rho_{\max}(A) \quad = \quad \bigoplus_{k=1}^{n} \mathrm{tr}\,(A^k)^{\frac{1}{k}} \quad = \max_{1\le k\le n}\ \max_{i_1,\dots,i_k} \frac{A_{i_1 i_2} + \cdots + A_{i_k i_1}}{k} \enspace . \tag{23}$$

We have the following refinements in terms of inequalities [**31**, Chap IV], [**5**].

LEMMA 12 ("COLLATZ-WIELANDT PROPERTIES"). *For any $A \in (\mathbb{R}_{\max})^{n\times n}$,*

$$\rho_{\max}(A) = \max\{\lambda \in \mathbb{R}_{\max} \mid \exists u \in (\mathbb{R}_{\max})^n \setminus \{\mathbb{0}\},\quad Au \succeq \lambda u\} \enspace . \tag{24}$$

Moreover, if A is irreducible,

$$\rho_{\max}(A) = \min\{\lambda \in \mathbb{R}_{\max} \mid \exists u \in (\mathbb{R}_{\max})^n \setminus \{\mathbb{0}\},\quad Au \preceq \lambda u\} \enspace . \tag{25}$$

The characterization (25) implies in particular that, for an irreducible matrix A, $\rho_{\max}(A)$ is the optimal value of the linear program

$$\min \lambda \text{ s.t. } \forall i, j \;\; A_{ij} + u_j \leq u_i + \lambda \ .$$

This was already noticed by Cuninghame-Green [**23**]. The standard way to compute the maximal circuit mean $\rho_{\max}(A)$ is to use Karp algorithm [**45**], which runs in time $O(n^3)$. The specialization of Howard algorithm (see e.g. [**67**]) to deterministic Markov Decision Processes with average reward, yields an algorithm whose average execution time is in practice far below that of Karp algorithm, but no polynomial bound is known for the execution time of Howard algorithm. Howard algorithm is also well adapted to the semi-Markov variants (12).

Unlike in conventional Perron-Frobenius theory, an irreducible matrix may have several (non proportional) eigenvectors. The characterization of the eigenspace uses the notion of *critical graph*. An arc (i, j) is *critical* if it belongs to a circuit $(i_1, \dots, i_k)$ whose mean weight attains the max in (23). Then, the nodes i, j are *critical*. Critical nodes and arcs form the *critical graph*. A *critical class* is a strongly connected component of the critical graph. Let $\mathcal{C}_1^c, \dots, \mathcal{C}_r^c$ denote the critical classes. Let $\tilde{A} = \rho_{\max}^{-1}(A)A$ (i.e. $\tilde{A}_{ij} = -\rho_{\max}(A) + A_{ij}$). Using Proposition 10, the existence of $\tilde{A}^* \ (\stackrel{\text{def}}{=} (\tilde{A})^*)$ is guaranteed. If i is in a critical class, we call the column $\tilde{A}^*_{\cdot,i}$ of $\tilde{A}^*$ *critical*. The following result can be found e.g. in [**3, 24**].

THEOREM 13 (EIGENSPACE). *Let $A \in (\mathbb{R}_{\max})^{n\times n}$ denote an irreducible matrix. The critical columns of $\tilde{A}^*$ span the eigenspace of A. If we select only one column, arbitrarily, per critical class, we obtain a weak basis of the eigenspace.*

Thus, the cardinality of a weak basis is equal to the number of critical classes. For any two i, j within the same critical class, the critical columns $\tilde{A}^*_{\cdot,i}$ and $\tilde{A}^*_{\cdot,j}$ are proportional.

We next show how the eigenvalue $\rho_{\max}(A)$ and the eigenvectors determine the asymptotic behavior of A^k as $k \to \infty$. The *cyclicity* of a critical class $\mathcal{C}_s^c$ is by definition the g.c.d. of the lengths of its circuits. The *cyclicity* c of A is the l.c.m. of the cyclicities of its critical classes. Let us pick arbitrarily an index i_s within each critical class $\mathcal{C}_s^c$, for $s = 1, \dots, r$, and let v_s, w_s denote the column and row of index i_s of $\tilde{A}^*$ (v_s, w_s are right and left eigenvectors of A, respectively). The following result follows from [**3**].

THEOREM 14 (CYCLICITY). *Let $A \in (\mathbb{R}_{\max})^{n\times n}$ be an irreducible matrix. There is an integer K_0 such that*

$$k \geq K_0 \implies A^{k+c} = \rho_{\max}(A)^c A^k \ , \tag{26}$$

where c is the cyclicity of A. Moreover, if $c = 1$,

$$k \geq K_0 \implies A^k = \rho_{\max}(A)^k P, \quad \text{where} \quad P = \bigoplus_{s=1}^{r} v_s w_s \ . \tag{27}$$

The matrix P which satisfies $P^2 = P$, $AP = PA = \rho_{\max}(A)P$ is called the *spectral projector* of A. The cyclicity theorem, which writes $A_{ij}^{k+c} = \rho_{\max}(A) \times c + A_{ij}^k$ in conventional algebra, implies that $A^k x$ grows as $k \times \rho_{\max}(A)$, independently of $x \in (\mathbb{R}_{\max})^n$, and that a periodic regime is attained in finite time. The limit behavior is known a priori. Ultimately, the sequence $\rho_{\max}(A)^{-k}A^k$ visits periodically c accumulation points, which are $Q, AQ, \dots, A^{c-1}Q$, where Q is the spectral projector of A^c. The length of the transient behavior K_0 can be arbitrarily large. In terms of Markov Decision, Theorem 14 says that optimal long trajectories stay almost all the time on the critical graph (Turnpike theorem). Theorem 14 is illustrated in Fig. 4, which shows the images of a cat (a region of the $\mathbb{R}^2$ plane) by the iterates of A (A, A^2, A^3, etc.), B and C, where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \ . \tag{28}$$

We have $\rho_{\max}(A) = 2$. Since A has a unique critical circuit, the spectral projector P is rank one (its column and row spaces are lines). We find that $\tilde{A}^2 = P$: every point of the plane is sent in at most two steps to the

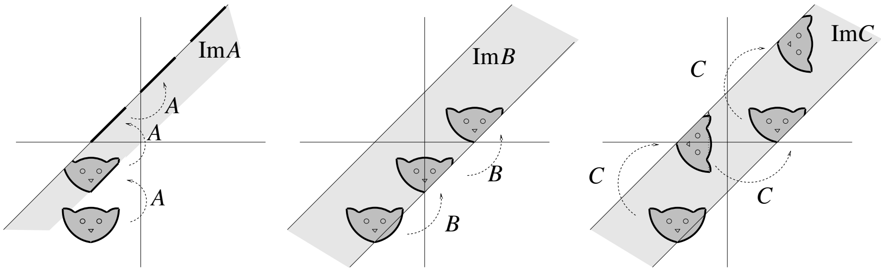


FIGURE 4. A cat in a (max, +) dynamics (see (28))

eigenline $y = 2 \otimes x = 2 + x$, then it is translated by $(2, 2)$ at each step. Similar interpretations exist for B and C.

Let us now consider a reducible matrix A. Given a class $\mathcal{C}$, we denote by $\rho_{\max}(\mathcal{C})$ the (max, +) eigenvalue of the restriction of the matrix A to $\mathcal{C}$. The *support* of a vector u is the set $\operatorname{supp} u = \{i \mid u_i \neq \mathbb{0}\}$. A set of nodes S is *closed* if $j \in S, i \xrightarrow{*} j$ implies $i \in S$. We say that a class $\mathcal{C} \subset S$ is *final in S* if there is no other downstream class in S.

THEOREM 15 (SPECTRUM OF REDUCIBLE MATRICES). *A matrix $A \in (\mathbb{R}_{\max})^{n\times n}$ has an eigenvector with support $S \subset \{1, \ldots , n\}$ and eigenvalue λ iff S is closed, λ is equal to $\rho_{\max}(\mathcal{C})$ for any class $\mathcal{C}$ that is final in S, and $\lambda \succeq \rho_{\max}(\mathcal{C}')$ for any other class $\mathcal{C}'$ in S.*

The proof can be found in [**82, 31**]. See also [**5**]. In particular, eigenvalues of initial classes are automatically eigenvalues of A. The maximal circuit mean $\rho_{\max}(A)$ (given by (23)) is also automatically an eigenvalue of A (but the associated eigenvector need not be finite). A weak basis of the eigenspace is given in [**31**, Chap. IV,1.3.4].

Example 16 (Taxicab eigenproblem). The matrix of the taxicab MDP, shown in Fig 1, has 2 classes, namely $\mathcal{C}_1 = \{c_1, a, c_2\}, \mathcal{C}_2 = \{c_3\}$. Since $\rho_{\max}(\mathcal{C}_2) = 2 \prec \rho_{\max}(\mathcal{C}_1) = 5$, there are no finite eigenvectors (which have support $S = \mathcal{C}_1 \cup \mathcal{C}_2$). The only other closed set is $S = \mathcal{C}_1$, which is initial. Thus $\rho_{\max}(A) = \rho_{\max}(\mathcal{C}_1) = 5$ is the only eigenvalue of A. Let A' denote the restriction of A to $\mathcal{C}_1$. There are two critical circuits (c_1) and (a, c_2), and thus two critical classes $\mathcal{C}^c_1 = \{c_1\}, \mathcal{C}^c_2 = \{a, c_2\}$. A weak basis of the eigenspace of A' is given by the columns c_1 and (e.g.) c_2 of

$$(\tilde{A}')^* = \begin{array}{c} \\ c_1 \\ a \\ c_2 \end{array} \begin{array}{c} \begin{array}{ccc} c_1 & a & c_2 \end{array} \\ \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \end{array}$$

Completing these two columns by a $\mathbb{0}$ in row 4, we obtain a basis of the eigenspace of A. The non existence of a finite eigenvector is obvious in terms of control. If such an eigenvector existed, by Fact 2, the optimal reward of the taxicab would be independent of the starting point. But, if the taxi driver starts from City 3, he remains blocked there with an income of 2 \$ per journey, whereas if he starts from any other node, he should clearly either run indefinitely in City 1, either shuttle from the airport to City 2, with an average income of 5 \$ per journey (these two policies can be obtained by applying Fact 2 to the MDP restricted to $\mathcal{C}_1$, taking the two above eigenvectors).

The following extension to the reducible case of the cyclicity theorem is worth being mentioned.

THEOREM 17 (CYCLICITY, REDUCIBLE CASE). *Let $A \in (\mathbb{R}_{\max})^{n\times n}$. There exist two integers K_0 and $c \geq 1$, and a family of scalars $\lambda_{ijl} \in \mathbb{R}_{\max}$, $1 \leq i, j \leq n$, $0 \leq l \leq c - 1$, such that*

$$k \geq K_0, \quad k \equiv l \mod c \implies A^{k+c}_{ij} = \lambda^c_{ijl} A^k_{ij} \ , \tag{29}$$

Characterizations exist for c and λ_{ijl}. The scalars λ_{ijl} are taken from the set of eigenvalues of the classes of A. If i, j belong to the same class $\mathcal{C}$, $\lambda_{ijl} = \rho_{\max}(\mathcal{C})$ for all l. If i, j do not belong to the same class, the theorem implies that the sequence $\frac{1}{k} \times A^k_{ij}$ may have distinct accumulation points, according to the congruence of k modulo c (see [**31**, Chap. VI,1.1.10]).

The cyclicity theorems for matrices are essentially equivalent to the characterization of rational series in one indeterminate with coefficient in $\mathbb{R}_{\max}$, as a merge of ultimately geometric series, see the paper of Gaubert in [**20**] and [**54**]. Transfer series and rational algebra techniques are particularly powerful for Discrete Event Systems. Timed Event Graphs can be represented by a remarkable (quotient) semiring of series with Boolean coefficients, in two commuting variables, called $\mathcal{M}^{\max}_{\min}[[\gamma, \delta]]$ (see [**3**, Chap. 5]). The indeterminates γ and δ have natural interpretations as *shifts* in dating and counting. The complete behavior of the system can be represented by simple —often small— commutative rational expressions [**3**],[**31**, Chap. VII–IX] (see also [**54**] in a more general context).

3.8. Symmetrization of the $(\max, +)$ **Semiring.** Let us try to imitate the familiar construction of $\mathbb{Z}$ from $\mathbb{N}$, for an arbitrary semiring $\mathcal{S}$. We build the set of couples $\mathcal{S}^2$, equipped with (componentwise) sum $(x', x'') \oplus (y', y'') = (x' \oplus y', x'' \oplus y'')$, and product $(x', x'') \otimes (y', y'') = (x'y' \oplus x''y'', x'y'' \oplus x''y')$. We introduce the *balance* relation

$$(x', x'')\nabla(y', y'') \iff x' \oplus y'' = x'' \oplus y' \ .$$

We have $\mathbb{Z} = \mathbb{N}^2/\nabla$, but for an idempotent semiring $\mathcal{S}$, the procedure stops, since ∇ is not transitive (e.g. $(\mathbb{1}, \mathbb{0})\nabla(\mathbb{1}, \mathbb{1})\nabla(\mathbb{0}, \mathbb{1})$, but $(\mathbb{1}, \mathbb{0}) \not\nabla(\mathbb{0}, \mathbb{1})$). If we renounce to quotient $\mathcal{S}^2$, we may still manipulate couples, with the $\ominus$ operation $\ominus(x', x'') = (x'', x')$. Indeed, since $\ominus$ satisfies the sign rules $\ominus\ominus x = x$, $\ominus(x \oplus y) = (\ominus x) \oplus (\ominus y)$, $\ominus(xy) = (\ominus x)y = x(\ominus y)$, and since $x\nabla y \iff x \ominus y\nabla\mathbb{0}$ (we set $x \ominus y \stackrel{\text{def}}{=} x \oplus (\ominus y)$), it is not difficult to see that *all the familiar identities valid in rings admit analogues in $\mathcal{S}^2$, replacing equalities by balances*. For instance, if $\mathcal{S}$ is commutative, we have for all matrices (of compatible size) with entries in $\mathcal{S}^2$ (determinants are defined as usual, with $\ominus$ instead of $-$):

$$\det(AB) \quad \nabla \quad \det A \ \det B, \tag{30}$$

$$P_A(A) \quad \nabla \quad \mathbb{0} \qquad \text{where } P_A(\lambda) = \det(A \ominus \lambda \mathrm{Id}) \ \ \text{(Cayley Hamilton)}. \tag{31}$$

Eqn 30 can be written directly in $\mathcal{S}$, introducing the positive and negative determinants $\det^+ A = \bigoplus_{\sigma \text{ even}} \bigotimes_{1\leq i\leq n} A_{i\sigma(i)}$, $\det^- A = \bigoplus_{\sigma \text{ odd}} \bigotimes_{1\leq i\leq n} A_{i\sigma(i)}$ (the sums are taken over even and odd permutations of $\{1, \dots, n\}$, respectively). The balance (30) is equivalent to the ordinary equality $\det^+ AB \oplus \det^+ A \ \det^- B \oplus \det^- A \ \det^+ B = \det^- AB \oplus \det^+ A \ \det^+ B \oplus \det^- A \ \det^- B$, but (30) is certainly more adapted to computations. Such identities can be proved combinatorially (showing a bijection between terms on both sides), or derived automatically from their ring analogues using a simple argument due to Reutenauer and Straubing [**68**, Proof of Lemma 2] (see also the *transfer principle* in [**31**, Ch. I]).

But in the $\mathbb{R}_{\max}$ case, one can do much better. Consider the following application of the Cayley-Hamilton theorem:

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}, \quad A^2 \ominus \mathrm{tr}\,(A)A \oplus \det A\nabla\mathbb{0}, \ \text{ i.e} \qquad A^2 \oplus 2\mathrm{Id} = 1A \oplus 7\mathrm{Id} \ .$$

Obviously, we may eliminate the 2Id term which will never saturate the identity (since $2 < 7$), and obtain $A^2 = 1A \oplus 7\mathrm{Id}$. Thus, to some extent $7 \ominus 2 = 7$. This can be formalized by introducing the congruence of semiring:

$$(x', x'')\ \mathcal{R}\ (y', y'') \Leftrightarrow (x' \neq x'', y' \neq y'' \text{ and } x' \oplus y'' = x'' \oplus y') \text{ or } (x', x'') = (y', y'').$$

The operations $\oplus, \ominus, \otimes$ and the relation ∇ are defined canonically on the quotient semiring, $\mathbb{S}_{\max} = \mathbb{R}^2_{\max}/\mathcal{R}$, which is called the *symmetrized semiring* of $\mathbb{R}_{\max}$. This symmetrization was invented independently by G. Hegedüs [**44**] and M. Plus [**65**].

In $\mathbb{S}_{\max}$, there are three kinds of equivalence classes; classes with an element of the form $(a, \mathbb{0})$, identified to $a \in \mathbb{R}_{\max}$, and called *positive*, classes with an element of the form $(\mathbb{0}, a)$ denoted $\ominus a$, called *negative*, classes with a single element (a, a), denoted $a^\bullet$ and called *balanced*, since $a^\bullet\nabla\mathbb{0}$ (for $a = \mathbb{0}$, the three above classes coincide, we will consider $\mathbb{0}$ as both a positive, negative, and balanced element).

We have the decomposition of $\mathbb{S}_{\max}$ in sets of positive, negative, and balanced elements, respectively

$$\mathbb{S}_{\max} = \mathbb{S}_{\max}^{\oplus} \cup \mathbb{S}_{\max}^{\ominus} \cup \mathbb{S}_{\max}^{\bullet} \ .$$

This should be compared with $\mathbb{Z} = \mathbb{Z}^+ \cup \mathbb{Z}^- \cup \{0\}$. For instance, $3 \ominus 2 = 3$, $2 \ominus 3 = \ominus 3$, but $3 \ominus 3 = 3^\bullet$. We say that an element is *signed* if it is positive or negative.

Obviously, if a system $Ax = b$ has a solution, the balance $Ax \nabla b$ has a solution. Conversely if $Ax \nabla b$ has a positive solution x, and if A, b are positive, it is not difficult to see that $Ax = b$. It remains to solve systems of linear balances. The main difficulty is that the balance relation is not transitive. As a result, $x \nabla a$ and $cx \nabla b$ do not imply $ca \nabla b$. However, when x is signed, the implication is true. This allows us to solve linear systems of balances by elimination, when the unknowns are signed.

THEOREM 18 (CRAMER FORMULÆ). *Let $A \in (\mathbb{S}_{\max})^{n\times n}$, and $b \in (\mathbb{S}_{\max})^n$. Every signed solution of $Ax \nabla b$ satisfies the Cramer condition $Dx_i \nabla D_i$, where D is the determinant of A and D_i is the i-th Cramer determinant[6]. Conversely, if D_i is signed for all i, and if D is signed and nonzero, then $x = (D^{-1} D_i)_{1 \le i \le n}$ is the unique signed solution.*

The proof can be found in [**65, 3**]. For the homogeneous system of n linear equations with n unknowns, $Ax \nabla \mathbb{0}$ has a signed non zero solution iff $\det A \nabla \mathbb{0}$ (see [**65, 31**]), which extends a result of Gondran and Minoux (see [**39**]).

Example 19. Let us solve the taxicab eigenproblem $Ax = 5x$ by elimination in $\mathbb{S}_{\max}$ (A is the matrix shown in Fig 1). We have

$$5^\bullet x_1 \oplus 4x_2 \oplus 7x_4 \quad \nabla \quad \mathbb{0} \tag{32a}$$

$$4x_1 \ominus 5x_2 \oplus 6x_3 \oplus 3x_4 \quad \nabla \quad \mathbb{0} \tag{32b}$$

$$4x_2 \ominus 5x_3 \quad \nabla \quad \mathbb{0} \tag{32c}$$

$$\ominus 5x_4 \quad \nabla \quad \mathbb{0} \ . \tag{32d}$$

The only signed solution of (32d) is $x_4 = \mathbb{0}$. By homogeneity, let us look for the solutions such that $x_3 = 0$. Then, using (32c), we get $4x_2 \nabla 5x_3 = 5$. Since we search a positive x_2, the balance can be replaced by an equality. Thus $x_2 = 1$. It remains to rewrite (32a),(32b): $5^\bullet x_1 \nabla \ominus 5$, $4x_1 \nabla 6^\bullet$, which is true for x_1 positive iff $0 \le x_1 \le 2$. The two extremal values give (up to a proportionality factor) the basis eigenvectors already computed in Ex. 19.

Determinants are not so easy to compute in $\mathbb{S}_{\max}$. Butkovič [**10**] showed that the computation of the determinant of a matrix with positive entries is polynomially equivalent (we have to solve an assignment problem) to the research of an even cycle in a (directed) graph, a problem which is not known to be polynomial. We do not know a non naive algorithm to compute the minor rank (=size of a maximal submatrix with unbalanced determinant) of a matrix in $(\mathbb{R}_{\max})^{n\times p}$. The situation is extremely strange: we have excellent polynomial iterative algorithms [**65, 31**] to find a signed solution of the square system $Ax \nabla b$ when $\det A \neq \mathbb{0}$, but we do not have polynomial algorithms to decide whether $Ax \nabla \mathbb{0}$ has a signed non zero solution (such algorithms would allow us to compute $\det A$ in polynomial time). Moreover, the theory partly collapses if one considers rectangular systems instead of square ones. The conditions of compatibility of $Ax \nabla \mathbb{0}$ when A is rectangular cannot be expressed in terms of determinants [**31**, Chap. III, 4.2.6].

Historical and Bibliographical Notes. The (max, +) algebra is not classical yet, but many researchers have worked on it (we counted at least 80), and it is difficult to make a short history without forgetting important references. We will just mention here main sources of inspiration. The first use of the (max, +) semiring can be traced back at least to the late fifties, and the theory grew in the sixties, with works of Cuninghame-Green, Vorobyev, Romanovskiĭ, and more generally of the Operations Research community (on path algebra). The first enterprise of systematic study of this algebra seems to be the seminal "Minimax algebra" of Cuninghame-Green [**23**]. A chapter on dioids can be found in Gondran et Minoux [**38**]. The

[6] Obtained by replacing the i-th column of A by b.

theory of linear independence using bideterminants, which is the ancester of symmetrization, was initiated by Gondran and Minoux (following Kuntzmann). See [**39**]. The last chapter of "Operatorial Methods" of Maslov [**61**] inaugurated the (max, $+$) operator and measure theory (motivated by semiclassical asymptotics). There is an "extremal algebra" tradition, mostly in East Europe, oriented towards algorithms and computational complexity. Results in this spirit can be found in the book of U. Zimmermann [**84**]. This tradition has been pursued, e.g. by Butkovič [**12**]. The *incline algebras* introduced by Cao, Kim and Roush [**13**] are idempotent semirings in which $a \oplus ab = a$. The tropical semiring was invented by Simon [**74**]. A number of language and semigroup oriented contributions are due to the tropical school (Simon, Hashiguchi, Mascle, Leung, Pin, Krob, Weber, . . .). See the survey of Pin in [**41**], [**76, 43, 57, 54, 52**], and the references therein. Since the beginning of the eighties, Discrete Event Systems, which were previously considered by distinct communities (queuing networks, scheduling, . . .), have been gathered into a common algebraic frame. "Synchronization and Linearity" by Baccelli, Cohen, Olsder, Quadrat [**3**] gives a comprehensive account of deterministic and stochastic (max,+) linear discrete event systems, together with recent algebraic results (such as symmetrization). Another recent text is the collection of articles edited by Maslov and Samborskiĭ [**60**] which is only the most visible part of the (considerable) work of the Idempotent Analysis school. A theory of probabilities in (max, $+$) algebra motivated by dynamic programming and large deviations, has been developed by Akian, Quadrat and Viot; and by Del Moral and Salut (see [**41**]). Recently, the (max, $+$) semiring has attracted attention from the linear algebra community (Bapat, Stanford, van den Driessche [**5**]). A survey with a very complete bibliography is the article of Maslov and Litvinov in [**41**]. Let us also mention the forthcoming book of Kolokoltsov and Maslov (an earlier version is in Russian [**59**]). The collection of articles edited by Gunawardena [**41**] will probably give the first fairly global overview of the different traditions on the subject.

CHAPTER 2

Exotic Semirings: Examples and General Results

1. Definitions and Zoology

A *semigroup* is a set equipped with an associative internal composition law. A *monoid* is a semigroup whose law has a neutral element. A semigroup is *commutative* if its law is commutative. A *semiring* is a set $\mathcal{S}$ equipped with two internal composition laws, called *addition* and *multiplication*, respectively, that satisfy the following axioms:

i. $\mathcal{S}$ is a commutative monoid for addition;
ii. $\mathcal{S}$ is a monoid for multiplication;
iii. multiplication distributes over addition;
iv. the neutral element for addition is absorbing for multiplication.

We will note[1] $(x, y) \mapsto x \oplus y$ the addition, $(x, y) \mapsto x \otimes y$ the multiplication, $\mathbb{0}$ the neutral element for addition, which is called *zero*, and $\mathbb{1}$ the neutral element for multiplication, which is called *unit*. A semiring is *commutative* if multiplication is commutative. A *semifield* is a semiring in which each non-zero element has an inverse for multiplication. A semiring is *idempotent* if addition is idempotent, i.e. if the identity $a \oplus a = a$ holds. Idempotent semirings are called *dioids* in [**3**]. A semiring is *zero-sum-free* if $a \oplus b = \mathbb{0} \implies a = b = \mathbb{0}$ (idempotent semirings are zero-sum-free, see Prop. 61 below). A semiring is *cancellative* if $a \oplus b = c \oplus b \implies a = c$. Of course, *rings* and *fields*, in which each element has an *opposite* (inverse for addition) are cancellative. For cancellative semirings, axiom *iv* is implied by axioms *i* and *iii*, as it can be shown easily[2].

A *morphism* from a semiring $\mathcal{S}$ to a semiring $\mathcal{S}'$ is a map $f : \mathcal{S} \to \mathcal{S}'$, which is both a morphism for the underlying additive monoid (i.e. $f(\mathbb{0}) = \mathbb{0}$, $f(a \oplus b) = f(a) \oplus f(b)$) and a morphism for the underlying multiplicative monoid (i.e. $f(\mathbb{1}) = \mathbb{1}$ and $f(a \otimes b) = f(a) \otimes f(b)$). When f is bijective, f is an *isomorphism*. We say that $\mathcal{S}$ is a *subsemiring* of $\mathcal{S}'$ if $\mathcal{S} \subset \mathcal{S}'$, if $\mathcal{S}$ contains the zero and the unit of $\mathcal{S}'$, and if $\mathcal{S}$ is stable by addition and multiplication. A subsemiring of a semifield, that is a semifield, is a *subsemifield*.

We will denote a semiring by a triple $(\mathcal{S}, \oplus, \otimes)$, since the elements $\mathbb{0}$ and $\mathbb{1}$ are uniquely defined from $(\mathcal{S}, \oplus, \otimes)$. We next tabulate some classical and less classical examples of semirings.

Example 20 ($(\mathbb{R}_+, +, \times)$ semifield). The set of real non-negative numbers, $\mathbb{R}_+$, equipped with the usual addition and multiplication, is a cancellative zero-sum-free semifield, in which $a \oplus b = a + b, a \otimes b = a \times b, \mathbb{0} = 0, \mathbb{1} = 1$.

[1] It may seem more natural to use the familiar symbols $+$ and $\times$ instead of $\oplus$ and $\otimes$, which usually denote direct sum and tensor product, respectively. Indeed, from the semiring theory point of view, using $+$, $\times$ for general semirings (and, for consistence, 0 and 1 for the zero and the unit) is the ideal choice, which is done e.g. in the survey of Gunawardena in [**41**]. But in some "applications" (e.g. to Discrete Event Systems, but also to other mathematical fields, such as optimal control) it is necessary to set $\oplus = \max$ and $\otimes = +$, where $+$ denotes here the conventional addition on a suitably completed set of real numbers (see. Ex. 23). In this context, adopting the generic notation $+$ and $\times$ for semiring addition and multiplication requires writing $+ = \max$, $\times = +$, from which we can apparently deduce that $\times = + = \max$, which has some antisocial consequences.

[2] Indeed, we have $\mathbb{0} \oplus a \otimes \mathbb{0} = a \otimes \mathbb{0} = a \otimes (\mathbb{0} \oplus \mathbb{0}) = a \otimes \mathbb{0} \oplus a \otimes \mathbb{0}$, and if $\mathcal{S}$ is cancellative, $\mathbb{0} \oplus a \otimes \mathbb{0} = a \otimes \mathbb{0} \oplus a \otimes \mathbb{0} \implies a \otimes \mathbb{0} = \mathbb{0}$, which shows that $\mathbb{0}$ is right absorbing. A dual argument shows that it is left absorbing.

Example 21 (max-times semifield). The set $\mathbb{R}_+$, equipped with the laws $a \oplus b = \max(a, b)$ and $a \otimes b = a \times b$ is a semifield, which is idempotent, for $\max(a, a) = a$. This structure is sometimes called *max-times semifield*.

Example 22 (Maslov-Pap semifields). Consider, for all positive real number p, the law $+_p$,

$$a +_p b = (a^p + b^p)^{\frac{1}{p}} \quad . \tag{33}$$

Then, $\mathbb{R}_{+_p} \stackrel{\text{def}}{=} (\mathbb{R}_+, +_p, \times)$ is a semifield, which is isomorphic to $(\mathbb{R}_+, +, \times)$, via the map $f(x) = x^p$. The max-times semiring $(\mathbb{R}_+, \max, \times)$ is a limit case of $\mathbb{R}_{+_p}$, for

$$\lim_{p \to +\infty} a +_p b = \max(a, b) \quad .$$

Example 23 (max-plus semifield). The set $\mathbb{R} \cup \{-\infty\}$, equipped with $\oplus = \max$, and $\otimes = +$ is an idempotent semifield, that we will denote by $\mathbb{R}_{\max}$. This semifield is called *max-plus semifield* or *max-plus semiring*, or, according to a different tradition, [**25**], *max algebra*, or *schedule algebra*. In $\mathbb{R}_{\max}$, $\mathbb{0} = -\infty$, $\mathbb{1} = 0$. The map $f(x) = e^x$ is an isomorphism from the max-plus semifield $\mathbb{R}_{\max}$ to the max-times semifield $(\mathbb{R}_+, \max, \times)$. The max-plus semifield has many interesting subsemirings. In particular $(\mathbb{Z} \cup \{-\infty\}, \max, +)$, $(\mathbb{Q} \cup \{-\infty\}, \max, +)$, $(\mathbb{N} \cup \{-\infty\}, \max, +)$, $((-\mathbb{N}) \cup \{-\infty\}, \max, +)$, are subsemirings of $\mathbb{R}_{\max}$, that we will denote by $\mathbb{Z}_{\max}$, $\mathbb{Q}_{\max}$, $\mathbb{N}_{\max}$, $(-\mathbb{N})_{\max}$, respectively. Of these four semirings, only $\mathbb{Q}_{\max}$ and $\mathbb{Z}_{\max}$ are semifields. More generally, each additive submonoid (resp. subgroup) M of the additive group $(\mathbb{R}, +)$ yields a subsemiring (resp. subsemifield) $(M \cup \{-\infty\}, \max, +)$ of $\mathbb{R}_{\max}$, that we will denote by $M_{\max}$.

Example 24 (min-plus semifield and tropical semirings). The *min-plus semifield* is the set $\mathbb{R} \cup \{+\infty\}$, equipped with $\oplus = \min$, and $\otimes = +$. It is denoted $\mathbb{R}_{\min}$. The map $x \mapsto -x$ is an isomorphism between $\mathbb{R}_{\min}$ and $\mathbb{R}_{\max}$. The subsemirings $\mathbb{Z}_{\min}$, $\mathbb{Q}_{\min}$, etc. are defined as in Ex. 23, mutatis mutandis. Particularly, the semiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +)$ is famous in the literature under the name of *tropical semiring* (see [**74, 75, 57, 54, 52, 53, 76**], and particularly the survey of Pin [**64**] and the presentation of Simon in these notes).

Let us recall that a *sup-semilattice* is a set E equipped with an order relation $\leq$, such that for all a, b, the set $\{a, b\}$ has a *least upper bound*[3], which is denoted $a \vee b$. An inf-semilattice is defined dually, by requiring that the set $\{a, b\}$ has a *greatest lower bound*, which is denoted $a \wedge b$. When E is both a sup and an inf-semilattice, it is a *lattice*.

Example 25 (Sup-semilattice ordered monoids are idempotent semirings). Consider a *sup-semilattice ordered* monoid M, i.e. a sup-semilattice M for an order relation $\leq$, equipped with a monoid law $\otimes$ which distributes over $\vee$. Let us complete M by an element $\perp$ (read "bottom"), that satisfies $\perp \leq a$ and $\perp \otimes a = a \otimes \perp = \perp$, for all $a \in M$. Then, $M_\vee \stackrel{\text{def}}{=} (M \cup \{\perp\}, \vee, \otimes)$ is an (idempotent) semiring.

Example 26 ($(\mathbb{N} \setminus \{0\})_{\mathrm{lcm}}$ semiring). The set $\mathbb{N} \setminus \{0\}$ of strictly positive integers, equipped with the order $a \leq b$ iff a divides b, is a sup-semilattice ordered monoid. In this case, $a \vee b = \mathrm{lcm}(a, b)$, and the semiring $(\mathbb{N} \setminus \{0\})_{\mathrm{lcm}}$ built in Ex. 25 can be seen as an "infinite dimensional" version of the max-plus semiring. Indeed, if we represent a positive integer by the product of powers of primes numbers $2^{\alpha_2} 3^{\alpha_3} 5^{\alpha_5} \ldots$, where α_p is an *almost zero*[4] sequence of non-negative integers, indexed by prime numbers p, then, the operation $\oplus = \vee$ corresponds to the pointwise max on sequences, and $\otimes$ to the pointwise addition of sequences.

[3]By definition, the least upper bound of a set X is the least element of the set $\{y \mid \forall x \in X,\ x \leq y\}$, if it exists (recall that a *least element* of a set Z is an element $z \in Z$ such that $z \leq w$, for all $w \in Z$; this notion should not be confused with that of *minimal element* of Z, which is an element $z \in Z$ such that $w \in Z$ and $w \leq z$ imply $w = z$; of course, a least element is minimal). The notions of greatest element and greatest lower bound are defined dually (by replacing $\leq$ by $\geq$).

[4] Recall that a sequence $\{y_i\}_{i \in I}$ is *almost zero* if the subset of $i \in I$ such that y_i is non-zero is finite. The definition of *almost zero functions* is identical (the notation y_i is just replaced by $y(i)$).

Example 27 (Lattice ordered groups are idempotent semifields). Take M, $\otimes$ and $\leq$ as in Ex. 25, and assume that M is a group for $\otimes$. Then, $M_\vee = (M \cup \{\bot\}, \vee, \otimes)$ is an idempotent semifield. Moreover, M is automatically an inf-semilattice, with $a \wedge b = (a^{-1} \vee b^{-1})^{-1}$, and multiplication also distributes with respect to $\wedge$. Hence, completing M by a greatest element $\top$ (for "top") which satisfies $\top \geq a$ and $\top \otimes a = a \otimes \top = \top$, for all $a \in M$, we obtain a dual idempotent semifield $M_\wedge \stackrel{\text{def}}{=} (M \cup \{\top\}, \wedge, \otimes)$, and the map $x \mapsto x^{-1}$ for $x \in M$, $\bot \mapsto \top$, is an isomorphism from $M_\vee$ to $M_\wedge$. Clearly, the max-plus and min-plus semifields, $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$, respectively, are obtained by applying this construction to the additive group or reals numbers, equipped with the usual order relation.

Example 28 (Bottleneck (or fuzzy) semirings, and distributive lattices). The *bottleneck semiring* is the set $[0, 1]$, equipped with $a \oplus b = \max(a, b)$ and $a \otimes b = \min(a, b)$. More generally, any distributive lattice[5] with a least element $\bot$ and a greatest element $\top$, equipped with the two laws $a \oplus b = a \vee b$, and $a \otimes b = a \wedge b$, is a commutative idempotent semiring, in which $\mathbb{0} = \bot$ and $\mathbb{1} = \top$.

Example 29 (Boolean semifield). The Boolean semifield $\mathbb{B} = \{\mathbb{0}, \mathbb{1}\}$ is a subsemiring of any idempotent semiring such that $\mathbb{0} \neq \mathbb{1}$ (see Exercise 39).

Example 30 (Semirings of matrices). If $(\mathcal{S}, \oplus, \otimes)$ is a semiring, then, for all positive integer n, the set of $n \times n$ matrices, denoted $\mathcal{S}^{n\times n}$, is a semiring for the laws $\oplus$ and $\otimes$ defined by $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$, $(A \otimes B)_{ij} = \bigoplus_{1\leq k\leq n} A_{ik} \otimes B_{kj}$, for all $A, B \in \mathcal{S}^{n\times n}$ and $1 \leq i, j \leq n$. The zero element of $\mathcal{S}^{n\times n}$ is the matrix whose entries are all zero, that we still denote by $\mathbb{0}$. The unit element is the *identity matrix*, with $\mathbb{1}$ on the diagonal and $\mathbb{0}$ elsewhere.

Example 31 (Semirings of Homomorphisms). If $\mathcal{X}$ is a commutative monoid for the law $\oplus$, we denote by $\mathrm{Hom}(\mathcal{X})$ the set of monoid morphisms $\mathcal{X} \to \mathcal{X}$. For all $f, g \in \mathrm{Hom}(\mathcal{X})$, we define $(f \oplus g)(a) = f(a) \oplus g(a)$, $(f \otimes g)(a) = f(g(a))$. The set $\mathrm{Hom}(\mathcal{X})$ is a semiring for these two laws, with zero element $a \mapsto \mathbb{0}$, and unit element $a \mapsto a$.

Example 32 (Semirings of Subsets of Monoids). Let M denote a monoid with law $\otimes$. We equip the set of subsets of M, $\mathcal{P}(M)$, with the law $A \otimes B = \{a \otimes b \mid a \in A,\ b \in B\}$. Then, $(\mathcal{P}(M), \cup, \otimes)$ is an idempotent semiring (with zero element $\emptyset$, and unit element $\{\mathbb{1}\}$), which is commutative iff M is commutative. The set $\mathcal{P}_f(M)$ of *finite* subsets of M is a subsemiring of $\mathcal{P}(M)$. The case when $M = \Sigma^*$ is the free monoid[6] over an alphabet Σ is particularly interesting. Then, it is not difficult to see that $\mathcal{P}_f(\Sigma^*)$ is the *free idempotent semiring* on Σ, i.e., that for all idempotent semirings $\mathcal{S}$, and for all maps $g : \Sigma \to \mathcal{S}$, there exists a unique semiring morphism $\tilde{g} : \mathcal{P}_f(\Sigma^*) \to \mathcal{S}$, such that $\tilde{g}(a) = g(a)$, for all a in Σ (when writing $\tilde{g}(a)$, we identify a to $\{a\}$). Similarly, $\mathcal{P}_f(\mathbb{N}^k)$ is the *free commutative idempotent semiring* on k generators[7]. Another remarkable example is $\mathcal{P}(\mathbb{R}^k)$, the set of subsets of the k-dimensional space, $\mathbb{R}^k$ being equipped with its usual additive structure. This idempotent semiring is used in particular in mathematical morphology [**73**].

Example 33 (Cones and Pareto Sets). Consider any subset $\mathcal{C}$ of $\mathbb{R}^k$ such that $\mathcal{C} + \mathcal{C} = \mathcal{C}$. Then, the set $\mathcal{P}(\mathbb{R}^k) + \mathcal{C}$ of subsets of $\mathbb{R}^k$ of the form $X + \mathcal{C}$, with $X \subset \mathbb{R}^k$, equipped with $\cup$ and $+$, is a semiring, with zero element $\emptyset$ and unit element $\mathcal{C}$. Such semirings are useful in multicriteria optimization, see [**72**], since, when $\mathcal{C} = \mathbb{R}^k_+$, the set of extremal points of $X + \mathcal{C}$ is a (possibly empty) *Pareto* set (i.e. a set whose elements are pairwise incomparable), which represents typically a Pareto-optimal value for a multicriteria optimization problem. The $\mathcal{M}^{\text{in}}_{\text{ax}}[[\gamma, \delta]]$ semiring, which is studied in [**19, 3**] (see the second presentation of Cohen in these notes), is a discrete variant of this construction, since it is isomorphic to $\mathcal{P}(\mathbb{Z}^2) + \mathbb{N} \times (-\mathbb{N})$.

Example 34 (Monoid semialgebras). Let M denote a monoid, whose law is denoted by concatenation. Let $\mathcal{S}$ denote a semiring. The $\mathcal{S}$*-semialgebra* of the monoid M is the set of *almost zero*[4] maps $s : M \to \mathcal{S}$, $m \mapsto s_m$. We denote by $\mathcal{S}[M]$ the set of these maps. We write a map of $\mathcal{S}[M]$ as a formal sum $s = \bigoplus_{m\in M} s_m m$.

[5] A lattice is distributive if the two equivalent identities hold: $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$, $(a \wedge b) \vee c = (a \wedge c) \vee (b \wedge c)$.

[6] I.e. the set of finite possibly empty words with letters in Σ, equipped with concatenation.

[7] Which is such that for all commutative idempotent semirings $\mathcal{S}$ and for all maps $g : \{e_1, \dots, e_k\} \to \mathcal{S}$ (e_i is the i-th basis vector of $\mathbb{N}^k$), there is a unique semiring morphism $\tilde{g} : \mathcal{P}_f(\mathbb{N}^k) \to \mathcal{S}$, such that $\tilde{g}(e_i) = g(e_i)$, for all $i = 1, \dots, k$.

We equip $\mathcal{S}[M]$ with the laws $(s \oplus s')_m = s_m \oplus s'_m$, and[8] $(s \otimes s')_m = \bigoplus_{uu'=m} s_u \otimes s'_{u'}$. These two laws, which are determined by the rule $am \otimes bm' = (a \otimes b)mm'$, with $a, b \in \mathcal{S}$, $m, m' \in M$, equip $\mathcal{S}[M]$ with a semiring structure. By taking $\mathcal{S} = \mathbb{B}$, we re-obtain the semiring $\mathcal{P}_f(M)$. Indeed, the *support* map, $\mathbb{B}[M] \to \mathcal{P}_f(M)$, $\operatorname{supp} s = \{m \in M \mid s_m \neq \mathbb{0}\}$ is an isomorphism of semirings.

Example 35 (Formal Polynomials vs. Polynomial Functions). When $M = \mathbb{N}^k$ (equipped with the usual addition), $\mathcal{S}[M]$ coincides with $\mathcal{S}[X_1, \dots, X_k]$, the semiring of formal polynomials with coefficients in $\mathcal{S}$ in k commuting indeterminates $X_1, \dots, X_k$. With a formal polynomial $P \in \mathcal{S}[X_1, \dots, X_k]$, $P = \bigoplus_{i_1,\dots,i_k \in \mathbb{N}} P_{i_1 \dots i_k} X_1^{i_1} \dots X_k^{i_k}$, we associate a *polynomial function*, $\hat{P} : \mathcal{S}^k \to \mathcal{S}$: $\hat{P}(x_1, \dots, x_k)$ is obtained by replacing X_i by x_i, for all $i = 1, \dots, k$, and by interpreting the formal sum and products in the expression of P as the addition and multiplication in $\mathcal{S}$. If $\mathcal{S}$ is commutative, the evaluation map $P \mapsto \hat{P}$ is a morphism of semirings. As it is well known, this morphism is bijective if $\mathcal{S}$ is an infinite field. This is not the case in general semirings. E.g., in $\mathbb{R}_{\max}[X]$ ($X = X_1$ is the only indeterminate), the two formal polynomials $P_1 = \mathbb{1} \oplus X \oplus X^2$, $P_2 = \mathbb{1} \oplus X^2$ are distinct, by definition. However, $\hat{P}_1(x) = \max(0, x, 2 \times x) = \max(0, 2 \times x) = \hat{P}_2(x)$, for all $x \in \mathbb{R} \cup \{-\infty\}$, which shows that $\hat{P}_1 = \hat{P}_2$. More generally, two formal polynomials in $\mathbb{R}_{\max}[X]$ yield the same polynomial function iff their coefficient sequences have the same concave hull, see [**3**, § 3.3.1.2]. The "fundamental theorem of max-plus algebra", due to Cuninghame-Green and Meijer [**26**], states that, given a polynomial function $\hat{P}$ of degree n over the max-plus semiring, there exists a unique $a \in \mathbb{R}_{\max}$ and a family $\{x_1, \dots, x_n\} \subset \mathbb{R}_{\max}$, unique up to a reordering, such that for all $x \in \mathbb{R}_{\max}$, $\hat{P}(x) = a(x \oplus x_1) \dots (x \oplus x_n)$. The x_i are called the *corners* of P.

Example 36 (Formal Series). If the monoid M is such that an element m has only a finite number of factorizations $m = uv$, we can extend the operations of Ex. 34 to the case of general maps $s : M \to \mathcal{S}$. We denote by $\mathcal{S}[[M]]$ this new semiring. When $M = \Sigma^*$ is the free monoid over an alphabet Σ, $\mathcal{S}[[M]]$ is the semiring of non-commutative formal series with coefficient in $\mathcal{S}$, which is classically denoted by $\mathcal{S}\langle\langle\Sigma\rangle\rangle$. When $M = \mathbb{N}^k$, equipped with the usual addition, we obtain the semiring of formal series in k commuting indeterminates, with coefficients in $\mathcal{S}$, which is usually denoted by $\mathcal{S}[[X_1, \dots, X_k]]$.

Example 37 (Semiring of max-jets*).* We denote by $\mathbb{J}_{\max}$ the semifield of couples of strictly positive real numbers, completed by a (zero) element 0, equipped with the laws:

$$(a, b) \oplus (a', b') = \begin{cases} (a + a', b) & \text{if } b = b', \\ (a, b) & \text{if } b > b', \\ (a', b') & \text{if } b < b', \end{cases} \qquad (a, b) \otimes (a', b') = (aa', bb') \ .$$

Intuitively, (a, b) codes an asymptotic expansion of the form $ab^p + o(b^p)$, when $p \to +\infty$. This zero-sum-free semifield of "max jets" can be seen as a common "extension" of the usual semifield of real nonnegative numbers, $(\mathbb{R}_+, +, \times)$ and of the max-times semiring. Indeed, on the one hand, the map $0 \mapsto 0$, $x \neq 0 \mapsto (x, 1)$, is an *injective* morphism from $(\mathbb{R}_+, +, \times)$ to $\mathbb{J}_{\max}$. On the other hand, the map $(a, b) \mapsto b$, $0 \mapsto 0$, is a *surjective* morphism from $\mathbb{J}_{\max}$ to $(\mathbb{R}_+, \max, \times)$. This semiring was apparently first introduced in [**30**], to compute asymptotic expansions of partition functions. It is used extensively in [**2**] to obtain large deviation type results, à la Wentzell-Freidlin, for the spectral elements of matrices with entries of the from $\exp(-A_{ij}/\epsilon)$ where ϵ is a small positive parameter.

The above examples should make it clear that essentially all the classical algebraic notions and constructions involving rings (polynomials, formal and numerical series, matrices, modules, direct and tensor products, etc.) still can be piled up in the context of general semirings (sometimes with some precautions). For this reason, we will use in the sequel the familiar notions and notations, in the context of general semirings, *without further notice*, unless a specific difficulty arises. E.g., if a and b are elements of a semiring $\mathcal{S}$, we will write ab instead of $a \otimes b$; a^k for $a \otimes a \otimes \cdots \otimes a$ (k-times);

[8] This sum is well defined since $s_u \otimes s'_{u'}$ is an almost zero function of (u, u').

if A and B are $n \times p$ and $p \times q$ matrices with entries in a semiring $\mathcal{S}$, $A \otimes B$ or AB will denote the matrix product of A and B, namely, the $n \times q$ matrix such that $(AB)_{ij} = \bigoplus_{1\le k\le p} A_{ik}B_{kj}$, etc.

The interpretation of the symbols should, in general, be clear from the context. E.g. if we write "$a, b \in \mathbb{R}_{\max}$" and, without specific comments, "$x = ab^2$", the value of x is $a + 2 \times b$ (with the usual notation). Indeed, by writing $a, b \in \mathbb{R}_{\max}$, we mean, strictly speaking, that a, b belongs to the *underlying set* of $\mathbb{R}_{\max}$, namely, $\mathbb{R} \cup \{-\infty\}$, but we also give informally a *type* indication (that the operations to be performed on a, b are, by default, in the max-plus semiring $\mathbb{R}_{\max}$). Similarly, if we write, later on, "$k \in \mathbb{N}$", and "$y = a^{\frac{1}{k}}$", the operation $\frac{1}{k}$ should be understood in the usual algebra (since we wrote $k \in \mathbb{N}$ and not e.g. $k \in \mathbb{N}_{\max}$). Hence y is the unique $z \in \mathbb{R}_{\max}$ such that $z^k = y$. With the usual notation, $z = y/k$.

Exercise 38. Compute the following expressions in the semiring $\mathbb{R}_{\max}$: $1 \otimes 3 =?$, $3^2 =?$, $\sqrt[3]{12} =?$, $2/3 =?$, $\sqrt{-1} =?$, $\left(\begin{smallmatrix} 0 & 0 \\ 3 & 2 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right) =?$ Do the identities $a^k a^l = a^{k+l}$, $(a^k)^l = a^{kl}$ hold for $a \in \mathbb{R}_{\max}$ and $k, l \in \mathbb{Q}$?.

Exercise 39. We will say that a semiring is *trivial* if $\mathbb{0} = \mathbb{1}$. Find all the trivial semirings.

Exercise 40. Show that a semiring $\mathcal{S}$ that is both cancellative and idempotent has a single element.

Exercise 41. This exercise is due to G. Duchamp. 1. Show that for all positive integers k, $\mathbb{R}$ is a field for the addition $+_{2k+1}$ (defined in Ex. 22) and for the usual multiplication. Show that this field is isomorphic to the usual field of real numbers. 2. Show that for all $a, b \in \mathbb{R}$, the pointwise limit law $+_\infty$, defined by $a +_\infty b \stackrel{\text{def}}{=} \lim_{k\to\infty} a +_{2k+1} b$ exists, and characterize it simply. 3. Show that the set $\mathbb{R}$, equipped with the two laws $+_\infty$ and $\times$, satisfies all the structure axioms of fields, except one. Which one?

Exercise 42 (Binomial formula). Show that if a, b are elements of an idempotent commutative semiring, then, for all integers k:

$$(a \oplus b)^k = a^k \oplus a^{k-1}b \oplus \cdots \oplus ab^{k-1} \oplus b^k \quad . \tag{34}$$

Exercise 43. Assume that $\mathcal{S}$ is a commutative idempotent semiring, that is *entire*, i.e. $ab = ac \implies b = c$ or $a = \mathbb{0}$. Show that for all $a, b \in \mathcal{S}$ and for all integers k:

$$(a \oplus b)^k = a^k \oplus b^k$$

(not easy without the hint at the end of the chapter).

Exercise 44. A semiring $\mathcal{S}$ is *radicable* if for all integers $p \ge 1$ and for all $b \in \mathcal{S}$, there is at least one x such that $x^p = b$, and we call x a p-th root of b. Show that if $\mathcal{S}$ is a radicable commutative idempotent semimield, b has a unique p-th root. Find a (noncommutative) radicable idempotent semifield in which the p-th root is not unique.

Exercise 45. Let k denote a positive integer. 1. Show that the set of compact convex subsets of $\mathbb{R}^k$, equipped with the laws $A \oplus B = \operatorname{conv}(A \cup B)$ (the convex hull of A and B, i.e. the intersection of all the convex supersets of A and B), and $A \otimes B = A + B$ is a semiring. 2. Is this semiring a subsemiring of $(\mathcal{P}(\mathbb{R}^k), \cup, +)$? 3. Show that this semiring is *entire*, i.e. that for all compact convex subsets A, B, C of $\mathbb{R}^k$, $A + B = A + C \implies B = C$ or $A = \emptyset$ (the last question is not easy without the first of the three successive hints at the end of the chapter).

Exercise 46. Let $\mathfrak{S}_n$ denote the group of permutations of $\{1, \ldots, n\}$. For all $i = 1, \ldots, n-1$, let τ_i denote the transposition of indices i and $i + 1$. Let $s = 1\tau_1 \oplus \cdots \oplus 1\tau_{n-1} \in \mathbb{N}_{\min}[\mathfrak{S}_n]$. Show that the sequence $s^{(k)} = s^0 \oplus \cdots \oplus s^k$ is constant, for k large enough. Compute the limit value, $s^{(\infty)}$, in the particular case when $n = 2$. In general, compute the support of the limit $s^{(\infty)}$, together with $\max_{g\in\mathfrak{S}_n} s_g^{(\infty)}$, and interprete these results.

2. Combinatorial Formulæ in Semirings

2.1. Matrices and Graphs. Let $\mathcal{S}$ denote a semiring. With an $n \times n$ matrix A with entries in $\mathcal{S}$, we associate a (directed) graph, with set of vertices $\mathcal{V} = \{1, \ldots, n\}$ and set of edges $\mathcal{E} = \{(i, j) \mid A_{ij} \neq \mathbb{0}\}$ (i.e. there is an arrow from i to j iff $A_{ij} \neq \mathbb{0}$).

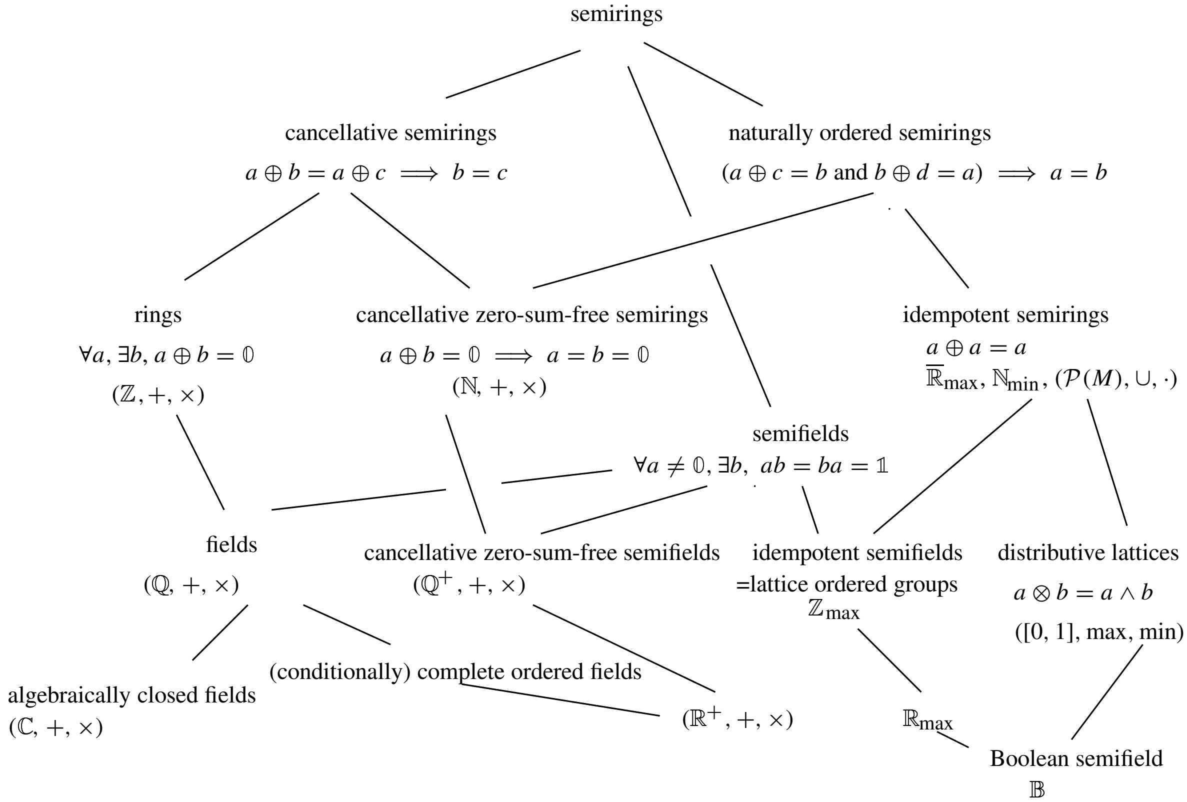


FIGURE 1. Hierarchy of Semirings

A *path* of *length* k from i to j is a finite sequence $p = (i_1, \dots, i_{k+1})$, such that $i_1 = i, i_{k+1} = j$, and $(i_1, i_2), \dots, (i_k, i_{k+1}) \in \mathcal{E}$. We say that p is a *circuit* if $i_{k+1} = i_1$. We *weight* of a path $p = (i_1, \dots, i_{k+1})$, is by definition $w(p) = A_{i_1 i_2} \dots A_{i_k i_{k+1}}$. If p is not a path, i.e. if $(i_l, i_{l+1}) \notin \mathcal{E}$ for some $1 \leq l \leq k$, we still define $w(p)$ by this formula. Then, $w(p) = \mathbb{0}$.

In the Discrete Event Systems literature, e.g. in [**3**] it is traditional to associate with a matrix the *opposite* digraph (in which arrows are reversed), whereas the direct orientation that we use here is traditional in Perron-Frobenius theory and Language theory.

The most important result of this chapter is also the simplest and probably oldest one: it just gives a graph interpretation to matrix product. This appealing observation should probably be incorporated to any even very elementary linear algebra course, for a number of matrix results have interesting or natural graph interpretations, and conversely.

THEOREM 47 (GRAPH INTERPRETATION OF MATRIX PRODUCT). *Let A denote a $n \times n$ matrix with entries in a semiring $\mathcal{S}$. For all $i, j \in \{1, \dots, n\}$, for all $k \in \mathbb{N}$, $(A^k)_{ij}$ is the sum of the weights of paths of length k from i to j.*

PROOF. The result is true by convention when $k = 0$. Indeed, if $i = j$, $(A^0)_{ij} = (A^0)_{ii} = \mathbb{1}$ and there is a unique path of length 0 from i to i, which has weight $\mathbb{1}$; and if $i \neq j$, $(A^0)_{ij} = \mathbb{0}$, there is no path of length zero from i to j (the sum of an empty family is $\mathbb{0}$). For $k = 1$, the result follows trivially from the definition of the graph of A. For $k = 2$, the results is just the matrix product definition: $(A^2)_{ij} = \bigoplus_{1 \leq l \leq n} A_{il} A_{lj} = \bigoplus_{1 \leq l \leq n} w(i, l, j)$, together with the observation that $w(i, l, j) = \mathbb{0}$ if (i, l, j) is not a path. The proof for $k > 2$ is similar. $\square$

In particular, a basic graph notion, *strong connectivity*, is also important for matrices. We say that i *has access* to j, and we write $i \xrightarrow{*} j$ if there is a path from i to j in the graph of A. We call *classes* of A the equivalence classes for the relation on $\{1,\dots,n\}$: $i\mathcal{R}j \iff (i \xrightarrow{*} j)$ and $(j \xrightarrow{*} i)$ (with the graph vocabulary, the classes are the *strongly connected components* of the graph of A). A matrix is *irreducible* iff it has a single class. Otherwise, it is reducible. In particular, a 1×1 matrix (even zero) is always irreducible. The proof of the following easy characterization is left to the reader.

PROPOSITION 48. *A $n\times n$ matrix A with entries in a semiring $\mathcal{S}$ is* reducible *iff there is permutation*[9] *matrix P and two strictly positive integers p,q, such that $n=p+q$ and*

$$A = P\begin{pmatrix} B & C\\ \mathbb{0} & D\end{pmatrix}P^{-1} \quad , \tag{35}$$

where B, C and D are matrices with entries in $\mathcal{S}$, of sizes $p\times p$, $p\times q$ and $q\times q$, respectively.

More generally, a reducible matrix A can be written as $A = PBP^{-1}$, where P is a permutation matrix, and B is a block triangular matrix, whose diagonal blocs are irreducible.

The following easy exercise shows that the behaviors of the powers of a reducible matrix is controlled by the behavior of powers of smaller matrices.

Exercise 49. Take A as in (35). Show that for all strictly positive integers k,

$$A^k = P\begin{pmatrix} B^k & \bigoplus_{i+j=k-1} B^iCD^j\\ \mathbb{0} & D^k\end{pmatrix}P^{-1} \quad ,$$

Exercise 50. Show that a strictly triangular square matrix A with entries in a semiring $\mathcal{S}$ is nilpotent (i.e. $A^k=\mathbb{0}$, for some k). When $\mathcal{S}$ is *zero-sum-free*, and has *no divisors of zero* (i.e. $ab=\mathbb{0} \implies a=\mathbb{0}$ or $b=\mathbb{0}$), show that A is nilpotent iff its graph has no circuits. Characterize graphically the index of nilpotency of A (least k such that $A^k=\mathbb{0}$).

Exercise 51 (Another Characterization of Irreducible Matrices). When $\mathcal{S}$ is zero-sum-free and has no divisors of zero, show that a $n\times n$ matrix A with entries in $\mathcal{S}$ is irreducible iff for all $i,j\in\{1,\dots,n\}$, $(\mathrm{Id}\oplus A)^{n-1}_{ij}\neq\mathbb{0}$ (Id denotes the identity matrix).

Exercise 52 (Diagonal Conjugacy). Let A, A' denote two square irreducible matrices with entries in a commutative semifield, that have the same graph, $\mathcal{G}$. Show that there exists a diagonal matrix D with non-zero diagonal entries such that $A = DA'D^{-1}$ iff all the circuits of the $\mathcal{G}$ have the same weight for A and A'. Extend this result to reducible matrices. Answer in [**29**] and [**42**].

Exercise 53. Give (without any computation) the eigenvalues and eigenvectors of the following matrix with entries in the field of complex numbers:

$$A = \begin{pmatrix} 1 & \alpha & \alpha^2\\ \alpha^{-1} & 1 & \alpha\\ \alpha^{-2} & \alpha^{-1} & 1\end{pmatrix} \quad .$$

2.2. Symmetrized Semirings, Determinants, Permanents, and Universal Combinatorial Formulæ.

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In classical linear algebra, determinants are particularly useful, particularly to express the compatibility conditions of linear systems, and to give convenient closed form for their solutions. Hence, it is very natural to ask whether we can use determinants in semirings. Of course, we cannot define directly determinants as usual, for the standard definition uses an *alternate* sum, which involves the *opposites* of some elements, which need not exist in general semirings.

[9]In any semiring, a $n\times n$ permutation matrix is of the form $P_{ij}=\mathbb{1}$ if $j=\sigma(i)$, $P_{ij}=\mathbb{0}$ otherwise, σ being a permutation of $\{1,\dots,n\}$.

A way to ignore the difficulty is to introduce the positive and negative *determinant*, $\det^+A$ and $\det^-A$, which are defined by

$$\det{}^+A = \bigoplus_{\sigma\in\mathfrak{S}_n^+} A_{1\sigma(1)}\dots A_{n\sigma(n)} \qquad \det{}^-A = \bigoplus_{\sigma\in\mathfrak{S}_n^-} A_{1\sigma(1)}\dots A_{n\sigma(n)} \enspace ,$$

where $\mathfrak{S}_n^+$ denotes the group of even permutations of $\{1,\dots,n\}$, and $\mathfrak{S}_n^- = \mathfrak{S}_n \setminus \mathfrak{S}_n^+$ denotes the set of odd permutations. These formulæ are meaningful for a square matrix A with entries in any semiring, although the main determinantal identities that we will display next are only valid commutative semirings.

We call always define the *permanent* perm A, which is obtained by writing the determinant sum, without minus sign:

$$\operatorname{perm} A = \det{}^+A \oplus \det{}^-A = \bigoplus_{\sigma\in\mathfrak{S}_n} A_{1\sigma(1)}\dots A_{n\sigma(n)} \enspace .$$

In the usual algebra, say when $\mathcal{S} = (\mathbb{R},+,\times)$, the permanent is a remarkable very useful quantity (see e.g. [**62, 4**]), which is much harder to compute that the determinant. As shown in the lecture of P. Butkovič in this school, in some exotic semirings, such as max-plus or bottleneck algebra, permanents are still of great importance, due to their relations with optimal assignment problems, and they are much easier to compute.

The main difficulty with permanents semirings is that few of the classical determinantal identities extend to permanents. A noticeable exception is *row expansion* (or dually column expansion), which holds for permanents of matrices with entries in a commutative semiring $\mathcal{S}$:

$$\operatorname{perm} A = \bigoplus_{1\leq j\leq n} A_{ij} \operatorname{perm} A(i|j) \enspace , \tag{36}$$

where $i \in \{1,\dots,n\}$ and $A(i|j)$ denotes the matrix obtained by deleting the i-th row and j-th column in A. This identity for permanents can be proved easily. We have the following analogous expansions for positive and negative determinants:

$$\det{}^+A = \begin{cases} A_{i1}\det{}^+A(i|1) \oplus A_{i2}\det{}^-A(i|2) \oplus A_{i3}\det{}^+A(i|3) \oplus \dots & \text{if } i \text{ is odd} \\ A_{i1}\det{}^-A(i|1) \oplus A_{i2}\det{}^+A(i|2) \oplus A_{i3}\det{}^-A(i|3) \oplus \dots & \text{if } i \text{ is even} \end{cases} \tag{37}$$

$$\det{}^-A = \begin{cases} A_{i1}\det{}^-A(i|1) \oplus A_{i2}\det{}^+A(i|2) \oplus A_{i3}\det{}^-A(i|3) \oplus \dots & \text{if } i \text{ is odd} \\ A_{i1}\det{}^+A(i|1) \oplus A_{i2}\det{}^-A(i|2) \oplus A_{i3}\det{}^+A(i|3) \oplus \dots & \text{if } i \text{ is even} \end{cases} \tag{38}$$

The identities (37) and (38) are just combinatorial tautologies: we can establish a bijection between terms on both sides of the equalities. Indeed, we can partition the set of even (resp. odd) permutations σ into n classes according to the value of $\sigma(i)$, and each of the n classes give precisely one of the n terms in the sum (37) (resp. (38)).

By comparison with usual algebra, the row expansions (37) and (38) are regrettably tedious, because these formulæ depend on the parity of indices. However, is possible to handle in a much simpler way such formulæ, but we will have to pay an additional price: the construction of a *symmetrized* semiring, in which the "minus" sign has a purely formal meaning.

We will call *symmetrized* semiring a semiring $(\mathcal{S}',\oplus,\otimes)$, equipped with a unary operation $\mathcal{S}' \to \mathcal{S}', a \mapsto \ominus a$, which satisfies the following identities: $\ominus\mathbb{0} = \mathbb{0}$, $\ominus(a\oplus b) = (\ominus a)\oplus(\ominus b)$, $\ominus(ab) = (\ominus a)b = a(\ominus b)$, $\ominus\ominus a = a$. Then, we call $\ominus$ the *symmetry* of $\mathcal{S}$. These rules allow us to use the familiar notations $a \ominus b \stackrel{\text{def}}{=} a \oplus (\ominus b)$, etc. In any symmetrized semiring $\mathcal{S}'$, we define determinants by the usual formula

$$\det A = \bigoplus_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) A_{1\sigma(1)}\dots A_{n\sigma(n)} \enspace ,$$

with $\operatorname{sgn}(\sigma) = \mathbb{1}$ if σ is an even permutation, and $\operatorname{sgn}(\sigma) = \ominus\mathbb{1}$ if σ is an odd permutation. Of course, rings are symmetrized semirings (in which $\ominus a$ is the opposite of a), but there are many other symmetrized semirings. Indeed, with any semiring $\mathcal{S}$, we associate the symmetrized semiring $\mathcal{S}^2$, whose elements are couples $(a', a'') \in \mathcal{S}^2$, equipped with the laws

$$(a', a'') \oplus (b', b'') = (a' \oplus b', a'' \oplus b'') \ , \quad (a', a'') \otimes (b', b'') = (a'b' \oplus a''b'', a'b'' \oplus a''b') \ ,$$
$$\ominus(a', a'') = (a'', a') \ .$$

The map $i : a' \mapsto (a', \mathbb{0})$ is an injective morphism of semirings: we have embedded $\mathcal{S}$ in a symmetrized semiring. We will call $\mathcal{S}^2$ the *free symmetrized semiring* of $\mathcal{S}$, because for all symmetrized semirings $\mathcal{S}''$ and all semiring morphism $\jmath : \mathcal{S} \to \mathcal{S}''$, there is a unique semiring morphism $\varphi : \mathcal{S}^2 \to \mathcal{S}''$ such that $\jmath(a') = \varphi(i(a'))$ and $\varphi(\ominus a) = \ominus\varphi(a)$, for all $a' \in \mathcal{S}$, $a \in \mathcal{S}^2$.

In the sequel, we will identify the element a' of $\mathcal{S}$ to the element $(a', \mathbb{0})$ of $\mathcal{S}^2$. Hence, if $a', a'' \in \mathcal{S}$, $a' \ominus a''$ denotes the couple $(a', a'') \in \mathcal{S}^2$.

If $\mathcal{S}'$ is a symmetrized semiring, and if $a, b \in \mathcal{S}'$, we say that *a balances b*, and we write $a \nabla b$, if $a \ominus b = b \ominus a$. E.g., in $\mathcal{S}^2$,

$$(a', a'')\nabla(b', b'') \iff a' \oplus b'' = a'' \oplus b' \ . \tag{39}$$

We will also use the notation ∇ for vectors and matrices, with an entrywise meaning.

In general, the balance relation is reflexive and symmetric, but it is *not* transitive. Indeed, let $\mathcal{S}$ denote a idempotent semiring, e.g. $\mathcal{S} = \mathbb{R}_{\max}$. We have, in $\mathcal{S}^2$, $(\mathbb{1}, \mathbb{0})\nabla(\mathbb{1}, \mathbb{1})$, for $\mathbb{1} \oplus \mathbb{1} = \mathbb{1} \oplus \mathbb{0}$, and by symmetry, $(\mathbb{0}, \mathbb{1})\nabla(\mathbb{1}, \mathbb{1})$. However, $(\mathbb{1}, \mathbb{0}) \not\nabla (\mathbb{0}, \mathbb{1})$, for $\mathbb{1} \oplus \mathbb{1} \neq \mathbb{0} \oplus \mathbb{0}$, unless $\mathbb{1} = \mathbb{0}$ in which case the semiring $\mathcal{S}$ is trivial.

We call *polynomial expression* over an alphabet $\Sigma = \{a_1, \dots, a_k\}$ a term P which is obtained by a finite number of application of the grammar rule $P \mapsto P \oplus P, P \otimes P, \ominus P, \mathbb{1}, a_1, \dots, a_k$, where $\mathbb{1}, \oplus, \otimes, \ominus$ being viewed as formal (constant) symbols. If $x_1, \dots, x_k$ are elements of a symmetrized semiring $\mathcal{S}$, we denote by $P(x_1, \dots, x_k)$ the element of $\mathcal{S}$ obtained by replacing a_i by x_i, for all $i = 1, \dots, k$, and by interpreting the symbols $\mathbb{1}$ and $\oplus, \otimes, \otimes$ as the unit and structure laws of $\mathcal{S}$. If P, Q are polynomials expressions over Σ, we say that the identity $P = Q$ (resp. the balance $P \nabla Q$) holds in a ring (resp. symmetrized semiring) $\mathcal{S}$ if for all $x_1, \dots, x_k \in \mathcal{S}$, $P(x_1, \dots, x_k) = Q(x_1, \dots, x_k)$ (resp. $P(x_1, \dots, x_k) \nabla Q(x_1, \dots, x_k)$).

E.g. $P =$"$(a \ominus b)(a \oplus b)$" and $Q =$"$a^2 \ominus b^2$" ($= a \otimes a \ominus b \otimes b$) and $R =$"$\mathbb{1}$" are polynomial expressions (over the alphabet $\Sigma = \{a, b\}$), but none of "$a\oplus$", "$a_1 \oplus a_2 \oplus \dots$"", and "$a^{-1}$" are polynomial expressions. In the symmetrized semiring $(\mathbb{R}_{\max})^2$, we have $P(2, 3) = (2 \ominus 3)(2 \oplus 3) = 5 \ominus 6$, and $Q(2, 3) = 2^2 \ominus 3^2 = 4 \ominus 6$. Hence, the identity $(a \ominus b)(a \oplus b) = a^2 \ominus b^2$ which is valid in rings is not valid in $(\mathbb{R}_{\max})^2$. However, $P(2, 3)\nabla Q(2, 3)$. This observation is general:

THEOREM 54 (TRANSFER PRINCIPLE [**31**, Ch. 1]). *For all polynomial expressions P, Q, if the identity $P = Q$ holds in rings, then the balance $P \nabla Q$ holds in semirings.*

This result is essentially a formalization of a very simple argument due Reutenauer and Straubing [**68**]. We will not prove it in general, but we will just prove one typical identity below, to show the technique.

As an illustration of this principle, we next list some useful identities. If $\mathcal{S}$ is a *commutative* symmetrized semiring, and if $A \in \mathcal{S}^{n \times n}$, we denote by A^{adj} the *transpose of the comatrix* of A, i.e. $A^{\mathrm{adj}}_{ij} = (\ominus\mathbb{1})^{i+j} \det A(j|i)$. We have:

$$\det A = \bigoplus_{1 \le j \le n} A_{ij} (\ominus\mathbb{1})^{i+j} \det A(i|j) \qquad \text{(Row Expansion)} \tag{40a}$$

Of course, $\det A = \det A^T$, and a similar property holds for columns. Moreover, for all matrices with entries in $\mathcal{S}$, partitioned in $(1 + (n - 1)) \times (1 + (n - 1))$ blocks,

$$\det \begin{pmatrix} x & Y \\ Z & T \end{pmatrix} = x \det T \ominus Y T^{\mathrm{adj}} Z \qquad \text{(Schur Complement)} \tag{40b}$$

If P is a $n \times n$ *monomial*[10] matrix with entries in $\mathcal{S}$, we have

$$\det AP = \det A \det P \quad . \tag{40c}$$

However, the determinant is not a morphism, in general, and for an arbitrary $B \in \mathcal{S}^{n\times n}$, we can only write:

$$\det AB \nabla \det A \det B \quad . \tag{40d}$$

As originally observed in [**68**], the usual Cramer identities have the following semiring analogues:

$$A^{\mathrm{adj}} A \nabla \det A\, \mathrm{Id}, \qquad AA^{\mathrm{adj}} \nabla \det A\, \mathrm{Id} \quad . \tag{40e}$$

The famous semiring version of the Cayley-Hamilton theorem, which was first noticed by Rutherford [**70**], and rediscovered by Straubing [**77**], can be expressed as

$$P_A(A) \nabla \mathbb{0} \qquad \text{where } P_A(X) \stackrel{\text{def}}{=} \det(A \ominus X\mathrm{Id}) \tag{40f}$$

Finally, for all matrices $A_1, \ldots, A_{2n} \in \mathcal{S}^{n\times n}$,

$$\bigoplus_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn}(\sigma) A_{\sigma(1)} \ldots A_{\sigma(2n)} \nabla \mathbb{0} \qquad \text{(Amitsur-Levitzski)}, \tag{40g}$$

Of course, all the balances in (40) follow readily from the transfer theorem, since the commutative ring-analogues of these results are valid.

Rather than proving the transfer theorem itself, we will just show that $\det AB \nabla \det A \det B$, using the general technique of [**68**].

PROOF OF (40d). Let $X = \{a'_{ij}, b'_{kl}, 1 \le i, j, k, l \le n$, denote a family of distinct commuting indeterminates, and consider the semiring of (formal) commutative polynomials with coefficients in $\mathbb{N}$ and indeterminates $x \in X$: $\mathcal{S}' = \mathbb{N}[X] \subseteq \mathbb{Z}[X]$. Introduce the two matrices $A' = (a'_{ij})$, $B' = (b'_{kl})$ with entries in $\mathcal{S}'$. Since there is a (unique) morphism of commutative symmetrized semirings $(\mathbb{N}[X])^2 \to \mathcal{S}$, which sends a'_{ij} to A_{ij}, and b'_{ij} to B_{ij}, it is enough to show that the identity (40b) holds in $(\mathcal{S}')^2$. Then, using (39), we get

$$\det A' \det B' \nabla \det A'B' \iff \tag{41}$$

$$\det{}^+ A'B' \oplus \det{}^- A' \det{}^+ B' \oplus \det{}^+ A' \det{}^- B' = \det{}^- A'B' \oplus \det{}^+ A' \det{}^+ B' \oplus \det{}^- A' \det{}^- B' \tag{42}$$

In a ring, (42) is exactly the identity $\det A' \det B' = \det A'B'$ (written without minus sign). Hence, (42) holds in rings, and a fortiori, in the semiring $\mathbb{N}[X]$ which can be embedded in the ring $\mathbb{Z}[X]$. Hence, (41) holds in $(\mathcal{S}')^2 = (\mathbb{N}[X])^2$. □

It is also possible to give a direct combinatorial proof of (42), see Zeilberger [**83**].

Remark 55. With little additional effort, we could have derived a slightly more precise result in the above proof, namely:

$$\det{}^+ AB = \det{}^+ A \det{}^+ B \oplus \det{}^- A \det{}^- B \oplus T \tag{43a}$$

$$\det{}^- AB = \det{}^- A \det{}^+ B \oplus \det{}^+ A \det{}^- B \oplus T \quad , \tag{43b}$$

for some $T \in \mathcal{S}$. Indeed, the two expansions $\det^- A' \det^+ B' \oplus \det^+ A' \det^- B'$ and $\det^+ A' \det^+ B' \oplus \det^- A' \det^- B'$ are the sum of $(n!)$ monomials, which are all distinct. Hence, we can deduce from (42) that $\det^+ A'B'$ is the sum of all the monomials appearing the expansion of $\det^+ A' \det^+ B' \oplus \det^- A' \det^- B'$, together possibly with other monomials, with sum T'. Then, cancelling $\det^+ A' \det^+ B' \oplus \det^- A' \det^- B'$ on both sides of (42), we get $\det^- A'B' = \det^- A' \det^+ B' \oplus \det^+ A' \det^- B'$, which shows that (43) holds. □

[10] Recall that a matrix is *monomial* if it is the product of a permutation matrix and a diagonal matrix.

In general semirings, identities (40) are much less powerful than in rings or fields. Essentially, in fields, determinants provide necessary and sufficient conditions for the solvability of linear systems, whereas the corresponding conditions involving symmetrized semirings and balances are only necessary (see [**31**, Ch. 3]). However, these identities are sometimes enough: e.g. in [**68**], a strengthening of (40e) is used to show that in a commutative semiring, a square matrix has a left inverse iff it has a right inverse. These identities are often most useful when coupled with other assumptions. E.g., in [**34**], a semiring analogue of the Binet-Cauchy formula is used to prove that the minimal linear realization obtained in [**25**] for strictly convex rational sequences is also minimal for non-strictly convex sequences, and, in [**33**], the semiring version of Amitsur-Levitzki identity is indeed enough to prove (using a straightforward variant of an argument of Straubing) that the answer to the Burnside problem is positive for semigroups of matrices with entries in an idempotent semiring in which for all a, the set of elements b such that $a \oplus b = a$ is finite (e.g., $\mathbb{N}_{\max}$ is such a semiring).

Exercise 56 (Cramer's Formuæ). Show that for all $n \times n$ matrices A and for all n-dimensional column vectors b, x with entries in commutative semiring,

$$Ax = b \implies (\det A)\, x \nabla A^{\mathrm{adj}} b \tag{44}$$

(Cramer's condition). Application. Try to solve the following linear system in $\mathbb{R}_{\max}$

$$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

using Cramer's condition. Compare this method with the residuation-based algorithms given in the lectures of G. Cohen and P. Butkovič, in this school.

Exercise 57. Find the error in the following argument. If $Ax\nabla b$, then, left multiplying by A^{adj}, we get $A^{\mathrm{adj}}Ax\nabla A^{\mathrm{adj}}b$. Now, right multiplying the first balance in (40e) by x, we get $\det Ax\nabla A^{\mathrm{adj}}Ax$, hence, $\det A\, x\nabla A^{\mathrm{adj}}b$. This shows that Cramer's condition (44) holds not only for $Ax = b$, but also for $Ax\nabla b$.

Exercise 58. 1. Compute the characteristic polynomial $P_A(X) = \det(A \ominus X\mathrm{Id})$ of the following matrix with entries in the max-plus semiring.

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 1 \end{bmatrix}$$

2. Show that

$$A^2 = 1A \oplus 7\mathrm{Id} \ . \tag{45}$$

3. Deduce from (45) an explicit formula for A^k.

Exercise 59. Consider the formal expansion in $\mathcal{S}^2[X]$:

$$\det(X\mathrm{Id} \ominus A) = X^n \ominus \Lambda_1 X^{n-1} \oplus \Lambda_2 X^{n-2} \oplus \cdots \oplus (\ominus \mathbb{1})^n \Lambda_n \ .$$

1. Show that Λ_i is the sum of the determinants of all $i \times i$ *principal submatrices*[11] of A. 2. We call *support* of a circuit c, the set of nodes that appear in c, and we denote it S_c. We call *circuit covering* of weight i a set $\mathcal{C} = \{c_1, \ldots, c_k\}$, of circuits of the graph of A, such that $S_{c_1} \cup \ldots \cup S_{c_k}$ is a partition of a i-elements subset of $\{1, \ldots, n\}$. We *weight* of $\mathcal{C}$ is $w(\mathcal{C}) = w(c_1) \ldots w(c_k)$. The *cardinality* of $\mathcal{C}$ is just $|\mathcal{C}| = k$. Show that

$$\Lambda_i = \bigoplus_{\mathcal{C}} (\ominus \mathbb{1})^{|\mathcal{C}|-i} w(\mathcal{C}) \ , \tag{46}$$

where the sum is taken over circuit coverings of weight i. 3. Use (46) to give a combinatorial proof of Cayley-Hamilton theorem (answer in [**77**]).

[11] We call principal submatrix a matrix obtained by deleting *the same* row and columns in A.

Exercise 60 (Recognizable sequences are linear recurrent). We say that a sequence $s_0, s_1, \dots$ of elements of a semiring $\mathcal{S}$ is *linear recurrent* if there exists two finite sequences $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{S}$, with at least one a_i or b_i different from zero, such that

$$s_k \oplus \bigoplus_{i=1}^{n} a_i s_{k-i} = \bigoplus_{i=1}^{n} b_i s_{k-i} \quad , \forall k \geq n \quad . \tag{47}$$

We say that s is *recognizable* iff there exists an integer n, and $A \in \mathcal{S}^{n\times n}$, $b \in \mathcal{S}^{n\times 1}$, $c \in \mathcal{S}^{1\times n}$, such that $s_k = cA^k b$, for all k. 1. Show that if $\mathcal{S}$ is commutative, a recognizable sequence is linear recurrent, and that we can even take $a_1 = \mathbb{0}$ in (47) 2. Show that the two properties are equivalent iff $\mathcal{S}$ is a ring, and that this equivalence need not hold if $\mathcal{S}$ is a semiring.

3. Naturally Ordered Semirings

In any semiring, and more generally in any additive monoid, we can define the pre-order[12] relation

$$a \preceq b \iff \exists c, \quad b = a \oplus c \quad , \tag{48}$$

which is such that for all a, $\mathbb{0} \preceq a$. We will write $a \prec b \iff a \preceq b$ and $a \neq b$. The relation $\preceq$ need not be antisymmetric. E.g., in the field of real numbers, $\preceq$ is the complete relation (which satisfies $a \preceq b$, for all $a, b \in \mathbb{R}$). We will say that a semiring or a monoid is *naturally ordered* if $\preceq$ is an order relation, and in this case, we call $\preceq$ the *natural order*.

More generally, a *ordered semiring*, is a semiring equipped with a (partial) order relation $\leq$ that satisfies

$$a \leq b \implies a \oplus c \leq b \oplus c, \tag{49a}$$

$$\mathbb{0} \leq c \implies (ac \leq bc \text{ and } ca \leq cb) \quad . \tag{49b}$$

Of course, the natural order satisfies (49). In any ordered semiring, we have

$$a' \geq \mathbb{0},\ b \geq \mathbb{0}, \quad \text{and } a \geq b \text{ and } a' \geq b' \implies aa' \geq bb' \quad , \tag{50a}$$

$$c \geq d \geq \mathbb{0} \implies d^{-1} \leq c^{-1} \quad , \tag{50b}$$

if c and d have an inverse. Since, in a naturally ordered semiring, $a \geq \mathbb{0}$, for all a, we simply have:

$$a \succeq b \text{ and } a' \succeq b' \implies aa' \succeq bb' \quad . \tag{51}$$

Let us begin with a simple remark.

PROPOSITION 61. *A naturally ordered semiring is zero-sum-free.*

PROOF. If $a \oplus b = \mathbb{0}$, then, $a \preceq \mathbb{0}$. But, trivially, $\mathbb{0} \preceq a$. Since $\preceq$ is antisymmetric, we get $a = \mathbb{0}$, and, exchanging the roles of a and b, $b = \mathbb{0}$. □

The next two propositions, which exhibit two important classes of naturally ordered semirings, yield a partial converse to Proposition 61.

PROPOSITION 62. *A zero-sum-free cancellative semiring is naturally ordered.*

PROOF. We only have to check that $\preceq$ is antisymmetric. If $a \preceq b \preceq a$, we have $b = a \oplus c$ and $a = b \oplus d$, for some $c, d \in \mathcal{S}$. Then, substituting b in the second equation, we get $a = a \oplus c \oplus d$. Cancelling a, we get $c \oplus d = \mathbb{0}$, which, if $\mathcal{S}$ is zero-sum-free, implies that $c = d = \mathbb{0}$. Hence, $a = b$ and $\preceq$ is antisymmetric. □

E.g. $\mathbb{R}_{+_p} = (\mathbb{R}_+, +_p, \times)$ is a cancellative and zero-sum-free semiring, in which the order relation $\preceq$ coincides with the usual order on non-negative reals. The second class of semirings is yet more important for our purposes.

[12] A pre-order is a reflexive transitive relation.

PROPOSITION 63. *An idempotent semiring $\mathcal{S}$ is naturally ordered. Moreover,*

$$a \preceq b \iff a \oplus b = b \ , \tag{52}$$

and $a \oplus b$ coincides with the least upper bound of $\{a, b\}$.

PROOF. If $b = a \oplus c$ for some c, then, using the idempotency of addition, $a \oplus b = a \oplus a \oplus c = a \oplus c = b$, hence, $b = a \oplus b$. Conversely, if $b = a \oplus b$, $b = a \oplus c$ (with $c = b$). This shows that the relations (48) and (52) coincide. If $a \preceq b$ and $b \preceq a$, we have $b = a \oplus b$, and $a = a \oplus b$, hence, $a = b$. This shows that $\preceq$ is antisymmetric.

It remains to prove that $a \oplus b$ is the least upper bound of $\{a, b\}$. First, $a \preceq a \oplus b$, for $a \oplus b \oplus a = a \oplus b$, and by symmetry, $b \preceq a \oplus b$. Second, if $x \succeq a$ and $x \succeq b$, we have $x = a \oplus x$ and $x = b \oplus x$, hence, adding these two equalities and using the idempotency of addition, $x = a \oplus b \oplus x$, which shows that $a \oplus b \preceq x$. These two points show that $a \oplus b$ is the least element of the set $\{x \mid \ a \preceq x, \ b \preceq x\}$, i.e., the least upper bound of $\{a, b\}$. □

In an ordered semiring $\mathcal{S}$, there is a very simple way to define infinite sums of non-negative elements, just by setting

$$\bigoplus_{x \in X} x \stackrel{\text{def}}{=} \sup\{\bigoplus_{y \in Y} y \mid \ Y \subset X, \ Y \text{ finite}\} \ , \tag{53}$$

where sup denotes the least upper bound for the natural order, when when it exists. This is exactly the way usual sums of numerical series with non-negative terms are defined. Of course, in general, the family of partial sums $\{\bigoplus_{j \in J} x_j\}$ indexed by finite subsets $J \subset I$ need not have a least upper bound. In fact, even for numerical series with non-negative terms, the least upper bound (53) may not exist (when the series is divergent).

We say that the sum (53) is *bounded* when the set $\{\bigoplus_{y \in Y} y \mid \ Y \subset X, \ Y \text{ finite}\}$ has a majorant. The importance of the following assumption is well known in the theory of formal series, see e.g. [**55**].

DEFINITION 64 (COMPLETE SEMIRING). A naturally ordered semiring $\mathcal{S}$ is (conditionnaly) *complete* if any (bounded) set X has a least upper bound, and if product distributes over (bounded) infinite sums.

In the sequel, when speaking of (conditionally) complete semiring, we will always mean implicitely that it is naturally ordered.

By specializing the definition to $X = \{a, b\}$, we see that a conditionally complete semiring is sup-semilattice ordered.

Example 65 (Completed Max-plus Semiring). The max-plus semifield $\mathbb{R}_{\max}$ is conditionnaly complete, but not complete, because $X = \{1, 2, 3, \dots\}$ has no least upper bound. However, we can embed $\mathbb{R}_{\max}$ in a complete semiring, namely $(\mathbb{R} \cup \{-\infty, +\infty\}, \max, +)$. We call completed max-plus semiring this structure, and we denote it by $\overline{\mathbb{R}}_{\max}$. Note that in $\overline{\mathbb{R}}_{\max}$, the rule $-\infty + +\infty = -\infty$ is forced by the axiom $\mathbb{0} \otimes a = \mathbb{0}$.

Example 66. The tropical semiring $\mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +)$ is complete.

Example 67. The semiring $(\mathbb{R}_+, +, \times)$ is conditionnaly complete. It can be embedded in the complete semiring $(\mathbb{R}_+ \cup \{\infty\}, +, \times)$.

Example 68 (Top-completion). More generally, we call *top completion* a of a naturally ordered semiring $\mathcal{S}$ the semiring $\overline{\mathcal{S}} = \mathcal{S} \cup \{\top\}$, where $\top$ is an extra element, that satisfies $a \oplus \top = \top$, for all a, and $a \otimes \top = \top \otimes a$, for all $a \neq \mathbb{0}$. Clearly, the top completion of a conditionnaly complete semiring is complete.

Exercise 69. Characterize the finite idempotent semifields, and the complete semifields.

It can be shown easily that in a complete semiring, the infinite sums defined by (53) satisfy Fubini's rule, i.e., for all (possibly infinite) partition $X = \bigcup_{i \in I} X_i$, there holds,

$$\bigoplus_{x \in X} x = \bigoplus_{i \in I} \bigoplus_{y \in X_i} y \ \text{ (Fubini's rule)} \tag{54}$$

The interest in complete semirings arises mostly from the following much celebrated theorem, which "solves" in a natural sense linear equations, by means of *Kleene's star* operation

$$a^* \stackrel{\text{def}}{=} a^0 \oplus a \oplus a^2 \oplus \cdots$$

(which is by definition the sum of the set $\{a^k \mid k \in \mathbb{N}\}$, defined according to (53)).

THEOREM 70 (STAR THEOREM). *(i) Let a, b denote two elements of a complete semiring $\mathcal{S}$. Then,*

$$ax \oplus b \preceq x \implies a^*b \preceq x \quad , \tag{55}$$

and $\overline{x} = a^*b$ satisfies $x = ax \oplus b$. (ii) The same conclusions hold if a is a $n \times n$ matrix and x, b are n dimensional column vectors.

The theorem says, firstly, that a^*b is the least solution of the equation $x = ax \oplus b$, and secondly that it remains the least solution of the inequation $x \succeq ax \oplus b$.

PROOF. Substituting $x \succeq ax \oplus b$ in the right hand side of $x \succeq ax \oplus b$, we get $x \succeq b \oplus ab \oplus a^2x$, and by an immediate induction $x \succeq b \oplus ab \oplus \cdots \oplus a^k b \oplus a^{k+1}x$. This shows that $x \succeq \bigoplus_{j \in J} a^j b$, for all finite set J. Hence, using the definition of infinite sums and the fact that the product distributes over infinite sums, we get $x \succeq \bigoplus_{k \in \mathbb{N}} a^k b = a^*b$. It remains to check that $a^*b = aa^*b \oplus b$, which follows from the infinite distributivity and from Fubini's rule. Indeed, $aa^*b \oplus b = a\left(\bigoplus_{k \in \mathbb{N}} a^k b\right) \oplus b = \bigoplus_{k \in \mathbb{N}} a^{k+1}b \oplus a^0 b = a^*b$. □

Expressions involving sum, product, and stars satisfy many useful rational identities. See [**22, 51**]. We will just tabulate the following identities for further reference, which hold for a, b in a complete semiring $\mathcal{S}$:

$$a^* = (a^0 \oplus \cdots \oplus a^{c-1})(a^c)^* \tag{56}$$

$$(a \oplus b)^* = (a^*b)^*a^* \tag{57}$$

$$(a \oplus b)^* = a^*b^* \quad \text{if } \mathcal{S} \text{ is idempotent and } ab = ba \tag{58}$$

$$a^*a^* = a^* \quad \text{if } \mathcal{S} \text{ is idempotent} \tag{59}$$

$$(a^*)^* = a^* \quad \text{if } \mathcal{S} \text{ is idempotent.} \tag{60}$$

We conclude this section by a very classical block formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^* \oplus a^*b(ca^*b \oplus d)^*ca^* & a^*b(ca^*b \oplus d)^* \\ (ca^*b \oplus d)^*ca^* & (ca^*b \oplus d)^* \end{pmatrix} \tag{61}$$

4. Semimodules and Linear Maps

Let M denote a monoid, with law $(\alpha, \beta) \to \alpha\beta$ and unit $\mathbb{1}$. A left *action* of M on a set X is a map $M \times X \to X$, $(\alpha, x) \mapsto \alpha \cdot x$, that satisfies

$$(\alpha\beta)x = \alpha \cdot (\beta \cdot x) \quad \text{and} \quad \mathbb{1} \cdot x = x \quad . \tag{62}$$

A left *semimodule* $\mathcal{X}$ is a commutative monoid for a law $(x, y) \to x \oplus y$, called addition, with zero element $\mathbb{0}$, equipped with a left action $\mathcal{S} \times \mathcal{X} \to \mathcal{X}$, $(\alpha, x) \mapsto \alpha \cdot x$ for the multiplicative monoid structure of $\mathcal{S}$, that satisfies:

$$(\alpha \oplus \beta) \cdot x = \alpha \cdot x \oplus \beta \cdot x \tag{63a}$$

$$\alpha \cdot (x \oplus y) = \alpha \cdot x \oplus \alpha \cdot y \tag{63b}$$

$$\mathbb{0} \cdot x = \mathbb{0} \tag{63c}$$

$$\alpha \cdot \mathbb{0} = \mathbb{0} \quad . \tag{63d}$$

Right semimodules are defined dually (by using right actions $\mathcal{X} \times \mathcal{S} \to \mathcal{X}$, $(x, \alpha) \mapsto x \cdot \alpha$, which satisfy $(x \cdot \alpha) \cdot \beta = x \cdot \alpha\beta$). Of course, if $\mathcal{S}$ is a commutative semiring, the distinction between right and left semimodules is pointless.

The axioms (62),(63) mean precisely that the map κ, which to an element $\alpha \in \mathcal{S}$, associates the map $\kappa(\alpha) : \mathcal{S} \to \mathcal{S}$, $\kappa(\alpha)(x) = \alpha \cdot x$, is a morphism of semirings from $\mathcal{S}$ to the semiring of additive morphisms

$\mathcal{X} \to \mathcal{X}$ (see Ex. 31). Elements of semimodules are *vectors*. A left semimodule is *faithful* if the map κ is injective. If $\mathcal{X}, \mathcal{Y}$ are $\mathcal{S}$-semimodules, a *morphism* of (left) semimodules or left *linear map* $F : \mathcal{X} \to \mathcal{Y}$ is a morphism of additive monoids, that satisfies $F(\alpha \cdot x) = \alpha \cdot F(x)$, for $\alpha \in \mathcal{S}$ and $x \in \mathcal{X}$. We denote by $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ the set of left semimodule morphisms $\mathcal{X} \to \mathcal{Y}$. A left subsemimodule of a semimodule $\mathcal{X}$ is a submonoid $\mathcal{X}'$ of $\mathcal{X}$, such that $\alpha \cdot x' \in \mathcal{X}$, for all $\alpha \in \mathcal{S}$ and $x' \in \mathcal{X}'$.

In the sequel, we will simply speak of "semimodule" and "linear map" when the right or left character will be clear from the context. For simplicity, we will also write the action of a left semimodule $(\alpha, x) \mapsto \alpha x$ instead of $(\alpha, x) \mapsto \alpha \cdot x$, and we will adopt the dual convention for right semimodules.

A $\mathcal{S}$-semimodule $\mathcal{X}$ inherits trivially some properties of the semiring $\mathcal{S}$.

PROPOSITION 71. *A semimodule over an idempotent semiring has an idempotent addition.*

PROOF. If addition of $\mathcal{S}$ is idempotent, $\mathbb{1} \oplus \mathbb{1} = \mathbb{1}$, hence, $x \oplus x = \mathbb{1} \cdot x \oplus \mathbb{1} \cdot x = (\mathbb{1} \oplus \mathbb{1}) \cdot x = \mathbb{1} \cdot x = x$, for all vectors x. $\square$

For this reason, we will call *idempotent semimodule* a semimodule over an idempotent semiring. Of course, a similar property holds when addition is group law: a semimodule over a ring is automatically a module.

Example 72. The set of vectors $\mathcal{S}^n$, equipped with entrywise sum, is a left $\mathcal{S}$ semimodule for the action $\alpha \cdot x = (\alpha x_1, \dots, \alpha x_n)$ It is also a right $\mathcal{S}^{n \times n}$-semimodule for the action $x \cdot M = xM$ (matrix product), x being viewed as a row vector. The semiring of polynomials in one indeterminate with coefficients in $\mathcal{S}$, $\mathcal{S}[X]$, is a $\mathcal{S}[X]$-semimodule. All these semimodules are faithful. Subsemimodules of $\mathcal{S}[X]$ are called *ideals*. For instance, the set of formal polynomials of the form $P(\mathbb{1} \oplus X)$, with $P \in \mathbb{R}_{\max}[X]$, is an ideal of $\mathbb{R}_{\max}[X]$. This $\mathbb{R}_{\max}[X]$-semimodule is not faithful. Indeed, $(\mathbb{1} \oplus X)(\mathbb{1} \oplus X \oplus X^2) = (\mathbb{1} \oplus X)(\mathbb{1} \oplus X^2)$, hence, $\kappa(\mathbb{1} \oplus X^2) = \kappa(\mathbb{1} \oplus X \oplus X^2)$, which shows that κ is not injective.

A *generating family* of a semimodule $\mathcal{X}$ is a family $\{x_i\}_{i \in I}$ of elements of $\mathcal{X}$, such that for all $x \in \mathcal{X}$, there is an almost zero[13] family $\{\alpha_i\}_{i \in I}$ of elements of $\mathcal{S}$ such that $x = \bigoplus_{i \in I} \alpha_i x_i$. A semimodule is *finitely generated*, or *f.g.*, if it has a finite generating family. A family $\{x_i\}_{i \in I}$ is free if $\bigoplus_{i \in I} \alpha_i x_i = \bigoplus_{i \in I} \beta_i x_i \implies \alpha = \beta$, for all almost zero families $\alpha = \{\alpha_i\}_{i \in I}, \beta = \{\beta_i\}_{i \in I}$ of elements of $\mathcal{S}$. A free generating family is a *basis*. A semimodule which has a basis is *free*. By definition, in a free semimodule $\mathcal{X}$ with basis $\{x_i\}_{i \in I}$, an element $x \in \mathcal{X}$ can be written uniquely as $x = \bigoplus_{i \in I} \alpha_i x_i$, for some almost zero family of elements of $\mathcal{S}$. We call $\{\alpha_i\}_{i \in I}$ the *coordinates* of x, with respect to the base $\{x_i\}_{i \in I}$. The following easy "universal property" states that a linear map whose domain is a free semimodule is characterized by the image of the basis elements:

PROPOSITION 73. *Let $\mathcal{X}$ denote a free semimodule with basis $\{x_i\}_{i \in I}$. For all families $\{y_i\}_{i \in I}$ of elements of a semimodule $\mathcal{Y}$, there is a unique semiring morphism $F : \mathcal{X} \to \mathcal{Y}$, such that $F(x_i) = y_i$, for all $i \in I$, namely:*

$$F(\bigoplus_{i \in I} \alpha_i x_i) = \bigoplus_{i \in I} \alpha_i y_i \quad ,$$

and all the linear maps $\mathcal{X} \to \mathcal{Y}$ are obtained in this way. $\square$

In particular, if $\mathcal{X}$ and $\mathcal{Y}$ are free finitely generated semimodules with bases $\{x_1, \dots, x_n\}$, and $\{y_1, \dots, y_p\}$, respectively, the linear map F is characterized by the $n \times p$-matrix $M = (F(x_i)_j)$. Indeed, F sends the vector x with coordinates $u = \{\alpha_i\}_{1 \leq i \leq n}$ to the vector $F(x)$ which has coordinates $((uM)_1, \dots, (uM)_p)$ (concatenation denotes the matrix product). Hence, modulo a choice of bases of $\mathcal{X}$ and $\mathcal{Y}$, we can identify $\mathcal{X}$ and $\mathcal{Y}$ to $\mathcal{S}^{1 \times n}$ and $\mathcal{S}^{1 \times p}$, and linear maps $\mathcal{X} \to \mathcal{Y}$ to matrix transformations: $\mathcal{S}^{1 \times n} \to \mathcal{Y}^{1 \times p}$, $u \mapsto uM$. Of course, when speaking of right semimodules, we should consider maps of the form $v \mapsto Mv$, which are right linear. In fact, when $\mathcal{S}$ is commutative, is is more traditional to deal with column vectors and to associate with a matrix $M \in \mathcal{S}^{n \times p}$ the linear map $v \mapsto Mv$, $\mathcal{S}^{p \times 1} \to \mathcal{S}^{n \times 1}$.

[13] I.e., the set $\{i \in I \mid \alpha_i \neq \mathbb{0}\}$ is finite.

5. Images and Kernels

The *image* of a linear map $F : \mathcal{X} \to \mathcal{Y}$ is the semimodule $\mathrm{Im}F = \{F(x) \mid x \in \mathcal{X}\}$, as usual. A *congruence* over a semimodule $\mathcal{X}$ is an equivalence relation $\mathcal{R} \subset \mathcal{X} \times \mathcal{X}$, that satisfies $(x, y) \in \mathcal{R} \implies (x \oplus z, y \oplus z) \in \mathcal{R}$, $(x, y) \in \mathcal{R} \implies (\alpha x, \alpha y) \in \mathcal{R}$, for all $\alpha \in \mathcal{S}$, $x, y, z \in \mathcal{X}$. Then, the set of equivalence classes for the relation $\mathcal{R}$ is equipped with a quotient semimodule structure, that we denote by by $\mathcal{X}/\mathcal{R}$. The map $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{R}$, which to an element associates its equivalence class, is called the *canonical morphism*. The *kernel* of a linear map $F : \mathcal{X} \to \mathcal{Y}$ denoted $\mathrm{Ker}F$, is the congruence

$$\mathrm{Ker}F = \{(x, x') \in \mathcal{X}^2 \mid F(x) = F(x')\} \ . \tag{64}$$

With this definition of kernel, the following standard factorization result trivially holds.

PROPOSITION 74. *If F is a linear map from a semimodule $\mathcal{X}$ to a semimodule $\mathcal{Y}$, there exists a unique isomorphism $\tilde{F} : \mathcal{X}/\mathrm{Ker}F \to F(\mathcal{X})$, such that $F = \tilde{F} \circ \pi$, i.e.*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] \arrow[d, "\pi"'] & F(\mathcal{X}) \\ \mathcal{X}/\mathrm{Ker}F \arrow[ur, dotted, "\tilde{F}"'] & \end{tikzcd}$$

The classical definition of kernel,

$$\mathrm{ker}'F = \{x \in \mathcal{X} \mid F(x) = \mathbb{0}\} \ ,$$

is often trivial in general semimodules, for it often captures little of the structure of F (see Ex.77). In particular, a map with $\mathrm{Ker}'F = \{\mathbb{0}\}$ need not be injective. Of course, if $\mathcal{X}$ and $\mathcal{Y}$ are modules, then, $F(x) = F(x')$ iff $F(x - x') = 0$, which means that the congruence $\mathrm{Ker}F$ is determined by the subsemimodule $\mathrm{ker}'F$ of $\mathcal{X}$. The fact that congruences over a semimodule $\mathcal{X}$ are *not*, a priori, represented by subsemimodules of $\mathcal{X}$ is the main difficulty, by comparison with the familiar modules and vector spaces theory (we already encountered a similar difficulty when, to state determinant identities, we had to replace $\mathcal{S}$ by the symmetrized semiring $\mathcal{S}^2$, which is "twice larger").

In the sequel, we will use the notation $\mathrm{Im}A$ and $\mathrm{Ker}A$ for a $n \times p$ matrix A with entries in a semiring $\mathcal{S}$, A being identified to the (right) linear map $\mathcal{S}^p \to \mathcal{S}^n$, $x \mapsto Ax$ ($x \in \mathcal{S}^p$ being viewed as a column vector). I.e., $\mathrm{Im}A = \{Ax \mid x \in \mathcal{S}^p\}$, $\mathrm{Ker}A = \{(x, y) \mid Ax = Ay, \ \ x, y \in \mathcal{S}^p\}$.

We call $\mathrm{Im}A$ the *column space* of A, and $\mathrm{Im}A^T$ the *row space* of A.

Example 75 (Subsemimodules of $(\mathbb{R}_{\max})^2$). Several subsemimodules of $(\mathbb{R}_{\max})^2$ are depicted in Fig. 2. The semimodules can be easily identified as the *connected regions* displayed on these pictures. E.g., at the top of the first picture, we have shown the semimodule generated by the vector $\binom{0}{2}$, which is the set of vectors $\binom{\alpha}{2+\alpha}$, with $\alpha \in \mathbb{R} \cup \{-\infty\}$. Hence, the finite points of this semimodule are on an usual affine line of slope 1, with y-coordinate at the origin 2. The semimodule generated by the vector $\binom{0}{-\infty}$ is the affine line parallel of the x-axis, with y-coordinate at the origin $-\infty$. At the top of the second picture, we have shown the semimodule generated by the vectors $\binom{0}{2}$ and $\binom{-\infty}{0}$, which is an upper-half of the plane. At the bottom of the same picture, we have displayed another semimodule, namely the set $\{\binom{x}{y} \mid x > 1 \otimes y = 1 + y\} \cup \binom{\mathbb{0}}{\mathbb{0}}$. This semimodule is *not* finitely generated. Finally, three different semimodules with two finite generators are shown on the last picture.

Example 76 (Kernels in $(\mathbb{R}_{\max})^2$). If A is $n \times 2$ matrix with entries in $\mathbb{R}_{\max}$, we cannot represent geometrically the graph of $\mathrm{Ker}A$, which is a subset of $((\mathbb{R}_{\max})^2)^2 \simeq (\mathbb{R}_{\max})^4$. But it is possible to represent the partition of $(\mathbb{R}_{\max})^2$ into equivalence classes. E.g., if $A = (0 \ {-2})$, for all $u \in \mathbb{R}$, $A^{-1}(u) = \{(x, y) \mid \max(x, y - 2) = u\} = \{(u, y) \mid y \leq 2 + u\} \cup \{(x, 2 + u) \mid x \leq u\}$, and $A^{-1}(-\infty) = \binom{-\infty}{-\infty}$. We have shown the subpartition $\{A^{-1}(u)\}_{u \in \mathbb{R}}$ on the first picture of Fig. 3. The partition associated with the matrix $B = \begin{pmatrix} 0 & -2 \\ 0 & -\infty \end{pmatrix}$ is shown on the second picture. This partition can be computed by enumeration of cases

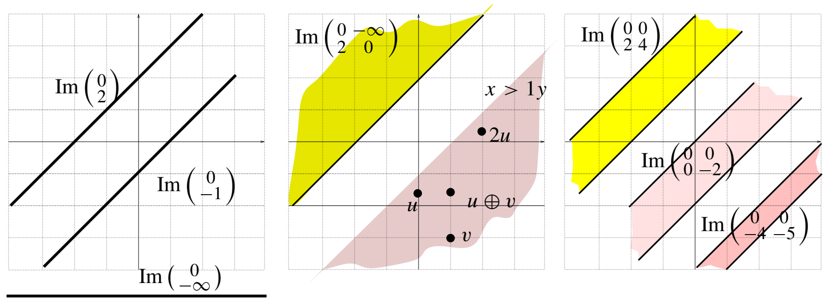


FIGURE 2. Subsemimodules of $(\mathbb{R}_{\max})^2$

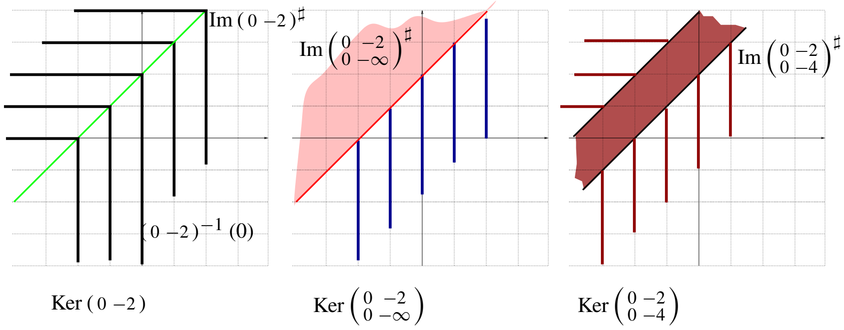


FIGURE 3. Kernels in dimension 2

(looking at where the max is attained), as in the above example. However, a very simple remark makes much easier this computation. Consider the *residuated operator of* B, $B^\sharp_i(z) = \min_j(-B_{ji} + z_j)$. This is a min-plus linear operator. We claim that: *(*) each equivalence class modulo* Ker$B$ *meets* Im$B^\sharp$ *at one and only one point*. Indeed, as shown in the presentation of G. Cohen in this school, the two following identities hold

$$B^\sharp \circ B \circ B^\sharp = B^\sharp \tag{65}$$

$$B \circ B^\sharp \circ B = B \quad . \tag{66}$$

From the second identity, we deduce that for all z, $Bz = Bz'$, with $z' = B^\sharp \circ B(z)$. From the first identity, we get that if $u = B^\sharp(v)$ and $u' = B^\sharp(v')$, and if $Bu = Bu'$, then, $u = u'$, which concludes the proof of (*). Thanks to remark (*), to represent the equivalence classes modulo Ker$B$, we first represent Im$B^\sharp$ (which is a $\mathbb{R}_{\min}$-semimodule). Then, it an easy exercise to guess the partition in equivalence classes from Im$B^\sharp$, since, in particular, points in the interior of Im$B^\sharp$ are equivalence classes of Ker$B$ reduced to one point.

Example 77 ("Ordinary" Max-Plus Kernels are Trivial). The "ordinary" kernel $\mathrm{Ker}'A = \{x \mid \ Ax = \mathbb{0}\}$ is reduced to $\{\mathbb{0}\}$, for the three matrices shown in Fig. 3.

6. Factorization of Linear Maps and Linear Extension Theorem

Can the inclusion of images or kernels be expressed algebraically? In the case of images, the answer is simple.

PROPOSITION 78. *Let $\mathcal{S}$ denote an arbitrary semiring. Consider a free $\mathcal{S}$-semimodule $\mathcal{F}$, two $\mathcal{S}$-semimodules $\mathcal{G}$, $\mathcal{H}$, and two linear maps $F : \mathcal{F} \to \mathcal{H}$, $G : \mathcal{G} \to \mathcal{H}$. The following assertions are equivalent:*

1. Im$F \subset$ ImG

2. *there exists a linear map $H : \mathcal{F} \to \mathcal{G}$ such that $F = G \circ H$, i.e.*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "F"] \arrow[d, dotted, "H"'] & \mathcal{H} \\ \mathcal{G} \arrow[ur, "G"] & \end{tikzcd}$$

PROOF. Trivially, (2) implies (1). Conversely, taking a basis of $\mathcal{F}$, $\{u_i\}_{i\in I}$, for all $i \in I$, there exists $h_i \in \mathrm{Im}G$ such that $F(u_i) = G(h_i)$. Since $\mathcal{F}$ is free, by Proposition 73, setting $H(u_i) = h_i$, for all $i \in I$, we define a linear map $H : \mathcal{F} \to \mathcal{G}$, which satisfies $F = G \circ H$. □

In other words, free semimodules are projective, see [**37**, Ch. 15].

The analogue of Proposition 78 on kernels requires some conditions on the semiring, as shown by the following counterexample.

Example 79. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ denote three free semimodules, and consider two maps $F : \mathcal{H} \to \mathcal{F}$, $G : \mathcal{H} \to \mathcal{G}$. The inclusion $\mathrm{Ker}G \subset \mathrm{Ker}F$ need not imply the existence of a linear map $H : \mathcal{G} \to \mathcal{F}$ such that $F = H \circ G$. Consider the semiring $\mathbb{N}_{\max} = (\mathbb{N} \cup \{-\infty\}, \max, +)$ equipped with the laws $\oplus = \max$, $\otimes = +$, $\mathcal{F} = \mathcal{G} = \mathcal{H} = \mathbb{N}_{\max}$, $G(x) = x + 2$, $F(x) = x + 1$. We have $\mathrm{Ker}G \subset \mathrm{Ker}F$ (in fact, $\mathrm{Ker}G = \mathrm{Ker}F = \{(x, x) \mid x \in \mathbb{N}_{\max}\}$) but there exists no linear map H such that $F = H \circ G$. Indeed, any linear map $H : \mathbb{N}_{\max} \to \mathbb{N}_{\max}$ writes $H(x) = x + a$ where $a = H(0) \in \mathbb{N}_{\max}$. We obtain $F(0) = 1 = 2 + a$: a contradiction.

DEFINITION 80 (FACTORIZATION). We say that the semiring $\mathcal{S}$ has the *factorization property* if for all free f.g. $\mathcal{S}$-semimodules, $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and for all maps $F : \mathcal{H} \to \mathcal{F}$, $G : \mathcal{H} \to \mathcal{G}$. The following assertions are equivalent:

1. $\mathrm{Ker}G \subset \mathrm{Ker}F$
2. there exists a linear map $H : \mathcal{G} \to \mathcal{F}$ such that $F = H \circ G$, i.e.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{F} \\ & \mathcal{G} \arrow[u, dotted, "H"'] \end{tikzcd}$$

Of course, trivially, $2 \Longrightarrow 1$ in any semiring.

DEFINITION 81 (LINEAR EXTENSION). We say that the semiring $\mathcal{S}$ has the *linear extension property* if for all free f.g. $\mathcal{S}$-semimodules $\mathcal{F}, \mathcal{G}$, for all f.g. subsemimodules $\mathcal{H} \subset \mathcal{G}$, and for all $F \in \mathrm{Hom}(\mathcal{H}, \mathcal{F})$, there exists $H \in \mathrm{Hom}(\mathcal{G}, \mathcal{F})$ such that $\forall x \in \mathcal{H},\ H(x) = F(x)$.

THEOREM 82. *In any semiring, the linear extension and the factorization property are equivalent.*

PROOF. Take $F, G, H, \mathcal{F}, \mathcal{G}, \mathcal{H}$ as in Definition 80. Assume that $\mathrm{Ker}G \subset \mathrm{Ker}F$. Then, there exists a map $K \in \mathrm{Hom}(\mathrm{Im}G, \mathcal{F})$ such that $K(G(x)) = F(x)$, for all $x \in \mathcal{H}$. Indeed, for any $y = G(x) \in \mathrm{Im}G$, define $K(y) = F(x)$. Since $\mathrm{Ker}G \subset \mathrm{Ker}F$, the value $K(y)$ is independent of the choice of x such $y = F(x)$. Clearly, the map K is linear, and it is defined on $\mathrm{Im}G$ which is a f.g. subsemimodule of $\mathcal{G}$. If the linear extension property holds, K admits a linear extension $H \in \mathrm{Hom}(\mathcal{G}, \mathcal{F})$. For all $x \in \mathcal{H}$, we have $H \circ G(x) = K(G(x)) = F(x)$, hence $H \circ G = F$.

Conversely, the linear extension property is just the specialization of the factorization property to the canonical injection $G : \mathcal{H} \to \mathcal{G}$. □

When the linear extension property holds, any linear map between two f.g. subsemimodules of free semimodules is represented by a matrix, which is of course particularly convenient from the computational point of view. The conditions on the semiring $\mathcal{S}$ for the linear extension property to hold are not well understood. This property does not hold in $(\mathbb{R}_+, +, \times)$ (e.g. the map $F : (x, y) \mapsto x - y$ is a $\mathbb{R}_+$-linear map

$\{(x, y) \mid x, y \geq 0, x \geq y\} \to \mathbb{R}_+$, and $\{(x, y) \mid x, y \geq 0, x \geq y\}$ is f.g., but F does not admit a linear extension). In the max-plus case, however, the answer is positive, as shown by the following theorem which is an extension of a result stated by Kim for Boolean matrices [**46**] (a slightly different extension to "incline algebras" can be found in [**13**])

THEOREM 83 (LINEAR EXTENSION [**31, 18**]). *The linear extension property holds in an idempotent semifield, which is a distributive*[14] *lattice for the natural order structure.*

In particular, the answer is positive if $\mathcal{S} = \mathbb{B}$ or $\mathcal{S} = \mathbb{R}_{\max}$.

The proof just consists in showing that the *maximal linear subextension* of a linear map F is a linear extension. This requires precisely the distributivity of $\wedge$ and $\vee = \bigoplus$. A similar idea is involved in the proof of the *density theorem* for idempotent measures, or "Riesz representation theorem" for idempotent linear forms of Kolokoltsov and Maslov [**48, 49**], and Akian [**1**]. The general version of the density theorem requires $\mathcal{S}$ to be a dually continuous lattice [**36**] for the natural order, which is a strengthening of the distributivity property that we use here, see [**1**].

7. Finiteness Theorems for Semimodules

The general system of right-linear equations over a semiring $\mathcal{S}$ can be written as $Ax = Bx$, where A and B are $n \times p$ matrices with entries with $\mathcal{S}$. We ask: *is the set of solutions of $Ax = Bx$ a* f.g. *semimodule*? *Are f.g. semimodules stable by operations such that intersection, inverse image, etc.*

A positive answer to these questions can be derived from a basic assumption, that we propose to christen "finiteness property". We call *hyperplane* of $\mathcal{S}^p$ the set of solutions of a single equation $ax = bx$, with $a, b \in \mathcal{S}^{1\times p}$.

DEFINITION 84 (FINITENESS PROPERTY). We say that the semiring $\mathcal{S}$ satisfies the *finiteness property* if for all integers p, hyperplanes of $\mathcal{S}^p$ are finitely generated.

The following general result is an immediate extension of a theorem of Butkovič and Hegedüs [**11**] in the max-plus case, which was rediscovered in [**31**, Chap. III]. It is a "transfer theorem": if an hyperplane is finitely generated, then, any finite intersection of hyperplanes is finitely generated.

THEOREM 85 (ELIMINATION OF EQUALITIES IN SEMIRINGS). *If the semiring $\mathcal{S}$ satisfies the finiteness property, then, for all integers n, p and for all matrices $A, B \in \mathcal{S}^{n\times p}$, the set of solutions of $Ax = Bx$ is a finitely generated semimodule.*

PROOF. We will prove by induction on n that the following property holds:

$$\mathcal{P}_n : \quad \forall p \geq 1, \ \forall A, B \in \mathcal{S}^{n\times p}, \ \exists q \in \mathbb{N}, \ \exists C \in \mathcal{S}^{p\times q}, \ \{x \in \mathcal{S}^p \mid Ax = Bx\} = \mathrm{Im}C \ . \tag{67}$$

This means precisely that $\{x \in \mathcal{S}^p \mid Ax = Bx\}$ is a finitely generated semimodule. The property $\mathcal{P}_1$ is exactly the finiteness property. So, let us assume that $n > 1$, and consider the $(1 + (n - 1)) \times p$ partition of A, B

$$A = \begin{pmatrix} a \\ A' \end{pmatrix} \qquad B = \begin{pmatrix} b \\ B' \end{pmatrix} \ .$$

Using $\mathcal{P}_1$ with a and b instead of A and B, we find an integer q and a matrix $D \in \mathcal{S}^{p\times q}$ such that $ax = bx \iff x = \mathrm{Im}D$. Then:

$$Ax = Bx \iff (ax = bx \text{ and } A'x = B'x) \iff (\exists y \in \mathcal{S}^q, \ x = Dy, \text{ and } A'Dy = B'Dy) \ .$$

Using $\mathcal{P}_{n-1}$ with $A'D$ and $B'D$ instead of A, B, we find an integer r and a matrix $F \in \mathcal{S}^{q\times r}$ such that $A'Dy = B'Dy \iff y \in \mathrm{Im}F$. Thus, $Ax = Bx \iff x \in \mathrm{Im}DF$, which shows that $\mathcal{P}_n$ holds. ☐

Trivially, finitely generated semimodules are stable by sum, by cartesian product, by quotient by a congruence, and by direct morphism. The following additional closure properties are consequences of the finiteness property.

[14] Recall that an idempotent semifield is a lattice for the natural order relation (see. Ex. 27).

COROLLARY 86 (CLOSURE PROPERTIES). *If $\mathcal{S}$ satisfies the finiteness property, then, for all integers n, p, q and for all matrices $A, C \in \mathcal{S}^{n\times p}$, $B, D \in \mathcal{S}^{n\times q}$, the following semimodules all are finitely generated:*

$$\mathrm{Im}A \cap \mathrm{Im}B \qquad \text{(intersection)} \tag{68}$$

$$\{x \in \mathcal{S}^p \mid \exists u \in \mathcal{S}^q,\quad Ax \oplus Du = Cx \oplus Bu\} \qquad \text{(elimination)} \tag{69}$$

$$A^{-1}(\mathrm{Im}B) = \{x \in \mathcal{S}^p \mid Ax \in \mathrm{Im}B\} \qquad \text{(inverse image)} \tag{70}$$

$$\mathrm{Im}A \ominus \mathrm{Im}B \stackrel{\text{def}}{=} \{x \in \mathcal{S}^n \mid \exists y \in \mathrm{Im}B,\ x \oplus y \in \mathrm{Im}A\} \qquad \text{(difference)} \tag{71}$$

Of course, the difference operator $\circleddash$ has nothing to do with the operator $\ominus$ of symmetrized semirings used in§2.2.

PROOF. By Theorem 85, $Ax = Bu \iff (A, \mathbb{0})\binom{x}{u} = (\mathbb{0}, B)\binom{x}{u} \iff \binom{x}{u} \in \mathrm{Im}G$, for some matrix G. Hence, $\mathrm{Im}A \cap \mathrm{Im}B = \mathrm{Im}(A, \mathbb{0})G$, which shows that $\mathrm{Im}A \cap \mathrm{Im}B$ is f.g.

By Theorem 85, $Ax \oplus Cu = Dx \oplus Bu \iff (A, C)\binom{x}{u} = (D, B)\binom{x}{u} \iff \binom{x}{u} \in \mathrm{Im}G$ for some matrix G. Thus, $\exists u$, $Ax \oplus Cu = Dx \oplus Bu$ iff $x \in \mathrm{Im}(\mathrm{Id}, \mathbb{0})G$, which shows that the semimodule (69) is f.g.

By taking $C = \mathbb{0}$ and $D = \mathbb{0}$ in (69), we get that $A^{-1}(\mathrm{Im}B)$ is f.g.

Finally, $\exists y \in \mathrm{Im}B,\ x \oplus y \in \mathrm{Im}A$ is equivalent to $\exists u, z,\quad x \oplus Bu = Az$, and the set of x that satisfy this property is clearly of the form (69). Hence, $\mathrm{Im}A \ominus \mathrm{Im}B$ is f.g. □

Obviously, fields satisfy the finiteness property. For the $(\mathbb{R}_+, +, \times)$ semiring, the finiteness property is a well known (easy) result.

PROPOSITION 87 ([**11**],[**31**, Ch. 3]). *The semiring $\mathbb{R}_{\max}$ satisfies the finiteness property.*

The fact that hyperplanes of $(\mathbb{R}_{\max})^q$ are finitely generated can be checked by elementary means (for instance, the set of solutions of $x \oplus y = y$ is generated by the two vectors $\binom{\mathbb{1}}{\mathbb{1}}$ and $\binom{\mathbb{0}}{\mathbb{1}}$). The number of generators of a minimal generating family of the set of solutions can be of order q^2. Due to this, in the $\mathbb{R}_{\max}$ case, the algorithm involved of the proof of Theorem 85 has a doubly exponential complexity. However, it is possible to incorporate the construction of minimal generating families in the algorithm, and to apply some clever strategy of elimination, which much reduces the execution time. The making (and complexity analysis) of efficient algorithms for $Ax = Bx$ is a major open problem. When only a single finite solution is needed, there is a remarkable reduction, due to Walkup and Borriello (in [**41**]). Walkup and Borriello observe that $Ax = Bx$ is equivalent to $x \leq A^\sharp Bx \wedge B^\sharp Ax$. But $F(x) = A^\sharp Bx \wedge B^\sharp Ax$ is a min-max function, and there is an $x \in \mathbb{R}^n$ such that $q \leq F(x)$ iff the *cycle time* $\chi(F)$ has all its coordinates greater than 0 (see the presentation of J. Gunawardena in these notes). Computing the cycle time $\chi(F)$ can be done very efficiently, and, if it exists, a solution $x \in \mathbb{R}^n$ of $x \leq F(x)$ can be computed via the min-max Howard algorithm presented in [**15, 35**]. Although this algorithm behaves remarquably well, in practice, we do not know whether its worst case execution time is polynomial.

We conclude this section by illustrating these results with subsemimodules of $(\mathbb{R}_{\max})^3$, which can be vizualized easily using a representation which was first used by Mairesse [**58**]. We represent the set of vectors with finite entries of a semimodule $\mathcal{V} \subset (\mathbb{R}_{\max})^3$ by the subset of $\mathbb{R}^2$, obtained by projecting $\mathcal{V}$ orthogonally, on any plane orthogonal to $(1, 1, 1)$. Since $\mathcal{V}$ is invariant by multiplication by any scalar λ, i.e. by the usual addition of the vector $(\lambda, \lambda, \lambda)$, the semimodule $\mathcal{V}$ is well determined by its projection. We only loose the points with $\mathbb{0}$ entries which are sent to some infinite end of the $\mathbb{R}^2$ plane. E.g., the blue(grey) region on the picture at the right of fig. 4 represents the semimodule generated by the vectors

$$u_1 = \begin{pmatrix}3\\0\\0\end{pmatrix},\ u_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\ u_3 = \begin{pmatrix}0\\0\\2\end{pmatrix},\ u_4 = \begin{pmatrix}2\\0\\1\end{pmatrix}.$$

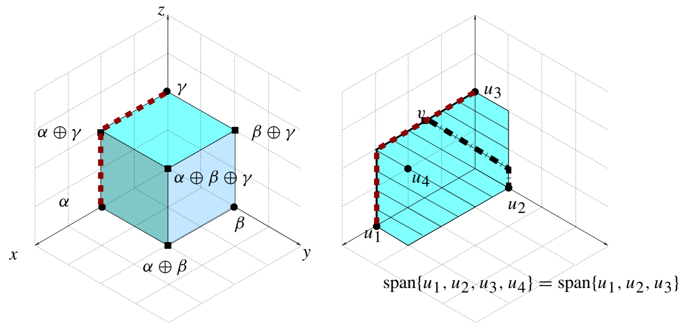


FIGURE 4. How to Draw a Subsemimodule of $(\mathbb{R}_{\max})^3$

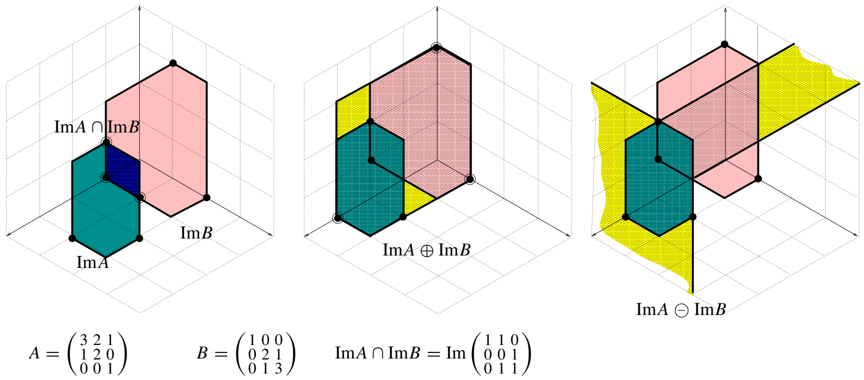


FIGURE 5. Intersection, Sum, and Difference of subsemimodules of $(\mathbb{R}_{\max})^3$.

The real semimodule is just the cylinder in the usual space, with basis this blue set, and with direction the line orthogonal to the present sheet of paper.

Remark 88. There is a very simple but tricky way to draw this representation from u_1, u_2, u_3, u_4, *without computation*. The procedure is indeed exactly similar to the computations of the usual convex hull of a finite set. First, the reader should (easily) convince himself that, on the simpler picture at the left of Fig. 4, the semimodule generated by α and γ is the dashed broken line which joins $\alpha, \alpha \oplus \gamma$ and γ. Then, it is not difficult to see that the semimodule generated by u_1, u_3, which is in a similar position, is the red dashed broken line from u_1 to u_3 displayed at the right of Fig. 4. To compute the semimodule generated by u_1, u_3, u_2, we have, for all v in the semimodule generated by u_1 and u_3, to draw the semimodule generated by v and u_2 (black –dashed– broken line joining v and u_2), and to take the union of these broken lines, which gives precisely the blue region.

Using this representation, it is fairly easy to compute —graphically— the intersection, sum, and difference of semimodules, as shown on Fig. 5.

Exercise 89. Show that the semimodules $\mathrm{Im}A \cap \mathrm{Im}B$ and $\mathrm{Im}A \ominus \mathrm{Im}B$ depicted in Fig. 5 are correct, using the algorithms of the proof of Corollary 86.

8. Minimal Generating Families, Convex Geometries, and Max-Plus Projective Geometry

A *convex cone* of $\mathbb{R}_+^n$ is a $\mathbb{R}_+$-subsemimodule, i.e. a set $\mathcal{C} \subset \mathbb{R}_+^n$ such that $0 \in \mathcal{C}$, and, $\forall u, v \in \mathcal{C}, \forall \lambda \in \mathbb{R}_+$, $u + v \in \mathcal{C}, \lambda u \in \mathcal{C}$. A *polyhedral cone* is a convex cone that is finitely generated as a $\mathbb{R}_+$-semimodule. It follows easily from the classical Krein-Milman theorem that a polyhedral cone $\mathcal{C}$ admits a minimal generating family, which is obtained by selecting exactly one vector in each *extremal ray* of $\mathcal{C}$, and that all the minimal generating families are obtained in this way. The same property holds for the max-plus semiring. In fact, it is not more difficult, and it probably sheds more light on the nature of the argument, to prove a similar property for a larger class of semirings, using axioms which encompass both $(\mathbb{R}_+, +, \times)$ and $\mathbb{R}_{\max}$. Hence, a version of the usual Krein-Milman theorem for polyhedral cones will be reobtained as a specialization of the results of this section.

Let us first recall some classical results about abstract convex geometries, which are taken from [**50**]. We say that a *finite* set E equipped with a map $\tau : \mathcal{P}(E) \to \mathcal{P}(E)$ is a *finite convex geometry* if

$$\tau(\emptyset) = \emptyset \tag{72a}$$

$$X \subset Y \implies \tau(X) \subset \tau(Y) \tag{72b}$$

$$X \subset \tau(X) \tag{72c}$$

$$\tau(\tau(X)) = \tau(X) \ , \tag{72d}$$

and if the following *anti-exchange property*[15] holds:

$$y, z \notin \tau(X),\ y \neq z,\ y \in \tau(X \cup z) \implies z \notin \tau(X \cup y) \ . \tag{73}$$

Of course, τ is an abstraction of the usual "convex hull" operator. We say that x is an *extremal element* or extremal point of X if $x \notin \tau(X \setminus x)$. A set X is *convex* if $X = \tau(X)$. We say that Y is a *spanning set* of X if $\tau(Y) = X$. We will need the following general result.

THEOREM 90 (ABSTRACT KREIN MILMAN THEOREM [**50**]). *Let (E, τ) denote a finite convex geometry. A convex subset $X \subset E$ has a unique minimal spanning set, whose elements are precisely the extremal elements of X.*

In order to apply these notions to semimodules, we need to introduce the *projective space*, which in the max-plus case, already appeared in [**47, 32, 58**].

DEFINITION 91 (PROJECTIVE SPACE). We say that two vectors x, y of a $\mathcal{S}$-semimodule $\mathcal{X}$ are *equivalent*, and we write $x \sim y$ if there exists an invertible scalar $\lambda \in \mathcal{S}$ such that $x = \lambda y$. The quotient of $\mathcal{X} \setminus \mathbb{0}$ by the equivalence relation $\sim$ is the *projective space* of $\mathcal{X}$, that we denote by $\mathbb{P}\mathcal{X}$ An element of $\mathbb{P}\mathcal{X}$ is a *ray*.

The map span : $\mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ passes to the quotient: $\mathcal{P}(\mathbb{P}\mathcal{X}) \to \mathcal{P}(\mathbb{P}\mathcal{X})$. If $\overline{X}$ is a set of rays, and if X is a set of representatives of the elements of $\overline{X}$, we still denote by $\mathrm{span}\overline{X} \subset \mathbb{P}\mathcal{X}$ the set of equivalence classes of elements of $\mathrm{span}X$. Clearly, the operator span : $\mathcal{P}(\mathbb{P}\mathcal{X}) \to \mathcal{P}(\mathbb{P}\mathcal{X})$ satisfies the properties (72).

DEFINITION 92 (GOOD CONVEX SEMIMODULES). If $\mathcal{S}$ is a naturally ordered semiring, we say that a $\mathcal{S}$-semimodule $\mathcal{X}$ is *good convex* if the following properties hold:

1. $\mathcal{X}$ is naturally ordered[16];
2. $\forall\lambda \in \mathcal{S}, \forall x \in \mathcal{X} \setminus \mathbb{0}, x \succeq \lambda x \implies \mathbb{1} \succeq \lambda$;
3. $\forall\lambda \in \mathcal{S}, \forall x, b \in \mathcal{X}$,

$$(\mathbb{1} \succ \lambda \text{ and } x = \lambda x \oplus b) \implies \exists\beta \in \mathcal{S}, x = \beta b \ . \tag{74}$$

We say that a semiring $\mathcal{S}$ is good convex if the $\mathcal{S}$-semimodule $\mathcal{S}$ is good convex.

Remark 93. 1. The semirings $(\mathbb{R}_+, +, \times)$ and $\mathbb{R}_{\max}$ are good convex. 2. For all integers n, the free semimodule $\mathcal{S}^n$ is good convex iff $\mathcal{S}$ is good convex (the conditions of Definition 92 hold componentwise). 3. A subsemimodule of a good convex semimodule is good convex. 4. A less trivial example of good convex semiring is $\mathbb{R}_{\max}[X]$ (the semiring of polynomials in one indeterminate with coefficients in $\mathbb{R}_{\max}$).

THEOREM 94 (ANTI-EXCHANGE PROPERTY). *If $\mathcal{X}$ is a good convex semimodule, then, for all $\overline{X} \subset \mathbb{P}\mathcal{X}$, and for all $\overline{y}, \overline{z} \in \mathbb{P}\mathcal{X}$, there holds*

$$\overline{y}, \overline{z} \notin \mathrm{span}(\overline{X}),\ \overline{y} \in \mathrm{span}(\overline{X} \cup \overline{z}),\ \overline{y} \neq \overline{z} \implies \overline{z} \notin \mathrm{span}(\overline{X} \cup \overline{y}) \ . \tag{75}$$

[15] The anti-exchange property is the opposite of the more familiar exchange property which holds when τ is interpreted as the "linear span" operator (in vector spaces): $y \notin \tau(X), y \in \tau(X \cup z) \implies z \in \tau(X \cup y)$.

[16] I.e. the relation $\preceq$ on $\mathcal{X}$ defined by (48) is an order relation.

PROOF. Let us assume, by contradiction, that the hypotheses of (75) are satisfied, and that $\overline{z} \in \mathrm{span}(\overline{X} \cup \overline{y})$. Choose $z \in \overline{z}$ and $y \in \overline{y}$. We can find $x \in \mathrm{span}(X)$ and $\lambda \in \mathcal{S}$ such that

$$z = x \oplus \lambda y \quad . \tag{76}$$

Since $\overline{y} \in \mathrm{span}(\overline{X} \cup \overline{z})$, a symmetrical property holds for y, namely

$$y = x' \oplus \mu z \quad , \tag{77}$$

for some $x' \in \mathrm{span}(X)$ and $\mu \in \mathcal{S}$. Substituting (77) in (76), we get

$$z = x \oplus \lambda x' \oplus \lambda\mu z \quad . \tag{78}$$

This implies that $z \succeq \lambda\mu z$. Then, using the second axiom of Definition 92, we get $\mathbb{1} \succeq \lambda\mu$. If the strict inequality holds, then, using the last axiom of Definition 92 and (78), we get $z = \beta(x \oplus \lambda x')$ for some $\beta \in \mathcal{S}$, hence $z \in \mathrm{span}(X)$: a contradiction. Thus, $\lambda\mu = \mathbb{1}$, and, exchanging the role of y and z in the above argument, $\mu\lambda = \mathbb{1}$, which shows that λ and μ are invertible. Moreover, $z \succeq \lambda y \succeq \mu\lambda z = z$. Since, by the first axiom of Definition 92, $\mathcal{X}$ is naturally ordered, the relation $\succeq$ is antisymmetric. Hence, $z = \lambda y$, which shows that $\overline{z} = \overline{y}$: a contradiction. □

As in the case of finite convex geometries, we say that a set of rays $\overline{X}$ is *convex* if $\overline{X} = \mathrm{span}(\overline{X})$. A *spanning set* of $\overline{X}$ is a set of rays $\overline{Y}$ such that $\mathrm{span}(\overline{Y}) = \overline{X}$. We say that a convex set $\overline{X}$ is *finitely generated* if it has a finite spanning set.

COROLLARY 95 (KREIN-MILMAN THEOREM FOR GOOD CONVEX SEMIMODULES). *A finitely generated convex set of rays of a good convex semimodule has a unique finite minimal spanning set.*

We could easily modify the proof of the abstract Krein-Milman theorem to make it work in this context. This raises more generally the interesting question of extending this abstract theorem to some reasonable classes of infinite structures. But it is also possible to reduce formally corollary 95 to the abstract Krein-Milman theorem, as follows.

PROOF. Proof of Corollary 95 For all finite spanning sets $\overline{E}$ of $\overline{X}$, consider $\tau : \mathcal{P}(\overline{E}) \to \mathcal{P}(\overline{E})$, $\tau(\overline{U}) = \mathrm{span}(\overline{U}) \cap \overline{E}$. We claim that

$$\begin{aligned} &(i) \quad (\overline{E}, \tau) \text{ is a finite convex geometry} \\ &(ii) \quad \tau(\overline{U}) = \overline{E} \iff \mathrm{span}(\overline{U}) = \overline{X} \end{aligned}$$

The properties *(ia)*: $\tau(\emptyset) = \emptyset$; *(ib)*: $\overline{U} \subset \overline{V} \implies \tau(\overline{U}) =\subset \tau(\overline{V})$; and *(ic)*: $U \subset \tau(\overline{U})$ are clear. Since the map $\overline{U} \mapsto \mathrm{span}(\overline{U})$ is monotone for inclusion, we have $\tau(\tau(\overline{U})) = \mathrm{span}(\mathrm{span}(\overline{U}) \cap \overline{E}) \cap \overline{E} \subset \mathrm{span}(\mathrm{span}(\overline{U})) \cap \mathrm{span}(\overline{E}) \cap \overline{E} = \tau(\overline{U})$. Since, by *(ic)*, $\tau(\tau(\overline{U})) \supset \tau(\overline{U})$, we have *(id)*: $\tau(\tau(\overline{U})) = \tau(\overline{U})$. Finally, the anti-exchange property for τ is an immediate consequence of the anti-exchange property for $\overline{X}$, which completes the proof of *(i)*. If $\mathrm{span}(\overline{U}) = \overline{X}$, then, $\tau(\overline{U}) = \overline{E}$. Conversely, if $\tau(\overline{U}) = \overline{E}$, we have $\mathrm{span}(\overline{U}) \cap \overline{E} = \overline{E}$, hence, $\mathrm{span}(\overline{U}) \supset \overline{E}$, and $\overline{X} \supset \mathrm{span}(U) = \mathrm{span}(\mathrm{span}(\overline{U})) \supset \mathrm{span}(\overline{E}) = \overline{X}$, which shows *(ii)*. Then, the abstract Krein-Milman theorem shows the *existence* of a minimal $\overline{Y} \subset \overline{E}$ such that $\tau(\overline{Y}) = \tau(\overline{E}) = \overline{X}$, and, thanks to Property *(ii)*, $\overline{Y}$ is a minimal spanning set of $\overline{X}$. Take another finite minimal spanning set of $\overline{X}$, $\overline{Z}$, and consider $\overline{E} = \overline{Y} \cup \overline{Z}$. There is a minimal $\overline{T} \subset \overline{E}$ such that $\tau(\overline{T}) = \overline{E}$. Since $\tau(\overline{Z}) = \overline{E}$, by minimality of $\overline{T}$, $\overline{T} \subset \overline{Z}$. By minimality of $\overline{Z}$, $\overline{Z} \subset \overline{T}$. Hence $\overline{Z} = \overline{T}$ and by symmetry $\overline{Y} = \overline{T} = \overline{Z}$, which shows that $\overline{Y}$ is the unique minimal finite spanning set of $\overline{X}$. □

As an immediate consequence of Corollary 95, we obtain the *basis theorem* which was proved by Moller [**63**] and Wagneur [**79**] essentially when $\mathcal{S} = \mathbb{R}_{\max}$.

COROLLARY 96 (BASIS THEOREM). *If $\{u_i\}_{1 \le i \le k}$ and $\{v_j\}_{1 \le j \le l}$ are minimal generating families of a good convex $\mathcal{S}$-semimodule $\mathcal{X}$, then, $k = l$, and there is a permutation σ of $\{1, \dots, k\}$ and a family $\{\lambda_i\}_{1 \le i \le k}$ of invertible elements of $\mathcal{S}$ such that $u_i = \lambda_i v_{\sigma(i)}$, for all $i \in I$.* □

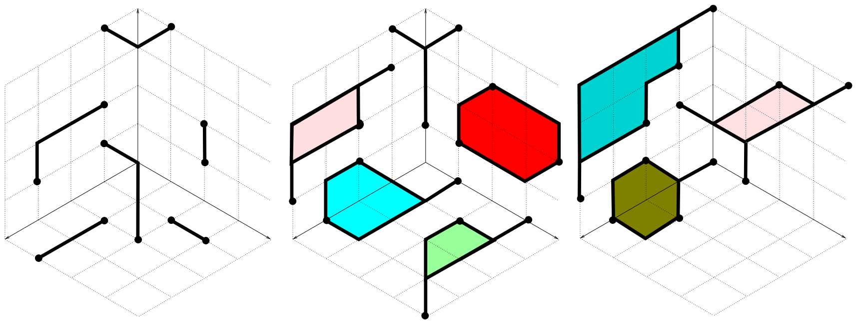


FIGURE 6. Subsemimodules of $(\mathbb{R}_{\max})^3$

Thus, two minimal generating families have the same cardinality. We call this cardinality the *weak rank* of the semimodule, denoted $\mathrm{rk}_w \mathcal{V}$. The *weak column rank* (resp. *weak row rank*) of a matrix A is the weak rank of its column (resp. row) space $\mathrm{Im} A$ (resp. $\mathrm{Im} A^T$). Unlike in usual algebra, the weak row rank in general differs from the weak column rank (this is already the case for Boolean matrices [**46**]).

We conclude this section on max-plus "convexity" by a max-plus analogue of Caratheodory theorem, which usually states that in dimension n, a point which is in the convex hull of a set X is in the convex hull of at most $n+1$ points of X.

THEOREM 97 (MAX-PLUS CARATHEODORY THEOREM). *If $X \subset (\mathbb{R}_{\max})^n$, and if $x \in \mathrm{span}(X)$, then, there is a subset Y of X, of cardinality at most n, such that $x \in \mathrm{span}(Y)$.*

PROOF. If $x \in \mathrm{span}(X)$, we can find $u^1, \dots, u^k \in X$, $\lambda_1, \dots, \lambda_k \in \mathcal{S}$ such that $x = \bigoplus_{1 \le i \le k} \lambda_i u^i$. For all $1 \le j \le n$, we define $I(j) = \{i \mid 1 \le k \le n,\ x_i = \lambda_i u^i_j\}$. Since $\mathbb{R}_{\max}$ is linearly ordered, all the $I(j)$ are non-empty. Choosing exactly one element i_j in $I(j)$, we obtain a family $u^{i_1}, \dots, u^{i_n}$ such that $x = \lambda_{i_1} u^{i_1} \oplus \dots \oplus \lambda_{i_n} u^{i_n}$. □

Example 98. Semimodules of weak rank 2, 3, 4 are represented on the three pictures of Fig. 6, respectively. The elements of minimal generating families are the extremal (bold) points of these compact connected regions.

9. Equivalence of matrices and Green classes

DEFINITION 99. The *Green relations* $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{J}$ on the set of matrices with entries in an arbitrary semiring $\mathcal{S}$, are defined as follows:

$$A \,\mathcal{L}\, B \iff \exists U, V,\ A = UB,\ B = VA\ , \tag{79a}$$

$$A \,\mathcal{R}\, B \iff \exists U, V,\ A = BU,\ B = AV\ , \tag{79b}$$

$$A \,\mathcal{D}\, B \iff \exists C,\ A \,\mathcal{L}\, C \text{ and } C \,\mathcal{R}\, B\ , \tag{79c}$$

$$A \,\mathcal{J}\, B \iff \exists U, V, X, Y,\ A = UBV,\ B = XAY\ , \tag{79d}$$

where A, B, U, V, X, Y are (possibly rectangular) matrices with entries in $\mathcal{S}$.

These relations are classical (and of great importance) in monoid and semigroup theory, see e.g. [**56**]. Of course, the set of matrices with entries in $\mathcal{S}$ is *not* a multiplicative semigroup, because the product of two matrices is not defined if the size is incompatible. This will not create difficulties in the sequel.

Clearly, the Green relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ are equivalence relations. Classically [**56**], $A \,\mathcal{D}\, B$ iff $\exists C$, $A \,\mathcal{R}\, C$ and $C \,\mathcal{L}\, B$. That is, $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, where $\circ$ denotes the composition of relations. This implies that $\mathcal{D}$ is also an equivalence relation. There exist another Green relation, namely, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, that we will not use here.

THEOREM 100. *Let A, B denote two matrices with entries in $\mathbb{R}_{\max}$. We have:*

1. $A \mathcal{R} B \iff \mathrm{Im}A = \mathrm{Im}B \iff \mathrm{Ker}A^T = \mathrm{Ker}B^T$*;*
2. $A \mathcal{L} B \iff \mathrm{Im}A^T = \mathrm{Im}B^T \iff \mathrm{Ker}A = \mathrm{Ker}B$*'*
3. $A \mathcal{D} B \iff \mathrm{Im}A$ *and* $\mathrm{Im}B$ *are isomorphic* $\iff \mathrm{Im}A^T$ *and* $\mathrm{Im}B^T$ *are isomorphic.*

PROOF. The first parts of item 1 and 2 follows from Prop. 78 (this holds in any semiring $\mathcal{S}$, and not only in $\mathbb{R}_{\max}$).

We have $\mathrm{Ker}A = (\mathrm{Im}A^T)^\perp$ (see 3.5 of Chap. 1 for the definition). In any semiring $\mathcal{S}$ with satisfies the duality theorem 9 of Chap. 1, and in particular in $\mathcal{S} = \mathbb{R}_{\max}$, $\mathrm{Ker}A = \mathrm{Ker}B \implies ((\mathrm{Im}A^T)^\perp)^\top = ((\mathrm{Im}B^T)^\perp)^\top \implies \mathrm{Im}A^T = \mathrm{Im}B^T \implies A \mathcal{L} B$. The other implication is obvious. We have shown the second equivalences in items 1 and 2

Let us prove Part 3. If $A \mathcal{L} C$ and $C \mathcal{R} B$, then, using assertion 1 of the theorem, $\mathrm{Im}B = \mathrm{Im}C$. We show that $\mathrm{Im}C$ is isomorphic to $\mathrm{Im}A$. We have $A = UC$ and $C = VA$. The linear maps $U' : \mathrm{Im}A \to \mathrm{Im}C, x \mapsto Ux$, $V' : \mathrm{Im}C \to \mathrm{Im}A$, are mutually inverse. Thus, $\mathrm{Im}A \simeq \mathrm{Im}C$. Conversely, assume that $\mathrm{Im}A \simeq \mathrm{Im}B$. Then, there exist two linear maps $U' \in \mathrm{Hom}(\mathrm{Im}A, \mathrm{Im}B)$, $V' \in \mathrm{Hom}(\mathrm{Im}B, \mathrm{Im}A)$ which are mutually inverse. Using the Linear Extension Theorem 83, we find two matrices U, V such that $\mathrm{Im}UA = \mathrm{Im}B, \mathrm{Im}VB = \mathrm{Im}A$. Since $VUA = A$, $A \mathcal{L} UA$. Since $\mathrm{Im}UA = \mathrm{Im}B$, by Part 1 of the theorem, we get $UA \mathcal{R} B$. Thus, $A \mathcal{D} B$. □

We say that a matrix A is *column compressed* if the set of its columns is a minimal generating family of its column space, $\mathrm{Im}A$. We say that A is *row compressed* if A^T is column compressed. The following lemma is a variant of a well known result for Boolean matrices.

LEMMA 101 (RAO "NORMAL FORMS"). *Let A denote a matrix with entries in* $\mathbb{R}_{\max}$.

1. *We can find a row compressed matrix* $A' \mathcal{L} A$ *and a matrix* X *such that* $A = P \begin{pmatrix} A' \\ XA' \end{pmatrix}$ *for some permutation matrix* P.
2. *We can find a column compressed matrix* $A'' \mathcal{R} A$ *and a matrix* Y *such that* $A = \begin{pmatrix} A'' & A''Y \end{pmatrix} Q$ *for some permutation matrix* Q.
3. *We can find a matrix* $\overline{A} \mathcal{D} A$*, which is both row and column compressed, and two matrices* X, Y *such that*

$$A = P \begin{pmatrix} \overline{A} & \overline{A}Y \\ X\overline{A} & X\overline{A}Y \end{pmatrix} Q \ , \tag{80}$$

for some permutation matrices P, Q.

We call (80) a Rao "normal form" of A.

PROOF. By Corollary 95, we can find a subset of the rows of A that is a minimal generating family of $\mathrm{Im}A^T$. The matrix A' composed only of these rows satisfies statement 1. The proof of statement 2 is dual. Applying statement 2 to A', we get $A' = \begin{pmatrix} \overline{A} & \overline{A}Y \end{pmatrix} Q$, where Q is a permutation matrix, and $A' \mathcal{R} \overline{A}$. Thus, we obtain the factorization (80), and since $A \mathcal{L} A' \mathcal{R} \overline{A}$, $A \mathcal{D} \overline{A}$. □

THEOREM 102. *Let* A, B *denote two matrices with entries in* $\mathbb{R}_{\max}$.

1. *If* A, B *are row compressed, then* $A \mathcal{L} B$ *iff* $A = PB$ *for some invertible monomial matrix* P*;*
2. *If* A, B *are column compressed, then* $A \mathcal{R} B$ *iff* $A = BQ$ *for some invertible monomial matrix* Q*;*
3. *If* A, B *are both row and column compressed, then* $A \mathcal{D} B$ *iff* $A = PBQ$ *for some invertible monomial matrices* P, Q.

PROOF. Statements 1 and 2 are immediate consequences of Corollary 96.

Trivially, if $A = PBQ$ for some invertible monomial matrices P, Q, then $A \mathcal{D} B$. Conversely, if $A \mathcal{D} B$, by Theorem 100, $\mathrm{Im}A$ and $\mathrm{Im}B$ are isomorphic, hence, their finite minimal generating families have the same cardinality, and if A, B are column compressed, they have the same number of columns. Dually, A and B have the same number of rows. Since $A \mathcal{D} B$, we have $A \mathcal{L} C$ and $C \mathcal{R} B$ for some C, which has the same size as A, B. Since $\mathrm{Im}A^T = \mathrm{Im}C^T$, C is row compressed, and, using statement 1,

$A = PC$ for some invertible monomial matrix P. Dually, $C = BQ$, for some invertible monomial matrix Q. Thus, $A = PCQ$. □

In [**80, 81, 66**], Wagneur studied equivalence classes of matrices modulo the relation $A \equiv B$ if $A = PBQ$ for some invertible monomial matrices P, Q. Lemma 101 and Theorem (102) shows that the heart of the difficulty, in characterizing matrices with isomorphic row or column spaces, consists precisely in testing whether the Wagneur's relation $A \equiv B$ holds.

Thanks to residuation, we can test in polynomial time if $A \mathcal{L} B$ or $A \mathcal{R} B$. Indeed, we only have to check whether $\mathrm{Im} A^T = \mathrm{Im} B^T$ or $\mathrm{Im} A = \mathrm{Im} B$. See the presentations of G. Cohen and P. Butkovic in this school. However, no efficient algorithm to test whether $A \mathcal{D} B$ or whether $A \equiv B$ is currently known. Testing naively the existence of monomials matrices P, Q such that $A = PBQ$ has of course an unacceptable complexity.

Counter Example 103 ($\mathcal{D} \neq \mathcal{J}$). For matrices with entries in $\mathbb{R}_{\max}$, $\mathcal{D} \neq \mathcal{J}$. The situation is analogous to the case of semigroups of non-negative matrices, and we can adapt the counter example given in [**6**, Ch. 3,p. 73], as follows. For all $x, y, z \in \mathbb{R}_{\max}$, consider the matrix

$$M_{xyz} = \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ x & \mathbb{1} & \mathbb{0} \\ y & z & \mathbb{1} \end{pmatrix}$$

Clearly, M_{xyz} is both row and column compressed. Hence, using Theorem 102, it is easy to check that, if x, y, z, x', y, z' all are different from $\mathbb{0}$,

$$M_{xyz} \,\mathcal{D}\, M_{x'y'z'} \iff \frac{y}{xz} = \frac{y'}{x'z'} \ .$$

Hence, $A = M_{2,\mathbb{1},-2} \not\mathcal{D}\; B = M_{2,\mathbb{1},\mathbb{1}}$. However, $AM_{\mathbb{0},\mathbb{0},\mathbb{1}} = B$, and

$$\begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{1} & \mathbb{0} & -2 \end{pmatrix} B \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & 2 \end{pmatrix} = A \ ,$$

hence, $A \mathcal{J} B$.

A matrix A is *regular* if it has a g-inverse, i.e. if $A = AXA$ for some matrix X. If B and C are $n \times p$ and $q \times n$ matrices, we say that $\mathrm{Im} B$ and $\mathrm{Ker} C$ are *direct factors* if (*) for all $x \in (\mathbb{R}_{\max})^n$, there exists a unique $z \in \mathrm{Im} B$ such that $Cz = Cx$. Then, the map $P : x \mapsto z$ is linear and satisfies $P = P^2$. It is called the *projector* on $\mathrm{Im} B$ parallel to $\mathrm{Ker} C$. It is proved in [**18**] that $\mathrm{Im} B$ and $\mathrm{Ker} C$ are direct factors iff there exist two matrices K, L such that

$$B = LCB \quad \text{and} \quad C = CBK \ . \tag{81}$$

These conditions can be tested easily via residuation. We say that $\mathrm{Im} B$ admits a direct factor if the condition (*) holds for some matrix C. Conversely, we say the $\mathrm{Ker} C$ admits a direct factor if the condition (*) holds for some matrix B.

THEOREM 104 (REGULAR ELEMENTS = IMAGES OF PROJECTORS [**18**]). *If $A \in (\mathbb{R}_{\max})^{n \times p}$, then, the following conditions are equivalent:*

1. $\mathrm{Im} A$ *admits a direct factor;*
2. $\mathrm{Ker} A$ *admits a direct factor;*
3. $\mathrm{Im} A = \mathrm{Im} P$ *for some matrix* $P = P^2$*;*
4. A *is regular.*

PROOF. The equivalences $1 \iff 4 \iff 3$ are proved in [**18**]. The equivalence $4 \iff 2$ follows from the equivalence $1 \iff 4$ and from the symmetry of the characterization (81). □

The results of [**18**] hold more generally in distributive idempotent semifields, and not only in $\mathbb{R}_{\max}$.

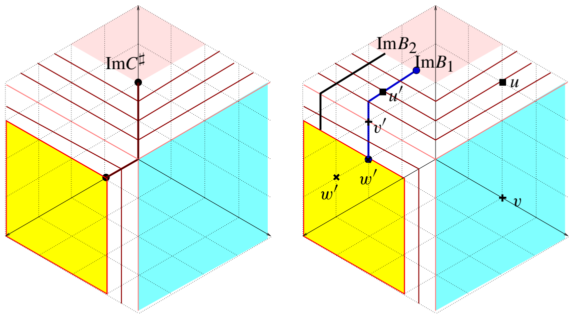


FIGURE 7. Projection on an Image parallel to a Kernel, in $(\mathbb{R}_{\max})^3$.

Example 105. The picture at left of Fig. 7 represents the partition in equivalence classes for the congruence Ker$C$, with $C = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \end{smallmatrix}\right)$. On the other picture, we have represented the semimodules ImB_1 and ImB_2, with $B_1 = \begin{pmatrix} 1 & 0 \\ -1 & -0.5 \\ 0 & 2 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 1 & 0 \\ -2.5 & -1.5 \\ 0 & 2 \end{pmatrix}$. Since any equivalence class modulo Ker$C$ meets ImB_1 at exactly one point, Ker$C$ and ImB_1 are direct factors. E.g., we have represented the projections u', v', w' of three points u, v, w, on ImB_1, parallel to Ker$C$. The image ImB_2 is also a direct factor to Ker$C$.

10. Hints and Answers to Exercises

10.1. Hints for Exercises.

Hint for Exercise 43. Compute $(a \oplus b)^2 \otimes (a \oplus b)$ and $(a \oplus b)^3$ using the Binomial formula.

First Hint for Exercise 45. Associate with each compact convex subset of $\mathbb{R}^k$ the map $\delta^*_A : \mathbb{R}^k \to \mathbb{R}$, defined by $\delta^*_A(p) = \sup_{a \in A} \sum_{1 \le i \le k} p_i \times a_i$,. Show that $\delta^*_{A+B} = \delta^*_A + \delta^*_B$, and that if A is non-empty, $\delta^*_A(p)$ is finite, for all p.

Second Hint for Exercise 45. Show that $\delta^*_A = \delta^*_B$ implies $A = B$, when A, B are compact convex sets.

Last Hint for Exercise 45. Use Hahn-Banach theorem.

Hint for Exercise 53. Compute the weights of the circuits of A, and look at Exercise 52.

10.2. Partial Answers to Exercises.

Answer to Exercise 38. $1 \otimes 3 = 4$; $3^2 = 6$; $\sqrt[3]{12} = 4$, for $4^3 = 12$; $2/3 = -1$, for $3 \otimes (-1) = 2$, $\sqrt{-1} = -0.5$ (we cannot write $-1/2$ which could be confused with -3), $\left(\begin{smallmatrix} 0 & \mathbb{0} \\ 3 & 2 \end{smallmatrix}\right) \otimes \left(\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 5 \\ 8 \end{smallmatrix}\right)$. The identities $a^k a^l = a^{k+l}$ and $(a^k)^l = a^{kl} (= a^{k \times l})$ clearly holds.

Answer to Exercise 39. Since $\mathbb{0}$ is absorbing, $a \otimes \mathbb{0} = \mathbb{0}$, but, if $\mathbb{0} = \mathbb{1}$, $a \otimes \mathbb{0} = a \otimes \mathbb{1} = a$. Hence, $a = \mathbb{0}$. This shows that $\mathcal{S}$ is reduced to a single element, $\mathbb{0} = \mathbb{1}$.

Answer to Exercise 41. We have $x +_\infty y = x$ if $|x| > |y|$, by symmetry, $x +_\infty y = y$ if $|x| < |y|$, and $x +_\infty x = x$, and $x +_\infty (-x) = 0$. This law is non-associative. E.g. $2 = (3 +_\infty (-3)) + 2 \neq 3 +_\infty (-3 +_\infty 2) = 0$.

Answer to Exercise 46. Answer: each permutation $g \in \mathfrak{S}_n$ is a finite product of transpositions $\tau_1, \ldots, \tau_{n-1}$, and $s^{(\infty)}_g$ gives the minimal length of such a product. E.g., when $n = 2$, $s^{(\infty)} = e \oplus 1\tau_1 \oplus 1\tau_2 \oplus 2\tau_1\tau_2 \oplus 2\tau_2\tau_1$. More generally, $\operatorname{supp} s^{(\infty)} = \mathfrak{S}_n$ and $\max_{g \in \mathfrak{S}_n} s^{(\infty)}_g = n - 1$.

Answer to Exercise 69. The only complete semifields are the trivial one, in which $\mathbb{0} = \mathbb{1}$, and the Boolean semifield $\mathbb{B} = \{\mathbb{0}, \mathbb{1}\}$. This characterization holds a fortiori for finite idempotent semifields, which are trivially complete, Indeed, in a complete semiring $\mathcal{S}$, we can define $\top = \bigoplus_{x \in \mathcal{S}} x$. For all finite subset Y of $\mathcal{S}$, we have, by definition, $\bigoplus_{y \in Y} y \le \top$. Then, using (50a), we get $\bigoplus_{y \in Y} y\mathbb{1} \otimes \left(\bigoplus_{y \in Y} y\right) \preceq \top \otimes \top$. Since this inequality holds for all finite $Y \subset \mathcal{S}$, we get $\top \preceq \top \otimes \top$, but $\top$ is the greatest element of $\mathcal{S}$, and thus, $\top = \top \otimes \top$. If $\top = \mathbb{0}$, we have, for all $a \in \mathcal{S}$, $\mathbb{0} \preceq a \preceq \mathbb{0}$, which shows that $\mathcal{S} = \mathbb{0}$. If $\top \neq \mathbb{0}$, we deduce that $\top = \mathbb{1}$. Hence, for all $a \neq \mathbb{0}$, $a \preceq \mathbb{1}$. But, then, $a = (a^{-1})^{-1}$, and by (50b), $\mathbb{1} \preceq a$. Hence, $a = \mathbb{1}$. This shows that $\mathcal{S} = \mathbb{B}$.

The max-plus policy improvement algorithm

In this appendix, we present a policy iteration algorithm, which computes the *cycle time vector* [**15**] of max-plus linear dynamical systems, and, a fortiori, the unique eigenvalue of an irreducible matrix with entries in the max-plus algebra. An iteration of the algorithm requires a linear time and a linear memory space. The average number of iterations N_H of the algorithm is experimentally[17] remarkably small, say $N_H \simeq 50$ for a multigraph with 10^6 nodes and 5×10^6 edges. The worst case number of iterations is uncertain. It does not seem to exceed substantially in magnitude the number of nodes, but we do not know a proof of this fact. The materials of this appendix are detailed in [**14**]. Classical Howard's policy iteration algorithms are surveyed in [**67**]. See [**15, 35**] for extensions to min-max functions.

The input of the algorithm is a directed multigraph $\mathcal{G}$ with set of nodes $\mathcal{N}$ and set of edges $\mathcal{E}$, equipped with two valuations $w : \mathcal{E} \to \mathbb{R}$, $\tau : \mathcal{E} \to \mathbb{R}_+$. We denote by e^- (resp. e^+) the initial (resp. terminal) node of an edge e. We assume that each node as at least one outgoing edge, and that there are no circuits c such that $\tau(e) = 0$ for all $e \in c$. The max-plus policy iteration algorithm returns two vectors η and x in $\mathbb{R}^{\mathcal{N}}$, that satisfy, for all $i \in \mathcal{N}$,

$$\eta_i = \max_{e^-=i} \eta_{e^+} \tag{82a}$$

$$x_i = \max_{e^-=i,\ \eta_i=\eta_{e^+}} (w(e) - \tau(e) \times \eta_{e^+} + x_{e^+}) \ . \tag{82b}$$

If the multigraph is strongly connected, then, $\eta = (\lambda, \dots, \lambda)$, for some $\lambda \in \mathbb{R}$, and we have

$$\lambda = \max_c \frac{\sum_{e\in c} w(e)}{\sum_{e\in c} \tau(e)} \ , \tag{83}$$

where the max is taken over the circuits c of the graph. In general, $\eta = \lim_{k\to\infty} z(k)/k$, where $z(k) \in \mathbb{R}^{\mathcal{N}}$ is any trajectory of the max-plus linear recurrent dynamical system:

$$z_i(k) = \max_{e^-=i} (w(e) + z_{e^+}(k - \tau(e))) \ , \ \forall k \geq 0, \forall i \in \mathcal{N}$$

A *policy* is a map $\pi : \mathcal{N} \to \mathcal{E}$, such that, for all $i \in \mathcal{N}$, $i \in \pi(i)^-$. The *graph* of π is the subgraph of $\mathcal{G}$ with edges $\pi(i)$, for $i \in \mathcal{N}$. Given a policy π, the following algorithm returns two vectors η^π, x^π that satisfy, for all $i \in \mathcal{N}$,

$$\begin{aligned} \eta^\pi_i &= \eta^\pi_{\pi(i)^+} \\ x^\pi_i &= w(\pi(i)) - \tau(\pi(i)) \times \eta^\pi_{\pi(i)^+} + x^\pi_{\pi(i)^+} \end{aligned}$$

ALGORITHM 106 (VALUE DETERMINATION). 1. Find a circuit c in the graph of π.

2. Set $\overline{\eta} = (\sum_{e\in c} w(e))/(\sum_{e\in c} \tau(e))$.
3. Select an arbitrary node i in c, set $\eta^\pi_i = \overline{\eta}$ and set x_i to an arbitrary value, say $x_i = 0$.
4. Visiting all the nodes j that have access to i in backward topological order, set

$$\eta^\pi_j = \overline{\eta}, \quad \text{and} \quad x_j = w(\pi(j)) - \overline{\eta} \times \tau(\pi(j)) + x_{\pi(j)^+} \tag{84}$$

5. If there is a nonempty set C of nodes that do not have access to i, find a circuit in C and go to step 2.

[17] An implementation of the algorithm, in C, can be found at the address http://amadeus.inria.fr/gaubert/howard

ALGORITHM 107 (MAX-PLUS POLICY ITERATION). 1. *Initialization.* Select an arbitrary policy π_1. Compute $\eta^1 \overset{\text{def}}{=} \eta^{\pi_1}$ and $x^1 \overset{\text{def}}{=} x^{\pi_1}$ via Algorithm 106. Set $k = 1$

2. *Policy improvement.* Let[18]

$$J = \{i \mid \max_{e^- = i} \eta^k_{e^+} > \eta^k_i\}, \quad K(i) = \mathop{\arg\max}_{e \text{ such that } e^- = i} \eta^k_{e^+}$$

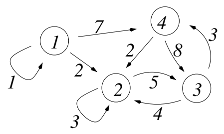
$$I = \{i \mid \max_{e \in K(i)} (w(e) - \tau(e)\eta^k_{e^+} + x^k_{e^+}) > x^k_i\}, \quad L(i) = \mathop{\arg\max}_{e \in K(i)} (w(e) - \tau(e)\eta^k_{e^+} + x^k_{e^+}).$$

3. If $I = J = \emptyset$, (η^k, x^k) is a solution of (82). Stop.
4. If $J \neq \emptyset$, set, for $i \in J$, $\pi_{k+1}(i) =$ any e in $K(i)$, and for $i \notin J$, $\pi_{k+1}(i) = \pi_k(i)$.
5. If $J = \emptyset$ but $I \neq \emptyset$, set for $i \in I$, $\pi_{k+1}(i) =$ any $e \in L(i)$, and for $i \notin I$, $\pi_{k+1}(i) = \pi_k(i)$.
6. *Value determination.* Find $\eta^{k+1} \overset{\text{def}}{=} \eta^{\pi_{k+1}}$, $x^{k+1} \overset{\text{def}}{=} x^{\pi_{k+1}}$), taking the special value $x^{k+1}_i = x^k_i$ in step 3 of Algorithm 106.
7. Increment k by one and go to step 2.

The algorithm builds a sequence (η^k, x^k) that is strictly increasing for the lexicographic order on $(\mathbb{R}^{\mathcal{N}})^2$, which is defined by $(\eta, x) <_{\text{lex}} (\eta, x')$ if $\eta < \eta'$ or $\eta = \eta'$ and $x < x'$. One iteration of the algorithm requires $O(|\mathcal{E}|)$ time and $O(|\mathcal{N}|)$ space. The termination is proved in [**28**] in a more general context.

Let us compute the eigenvalue of the matrix displayed in Fig. 8 using Algorithm 107. Here, $\tau(e) = 1$ for all e, and the multigraph $\mathcal{G}$ coincides with the graph of A. The following run of the algorithm is visualized at

$$A = \begin{bmatrix} 1 & 2 & \mathbb{0} & 7 \\ \mathbb{0} & 3 & 5 & \mathbb{0} \\ \mathbb{0} & 4 & \mathbb{0} & 3 \\ \mathbb{0} & 2 & 8 & \mathbb{0} \end{bmatrix}$$



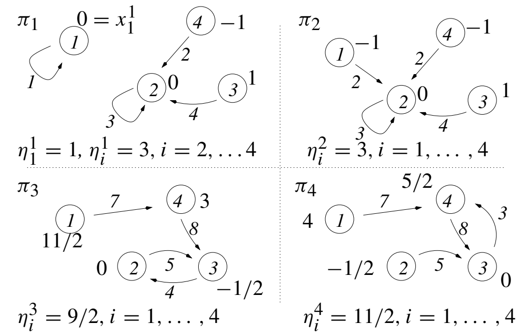


FIGURE 8. A matrix, its graph, and the policies built by the policy improvement algorithm

the right of Fig. 8. We choose the initial policy π_1: $1 \to 1$, $i \to 2$, for $i = 2, 3, 4$. Applying Algorithm 106, we find a circuit $c_1 : 1 \to 1$, with $\overline{\eta} = w(c_1)/\tau(c_1) = 1$. We set $\eta^1_1 = 1$, $x^1_1 = 0$. Since 1 is the only node which has access to 1, we apply Algorithm 106 to the remaining subgraph with nodes 2, 3, 4. We find the circuit $c_2 : 2 \to 2$ and set $\overline{\eta} = w(c_2)/\tau(c_2) = 3$, $\eta^1_2 = 3$ and $x^1_2 = 0$. Since 3, 4 have access to 2, we set $\eta^1_i = 3$ for $i = 3, 4$. Then, an application of (84) yields $x^1_3 = 4 - 3 + x^1_2$, $x^1_4 = 2 - 3 + x^1_2$. To summarize:

$$\eta^1 = \begin{bmatrix} 1 & 3 & 3 & 3 \end{bmatrix}^T, \quad x^1 = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}^T \ .$$

We improve the policy: $J = \{1\} \neq \emptyset$, and $\pi_2 : i \to 2$, for $i = 1, 2, 3, 4$. Only the entry 1 of x^1 and η^1 has to be modified, which yields

$$\eta^2 = \begin{bmatrix} 3 & 3 & 3 & 3 \end{bmatrix}^T, \quad x^2 = \begin{bmatrix} -1 & 0 & 1 & -1 \end{bmatrix}^T \ .$$

We next tabulate with less details the end of the run of the algorithm. *Policy improvement,* $J = \emptyset$. $\pi_3 : 1 \to 4, 2 \to 3, 3 \to 2, 4 \to 3$. *Algorithm 106. Value determination.* Circuit found, $c : 3 \to 2 \to 3$, $\overline{\eta} = (w(2, 3) + w(3, 2))/2 = 9/2$.

$$\eta^3 = \begin{bmatrix} \frac{9}{2} & \frac{9}{2} & \frac{9}{2} & \frac{9}{2} \end{bmatrix}^T, \quad x^3 = \begin{bmatrix} \frac{11}{2} & 0 & -\frac{1}{2} & 3 \end{bmatrix}^T \ .$$

[18]Recall that by $\arg\max_{e \in \mathcal{E}} f(e)$, we mean as usual the set of elements $m \in \mathcal{E}$ such that $f(m) = \max_{e \in \mathcal{E}} f(e)$.

Policy improvement, $J = \emptyset$. The only change is $\pi_4(3) = 4$. *Algorithm 106. Value determination.* Circuit found, $c : 3 \to 4 \to 3$, $\overline{\eta} = (w(3,4) + w(4,3))/2 = 11/2$.

$$\eta^4 = \begin{bmatrix} \frac{11}{2} & \frac{11}{2} & \frac{11}{2} & \frac{11}{2} \end{bmatrix}^T, \; x^4 = \begin{bmatrix} 4 & -\frac{1}{2} & 0 & \frac{5}{2} \end{bmatrix}^T .$$

Stop. $11/2$ is an eigenvalue of A, and x^4 is an eigenvector.

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