Products of Random Rectangular Matrices

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Abstract

We study the asymptotic behaviour of points under matrix cocyles generated by rectangular matrices. In particular we prove a random Perron-Frobenius and a Multiplicative Ergodic Theorem. We also provide an example where such products of random rectangular matrices arise in the theory of random walks in random environments and where the Multiplicative Ergodic Theorem can be used to investigate recurrence problems.

Key words: random dynamical system, products of random matrices, random walks in random environments, Perron-Frobenius theory, multiplicative ergodic theory.

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Introduction

Matrix products might define the simplest dynamical systems possible, but nevertheless they still play an important role in investigations of dynamics. They arise naturally as linearizations of smooth systems and also as transition matrices in population dynamics or economic models. Moreover they can be found in the form of stochastic matrices in the theory of random walks and Markov chains. In the trivial case of just one matrix A, linear algebra allows a complete description and characterization of the dynamics given by the iteration of the matrix in terms of the eigenvalues and eigenspaces of A. For the case of inhomogeneous matrix products $A_n A_{n-1} \dots A_1$ the situation immediately becomes very complicated. In the case of products of random matrices, where the A_i form a stochastic process, there is at least a possibility to describe the asymptotic behaviour of the dynamics using ergodic theory. This requires that the matrices are chosen in a stationary way according to some probability measure.

The history of ergodic theory for products of random matrices really starts with a result of Furstenberg and Kesten [8] on the dominating growth rate which can be described as follows. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and ϑ is a $\mathbb{P}$-preserving transformation consider a random variable $A : \Omega \to \mathbb{R}^{d \times d}$, $d \in \mathbb{Z}^+$ such that $\log^+ \|A(.)\| \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a measurable ϑ-invariant function $\lambda_1 : \Omega \to \mathbb{R} \cup \{-\infty\}$ with $\lambda_1^+ \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$

$$\lim_{n \to \infty} \frac{1}{n} \log \|A(\vartheta^{n-1}\omega) \dots A(\vartheta\omega) A(\omega)\| = \lambda_1(\omega).$$

λ_1 presents the maximal asymptotic growth rate for almost all products of the random matrices A. Other growth rates can only be realized by restrictions to lower dimensional subspaces of $\mathbb{R}^d$. The celebrated Multiplicative Ergodic Theorem of Oseledets [15] guarantees the existence of the smaller growth rates λ_i and the corresponding subspaces. To be more precise, under the same conditions as in the Furstenberg-Kesten Theorem $\mathbb{P}$-almost surely ϑ-invariant measurable functions $s : \Omega \to \mathbb{Z}^+$,

$\lambda_i : \Omega \to \mathbb{R} \cup \{-\infty\}$, $i = 1, \ldots, s$ with $\lambda_1(\omega) > \lambda_2(\omega) > \ldots > \lambda_{s(\omega)}(\omega)$ and measurable linear subspaces $\mathcal{V}_i$, $i = 1, \ldots, s$ with $\{0\} = \mathcal{V}_{s(\omega)}(\omega) \subset \ldots \subset \mathcal{V}_1(\omega) \subset \mathcal{V}_0(\omega) = \mathbb{R}^d$ can be found such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A(\vartheta^{n-1}\omega) \ldots A(\vartheta\omega)A(\omega)x\| = \lambda_i(\omega) \text{ for all } x \in \mathcal{V}_{i-1}(\omega) \setminus \mathcal{V}_i(\omega),$$

and $A(\omega)\mathcal{V}_i(\omega) \subset \mathcal{V}_i(\vartheta\omega)$. Thus this result presents a substitute of linear algebra for the investigation of the asymptotic behaviour of products of random matrices. Note that the functions s and λ_i are necessarily constant, if the measure $\mathbb{P}$ is ergodic.

The results mentioned so far are all concerned with quadratic matrices. Indeed, random matrices appearing in ergodic theory as linearizations of smooth mappings are expected to be quadratic. In the case of (random) transition matrices rectangular matrices do make sense. In biological models the change of dimension can be caused by a variable environment involving a change of number of components in the total ecological system. We will show in Section 4 how random rectangular matrices arise naturally as companion matrices for recurrence problems in the theory of random walks in random environments, and how a multiplicative ergodic theorem for random rectangular matrices can be used. Note that for studying transition probabilities of Markov chains in random stationary environments there exists a Perron-Frobenius type of theorem for random positive operators (cf. Kifer [13]). We will consider the particular case of primitive non-negative random rectangular matrices for which we weaken the integrability condition for multiplicative ergodic theory and find additional information on the subspace $E_1(\omega)$ by establishing a random version of the Perron-Frobenius Theorem.

There exists a huge literature on products of positive random matrices and a random Perron-Frobenius theory (see for example the discussion in Arnold, Demetrius and Gundlach [2, Section 3]). Also up to our knowledge the most general Perron-Frobenius Theorem for products of positive random matrices can be found in [2]. Namely, if $\mathbb{P}$ is ergodic and the random matrix $A = (a_{ij})$ takes values only in the semi-group of positive $d \times d$ matrices and $m = \min_{1\le i,j\le d} a_{ij}$, $M = \max_{1\le i,j\le d} a_{ij}$ satisfy $\log^+ \frac{1}{m} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $\log^+ M \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ then $\mathbb{P}$-almost surely there exists a splitting $\mathbb{R}^d = \mathcal{V}_1(\omega) \oplus W(\omega)$ such that $\mathcal{V}_1$ (as in the Multiplicative Ergodic Theorem) is one-dimensional and spanned by a random positive vector u satisfying $A(\omega)u(\omega) = q(\omega)u(\vartheta\omega)$ for a positive random scalar q with $\log q \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E} \log q = \lambda_1$. Moreover $A(\omega)W(\omega) \subset W(\vartheta\omega)$. This result does not only hold for positive random matrices, but obviously also for strongly primitive non-negative random matrices. For the latter there exists a fixed number $p \in \mathbb{Z}^+$ such that $\mathbb{P}$-a.s. $A(\vartheta^{p-1}\omega) \ldots A(\omega)$ is positive. Under this primitivity condition and a similar integrability condition the same result can be extended to products of random rectangular matrices via a so-called Ruelle-Perron-Frobenius Theorem for random subshifts of finite type (see Bogenschütz and Gundlach [5, Theorem 5.1]). Namely let $D = \{D(\omega) = (d_{ij}(\omega))\}$ be a measurable family of non-negative $d(\vartheta\omega) \times d(\omega)$-matrices for a random variable $d : \Omega \to \mathbb{Z}^+$ such that each $D(\omega)$ has at least one non-zero entry in each row and each column, and $\tilde{M}$ be defined by $\tilde{M}(\omega) = \max_{1\le i\le d(\vartheta\omega)} \sum_{j=1}^{d(\omega)} d_{ij}(\omega)$, m as above. Then if $\log^+ \frac{1}{m} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $\log^+ \tilde{M} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ the assertions of the random Perron-Frobenius Theorem remain true for the products $D(\vartheta^{n-1}\omega) \ldots D(\vartheta\omega)D(\omega)$ of the random rectangular matrices D.

As could be shown (cf. Khanin and Kifer [12] and Gundlach [9]) that the primitivity condition for the random Ruelle-Perron-Frobenius Theorem for random subshifts of finite type could be weakened by replacing the constant p by a random variable p, it is natural to expect a similar weakening of conditions also for the random Perron-Frobenius Theorem for products of random rectangular matrices. Thus in this paper we provide the most general version of such a result together with the Multiplicative Ergodic Theorem corresponding to this set-up. The latter was proved by the second author under the supervision of the first in [19] and here we present a short version of it. In detail we describe a case of a random walk in a random environment where random rectangular matrices arise as companion matrices and where the Multiplicative Ergodic Theorem can be used for an analysis.

The paper is organized as follows. In Section 1 we introduce the notion of bundle random dynamical systems which provides the right set-up to deal with products of random rectangular matrices. In Section 2 we present and prove a version of the Perron-Frobenius Theorem for products of random rectangular matrices, while in the successing section the analogous will be done for the Multiplicative

Ergodic Theorem. Section 4 is devoted to random walks in random environments which can be described with the help of random rectangular matrices and where the Multiplicative Ergodic Theorem can be used to characterize recurrence. The paper ends with a special example for this situation.

1 Random Bundle Systems

The main problem in dealing with asymptotic properties of products of random rectangular matrices lies in the different dimensions of the Euclidean spaces the matrices are acting on. We are going to introduce now a set-up which allows to deal with products of random rectangular matrices, the measurability problems involved and their asymptotic properties in a nice way. For this purpose we consider an abstract dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ with a measure preserving transformation ϑ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will always assume that ϑ is ergodic and measurably invertible. In the case of Ω being a Lebesgue space these two properties may be assumed without loss of generality, the first one due to the existence of natural extensions (see Cornfeld, Fomin and Sinai [7, p. 239]), the second one due to the decomposition into ergodic components (see Cornfeld, Fomin and Sinai [7, Ch. 1,§ 3]). It will be crucial for our investigations that ϑ is invertible, though the actual system of interest will be non-invertible.

Now consider a metric space (X, d) with Borel-σ-algebra $\mathcal{B}$ and the product $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$. Using the projection Pr_Ω from $\Omega \times X$ to Ω we can view $\Omega \times X$ as a (trivial) measurable bundle. Moreover we call a set $E \in \mathcal{F} \otimes \mathcal{B}$ with $\mathrm{Pr}_\Omega E = \Omega$ a measurable bundle over Ω with closed fibres $E_\omega := \{x \in X : (\omega, x) \in E\}$. In case of X being separable the measurability condition can be replaced by the requirement that the distance function $\omega \mapsto d(x, E_\omega)$ is $\mathcal{F}$, $\mathcal{B}_{\mathbb{R}^+}$ measurable for all $x \in X$, where $\mathcal{B}_{\mathbb{R}^+}$ is the Borel-σ-algebra on $\mathbb{R}^+$. Then $E = \{(\omega, x) \in \Omega \times X : d(x, E_\omega) = 0\} \in \mathcal{F} \otimes \mathcal{B}$ follows from Castaing and Valadier [6, Lemma III.14], since $x \mapsto d(x, E_\omega)$ is continuous.

Definition 1.1 *Let E be a measurable bundle over Ω with fibres in X. A (one-sided, discrete time) bundle random dynamical system (abbreviated: bundle RDS) on E over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a mapping*

$$\varphi : \mathbb{N} \times E \to X, \qquad (n, \omega, x) \mapsto \varphi(n, \omega)x$$

such that

(i) $(\vartheta\omega, \varphi(1, \omega)x) \in E$ *for all* $(\omega, x) \in E$.

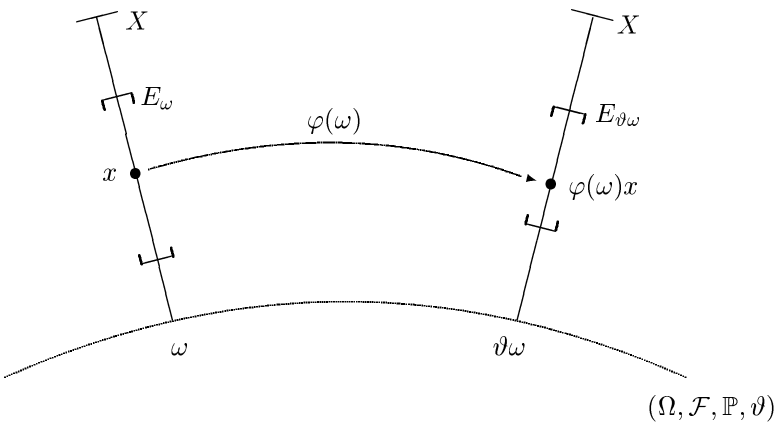
(ii) $(\omega, x) \mapsto \varphi(n, \omega)x$ *maps measurably from E to X for all $n \in \mathbb{N}$.*

(iii) $\qquad \varphi(0, \omega) = \mathrm{id} \qquad$ *for all* $\omega \in \Omega$.

(iv) The cocycle property *is satisfied:*

$$\varphi(n + m, \omega) = \varphi(n, \vartheta^m\omega) \circ \varphi(m, \omega) \qquad \text{for all } n, m \in \mathbb{N},\ \omega \in \Omega.$$

If φ is defined on the trivial bundle $E = \Omega \times X$ one simply talks of a random dynamical system (RDS). We denote the time-one mapping by $\varphi(\omega) := \varphi(1, \omega)$.



Note that it would be sufficient to formulate the conditions of Definition 1.1 for $\mathbb{P}$-almost all $\omega \in \Omega$ instead of all $\omega \in \Omega$, but then we could replace Ω by a set $\tilde{\Omega} \subset \Omega$ of full measure such that the conditions are fulfilled for all $\omega \in \tilde{\Omega}$. So for convenience we restrict our attention without loss of generality always to the case that all conditions are satisfied for all $\omega \in \tilde{\Omega}$.

In the following we will restrict our attention to the special choice of the space of all real, quadratic summable sequences $X := \ell^2_{\mathbb{R}}(\mathbb{N}) = \{x \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ equipped with the standard inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ and the standard basis $(e_n)_{n\in\mathbb{N}}$, $e_n = (0, \ldots, 1, 0, \ldots)$. This is a separable Hilbert space with induced norm $\|.\|_2$ and metric δ. We denote the corresponding Borel-σ-algebra by $\mathcal{B}_X$. Moreover we define for $n \in \mathbb{N}$ n-dimensional subspaces of X by

$$X^{(n)} = \{x = (x_1, x_2, \ldots) \in X : x_i = 0 \text{ for all } i \geq n+1\}.$$

These $X^{(n)}$ are closed, hence in $\mathcal{B}_X$, and they define separable Hilbert spaces with respect to the induced inner product.

Let us consider now a random variable $d : \Omega \to \mathbb{N}$ with $d \geq 2$. Then the mapping

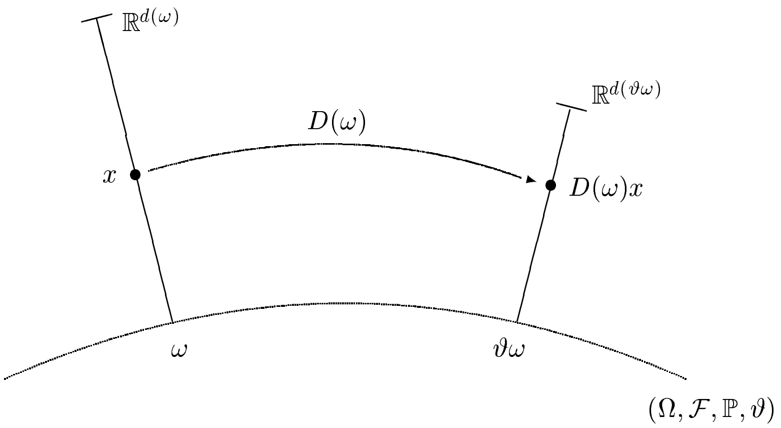
$$\begin{aligned} \omega \quad &\mapsto \quad \operatorname{dist}\left(x, X^{(d(\omega))}\right) = \inf\left\{\delta(x,y) : y \in X^{(d(\omega))}\right\} \\ &= \quad \left(\sum_{i=1}^{\infty} |x_i|^2 \cdot \mathbb{1}_{\{\omega : i > d(\omega)\}}(\omega)\right)^{1/2} < \infty \text{ for all } x \in X, \omega \in \Omega \end{aligned} \tag{1.1}$$

is measurable in ω for all $x \in X$. Thus

$$E := \{\, (\omega, x) \in \Omega \times X : x \in X^{(d(\omega))}\}$$

defines a measurable bundle with fibres $E_\omega = X^{(d(\omega))}$.

Now corresponding to the random variable d we consider random $d(\vartheta\omega) \times d(\omega)$-matrices $D(\omega) = (d_{ij}(\omega))$ (i.e. $d_{ij} : \Omega \mapsto \mathbb{R}$ is $\mathcal{F}$-measurable for all i, j)



and the product

$$D(n,\omega) = \begin{cases} D(\vartheta^{n-1}\omega)\cdot\ldots\cdot D(\omega) & n \geq 1, \\ \mathrm{id}_{\mathbb{R}^{d(\omega)}} & n = 0. \end{cases} \tag{1.2}$$

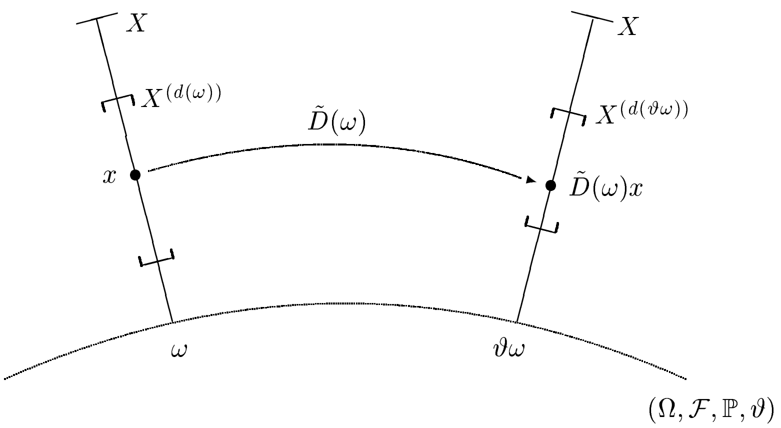
We define corresponding random bounded linear (i.e. continuous) operators $\tilde{D} = \{\tilde{D}(\omega) : E_\omega \to E_{\vartheta\omega}\}$ by

$$\tilde{D}(\omega) = \begin{bmatrix} D(\omega) & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.3}$$

Note that $\omega \mapsto \tilde{D}(\omega)x$ is $\mathcal{F},\mathcal{B}_X$-measurable for all $x \in X$ if and only if d_{ij} is $\mathcal{F},\mathcal{B}_{\mathbb{R}}$-measurable for all i,j, in which case $(\omega,x) \mapsto \tilde{D}(\omega)x$ is $\mathcal{F}\otimes\mathcal{B}_X,\mathcal{B}_X$-measurable by Castaing and Valadier [6, Lemma III.14] and also measurable with respect to the trace-σ-algebra $E \cap (\mathcal{F}\otimes\mathcal{B}_X)$. Thus the cocycle defined by

$$\tilde{D}(n,\omega) = \begin{cases} \tilde{D}(\vartheta^{n-1}\omega)\cdot\ldots\cdot\tilde{D}(\omega), & n \geq 1, \\ \mathrm{id}_X, & n = 0, \end{cases} \tag{1.4}$$

is a bundle random dynamical system in the sense of Definition 1.1 on the measurable bundle E.



It is obvious that there is a 1-1 correspondence between (1.2) and (1.4). We therefore identify the systems defined by these two products, which clears up all measurability problems. In the following we will prefer to use (1.2) exclusively. One of the advantages of this choice is that we can work on (finite-dimensional) Euclidean spaces only. In the case of positive random matrices we therefore can use reductions to finite-dimensional unit spheres, where a special and very helpful metric, the so-called Hilbert metric is available.

2 A Random Perron-Frobenius Theorem

In this section we restrict our attention to non-negative random rectangular matrices D. Let $\mathbb{R}^d_+$ be the subset of $\mathbb{R}^d$ consisting of strictly positive vectors, $S^{d-1} \subset \mathbb{R}^d$ the $(d-1)$-dimensional unit sphere and $S^{d-1}_+ = \mathbb{R}^d_+ \cap S^{d-1}$. On S^{d-1}_+ one can use the Hilbert metric (projective distance) d, which is defined by

$$d(x,y) := \log \frac{\max_{1\leq i\leq d} \frac{x_i}{y_i}}{\min_{1\leq i\leq d} \frac{x_i}{y_i}}, \quad x,y \in \mathbb{R}^d_+$$

and is indeed a finite metric on S^{d-1}_+. It makes (S^{d-1}_+, d) a complete metric space. We have

$$\|x-y\| \leq \exp(d(x,y)) - 1.$$

In particular, every Cauchy sequence in (S_+^{d-1}, d) is also a Cauchy sequence in $(S_+^{d-1}, \|.\|)$. Moreover the so-called Birkhoff's contraction coefficient $\tau(D)$ of a non-negative $c \times d$-matrix D can be defined as

$$\tau(D) := \sup_{x,y \in S_+^{d-1},\, x \neq y} \frac{d(Dx, Dy)}{d(x,y)},$$

where the projective distance is once used on S_+^{d-1} and once on $\mathbb{R}_+^c$. As we assume clever readers, we renounce the use of an additional index for the respective spaces. Birkhoff's contraction coefficient satisfies for strictly positive matrices $A \in \mathbb{R}_+^{d\times c}$ and $B \in \mathbb{R}_+^{c \times b}$

(i) $0 \leq \tau(A) < 1$,

(ii) $\tau(AB) \leq \tau(A)\tau(B)$.

Thus positive matrices D automatically contract all d-distances in S_+^{d-1},

$$d(Dx, Dy) \leq \tau(D) d(x,y) < d(x,y).$$

As shown in Wojtkowski [21] the Hilbert metric is not a Riemannian metric, but is the only, up to a scalar factor, Finsler metric which makes the action of any positive matrix on S_+^{d-1} a contraction. For a proof of the other facts we refer to Birkhoff [3] and [4] and Seneta [17] and leave it to the reader to do the trivial extension from the quadratic to the rectangular case.

Now consider the matrix cocycle (1.2). Using polar coordinates $s = x/\|x\|$, $r = \|x\|$ the action of $D(n,\omega)$ on $\mathbb{R}^d \setminus \{0\}$, described by the random difference equation $x_{n+1} = D(\theta^n\omega)x_n$, can be split into

$$s_{n+1} = \bar{D}(\theta^n\omega)s_n, \quad r_{n+1} = \|D(\theta^n\omega)s_n\| r_n,$$

where $\bar{D}(\omega) : S^{d(\omega)-1} \to S^{d(\vartheta\omega)-1}$ is defined by $s \mapsto \bar{D}(\omega)s = D(\omega)s/\|D(\omega)s\|$. In particular, the linear cocycle $D(n,.)$ induces via the canonical projections $p(\omega) : \mathbb{R}^{d(\omega)} \setminus \{0\} \to S^{d(\omega)-1}$ a nonlinear cocycle $\bar{D}(n,.)$ on S^{d-1} given by

$$\bar{D}(n,\omega) = \bar{D}(\vartheta^{n-1}\omega) \circ \ldots \circ \bar{D}(\omega)$$

with $p(\vartheta^n\omega) \circ D(n,\omega) = \bar{D}(n,\omega) \circ p(\omega)$ which leaves S_+^{d-1} invariant. The following lemma can be obtained analogously to the case of random quadratic matrices (cf. Arnold, Demetrius and Gundlach [2, Lemma 3.4].

Lemma 2.1 *Let D be a positive random matrix. Then there exists a ϑ-invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure such that for all $\omega \in \tilde{\Omega}$, $x, y \in S_+^{d(\omega)-1}$*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\bar{D}(n,\omega)x - \bar{D}(n,\omega)y\| \leq \limsup_{n\to\infty} \frac{1}{n} \log d(D(n,\omega)x, D(n,\omega)y) \leq \mathbb{E} \log \tau(D) \in [-\infty, 0).$$

This exponential clustering property can be also obtained under weaker conditions than the positivity of the random matrix. So in the following we will just consider non-negative random matrices D.

Definition 2.2 *A non-negative matrix D is called row-allowable, if it has a positive entry in each row, and column-allowable, if it has a positive entry in each column. A non-negative matrix D is said to be primitive, if there exists a random variable M adopting values in $\mathbb{N}$ such that the $d(\vartheta^{M(\omega)}\omega) \times d(\omega)$-matrix $D(M(\omega),\omega)$ is positive for all $\omega \in \Omega$. It is called uniformly primitive, if M is constant.*

If the non-negative matrix D is row-allowable and column-allowable, then the positivity of the matrix $D(N,\omega)$ for some $N \in \mathbb{N}$ implies the positivity of the matrices $D(N+n,\omega)$ and $D(N+n, \vartheta^{-n}\omega)$ for $n \in \mathbb{N}$. Furthermore for column-allowable matrices D we have $0 \leq \tau(D) \leq 1$. Because of these properties we will assume for the rest of this section that the primitive random matrices under consideration are all row- and column-allowable. Incidentally this property is one of the reasons why rectangular matrices of interest are not extended to quadratic matrices by adding zero rows or columns.

Of course Lemma 2.1 can easily be extended to the case of uniformly primitive non-negative matrices, if we replace the convergence rate by $\frac{1}{M}\mathbb{E}\log\tau(A)$. In the general primitivity case we consider for $k \in \mathbb{N}_+$ the subsets

$$\Omega_k := \{\omega \in \Omega : M(\omega) \leq k\}.$$

If k is sufficiently large, then $\mathbb{P}(\Omega_k) > 0$ and we can define a return time for Ω_k in the form

$$t_k(\omega) = k + r_k(\omega), \quad r_k(\omega) > 0$$

as well as an induced transformation $\hat{\vartheta}_k : \Omega_k \to \Omega_k$ which preserves the $\mathbb{P}$-induced measure on Ω_k such that the resulting abstract dynamical system is ergodic and the random variables t_k and r_k are integrable (cf. Cornfeld, Fomin and Sinai [7]). Moreover it follows from the Ergodic Theorem that for $\mathbb{P}$-almost all ω the trajectories $\vartheta^n\omega$ enter Ω_k.

Lemma 2.3 *Let D be a primitive random matrix. Then there exists a ϑ-invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure and a number $\kappa < 0$ such that for all $\omega \in \tilde{\Omega}$, $x, y \in S_+^{d(\omega)-1}$*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\bar{D}(n,\omega)x - \bar{D}(n,\omega)y\| \leq \limsup_{n\to\infty} \frac{1}{n} \log d(D(n,\omega)x, D(n,\omega)y) \leq \kappa.$$

Proof. We fix a k such that $\mathbb{P}(\Omega_k) > 0$ and restrict our attention to $\omega \in \Omega_k$, as $\mathbb{P}$-almost all ω will return infinitely often to Ω_k. We represent any $n \in \mathbb{N}_+$ uniquely by

$$n = kq(n,\omega) + \sum_{i=0}^{q(n,\omega)-1} r_k(\hat{\vartheta}^i\omega) + R(n,\omega) \qquad \text{with } R(n,\omega) < k + r_k(\hat{\vartheta}^{q(n,\omega)}\omega).$$

Here $q(n,\omega)$ counts the number of times the trajectory of length n started in $\omega \in \Omega_k$ has returned to Ω_k. Then

$$D(n,\omega) = D(R(n,\omega), \hat{\vartheta}^{q(n,\omega)}\omega)\hat{D}(q(n,\omega),\omega)$$

where $\hat{D}$ denotes the random positive matrix corresponding to the induced transformation $\hat{\vartheta}_k$ by taking the product of t_k random matrices $D(t_k(\omega),\omega)$. We can apply Lemma 2.1 to $\hat{D}$. Therefore

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log d(D(n,\omega)x, D(n,\omega)y) &\leq \limsup_{n\to\infty} \frac{q(n,\omega)}{nq(n,\omega)} \log d(\hat{D}(q(n,\omega),\omega)x, \hat{D}(q(n,\omega),\omega)y) \\ &\leq \limsup_{n\to\infty} \frac{1}{k + \frac{1}{q(n,\omega)}\sum_{i=0}^{q(n,\omega)-1} r_k(\hat{\vartheta}^i\omega)} \mathbb{E}\log\tau(\hat{D}) < 0.\end{aligned}$$

The limit on the right hand side exists due to Birkhoff's Ergodic Theorem. So due to the ergodicity of the induced transformation, Poincaré's Recurrence Theorem and the integrability of r_k the result follows immediately. The left hand side of the equation in the lemma follows from the definition of the Hilbert metric. □

We can use the invertibility of $\hat{\vartheta}_k$ to prove analogously to Arnold, Demetrius and Gundlach [2, Lemma 3.5] the following result.

Lemma 2.4 *Let D be a primitive random matrix and $x_0 : \omega \mapsto x_0(\omega) \in S^{d(\omega)-1}$ be a random vector such that $\log^+ d(D(.)x_0(.), x_0(.)) \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Then there is a ϑ-invariant set $\tilde{\Omega} \subset \Omega$ of full $\mathbb{P}$-measure (not depending on x_0) such that*

$$\limsup_{n\to\infty} \frac{1}{n} \log d(\bar{D}(n,\vartheta^{-n}\omega)x_0(\vartheta^{-n}\omega), \bar{D}(n+1,\vartheta^{-(n+1)}\omega)x_0(\vartheta^{-(n+1)}\omega)) \leq \kappa < 0.$$

In particular, $(\bar{D}(n,\vartheta^{-n}\omega)x_0(\vartheta^{-n}\omega))$ is exponentially Cauchy with respect to the projective metric and hence also with respect to the Euclidean metric.

A popular choice for x_0 is $\mathbb{1}$ given by

$$(\mathbb{1}(\omega))_i = \frac{1}{\sqrt{d(\omega)}} \quad \text{for } i = 1, \dots, d(\omega), \quad \omega \in \Omega$$

which reduces the integrability condition in Lemma 2.4 to an integrability condition on $D = (d_{ij})$ only, namely

$$\log \frac{\max_i \sum_j d_{ij}(.)}{\min_i \sum_j d_{ij}(.)} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.1}$$

Under this condition the limit

$$u(\omega) := \lim_{n\to\infty} \bar{D}(n, \vartheta^{-n}\omega)\mathbb{1}(\vartheta^{-n}\omega)$$

exists $\mathbb{P}$-a.s. by Lemma 2.4 and defines a random variable u which is measurable with respect to the 'past' σ-algebra $\mathcal{F}^-$ generated by the random variables $(D(\vartheta^{-n}\omega))_{n\in\mathbb{N}}$. Analogously to Arnold, Demetrius and Gundlach [2, Proposition 3.6] one can prove that u is the unique stationary solution of $\bar{D}$, i.e.

$$\bar{D}(\omega)u(\omega) = u(\vartheta\omega) \quad \mathbb{P}-\text{a.s.}$$

and that with the help of the random variable q defined by

$$q(\omega) = \|D(\omega)u(\omega)\| > 0$$

this property can be carried over to the linear cocycle $D(n, .)$ such that

$$D(\omega)u(\omega) = q(\omega)u(\omega) \quad \mathbb{P}-\text{a.s.}$$

Moreover u is attractive in the sense that $\mathbb{P}$-a.s. for all $x \in S^{d(\omega)-1}$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\bar{D}(n,\omega)x - u(\vartheta^n\omega)\| < 0.$$

Analogous results can be obtained for the transposed linear cocycle $D^*(\vartheta^{-n+1}\omega)\dots D^*(\vartheta^{-1}\omega)D^*(\omega)$ over ϑ^{-1} under the integrability condition

$$\log \frac{\max_j \sum_i d_{ij}(.)}{\min_j \sum_i d_{ij}(.)} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{2.2}$$

Thus the important assertions can be summarized as follows.

Theorem 2.5 (Random Perron-Frobenius Theorem) *Assume that the random primitive matrix D satisfies the integrability conditions (2.1) and (2.2). Then there exist unique positive random unit vectors u and v and positive random scalars q and q^* such that $\mathbb{P}$-a.s.*

(i) $D(\omega)u(\omega) = q(\omega)u(\vartheta\omega)$,

(ii) $D^*(\omega)v(\vartheta\omega) = q^*(\omega)v(\omega)$,

(iii) $\frac{q^*(\omega)}{q(\omega)} = \frac{\langle u(\vartheta\omega), v(\vartheta\omega)\rangle}{\langle u(\omega), v(\omega)\rangle}$ *where $\langle ., . \rangle$ denotes the standard inner product in $\mathbb{R}^{d(\omega)}$,*

(iv) $\mathbb{R}^{d(\omega)} = \mathbb{R}u(\omega) \oplus W(\omega)$ *with* $W(\omega) = \{x \in \mathbb{R}^{d(\omega)} : \langle x, v(\omega)\rangle = 0\} \supset \ker D(\omega)$ *is an invariant splitting in the sense that* $D(\omega)W(\omega) \subset W(\omega)$ *and* $D(\omega)u(\omega)\mathbb{R} \subset u(\vartheta\omega)\mathbb{R}$,

(v) for each $x \in \mathbb{R}^{d(\omega)}$

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| \frac{D(n,\omega)x}{q_n(\omega)} - \frac{\langle x, v(\omega)\rangle}{\langle u(\omega), v(\omega)\rangle} u(\vartheta^n\omega) \right\| \le \rho < 0$$

where $q_n(\omega) = q(\vartheta^{n-1}\omega)\dots q(\omega)$ *is a scalar cocycle and* $\rho \in [-\infty, 0)$ *is a constant which only depends on D,* *(vi)* $\log q$, $\log q^*$, $\log\langle u,v\rangle \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ *with* $\mathbb{E}\log q^* = \mathbb{E}\log q$, *if* $\log d$, $\log^+ M$, $\log^+ \frac{1}{m} \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})$ *where* $M(\omega) = \max\{a_{ij}(\omega) : 1 \le i \le d(\omega),\ 1 \le j \le d(\vartheta\omega)\}$ *and* $m(\omega) = \min\{a_{ij}(\omega) > 0 : 1 \le i \le d(\omega),\ 1 \le j \le d(\vartheta\omega)\}$. *In this case we have*

$$x \notin W(\omega) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \log \|D(n,\omega)x\| = \mathbb{E}\log q = \lambda,$$

$$x \in W(\omega) \Rightarrow \limsup_{n\to\infty} \frac{1}{n} \log \|D(n,\omega)x\| < \lambda.$$

The last assertion of Theorem 2.5 allows a specification of the Multiplicative Theorem (see next section) for random primitive matrices: the subspace corresponding to the largest exponential growth rate (the top Lyapunov exponent) λ is one-dimensional. Moreover (iv) yields the invariance of the subspaces corresponding to the Lyapunov exponents different from the largest one.

Note that a version of this theorem for uniformly primitive matrices D can be found in Bogenschütz and Gundlach [5, Theorem 5.1] where it was deduced from a random Ruelle-Perron-Frobenius Theorem. Such a theorem is also available for the primitive case (cf. Gundlach [9, Theorem 4.4]), but the integrability conditions coming from random Ruelle-Perron-Frobenius Theorems are stronger than the ones we have used here.

3 The Multiplicative Ergodic Theorem

In this section we will consider the spaces $\mathbb{R}^{d(\omega)}$ as fibres of the bundle E over the abstract dynamical system $(\Omega,\mathcal{F},\mathbb{P},\vartheta)$ endowed with the corresponding Euclidean norm $\|.\|_\omega$. We stick to the notations of Section 1 and investigate the asymptotic behaviour of the random rectangular matrices D. Then

$$\|D\|_{\omega,\vartheta\omega} = \sup_{x\in\mathbb{R}^{d(\omega)}\setminus\{0\}} \frac{\|Dx\|_{\vartheta\omega}}{\|x\|_\omega} = \sup_{\|x\|_\omega=1} \|Dx\|_{\vartheta\omega}$$

defines the induced operator norm on the fibers.

The Multiplicative Ergodic Theorem requires the following integrability condition on the matrices $D(\omega)$

$$\log^+ \|D(\,.\,)\|_{\,.,\vartheta.} \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}). \tag{3.1}$$

Lemma 3.1 *For the random matrix D set $M(\omega) = \max_{i,j} |d_{ij}(\omega)|$. Then*

$$\log^+ d,\ \log^+ M \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}) \ \Rightarrow\ \log^+ \|D(\,.\,)\|_{\,.,\vartheta.} \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P}).$$

If $\mathbb{E}\log^+ d < \infty$ is not fulfilled or one wishes to consider the case $\mathbb{P}\{\,\omega : d(\omega) = \infty\,\} > 0$, some stronger integrability conditions on the family D are required. If $D(\omega)$ is interpreted as a linear operator on $\ell^2_{\mathbb{R}}(\mathbb{N})$ then $\|.\|_{\,.,\vartheta.}$ is just the norm of the Banach algebra of linear bounded operators. Regarded to this norm the integrability condition is in fact a stronger condition to the entries of each $D(\omega)$ then the one of Lemma 3.1.

Theorem 3.2 (Multiplicative Ergodic Theorem, MET) *For a random rectangular matrix D satisfying (3.1) the following statements hold:*

(A) *There exists an absorbing set $\tilde\Omega \in \mathcal{F}$, i.e. $\tilde\Omega \subset \vartheta^{-1}\tilde\Omega$, of full measure such that for all $\omega \in \tilde\Omega$ the following is true:*

(i) *The limit* $\lim_{n\to\infty}(D(n,\omega)^* D(n,\omega))^{1/2n} =: \Delta(\omega)$ *exists.*

(ii) *Let* $0 < e^{\lambda_{p(\omega)}(\omega)} < \ldots < e^{\lambda_1(\omega)}$ *the different nonzero eigenvalues of* $\Delta(\omega)$ ($\lambda_{p(\omega)+1}(\omega) := -\infty$, *possibly* $p(\omega) = 0$), $\mathcal{U}_{p(\omega)+1}(\omega) = \ker \Delta(\omega)$ *and let* $\mathcal{U}_{p(\omega)}(\omega), \ldots, \mathcal{U}_1(\omega)$ *the corresponding eigenspaces with multiplicities* $m_i(\omega) = \dim \mathcal{U}_i(\omega), i = 1, \ldots, p(\omega) + 1$. *Then*

$$p(\vartheta\omega) = p(\omega), \qquad \lambda_i(\vartheta\omega) = \lambda_i(\omega)$$

for $i = 1, \ldots, p(\omega) + 1$ *and*

$$m_i(\vartheta\omega) = m_i(\omega)$$

for $i = 1, \ldots, p(\omega)$.

(iii) *Let* $\mathcal{V}_k(\omega) = \ker \Delta(\omega) \oplus \mathcal{U}_{p(\omega)}(\omega) \oplus \ldots \oplus \mathcal{U}_k(\omega), k = 1, \ldots, p(\omega)$ *and* $\mathcal{V}_{p(\omega)+1}(\omega) = \ker \Delta(\omega)$ *such that*

$$\{0\} \subseteq \mathcal{V}_{p(\omega)+1}(\omega) \subset \mathcal{V}_{p(\omega)}(\omega) \subset \ldots \subset \mathcal{V}_1(\omega) = \mathbb{R}^{d(\omega)}$$

defines a filtration of $\mathbb{R}^{d(\omega)}$. *For each* $x \in \mathbb{R}^{d(\omega)}$ *the Lyapunov exponent*

$$\lambda(\omega, x) = \lim_{n\to\infty} \frac{1}{n} \log \|D(n, \omega)x\|_{\vartheta^n\omega}$$

exists, and

$$\lambda(\omega, x) = \lambda_k(\omega) \iff x \in \mathcal{V}_k(\omega) \backslash \mathcal{V}_{k+1}(\omega)$$

for $k = 1, \ldots, p(\omega)$ *and*

$$\lambda(\omega, x) = -\infty \iff x \in \mathcal{V}_{p(\omega)+1}(\omega).$$

(iv) *For each* $x \in \mathbb{R}^{d(\omega)} \backslash \{0\}$

$$\lambda(\vartheta\omega, D(\omega)x) = \lambda(\omega, x),$$

such that

$$D(\omega)\mathcal{V}_k(\omega) \subset \mathcal{V}_k(\vartheta\omega)$$

for $k = 1, \ldots, p(\omega) + 1$, *which means*

$$D(\omega) \ker \Delta(\omega) \subset \ker \Delta(\vartheta\omega)$$

in particular, if $k = p(\omega) + 1$.

(B) *The mapping* $\omega \mapsto p(\omega) \in \mathbb{Z}^+$ *and the mappings* $\omega \mapsto \lambda_i(\omega) \in \mathbb{R} \cup \{-\infty\}$ *(measurably extended from* $\tilde{\Omega}$ *to* Ω*) and* $\omega \mapsto m_i(\omega) \in \mathbb{Z}^+$ *(measurably extended from* $\{\omega : p(\omega) + 1 \geq i\} \in \mathcal{F}$ *to* Ω*) are measurable for* $i = 1, \ldots, p(\omega) + 1$.

Also, the mappings $\omega \mapsto \mathcal{U}_i(\omega) \in \bigcup_{d=2}^{\infty} \bigcup_{k=0}^{d} Gr\,(k, d)$ *and* $\omega \mapsto \mathcal{V}_i(\omega) \in \bigcup_{d=2}^{\infty} \bigcup_{k=0}^{d} Gr\,(k, d)$, *where* $Gr\,(k, d)$ *denotes the Grassmann manifold of* k*-dimensional subspaces of* $\mathbb{R}^d$ *and* $Gr(0, d) := \{0\}, 0 \in \mathbb{R}^d$, *(measurably extended from* $\{\omega : p(\omega) + 1 \geq i\} \in \mathcal{F}$ *to* Ω*) are measurable for* $i = 1, \ldots, p(\omega) + 1$.

(C) *If the dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ *is ergodic, then the mappings* $p(\,.\,)$, $\lambda_i(\,.\,)$, $i = 1, \ldots, p(\,.\,) + 1$ *and* $m_i(\,.\,)$, $i = 1, \ldots, p(\,.\,)$ *are constant.*

Remark 3.3 *In the case of products of square matrices the MET (cf. Arnold [1], Oseledets [15] or Walters [20]) guarantees the invariance of the dimensions of the subspaces for the corresponding Lyapunov exponents. It is obvious, that this cannot be fulfilled here for all subspaces since* $d(\omega)$ *is not constant. But from the proof of the Furstenberg-Kesten Theorem for the case of products of random rectangular matrices (cf. Steinkamp [19]) one gets the dimension of the subspace for the Lyapunov exponent* $-\infty$*, which is given by*

$$m_{p(\omega)+1}(\omega) = d(\omega) - \sum_{i=1}^{p(\omega)} m_i(\omega)$$

for each $\omega \in \tilde{\Omega}$. *That means, that the dimension of* $\ker \Delta(\omega)$ *is the only one which is* not ϑ*-invariant.*

There are two possibilities to proof the MET for random rectangular matrices. The first way is quite analogous to the proof of the MET for square matrices (cf. Arnold [1]). This approach follows the classical line via singular value decomposition, now for rectangular matrices, the subadditive ergodic theorem (cf. Kingman [14]) and the use of exterior algebra to obtain a Furstenberg-Kesten Theorem to proof the existence of the Lyapunov exponents for this case. Also, as in the square case now the hard work begins, but it turns out, that the proof for the square case can be used with some changes, which are concerned with the use of norms on the bundle. An explicit proof following these lines can be found in Steinkamp [19].

Proof. The second way which is presented here, is to use a multiplicative ergodic theorem for compact operators on a separable Banach space, which is due to Ruelle [16, Corollary 2.2]. We consider random operators $\tilde{D}$ corresponding to D as in Section 1. $\tilde{D}(\omega)$ is a compact operator for each $\omega \in \Omega$ on the separable Banach space $\ell^2_{\mathbb{R}}(\mathbb{N})$ and

$$\tilde{D}(n,\omega) = \mathrm{diag}(D(n,\omega), 0, \ldots),$$

and $\tilde{D}(n,\omega)^* \tilde{D}(n,\omega) = \mathrm{diag}(D(n,\omega)^* D(n,\omega), 0, \ldots)$ with $D(n,\omega)^* D(n,\omega) \in \mathbb{R}^{d(\omega)\times d(\omega)}$ for each $n \in \mathbb{N}$. With Ruelle's MET one obtains the existence of

$$\Lambda(\omega) = \lim_{n\to\infty} (\tilde{D}(n,\omega)^* \tilde{D}(n,\omega))^{1/2n} = \lim_{n\to\infty} \mathrm{diag}((D(n,\omega)^* D(n,\omega))^{1/2n}, 0, \ldots)$$

and hence the existence of

$$\Delta(\omega) = \lim_{n\to\infty} ((D(n,\omega)^* D(n,\omega))^{1/2n}.$$

Write

$$\begin{aligned} \Lambda(\omega) &= \mathrm{diag}(U^* \mathrm{diag}(e^{\lambda_1(\omega)}, \ldots, e^{\lambda_{d(\omega)}(\omega)}) U, 0, \ldots) \\ &= \mathrm{diag}(U^*, 0, \ldots) \mathrm{diag}(e^{\lambda_1(\omega)}, \ldots, e^{\lambda_{d(\omega)}(\omega)}, 0, \ldots) \mathrm{diag}(U, 0, \ldots), \end{aligned} \tag{3.2}$$

with $U \in \mathcal{O}(d(\omega), \mathbb{R})$ an orthogonal $d(\omega) \times d(\omega)$ matrix.

The assertions which go beyond Ruelle's theorem are

- the number $p(\omega)$ of Lyapunov exponents $\neq -\infty$ is finite, explicitly $0 \le p(\omega) \le d(\omega) < \infty \;\; \forall\, \omega \in \tilde{\Omega}$,
- the dimensions $\dim \mathcal{U}_i(\omega), i = 1, \ldots, p(\omega)$, of the corresponding eigenspaces of $\Lambda(\omega)$ are finite also. The only infinite dimensional subspace from Ruelle's theorem is the one corresponding to Lyapunov exponent $-\infty$: $\mathcal{V}_{p(\omega)+1}(\omega) = \ker \Lambda(\omega)$.

But both are consequences of (3.2). The assertions of the theorem follow from the embedding of $\mathbb{R}^{d(\omega)}$ in $\ell^2_{\mathbb{R}}(\mathbb{N})$. □

4 Random Walks in Random Environments

We are going to consider now discrete-time stochastic jump processes $\{X_n\}_{n\in\mathbb{N}}$ on $\mathbb{Z}$ to which we can apply the multiplicative ergodic theory developed in the previous section in order to analyze recurrence questions.

4.1 Definition

Let $X = \mathbb{Z}^{\mathbb{N}} = \{x = (x_1, x_2, \ldots) : x_i \in \mathbb{Z}, i \in \mathbb{N}\}$ be the set of all integer sequences and $\mathcal{F}$ the σ-algebra generated by the cylinder sets in X. A random walk in random environment (abbrv. RWIRE) is a discrete-time stochastic process $\{X_n\}_{n\in\mathbb{N}}$ with state space $\mathbb{Z}$. The measure $\mathbb{P}\{\,.\,\}$ on $(X, \mathcal{F})$ which defines this process is constructed in two steps. The first step is to define a random environment. We fix $R, L \in \mathbb{N}$, let G denote the set of all probability measures on $\{-L, \ldots, R\}$ and μ a probability measure on G.

Definition 4.1 *A* random environment *defined by μ is an independent, identically distributed (i.i.d.) sequence of G-valued random variables $\{e(z,\,.\,)\}_{z\in\mathbb{Z}}$ with common distribution μ. An* environment *is a realization of the random environment e. Let $R_z := \max\{a \in \mathbb{Z}^+ : e(z,a) > 0\}$ and $L_z := -\min\{a \in \mathbb{Z}^- : e(z,a) > 0\}$ for $z \in \mathbb{Z}$.*

To omit some trivial cases we restrict ourselves to a measure μ which satisfies

$$\mu\{g \in G : \exists\, a < 0 < b \text{ with } g(a)g(b) > 0\} = 1, \tag{4.1}$$

i.e. $\{X_n\}_{n\in\mathbb{N}}$ can jump to the left and to the right from every state.

The second step is to define a measure $\mathbb{P}\{\,.\,\}$ corresponding to e, so that $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain on $\mathbb{Z}$ starting from 0 with respect to the measure $\mathbb{P}\{\,.\,\}$. Let $\mathbb{P}^{e,j}\{\,.\,\}$ be a family of random measures on $(X, \mathcal{F})$ with

$$\mathbb{P}^{e,j}\{X_0 = j\} = 1,$$

$$\mathbb{P}^{e,j}\{X_{n+1} = z_n + a | X_0 = j, X_1 = z_1, \ldots, X_n = z_n\} = \mathbb{P}^{e,j}\{X_{n+1} = z_n + a | X_n = z_n\} = e(z_n, a).$$

Thus $\{X_n\}_{n\in\mathbb{N}}$ is a stationary Markov chain starting in j with respect to the measure $\mathbb{P}^{e,j}$, whose transition matrix is independent of j. Now define $\mathbb{P}^e\{\,.\,\} := \mathbb{P}^{e,0}\{\,.\,\}$. Then

$$\mathbb{P}^e\{X_0 = 0\} = 1,$$

$$\mathbb{P}^e\{X_{n+1} = z_n + a | X_0 = 0, X_1 = z_1, \ldots, X_n = z_n\} = \mathbb{P}^e\{X_{n+1} = z_n + a | X_n = z_n\} = e(z_n, a). \tag{4.2}$$

To finish the definition we take the expectation of the random measures $\mathbb{P}^e$ with respect to μ

$$\mathbb{P}\{\,.\,\} := \mathbb{E}(\mathbb{P}^e\{\,.\,\}). \tag{4.3}$$

Therefore $\{X_n\}_{n\in\mathbb{N}}$ is a Markov chain with respect to the measure $\mathbb{P}\{\,.\,|e\}$, i.e. once $\{e(z,\,.\,)\}_{z\in\mathbb{Z}}$ is fixed $\{X_n\}_{n\in\mathbb{N}}$ is simply a Markov chain on $\mathbb{Z}$. (4.2) and (4.3) show that the RWIRE is determined by the measure μ.

Note that randomness enters the scene at two different points. First, in choosing the random environment $\{e(z,\,.\,)\}_{z\in\mathbb{Z}}$ which gives us a random measure $\mathbb{P}^e\{\,.\,\}$ on $(X, \mathcal{F})$, and second, if this measure is choosen we let the process $\{X_n\}_{n\in\mathbb{N}}$ move under the fixed transition probability $e(z,.)$ in z for a given realization of $\{e(z,\,.\,)\}_{z\in\mathbb{Z}}$.

The case $R = L = 1$ with $e(z,-1) + e(z,+1) = 1$ (simple RWIRE) is studied extensively. In Solomon [18, Lemma 1.5] one finds the following

Theorem 4.2 *For a simple RWIRE on $\mathbb{Z}$ with $e(0,-1) + e(0,+1) = 1$ a.s. let $d = \mathbb{E}\log \frac{e(0,1)}{e(0,-1)}$. Then*

(i) $d > 0 \quad \Rightarrow \quad \lim_{n\to\infty} X_n = +\infty \quad$ *a.s.,*

(ii) $d = 0 \quad \Rightarrow \quad -\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = +\infty \quad$ *a.s.,*

(iii) $d < 0 \quad \Rightarrow \quad \lim_{n\to\infty} X_n = -\infty \quad$ *a.s.*

RWIRE on countable abelian groups, especially $\mathbb{Z}^2$, are studied by Kalikow [10], who stated sufficient conditions for $\ell(X_n) \to +\infty\,(n \to +\infty)$ where ℓ is a linear functional on the abelian group (cf. [10, Theorem 1]). This generalization allows the consideration of cases which go beyond a simple RWIRE . In Key [11] the following Zero-One Law is found which generalizes the theorem of Solomon.

Theorem 4.3 (Zero-One Law) *Let μ satisfy (4.1). Then one of the following possibilities holds:*

(i) $\lim_{n\to\infty} X_n = +\infty \quad$ *a.s.,*

(ii) $-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = +\infty \quad$ *a.s.,*

(iii) $\lim_{n\to\infty} X_n = -\infty \quad a.s.$

For a proof see Key [11, Theorem 11].

Definition 4.4 *A RWIRE is called* recurrent *if case (ii) in Theorem 4.3 occurs. For the case, that (i) or (iii) holds, the RWIRE is called* transient.

Further, under the condition

$$R_z = R \text{ and } L_z = L \text{ a.s. } \forall z \in \mathbb{Z} \tag{4.4}$$

Key deduces recurrence and transience criteria of such RWIRE in terms of the Lyapunov exponents of a certain sequence of random matrices using the MET of Oseledets. The aim of this work is to weaken condition (4.4) to $1 \leq R_z \leq R$ and $1 \leq L_z \leq L$ a.s. $\forall z \in \mathbb{Z}$. This leads to a sequence of random rectangular matrices for which Theorem 3.2 applies.

Definition 4.5 *Let $A \subset \mathbb{Z}$ and $\tau = \inf\{n > 0 : X_n \in A\}$. The process $\{X_n\}_{n\in\mathbb{N}}$ is said to* enter A *if $\tau < \infty$.* $\{X_n\}_{n\in\mathbb{N}}$ enters A at a *if τ is finite and $X_\tau = a$ for $a \in A$.*

Definition 4.6 *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a Markov chain on $\mathbb{Z}^- \cup (0,\infty)$ and transition probability $Q(x,y) = (e(x, y-x))$. For every $y \in \mathbb{Z}^- \cup (0,\infty)$ let*

$$f(y) = \begin{cases} \mathbb{P}\{\{Y_n\}_{n\in\mathbb{N}} \text{ enters } (0,\infty) \,|\, Y_0 = y\}, & y \leq 0, \\ 1, & y = 1. \end{cases}$$

Therefore $f(y)$ is the probability that $\{Y_n\}_{n\in\mathbb{N}}$ ever reaches $(0,\infty)$ if it starts in $y \in \mathbb{Z}^-$. The proof of Theorem 4.3 does not need condition (4.4), so that it can be used in our case. We do not prove the Zero-One Law, but we state the main theorem in the proof which is important for the following considerations.

Theorem 4.7 *Suppose (4.1) holds. Then there are positive constants $c_1, c_2 > 0$, such that either*

$$\mathbb{P}^{e,z}\{\{X_n\}_{n\in\mathbb{N}} \text{ enters } (0,\infty)\} = 1 \quad a.s. \quad \forall z \in \mathbb{Z} \tag{4.5}$$

or

$$\limsup_{k\to\infty} \frac{1}{k} \log \mathbb{P}^{e,-k}\{\{X_n\}_{n\in\mathbb{N}} \text{ enters } (0,\infty)\} \leq -c_1 \quad a.s., \tag{4.6}$$

either

$$\mathbb{P}^{e,z}\{\{X_n\}_{n\in\mathbb{N}} \text{ enters } (-\infty,0)\} = 1 \quad a.s. \quad \forall z \in \mathbb{Z} \tag{4.7}$$

or

$$\limsup_{k\to\infty} \frac{1}{k} \log \mathbb{P}^{e,k}\{\{X_n\}_{n\in\mathbb{N}} \text{ enters } (-\infty,0)\} \leq -c_2 \quad a.s. \tag{4.8}$$

For a proof see Key [11, Theorem 17].

4.2 Recurrence and transience criteria

Now we deduce criteria to distinguish between the three possibilities of Theorem 4.3. To do this we construct a sequence of certain random matrices to derive inductively the probability $f(y)$ for returning to $(0,\infty)$ for every $y \in \mathbb{Z}^-$. Unfortunately, it has turned out that only the case $R_z \in \{1,2\}$ and $L_z = 1$ a.s. for all $z \in \mathbb{Z}$ can be treated with Key's approach, so that from now on we concentrate our considerations on a measure μ on G which satisfies

$$R_y \in \{1,2\} \quad \text{and} \quad L_y = 1 \text{ for } y \in \mathbb{Z} \tag{4.9}$$

with R_y not constant, which implies (4.1). From the definition of $f(y)$ and the Chapman-Kolmogorov equation one obtains

$$f(y) \;=\; \sum_{z\in\mathbb{Z}} \mathbb{P}^{\,e,y}\{X_{n+1}=z\}f(z) = \sum_{a=-1}^{R_y} e(y,a)f(y+a)$$

which implies

$$f(y-1) \;=\; \sum_{a=0}^{R_y} \frac{\delta_{0,a} - e(y,a)}{e(y,-1)} \cdot f(y+a). \tag{4.10}$$

Let us introduce for every $y \in \mathbb{Z}^-$

$$\mathbf{f}_y := \begin{bmatrix} f(y+R_y) \\ \vdots \\ f(y) \end{bmatrix} \in \mathbb{R}^{R_y+1}$$

and the matrix $A_y \in \mathbb{R}^{(R_{y-1}+1)\times(R_y+1)}$ by

$$A_y = \begin{cases} \begin{bmatrix} 0 & 1 \\ \frac{-e(y,1)}{e(y,-1)} & \frac{1-e(y,0)}{e(y,-1)} \end{bmatrix} & \text{if } R_{y-1}=1, R_y=1, \\[2ex] \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-e(y,1)}{e(y,-1)} & \frac{1-e(y,0)}{e(y,-1)} \end{bmatrix} & \text{if } R_{y-1}=2, R_y=1, \\[2ex] \begin{bmatrix} 0 & 0 & 1 \\ \frac{-e(y,2)}{e(y,-1)} & \frac{-e(y,1)}{e(y,-1)} & \frac{1-e(y,0)}{e(y,-1)} \end{bmatrix} & \text{if } R_{y-1}=1, R_y=2, \\[2ex] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-e(y,2)}{e(y,-1)} & \frac{-e(y,1)}{e(y,-1)} & \frac{1-e(y,0)}{e(y,-1)} \end{bmatrix} & \text{if } R_{y-1}=2, R_y=2. \end{cases}$$

Now define the following product of random matrices

$$A_y \cdot \ \ldots \ \cdot A_{-1}A_0, \ y \le 0. \tag{4.11}$$

Then (4.10) implies

$$\mathbf{f}_{y-1} = A_y\,\mathbf{f}_y = A_y \cdot \ \ldots \ \cdot A_{-1}A_0\,\mathbf{f}_0 \tag{4.12}$$

for each $y \le 0$.

Note that (4.9) implies the irreducibility of $\{X_n\}_{n\in\mathbb{N}}$ for every realization of $\{e(z,.)\}_{z\in\mathbb{Z}}$ so that the asymptotic behaviour of $\{X_n\}_{n\in\mathbb{N}}$ does not depend on the initial state. To simplify the following considerations we choose the initial state of $\{X_n\}_{n\in\mathbb{N}}$ so that $R_0 = L_0 = 1$ and $R_{-1} = L_{-1} = 1$. Hence the state space for the cocycle (4.11) has dimension 2.

Lemma 4.8 *Suppose that*

$$\mathbb{E}\log e(y,-1) > -\infty \tag{4.13}$$

and

$$\mathbb{E}\log e(y,R_y) > -\infty \tag{4.14}$$

hold, where

$$\mathbb{E}\log e(y,R_y) = \int\limits_{\{e(y,.)\in G: R_y=1\}} \log e(y,1)\, d\mu + \int\limits_{\{e(y,.)\in G: R_y=2\}} \log e(y,2)\, d\mu > -\infty.$$

Then (4.11) satisfies the assumptions of the MET (Theorem 3.2).

Proof. The matrices A_y depend on the environments $e(y,\,.\,)$ and $e(y-1,\,.\,)$ in the points y and $y-1$ only, such that $\{A_{-y}\}_{y=0,1,\ldots}$ is a stationary sequence. The use of a canonical dynamical system for the stochastic process $\{e(z,\,.\,)\}_{z\in\mathbb{Z}}$ (cf. Arnold [1, Appendix A.2, p.265]) justifies the application of Theorem 3.2, if the integrability condition for the matrices A_y is fulfilled.

Let M_y be the largest absolute value of the entries of A_y. Then it suffices to show the $\log^+$-integrabilty of M_y. With (4.1) and (4.13) one obtains $0 < e(y,-1) < 1$, which implies

$$M_y = \max\left\{1, \frac{e(y,R_y)}{e(y,-1)}, \ldots, \frac{1-e(y,0)}{e(y,-1)}\right\} \leq \max\left\{1, \frac{1}{e(y,-1)}\right\} = \frac{1}{e(y,-1)}.$$

Now $\log^+ M_y \in \mathcal{L}^1(\mu)$ follows from

$$0 \leq \log M_y \leq \log \frac{1}{e(y,-1)} = -\log e(y,-1)$$

and

$$-\log e(y,-1) \in \mathcal{L}^1(\mu) \iff \log e(y,-1) \in \mathcal{L}^1(\mu) \iff (4.13).$$

□

Now we are able to state the main theorem which characterizes the recurrence and transience properties of a RWIRE in terms of the Lyapunov exponents of (4.11).

Theorem 4.9 *Let $\{X_n\}_{n\in\mathbb{N}}$ a RWIRE on $\mathbb{Z}$ fulfilling (4.9) and assume*

$$\mathbb{E}\log e(y,-1) + \mathbb{E}\log e(y,R_y) > -\infty.$$

Let the numbers $\lambda_1 \geq \lambda_2 > -\infty$ be the Lyapunov exponents of the cocycle (4.11). Then

(i) $\lambda_1+\lambda_2 > 0 \quad \Rightarrow \quad \mathbb{P}\{\lim_{n\to\infty} X_n = +\infty\} = 1,$

(ii) $\lambda_1+\lambda_2 = 0 \quad \Rightarrow \quad \mathbb{P}\{-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = +\infty\} = 1,$

(iii) $\lambda_1+\lambda_2 < 0 \quad \Rightarrow \quad \mathbb{P}\{\lim_{n\to\infty} X_n = -\infty\} = 1.$

This theorem generalizes the result of Key [11, Theorem 22] for the special case which is considered here. In fact, the condition $e(y,R) > 0$ a.s. is not necessary anymore. The reason for considering only this rather restricted case is the fact that we derive the probability $f(y-1)$ for returning from state $y-1$ever to $(0,\infty)$ inductively from $f(y), f(y+1)$ and $f(y+2)$. If L_y varied randomly, then we would have to calculate $f(y-i)$ for $i \geq 2$ with some positive probability which, in our opinion, is impossible with this kind of random matrices, so that we have to fix the maximum jumpsize to the left. Maybe some more general R_y are possible, but this can cause that all Lyapunov-exponents to be $-\infty$ a.s. if the increments of R_y become too large.

4.3 Preliminary considerations

Let $\|.\|_k$ denote the euclidean norm on $\mathbb{R}^{R_k+1}$ and V defined by

$$V = \{v \in \mathbb{R}^2 : \lim_{n\to\infty} \frac{1}{n} \log \|A_{1-n} \cdot \ldots \cdot A_{-1} A_0\, v\|_{1-n} \leq 0\}$$

be the random subspace of $\mathbb{R}^2$ given by the direct sum of the Oseledets spaces of non-positive Lyapunov exponents. Let

$$\mathbb{1}_k = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^k.$$

Lemma 4.10 *(i) For each matrix A_y and the corresponding $\mathbb{1}$-vectors we have $A_y\, \mathbb{1} = \mathbb{1}$. Therefore the Lyapunov exponent for $\mathbb{1}$ is calculated as*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_{1-n} \cdot \ldots \cdot A_{-1} A_0\, \mathbb{1}\| = 0,$$

i.e. $\mathbb{1} \in V$ and $\dim V \geq 1$.
(ii) $\dim V = 1 \quad \Rightarrow \quad \mathbb{P}^{e,y}\{X_n$ enters $(0,\infty)\} = 1 \quad$ a.s. $\quad \forall y \in \mathbb{Z}$.

Proof. (i) follows directly from the definition of A_y. From (4.9) one obtains $\|\mathbf{f}_y\| \leq \sqrt{3}$ for $y \in \mathbb{Z}^-$, since the entries of $\mathbf{f}_y$ are probabilities. With (4.12) we have $\mathbf{f}_0 \in V$. Now $\dim V = 1$ and $\mathbb{1} \in V$ imply

$$\mathbf{f}_0 = \begin{bmatrix} 1 \\ f(0) \end{bmatrix} = \mathbb{1}, \text{ i.e.} \quad f(0) = 1.$$

$R_0 = L_0 = 1$ and part (i) of this Lemma imply $f(y) = 1$ for every $y \leq 0$. This proves (ii). □

Part (i) of this Lemma means that always at least one of the Lyapunov exponents is zero. The next proposition deals with the invertibility of some matrices which are needed in the following.

Proposition 4.11 *Let $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ be sequences of real numbers with $a_n < 0$, $b_n \leq 0$, $c_n > 0$ and $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$. Let further $b_0, c_0 \in \mathbb{R}$ with $b_0 < 0 < c_0$ and $b_0 + c_0 = 1$. Then for each $n \in \mathbb{N}$ the matrix*

$$B^{(n)} := \begin{bmatrix} 0 & 0 & 1 \\ a_n & b_n & c_n \end{bmatrix} P^{(n-1)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ b_0 & c_0 \end{bmatrix}$$

with

$$P^{(n-1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{n-1} & b_{n-1} & c_{n-1} \end{bmatrix} \cdot \ldots \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & b_1 & c_1 \end{bmatrix} \quad \text{and } P^{(0)} = id$$

is invertible, i.e. $B^{(n)} \in Gl(2,\mathbb{R})$.

Proof. This proof proceeds via induction. In the cases $n = 1, 2$ direct calculation gives $\det B^{(n)} > 0$. For arbitrary $n \in \mathbb{N}$ consider the product $\tilde{B}^{(n)}$ with extended first and extended last factor: Let

$$\tilde{B}^{(n)} := \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 \\ a_n & b_n & c_n \end{bmatrix} P^{(n-1)} \begin{bmatrix} \mathbf{1} & 1 & 0 \\ \mathbf{1} & 0 & 1 \\ \mathbf{1} & b_0 & c_0 \end{bmatrix}.$$

The first column of the right factor in this representation of $\tilde{B}^{(n)}$ is invariant under all further matrices of this product since $a_n + b_n + c_n = 1$, $n \in \mathbb{N}$. Therefore the first column of $\tilde{B}^{(n)}$ contains 1's only. Due to the shift part in the first two rows of the matrices, $B^{(n)}$ and $B^{(n-1)}$ can be found in $\tilde{B}^{(n)}$ as block matrices:

$$\tilde{B}^{(n)} = \begin{bmatrix} 1 & * & * \\ 1 & \multicolumn{2}{c}{B^{(n)}} \\ 1 & & \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_n & b_n & c_n \end{bmatrix} \begin{bmatrix} 1 & * & * \\ 1 & \multicolumn{2}{c}{B^{(n-1)}} \\ 1 & & \end{bmatrix} = \begin{bmatrix} 1 & \multicolumn{2}{c}{B^{(n-1)}} \\ 1 & & \\ 1 & * & * \end{bmatrix}.$$

Denote the entries of $B^{(n)}$ by $B^{(n)} = \begin{bmatrix} b_{11}^{(n)} & b_{12}^{(n)} \\ b_{21}^{(n)} & b_{22}^{(n)} \end{bmatrix}$. Therefore we have

$$b_{11}^{(n)} = b_{21}^{(n-1)},\ b_{12}^{(n)} = b_{22}^{(n-1)} \text{ and}$$

$$b_{11}^{(n+1)} = b_{21}^{(n)},\ b_{12}^{(n+1)} = b_{22}^{(n)}, \text{ resp.} \tag{4.15}$$

Further

$$\tilde{B}^{(n+1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_{n+1} & b_{n+1} & c_{n+1} \end{bmatrix} \tilde{B}^{(n)} = \begin{bmatrix} 1 & \multicolumn{2}{c}{B^{(n)}} \\ 1 & & \\ 1 & b_{21}^{(n+1)} & b_{22}^{(n+1)} \end{bmatrix}$$

gives

$$\begin{aligned} b_{21}^{(n+1)} &= a_{n+1} b_{11}^{(n-1)} + b_{n+1} b_{11}^{(n)} + c_{n+1} b_{21}^{(n)} \quad \text{and} \\ b_{22}^{(n+1)} &= a_{n+1} b_{12}^{(n-1)} + b_{n+1} b_{12}^{(n)} + c_{n+1} b_{22}^{(n)}. \end{aligned} \tag{4.16}$$

Since $b_0 + c_0 = 1$ we have

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & b_0 & c_0 \end{bmatrix} = \det \begin{bmatrix} 1 & 1+0 & 0 \\ 1 & 0+1 & 1 \\ 1 & b_0 + c_0 & c_0 \end{bmatrix} = 0$$

and therefore $\det \tilde{B}^{(n)} = 0$ follows. Evaluating $\det \tilde{B}^{(n)}$ one obtains:

$$0 = \det \tilde{B}^{(n)} = \det B^{(n)} + \det B^{(n-1)} - \det \begin{bmatrix} b_{11}^{(n-1)} & b_{12}^{(n-1)} \\ b_{21}^{(n)} & b_{22}^{(n)} \end{bmatrix}$$

and therefore the assertion

$$\det B^{(n)} > 0,\ \det B^{(n-1)} > 0 \quad \Rightarrow \quad \det \begin{bmatrix} b_{11}^{(n-1)} & b_{12}^{(n-1)} \\ b_{21}^{(n)} & b_{22}^{(n)} \end{bmatrix} > 0. \tag{4.17}$$

Let $\det B^{(n)} > 0$ and $\det B^{(n-1)} > 0$. Then

$$\begin{aligned} \det B^{(n+1)} &= b_{11}^{(n+1)} b_{22}^{(n+1)} - b_{12}^{(n+1)} b_{21}^{(n+1)} \overset{(4.15)}{=} b_{21}^{(n)} b_{22}^{(n+1)} - b_{22}^{(n)} b_{21}^{(n+1)} \\ &\overset{(4.16)}{=} b_{21}^{(n)} \left(a_{n+1} b_{12}^{(n-1)} + b_{n+1} b_{12}^{(n)} + c_{n+1} b_{22}^{(n)} \right) - \\ &\qquad - b_{22}^{(n)} \left(a_{n+1} b_{11}^{(n-1)} + b_{n+1} b_{11}^{(n)} + c_{n+1} b_{21}^{(n)} \right) \\ &= b_{n+1} \left(b_{12}^{(n)} b_{21}^{(n)} - b_{11}^{(n)} b_{22}^{(n)} \right) - a_{n+1} \left(b_{11}^{(n-1)} b_{22}^{(n)} - b_{12}^{(n-1)} b_{21}^{(n)} \right) \\ &= -b_{n+1} \det B^{(n)} - a_{n+1} \det \begin{bmatrix} b_{11}^{(n-1)} & b_{12}^{(n-1)} \\ b_{21}^{(n)} & b_{22}^{(n)} \end{bmatrix} \overset{(4.17)}{>} 0. \end{aligned}$$

□

Proposition 4.12 *Let λ_1, λ_2 be the Lyapunov exponents of (4.11). Then the assumptions of Theorem 4.9 imply $\lambda_1 + \lambda_2 > -\infty$.*

Proof. The proof is seperated into 3 steps.
Step 1 (Construction of a sequence $\{B_i\}_{i=0,1,\dots}$ of 2×2 matrices with the same Lyapunov exponents as $\{A_{-y}\}_{i=0,1,\dots}$): Consider the cocycle (4.11). Define a decreasing sequence of stopping times $y_0 > y_1 > y_2 > \dots$, $y_i \in \mathbb{Z}^-$ with

$$y_i = \begin{cases} 0, & i = 0, \\ \max\{y < y_{i-1} : R_y = L_y = 1\}, & i \geq 1. \end{cases}$$

Now define a sequence $\{B_i\}_{i=0,1,\dots}$ of 2×2 matrices by

$$\begin{aligned} B_0 &= A_{y_1+1} \cdot \ \dots \ \cdot A_0, \\ B_1 &= A_{y_2+1} \cdot \ \dots \ \cdot A_{y_1}, \\ &\vdots \\ B_i &= A_{y_{i+1}+1} \cdot \ \dots \ \cdot A_{y_i}. \end{aligned}$$

Thus, for each $i = 0, 1, \dots$ we have

$$A_{y_{i+1}+1} \cdot \ \dots \ \cdot A_{-1} A_0 = B_i \cdot \ \dots \ \cdot B_1 B_0. \tag{4.18}$$

Step 2 (Proving the invertibility of B_i for $i \in \mathbb{N}$): Each B_i consists of one of the following three types:

(i) If $y_i - y_{i+1} = 1$: From the definition of A_y follows $B_i = A_{y_i} \in Gl(2, \mathbb{R})$, since (4.13) and (4.14) imply $\frac{-e(y_i,1)}{e(y_i,-1)} \neq 0$.

(ii) If $y_i - y_{i+1} = 2$:

$$B_i = A_{y_{i+1}+1} A_{y_i} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{-e(y_{i+1}+1,2)}{e(y_{i+1}+1,-1)} & \frac{-e(y_{i+1}+1,1)}{e(y_{i+1}+1,-1)} & \frac{1-e(y_{i+1}+1,0)}{e(y_{i+1}+1,-1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-e(y_i,1)}{e(y_i,-1)} & \frac{1-e(y_i,0)}{e(y_i,-1)} \end{bmatrix}$$

(iii) If $y_i - y_{i+1} \geq 3$:

$$\begin{aligned} B_i &= A_{y_{i+1}+1} \cdot \ \dots \ \cdot A_{y_i} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{-e(y_{i+1}+1,2)}{e(y_{i+1}+1,-1)} & \frac{-e(y_{i+1}+1,1)}{e(y_{i+1}+1,-1)} & \frac{1-e(y_{i+1}+1,0)}{e(y_{i+1}+1,-1)} \end{bmatrix} \cdot \\ & \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix} \cdot \ \dots \ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}}_{(y_i - y_{i+1} - 2) \text{ matrices}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-e(y_i,1)}{e(y_i,-1)} & \frac{1-e(y_i,0)}{e(y_i,-1)} \end{bmatrix} \end{aligned}$$

Proposition 4.11 guarantees the invertibility of B_i in the cases (ii) and (iii).

Step 3: Finally, we have to prove that the sequence $\{B_i\}_{i=0,1,\dots}$ fulfills the assumptions of a MET for invertible square matrices (cf. Arnold [1, Theorem 3.5.1 (B)], Walters [20], e.g.). Again, we can use a canonical dynamical system as in Lemma 4.8, but here a slight modification is necessary. Let

$$T = \{\{e(y,\cdot)\}_y \in G^{\mathbb{Z}^-} : e(0,2) = 0\}$$

and

$$N\left(\{e(y,\cdot)\}_y\right) = -\max\{y < 0 : e(y,2) = 0\}.$$

Then the canonical dynamical system of Lemma 4.8 induces a transformation

$$\vartheta_T : T \to T, \{e(y,\cdot)\}_y \mapsto \vartheta^{N(\{e(y,\cdot)\}_y)}\{e(y,\cdot)\}_y$$

on T, which is the (random times) iterated shift map. For details see Steinkamp [19] or Cornfeld, Fomin, Sinai [7]. It remains to show the integrability conditions of the MET, which are

$$\log^+ \|B_i\| \in \mathcal{L}^1(\mu) \tag{4.19}$$

and

$$\log^+ \|B_i^{-1}\| \in \mathcal{L}^1(\mu). \tag{4.20}$$

For every fixed $i \in \mathbb{N}$ the matrix B_i is a product which consists of randomly many factors. Since $\{R_z\}_{z\in\mathbb{Z}}$, as $\{e(z, .)\}_{z\in\mathbb{Z}}$, is i.i.d. with $\mu\{R_0 = 1\} \cdot \mu\{R_0 = 2\} > 0$ the Borel-Cantelli-Lemma implies that the number N of factors of B_i is finite a.s. Let $m = \mu\{R_0 = 2\}$ then N has the distribution

$$\mu\{ N = k \} = (1 - m)\, m^{k-1},\ k \geq 1,$$

with expected value $\mathbb{E}N = \frac{1}{1-m} < \infty$.

Let M_y be the largest absolute value of the entries of A_y. Then

$$\log \|B_i\| = \log \|A_{y_{i+1}+1} \cdot \ \ldots \ \cdot A_{y_i}\| \leq \sum_{j=y_i}^{y_{i+1}+1} \log \|A_j\| \leq N \log M_{j_0}$$

for an index j_0. Wald's identity imply (4.19), since

$$\mathbb{E} \log^+ \|B_i\| \leq \mathbb{E} N \ \log^+ M_{j_0} = \mathbb{E}N \ \mathbb{E} \log^+ M_{j_0} \ < \infty$$

and $\log^+ M_j \in \mathcal{L}^1(\mu)$. Note that the A_y are not an i.i.d. sequence, but M_y are, since the largest absolute value of the entries of A_y does not depend on the format of the matrices.

From Lemma 4.10 we have $A_y \mathbb{1} = \mathbb{1}$ and hence $B_i \mathbb{1} = \mathbb{1}$ so that there are $u, v \in \mathbb{R}$ with

$$B_i = \begin{bmatrix} u & 1-u \\ v & 1-v \end{bmatrix}.$$

Let $M = \max\{|u|, |1-u|, |v|, |1-v|\}$ the largest absolute value of the entries of B_i. Then (4.19) imply $\log^+ M \in \mathcal{L}^1(\mu)$. From step 2 we have the invertibility of B_i, so that

$$B_i^{-1} = \frac{1}{\det B_i} \begin{bmatrix} 1-v & -1+u \\ -v & u \end{bmatrix}.$$

Since $\det B_i > 0$ (see the proof of Proposition 4.11) we have

$$\log^+ \|B_i^{-1}\| \leq \log^+ \frac{M}{\det B_i} \in \mathcal{L}^1(\mu),$$

if

$$\log^+ \frac{1}{\det B_i} \in \mathcal{L}^1(\mu).$$

From the proof of Proposition 4.11 we obtain the following estimate

$$\begin{aligned} \det B^{(n+1)} &= -b_{n+1} \det B^{(n)} - a_{n+1} \det \begin{bmatrix} b_{11}^{(n-1)} & b_{12}^{(n-1)} \\ b_{21}^{(n)} & b_{22}^{(n)} \end{bmatrix} \geq -a_{n+1} \left(\det B^{(n)} + \det B^{(n-1)}\right) \\ &\geq -a_{n+1} \det B^{(n)} \geq a_{n+1} a_n \det B^{(n-1)} \geq \ldots \geq (-1)^n \prod_{j=1}^{n} a_{j+1} \det B^{(1)} \\ &\geq (-1)^{n+1} \prod_{j=1}^{n+1} a_j \cdot c_0 > 0 \quad \forall n \in \mathbb{N}. \end{aligned}$$

B_i consists of N factors. Let

$$a_j = \frac{-e(y_i - j, R_{y_i - j})}{e(y_i - j, -1)} \quad \text{and} \quad c_0 = \frac{1 - e(y_i, 0)}{e(y_i, -1)}.$$

Then

$$\begin{aligned}\log\frac{1}{\det B_i} &\le \log\frac{1}{(-1)^N\prod_{j=1}^N a_j\,\cdot\, c_0} = \sum_{j=1}^N -\log(-a_j) \,-\, \log c_0\\ &= \sum_{j=1}^N -\log\frac{e(y_i-j,R_{y_i-j})}{e(y_i-j,-1)} - \log\frac{1-e(y_i,0)}{e(y_i,-1)}.\end{aligned}$$

Now, taking the expectation of both sides, again with Wald's identity one obtains

$$\mathbb{E}\log\frac{1}{\det B_i} \le \mathbb{E}N\cdot(\mathbb{E}\log e(y,-1) - \mathbb{E}\log e(y,R_y)) - \mathbb{E}\log\frac{1-e(y,0)}{e(y,-1)}$$

since $\{a_j\}_j$ is i.i.d.

(4.13) and (4.14) imply that each expectation on the right hand side is finite. $\mathbb{E}\log\frac{1-e(y,0)}{e(y,-1)} \in \mathcal{L}^1(\mu)$ follows from the proof of Lemma 4.8 so that

$$\log^+\frac{1}{\det B_i} \in \mathcal{L}^1(\mu)$$

which implies (4.20).

The sequences $\{A_{-y}\}_{y=0,1,\dots}$ and $\{B_i\}_{i=0,1,\dots}$ have the same Lyapunov exponents since (4.18). Now a MET for square invertible matrices implies $\lambda_1+\lambda_2 > -\infty$. □

4.4 Proof of the recurrence and transience criteria

The proof of Theorem 4.3 reveals that it is sufficient to show the following implications to proof Theorem 4.9:

- $\lambda_1+\lambda_2>0 \quad\Rightarrow\quad$ (4.5) and (4.8)
- $\lambda_1+\lambda_2=0 \quad\Rightarrow\quad$ (4.5) and (4.7)
- $\lambda_1+\lambda_2<0 \quad\Rightarrow\quad$ (4.6) and (4.7)

Now we consider the probabilities $\mathbb{P}^{e,y}\{\{X_n\}_{n\in\mathbb{N}}$ enters $(-\infty,0)\}$ for $y\ge 0$. In the same way as in Definition 4.6 let

$$g(y) = \begin{cases}\mathbb{P}^{e,y}\{\{X_n\}_{n\in\mathbb{N}} \text{ enters } (-\infty,0)\}, & y\ge 0,\\ 1, & y<0.\end{cases}$$

The event $\{\{X_n\}_{n\in\mathbb{N}}$ enters $(-\infty,0)\}$ is the same as $\{X_n\}_{n\in\mathbb{N}}$ will ever reach -1, if $X_0\ge 0$. Similar to (4.10) we have

$$g(y) = \sum_{a=-1}^{R_y} e(y,a)\,g(y+a), \qquad g(y-1) = \sum_{a=0}^{R_y}\frac{\delta_{0,a}-e(y,a)}{e(y,-1)}\cdot g(y+a).$$

Let $\mathbf{g}_y = \begin{bmatrix} g(y+R_y)\\ \vdots\\ g(y)\end{bmatrix}$. Since $R_{-1}=1$ and (4.11) for every $y\ge 0$ we have

$$\mathbf{g}_{-1} = \begin{bmatrix} g(0)\\ g(-1)\end{bmatrix} = A_0\,\mathbf{g}_0 = A_0\dots A_{y-1}A_y\,\mathbf{g}_y,$$

where $\mathbf{g}_y \in \mathbb{R}^{R_y+1}$. As in the proof of Proposition 4.12 we construct a (now increasing) sequence $y_0 < y_1 < y_2 < \dots$ with

$$y_i = \begin{cases} 0, & i = 0, \\ \min\{y > y_{i-1} : R_y = L_y = 1\}, & i \geq 1, \end{cases}$$

obtaining a sequence of invertible 2×2 matrices B_i, such that

$$A_0 \cdot \ \dots \ \cdot A_{y_i-1} A_{y_i} = B_0 \cdot \ \dots \ \cdot B_i =: \Phi_i$$

and therefore

$$\mathbf{g}_{y_i} = B_i^{-1} \cdot \ \dots \ \cdot B_0^{-1} \, \mathbf{g}_{-1} = \Phi_i^{-1} \, \mathbf{g}_{-1} \quad , i = 0, 1, \dots \ .$$

Now, the Lyapunov exponents of Φ_i^{-1} are the negative Lyapunov exponents of Φ_i (cf. Walters [20, Theorem 10.3]), which are $-\infty < -\lambda_1 \leq -\lambda_2$. Changing the roles of left and right the proof of $\lambda_1 + \lambda_2 > 0 \Rightarrow$ (4.5) is the same as the proof of $\lambda_1 + \lambda_2 < 0 \Rightarrow$ (4.7). Also the proof of $\lambda_1 + \lambda_2 > 0 \Rightarrow$ (4.8) is the same as the proof of $\lambda_1 + \lambda_2 < 0 \Rightarrow$ (4.6). Therefore it remains to show

(i) $\lambda_1 + \lambda_2 > 0 \quad \Rightarrow \quad$ (4.5)

(ii) $\lambda_1 + \lambda_2 < 0 \quad \Rightarrow \quad$ (4.6)

(iii) $\lambda_1 + \lambda_2 = 0 \quad \Rightarrow \quad$ (4.5) and (4.7)

Proof. (i) $\lambda_1 + \lambda_2 > 0$ implies $\dim V = 1$ since one of the Lyapunov exponents is always zero and therefore Lemma 4.10 implies (4.5).

(ii) Let

$$V' = \{v \in \mathbb{R}^2 : \lim_{n\to\infty} \frac{1}{n} \log \|A_{1-n} \cdot \ \dots \ \cdot A_{-1} A_0 \, v\| < 0\}.$$

$\lambda_1 + \lambda_2 < 0$ implies $\dim V' = 1$. Let $\mathbf{h}_0 \in V'$ the unique vector in V', whose first coordinate is 1. A solution $h(y)$ of

$$h(y) = \sum_{a=-1}^{R_y} e(y,a) \, h(y+a) \text{ for } y \leq 0 \ , \ h(1) = 1 \tag{4.21}$$

is given by

$$\mathbf{h}_y \ := \ \begin{bmatrix} h(y+R_y) \\ \vdots \\ h(y) \end{bmatrix} = A_y \cdot \ \dots \ \cdot A_0 \, \mathbf{h}_0 = A_y \cdot \ \dots \ \cdot A_0 \begin{bmatrix} h(1) \\ h(0) \end{bmatrix},$$

where $h(y)$ is the last coordinate of $\mathbf{h}_y$. From $\mathbf{h}_0 \in V'$ one obtains $\lim_{n\to\infty} h(-n) = 0$. Also $f(y)$ (see Definition 4.6) is, as $h(y)$, a bounded solution of (4.21).

Lemma 4.13 *$h(y) = f(y)$ for $y \leq 1$.*

The proof follows completely the one of Key [11, Claim 2, p. 545].

From Lemma 4.13 we obtain

$$\lim_{n\to\infty} f(-n) = \lim_{n\to\infty} h(-n) = 0$$

with an exponential convergence rate $\lambda_2 < 0$, since $\mathbf{h}_0 \in V'$, i.e.

$$\limsup_{k\to\infty} \frac{1}{k} \log \mathbb{P}^{\,e,-k}\{\{X_n\}_{n\in\mathbb{N}} \text{ enters } (0,\infty)\} \leq \lambda_2 \qquad \text{a.s.}$$

and therefore (4.6).

(iii) First note that (4.6) $\iff$ $\dim V' = 1$: From (4.6) we have $\mathbf{f}_0 \in V'$ and therefore $\dim V' = 1$ since $\dim V' \leq 1$. Conversely, we have $\dim V' = 1 \Rightarrow \lambda_1 + \lambda_2 < 0$, which implies (4.6) by the above.

Therefore (4.6) $\iff$ $\lambda_2 < 0$, since $\lambda_1 = 0$. In the same way we obtain (4.8) $\iff$ $\lambda_1 > 0$. The only possibility that $\lambda_1 + \lambda_2 = 0$ holds is neither (4.6) nor (4.8) holds. Therefore (4.5) and (4.7) hold. □

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