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# Convergence of inhomogenous products of matrices and coefficients of ergodicity

D.J. Hartfiel [a], Uriel G. Rothblum [b,*,1]

[a] *Mathematics Department, Texas A&M University, College Station, Texas 77843, USA*

[b] *Faculty of Industrial Engineering and Management, Technion – Israel Institute of Technology, Haifa 32000, Israel*

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**Abstract**

Given a square matrix $A$ and a norm $\| \ \|$, the coefficient of ergodicity of $A$ with respect to $\| \ \|$ is defined as max $\{\|x^{\mathrm{T}}A\| : x \in \mathbb{R}^n,\ \|x\| = 1,\ x^{\mathrm{T}}F = 0\}$ with $F$ as a matrix satisfying $AF = 0$. We demonstrate that for a bounded set of such matrices with all coefficients of ergodicity of the matrices in the set below 1, all sequences constructed through inhomogenous products of matrices from the set converge geometrically.


## 1. Introduction

Let $\Sigma$ be a set of $n \times n$ matrices. We say that $\Sigma$ has *left converging products*, or briefly, the LCP property, if for every sequence $A_1, A_2, \ldots$ of matrices from $\Sigma$ we have that the sequence $\{A_iA_{i-1} \ldots A_2A_1\}_{i=1,2,\ldots}$ is converging. Motivated by the theory of Markov chains which concerns the case where the matrices are stochastic, the convergence of products of nonnegative matrices is sometimes referred to as *ergodicity*; here, "inhomogenous" refers to products of matrices that do not necessarily coincide while "homogenous" refers to products of the same matrix, i.e., powers of a single matrix.

* Corresponding author.

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The goal of this paper is to obtain sufficient conditions for the LCP property through characteristics of the *individual* matrices in the underlying set of matrices. Earlier conditions for the convergence of inhomogenous products of nonnegative matrices were obtained in [1,2,4–7] and [9]. The earlier results focused on the eigenvalues and matrix norms of (finite) inhomogenous products of the multiplicand matrices and the derived (sufficient) conditions were not easy to verify.

## 2. Preliminaries

The *spectrum* of a square matrix $A$ (that is, the set of its eigenvalues of $A$) will be denoted $\sigma(A)$. The *spectral radius* of $A$, denoted $\rho(A)$ is given by max $\{|\lambda|: \lambda \in \sigma(A)\}$. The *index* of an eigenvalue $\lambda$ of $A$, denoted $v_A(\lambda)$, is the smallest integer $k$ such that the null spaces of $(A - \lambda 1)^k x$ and $(A - \lambda I)^{k+1}$ coincide.

It is well known that powers of a matrix $A \in \mathbb{R}^{n\times n}$ converge if and only if $A$ satisfies the following two conditions:

$$\rho(A) \leqslant 1, \tag{1}$$

$$\text{if } \lambda \in \sigma(A) \text{ and } |\lambda| = 1, \text{ then } \lambda = 1 \text{ and } v_A(\lambda) = 1. \tag{2}$$

Verification of the equivalence is trite if the matrix is in Jordan form, and follows for arbitrary matrices from the observation that the convergence of the powers as well as conditions (1) and (2) are invariant under equivalence-transformations. Conditions (1) and (2) together imply that:

$$\text{the sequence } A^k \text{ is bounded;} \tag{3a}$$

further, (3a) implies (1) and a weak version of Eq. (2) which asserts that each $\lambda \in \sigma(A)$ with $|\lambda| = 1$, satisfies $v_A(\lambda) = 1$ (but $\lambda \neq 1$ is possible). Again, these implications follow easily by considering the Jordan form and observing that the properties are invariant under equivalence-transformations.

Given a norm $\|\ \|$ on $\mathbb{R}^n$, we use the same notation for the corresponding norm on $\mathbb{R}^{1\times n}$ defined for $a \in \mathbb{R}^{1\times n}$ as $\|a^{\mathrm{T}}\|$, and for the derived matrix norm defined for $A \in \mathbb{R}^{n\times n}$ by $\|A\| \equiv \max\{\|Ax\|: x \in \mathbb{R}^n,\ \|x\| = 1\}$. As all norms on $\mathbb{R}^n$ are equivalent, condition (3) can be written as

$$\text{for some, or equivalently every, norm on } \mathbb{R}^n, \quad \sup_{k\in\{1,2,\dots\}} \|A^k\| < +\infty. \tag{3b}$$

Let $\Sigma$ be a set of $n \times n$ matrices. If $\Sigma$ has the LCP property then each matrix in $\Sigma$ must satisfy conditions (1), (2) and (3a). The following condition generalizes the validity of Eq. (3b) (which was restricted to individual matrices of $\Sigma$):

for some, or equivalently every, norm $\| \ \|$ on $\mathbb{R}^n$, $\sup_{\substack{k\in\{1,2,\ldots\} \\ A_1,\ldots,A_k\in\Sigma}} \|A_k \ldots A_1\|$

$$< +\infty. \tag{4a}$$

We also consider the condition:

all the matrices in $\Sigma$ have 1 as an eigenvalue and have a common set of independent nonnegative right eigenvectors corresponding to 1, say $f^1, \ldots f^r$. (4b)

It is shown in [2] how such common eigenvectors are available when the mapping of sequences of matrices into limits of corresponding (left) matrix-products is continuous. Given the vectors $f^1, \ldots, f^r$, we let $F \equiv (f^1, \ldots, f^r)$, that is, $F$ is the matrix with columns $f^1, \ldots, f^r$ in consecutive order; in particular, we have that $AF = F$. We observe that if the matrices of $\Sigma$ are nonnegative, Eq. (4a) is satisfied trivially when the matrices of $\Sigma$ have a common strictly positive eigenvector corresponding to 1, for example, this is the case when Eq. (4b) is satisfied with $\Sigma_{j=1}^r f^j$'s strictly positive.

## 3. Coefficients of ergodicity

Henceforth, we let $\Sigma$ be a set of $n \times n$ matrices satisfying Eq. (4b). In particular, linearly independent vectors $f^1, \ldots, f^r$ are assumed given, and we let $F$ be the matrix with columns $f^1, \ldots, f^r$ in consecutive order, so $F \equiv (f^1, \ldots, f^r)$. Condition (4b) asserts that $AF = F$ for each $A \in \Sigma$.

Let $\| \ \|$ be a norm of $\mathbb{R}^n$. For a matrix $A \in \Sigma$, the $\| \ \|$-*coefficient of ergodicity of* $A$, denoted $\tau_{\| \ \|}(A)$, is defined by

$$\tau_{\| \ \|}(A) \equiv \max_{\substack{x \in \mathbb{R}^n \\ \|x\|=1 \\ x^{\mathrm{T}}F=0}} \|x^{\mathrm{T}}A\|, \tag{5}$$

and the $\| \ \|$-*coefficient of ergodicity of* $\Sigma$, denoted $\tau_{\| \ \|}(\Sigma)$, is defined by

$$\tau_{\| \ \|}(\Sigma) \equiv \sup_{A\in\Sigma} \tau_{\| \ \|}(A). \tag{6}$$

We next interpret the coefficient of ergodicity of a matrix $A \in \mathbb{R}^{n\times n}$ as a matrix norm when $A$ is viewed as an operator on a subspace. Specifically, let $\| \ \|$ be a norm on $\mathbb{R}^n$ and let $N \equiv \text{null}\, F^{\mathrm{T}} = \{x \in \mathbb{R}^n\colon x^{\mathrm{T}}F = 0\}$. Now, if $AF = F$, then for every $x \in N$, $(x^{\mathrm{T}}A)F = x^{\mathrm{T}}(AF) = x^{\mathrm{T}}F = 0$, demonstrating that $N$ is invariant under $A$. Now consider the restriction of the norm $\| \ \|$ (which is defined on $\mathbb{R}^n$) to $N$; of course, we obtain a norm on $N$. We then have that $\tau_{\| \ \|}(A) = \max\{\|x^{\mathrm{T}}A\|\colon x \in N,\ \|x\| = 1\}$ is the matrix norm of $A$, when viewed as an operator on $N$, and the underlying norm is the restriction of $\| \ \|$ to $N$.

The above characterization of coefficients of ergodicity as norms of operators on subspaces motivates us to derive explicit representations for coefficients or ergodicity in terms of matrix norms. For that purpose, we introduce some further notation.

First, select fixed vectors $g^1,\dots,g^{n-r}$ so that $f^1,\dots,f^r,\ g^1,\dots,g^{n-r}$ form a basis of $\mathbb{R}^n$. With $G \equiv (g^1,\dots,g^{n-r}) \in \mathbb{R}^{n\times(n-r)}$, the matrix $P \equiv (f^1,\dots,f^r,\ g^1,\dots,g^{n-r}) = (F,G)$ is nonsingular and we partition its inverse $P^{-1}$ in the following way:

$$P^{-1} = \begin{pmatrix} H \\ J \end{pmatrix} \text{ with } H \in \mathbb{R}^{r\times n} \text{ and } J \in \mathbb{R}^{(n-r)\times n}. \tag{7}$$

Now, let $A \in \mathbb{R}^{n\times n}$ satisfy $AF = F$. As the columns of $P$ form a basis of $\mathbb{R}^n$, there exist column vector $v^1,\dots,v^{n-r}$ with $Ag^j = Pv^j$ for $j = 1,\dots,n-r$. As $AF = F$, we conclude the existence of a matrix $A' \in \mathbb{R}^{n\times n}$ with $AP = PA'$ and $A'$ having the decomposition

$$A' = \begin{pmatrix} I & B_A \\ 0 & N_A \end{pmatrix} \text{ with } B_A \in \mathbb{R}^{r\times(n-r)},\ \ N_A \in \mathbb{R}^{(n-r)\times(n-r)} \tag{8}$$

and $I$ as the $r \times r$ identity; of course, we have $A' = P^{-1}AP$.

**Lemma 1.** *Let $A \in \mathbb{R}^{n\times n}$ satisfy $AF = F$,* let *$A' = P^{-1}AP$ have the decomposition (8) and let $\|\ \|$ be a norm on $\mathbb{R}^n$. For $z \in \mathbb{R}^{n-r}$, let $\|z\|_J = \|z^{\mathrm{T}}J\|$. Then $\|\ \|_J$ is a norm on $\mathbb{R}^{n-r}$ and $\tau_{\|\ \|}(A) = \|N_A\|_J$.*

**Proof.** The fact that $\|\ \|_J$ is a norm on $\mathbb{R}^{n-r}$ is immediate from the fact that the rows of $P^{-1}$ are linearly independent, implying that so are the rows of its submatrix $J$.

To see that $\tau_{\|\ \|}(A) \leqslant \|N_A\|_J$, let $x \in \mathbb{R}^n$ satisfy $x^{\mathrm{T}}F = 0$ and $\|x\| = 1$. Then $x^{\mathrm{T}}P = x^{\mathrm{T}}(F,G) = (0, x^{\mathrm{T}}G)$,

$$\|x^{\mathrm{T}}G\|_J = \|x^{\mathrm{T}}GJ\| = \left\|(0,x^{\mathrm{T}}G)\begin{pmatrix} H \\ J \end{pmatrix}\right\| = \|(x^{\mathrm{T}}P)P^{-1}\| = \|x^{\mathrm{T}}\| = 1$$

and, as $x^{\mathrm{T}}F = 0$,

$$\begin{aligned}\|x^{\mathrm{T}}A\| &= \|x^{\mathrm{T}}PA'P^{-1}\| = \left\|x^{\mathrm{T}}(F,G)\begin{pmatrix} I & B_A \\ 0 & N_A \end{pmatrix}\begin{pmatrix} H \\ J \end{pmatrix}\right\| \\ &= \|x^{\mathrm{T}}(FH + FB_AJ + GN_AJ)\| = \|x^{\mathrm{T}}GN_AJ\| \leqslant \|x^{\mathrm{T}}G\|_J\|N_A\|_J \leqslant \|N_A\|_J.\end{aligned}$$

So, $\tau_{\|\ \|}(A) = \max\{x^{\mathrm{T}}A\|\colon x \in \mathbb{R}^n,\ \ x^{\mathrm{T}}F = 0,\ \ \|x\| = 1\} \leqslant \|N_A\|_J$.

For the reverse inequality $\|N_A\|_J \leqslant \tau_{\|\ \|}(A)$, let $z \in \mathbb{R}^{n-r}$ satisfy $\|z\|_J = 1$. Let $x \equiv J^{\mathrm{T}}z = (0,z^{\mathrm{T}})\binom{H}{J} = (0,z^{\mathrm{T}})P^{-1}$. Then $\|x\| = \|x^{\mathrm{T}}\| = \|z^{\mathrm{T}}J\| = \|z\|_J = 1$. Also, as $P^{-1}(F,G) = P^{-1}P = I \in \mathbb{R}^{n\times n}$, we have that $\binom{HF}{JF} = \binom{H}{J}F = P^{-1}F = \binom{I}{0} \in \mathbb{R}^{n\times r}$ assuring that $JF = 0 \in \mathbb{R}^{(n-r)\times r}$ and therefore $x^{\mathrm{T}}F = z^{\mathrm{T}}JF = 0$. It follows that $\|x^{\mathrm{T}}A\| \leqslant \tau_{\|\ \|}(A)$ and therefore

$$\|z^{\mathrm{T}}N_A\|_J = \|z^{\mathrm{T}}N_AJ\| = \|(0, z^{\mathrm{T}})\begin{pmatrix} I & B_A \\ 0 & N_A \end{pmatrix}\begin{pmatrix} H \\ J \end{pmatrix}\|$$
$$= \|(0, z^{\mathrm{T}})A'P^{-1}\| = \|[(0, z^{\mathrm{T}})P^{-1}A\| = \|[(0, z^{\mathrm{T}})\begin{pmatrix} H \\ J \end{pmatrix}A\|$$
$$= \|z^{\mathrm{T}}JA\| = \|x^{\mathrm{T}}A\| \leqslant \tau_{\|\ \|}(A).$$

So, $\|N_A\|_J = \max\{\|z^{\mathrm{T}}N_A\|_J : z \in \mathbb{R}^{n-r}, \|z\|_{J=1}\} \leqslant \tau_{\|\ \|}(A)$. $\square$

We next derive a sufficient condition for Eq. (4a) through uniform bounds on the individual matrices in $\Sigma$.

**Lemma 2.** *If $\Sigma$ is bounded and $\tau_{\|\ \|}(\Sigma) < 1$ for some norm $\|\ \|$* on $\mathbb{R}^n$, *then condition (4a) is satisfied.*

**Remark.** While boundedness of $\Sigma$ and condition (4a) are invariant under the choice of a particular norm (by the equivalence of norms over a finite-dimensional vector space), the inequality $\tau_{\|\ \|}(\Sigma) < 1$ depends on the selected norm and may hold for one norm but not for another; see examples in [8] for individual matrices.

**Proof.** As $\Sigma$ is assumed bounded, so are $\{A' = P^{-1}AP : A \in \Sigma\}$ and $\{B_A : A \in \Sigma\}$; in particular, the coordinates of the matrices in $\{B_A : A \in \Sigma\}$ are uniformly bounded, say by positive number $K$. Also, let $\tau \equiv \tau_{\|\ \|}(\Sigma)$.

Let $A_1, \ldots, A_k$ be a sequence of matrices in $\Sigma$. For $j = 1, \ldots, k$, then $A_jF = F$ and we let $A_j' = P^{-1}A_jP$ have decomposition (8) given by

$$A_j' = \begin{pmatrix} I & B_{A_j} \\ 0 & N_{A_j} \end{pmatrix} \text{ with } B_{A_j} \in \mathbb{R}^{r\times(n-r)} \text{ and } N_{A_j} \in \mathbb{R}^{(n-r)\times(n-r)}.$$

Now, as $A_k \ldots A_1F = F$, $(A_k \ldots A_1)' = P^{-1}(A_k \ldots A_1)^1P$, has the decomposition (8) and

$$\begin{pmatrix} I & B_{A_k\ldots A_1} \\ 0 & N_{A_k\ldots A_1} \end{pmatrix} = (A_k \ldots A_1)' = P^{-1}A_k \ldots A_1P$$
$$= (P^{-1}A_kP)(P^{-1}A_{k-1}P)\ldots(P^{-1}A_1P) = A_k' \ldots A_1'$$
$$= \begin{pmatrix} I & B_{A_k} \\ 0 & N_{A_k} \end{pmatrix} \cdots \begin{pmatrix} I & B_{A_1} \\ 0 & N_{A_1} \end{pmatrix} \tag{9}$$

with $B \equiv B_{A_k\ldots A_1}$, *then*

$$B = B_{A_1} + B_{A_2}N_{A_1} + B_{A_3}N_{A_2}N_{A_1} + \cdots + B_{A_k}N_{A_{k-1}} \ldots N_{A_1}. \tag{10}$$

For $j = 1, \ldots, k$, we get from a standard inequality of matrix-norms, Lemma 1 and the assertion $\tau = \tau_{\|\ \|}(\Sigma) < 1$ that

$$\|N_{A_j}\dots N_{A_1}\|_J \leqslant \prod_{i=1}^{j}\|N_{A_i}\|_J = \prod_{i=1}^{j}\tau_{\|\ \|}(A_i) \leqslant \tau^j; \tag{11}$$

and by the equivalence of all norms on $\mathbb{R}^{(n-r)\times(n-r)}$ we have that for some positive number $L$, the individual elements of $N_{A_j}\dots N_{A_1}$ are bounded by $L\tau^j$, with $L$ as a positive number which is independent of $j$. It follows that for $j=1,\dots,k$, $u=1,\dots,r$ and $v=1,\dots,n-r$,

$$|(B_{A_j}N_{A_{j-1}}\dots N_{A_1})_{uv}| \leqslant \sum_{s=1}^{n-r}|(B_{A_j})_{us}||(N_{A_{j-1}}\dots N_{A_1})_{sv}| \leqslant (n-r)KL\tau^{j-1}, \tag{12}$$

and

$$|B_{uv}| \leqslant \sum_{j=1}^{k}|(B_{A_j}N_{A_{j-1}\dots N_{A_1}})_{uv}| \leqslant \sum_{j=1}^{k}(n-r)KL\tau^{j-1} \leqslant \frac{(n-r)KL}{1-\tau}. \tag{13}$$

We conclude from Eq. (9), (11) and (13) that the elements $(A_kA_{k-1}\dots A_1)'$ are elementwise uniformly bounded by $(n-r)KL/(1-\tau)$, immediately implying that the elements of $A_kA_{k-1}\dots A_1 = P(A_kA_{k-1}\dots A_1)'\ P^{-1}$ are uniformly bounded. □

## 4. The main result

We are now ready for our main result which provides a sufficient condition for the LCP property in terms of coefficients of ergodicity.

**Theorem 1.** *If $\Sigma$ is bounded and $\tau_{\|\ \|}(\Sigma) < 1$ for some norm $\|\ \|$ on $\mathbb{R}^n$, then $\Sigma$ has the LCP property. Further, the convergence of any sequence $\{A_iA_{i-1}\dots A_2A_1\}_{i=1,2,\dots}$ of products of matrices from $\Sigma$ to its limit is geometric with $\tau_{\|\ \|}(\Sigma)$ as a bound on the geometric coefficient of convergence.*

**Proof.** By Lemma 2 we have that (4) holds and we let

$$K \equiv \sup_{\substack{k\in\{1,2,\dots\}\\ A_1,\dots,A_k\in\Sigma}} \|A_k\dots A_1\|. \tag{14}$$

We note that the proof of Lemma 2 demonstrates that the right-hand side of Eq. (14) is bounded with $\|\ \|$ as the $1_\infty$ norm on $\mathbb{R}^{n\times n}$; but by the equivalence of all norms over $\mathbb{R}^{n\times n}$ assures that we have boundedness with respect to the matrix norm induced by $\|\ \|$ as well. Of course, the constant $K$ depends on the selected norm.

Let $\tau \equiv \tau_{\|\ \|}(\Sigma)$. For every vector $x$ satisfying $x^{\mathrm{T}}F = 0$ and matrix $A$ in $\Sigma$, we have $\|x^{\mathrm{T}}A\| \leqslant \|x\|\tau$. In particular, for every finite product of matrices from $\Sigma$, say $U$, we have that $UF = F$ and therefore for each vector

$a \in \mathbb{R}^n$, $a^{\mathrm{T}}(U - I)F = 0$ and $\|a^{\mathrm{T}}(U - I)A\| \leqslant \|a^{\mathrm{T}}(U - I)\|\tau$. Also, the definition of $K$ in Eq. (14) assures that $\|a^{\mathrm{T}}U\| \leqslant \|a\|K$ for each vector $a$ in $\mathbb{R}^n$. Now, suppose a sequence $A_1, A_2, \ldots$ of matrices from $\Sigma$ is given. For positive integers $p$ and $q$ with $p \geqslant q$ let $A^{(p,q)} \equiv A_p A_{p-1} \ldots A_{q+2} A_{q+1}$; in particular, $A^{(p,0)} = A^{(p,q)} A^{(q,0)}$ and $A^{(p,0)} - A^{(q,0)} = [A^{(p,q)} - I]A^{(q,0)} = [A^{(p,q)} - I]A_q \ldots A_1$. It follows that for $a \in \mathbb{R}^n$

$$\begin{aligned} \|a^{\mathrm{T}}A^{(p,0)} - a^{\mathrm{T}}A^{(q,0)}\| &= \|a^{\mathrm{T}}[A^{(p,q)} - I]A_q \ldots A_1\| \\ &\leqslant \|a^{\mathrm{T}}[A^{(p,q)} - I]\|\tau^q \leqslant \|a\|(K + 1)\tau^q, \end{aligned} \tag{15}$$

implying that the sequence $\{a^{\mathrm{T}}A_i A_{i-1} \ldots A_2 A_1\}_{i=1,2,\ldots}$ is a Cauchy sequence, hence converging. The convergence of $\{A_i A_{i-1} \ldots A_2 A_1\}_{i=1,2,\ldots}$ now follows by letting a stand for each of the $n$ unit vectors in $\mathbb{R}^n$.

Finally, let $A \equiv \lim_{i \to \infty} A^{(i,0)}$. The validity of Eq. (15) for each vector $a \in \mathbb{R}^n$ and integers $p$ and $q$ with $p \geqslant q$ implies that for each $i \geqslant 0$, $\|a^{\mathrm{T}}[A^{(i,0)} - A]\| \leqslant \|a\|(K + 1)\tau^i$; thus, the convergence rate of $A^{(i,0)}$ to $A$ as $i \to +\infty$ is geometric with $\tau$ as a bound on geometric coefficient of convergence (the conclusion follows for the matrix norm induced by $\| \;\|$ and holds for any norm on $\mathbb{R}^{n \times n}$ by the equivalence of all norms on a finite dimensional vector space). □

Necessary conditions and sufficient conditions for a set of nonnegative matrices $\Sigma$ to have the LCP property are given in [2] in terms of eigenvalues and matrix-norms of products of matrices of $\Sigma$. Theorem 1 states a sufficient condition for the LCP property in terms of coefficients of ergodicity of the individual matrices of $\Sigma$.

The next result puts the result of Theorem 1 in perspective of the results of [2]. Its proof is omitted as it follows from the arguments of [8] (where the case of $F$ with a single column is considered).

**Theorem 2.** *Let $\| \;\|$ be a norm on $\mathbb{R}^n$ and let $F$ be a nonnegative matrix with $n$ rows. If $A$ is a square nonnegative matrix with spectral radius 1 satisfying $AF = F$, then $\tau_{\| \;\|}(A)$ is an upper bound on* $\max\{|\lambda|: \lambda \neq 1$ *is an eigenvalue of* $A\}$.

## 5. Examples

The following two examples appear in [2]. For their analysis, we let the underlying norm will be the $l_1$ norm which we denote $\| \;\|_1$. Also, we will use subscripts to denote rows of matrices, e.g., for a square matrix $A$, $A_i$ will stand for the $i$th row of $A$.

**Example 1.** Let

$$U = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad V = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

and $\Sigma = \{U, V\}$. It is easy to verify that $\Sigma$ satisfies Eq. (4b) with

$$F = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now, a vector $x \in \mathbb{R}^3$ satisfies $xF = 0$ if and only if $x_3 = 0$, and for such a vector and a $3 \times 3$ matrix $A$, $\|(x_1, x_2, 0)A\|_1 = \|x_1 A_1 + x_2 A_2\|$, implying that $\max\{x^{\mathrm{T}}A: x \in \mathbb{R}^3,\ \|x\|_1 = 1 \text{ and } x^{\mathrm{T}}F = 0\} = \max\{\|A_1\|, \|A_2\|\}$. So, $\tau_{\|\ \|_1}(U) = \frac{2}{3}$, $\tau_{\|\ \|_1}(V) = \frac{2}{3}$ and $\tau_{\|\ \|_1}(\Sigma) = \frac{2}{3}$. It now follows from Theorem 1 that $\Sigma$ has the LCP property.

**Example 2.** Let

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix},$$

$$V = \begin{pmatrix} 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 1/6\sqrt{3} & 1/2 & -1/6\sqrt{3} \\ -1/6\sqrt{3} & 1/2 & 1/6\sqrt{3} & 1/2 \end{pmatrix},$$

$$W = \begin{pmatrix} 1/2 & 1/6\sqrt{3} & 1/2 & -1/6\sqrt{3} \\ -1/6\sqrt{3} & 1/2 & 1/6\sqrt{3} & 1/2 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \end{pmatrix},$$

$$T = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\Sigma = \{U, V, W, T\}$. It is easy to verify that $\Sigma$ satisfies Eq. (4b) with

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The two vectors $u^1 = (1, 0, -1, 0)^{\mathrm{T}}$ and $u^2 = (0, 1, 0, -1)^{\mathrm{T}}$ form a basis for null $F^{\mathrm{T}} = \{x \in \mathbb{R}^4\colon x^{\mathrm{T}}F = 0\}$, and a vector $x$ in this set with the representation $x = \alpha u^1 + \beta u^2$ has norm 1 or less if and only if $|\alpha| + |\beta| \leqslant 1/2$. It follows that $C \equiv \{x \in \mathbb{R}^4\colon x^{\mathrm{T}}F = 0,\ \|x\|_1 \leqslant 1\}$ is convex with $u^1/2$, $-u^1/2$, $u^2/2$, $-u^2/2$ as its extreme vectors. Let $A \in \{U, V, W, T\}$. As the map $x \to \|x^{\mathrm{T}}A\|_1$ is convex, we have that $\max\{\|x^{\mathrm{T}}A\|_1\colon x \in C\}$ is attained at either of $C$'s extreme points (see [3]), and as the map is symmetric we conclude that the maximum is attained at either $u^1/2$ or $u^2/2$; thus, $\tau_{\|\ \|_1}(A) = \max\{\|(u^1)^{\mathrm{T}}A\|_1, \|(u^2)^{\mathrm{T}}A\|_1\} = \max\{\|A_1 - A_3\|_1/2,\ \|A_2 - A_4\|_1/2\}$. Explicit computation shows that $\tau_{\|\ \|_1}(U) = \tau_{\|\ \|_1}(T) = \frac{1}{3} < 1$ and $\tau_{\|\ \|_1}(V) = \tau_{\|\ \|_1}(W) = \frac{1}{6} + 1/6\sqrt{3} < 1$; hence, Theorem 1 implies that $\Sigma$ has the LCP property.

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