6
Convergence

In this chapter we look at some basic convergence results on infinite products of matrices. Some of these results are somewhat old, but perhaps not well known. Other results in this chapter are rather new.

6.1 Reduced Matrices

An $n \times n$ matrix M that has partitioned form

$$M = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where A is square, is *reduced.* In this section we show when infinite products of such matrices converge. To obtain such a result requires a few preliminaries.

If $\|\cdot\|_a$ and $\|\cdot\|_c$ are vector norms on F^{n_1} and F^{n_2}, respectively, we can define a norm on the $n_1 \times n_2$ matrices B using

$$\|B\|_b = \max_{x \neq 0} \frac{\|Bx\|_a}{\|x\|_c}. \tag{6.1}$$

For products, this norm behaves as follows.

Lemma 6.1 *Let B be an $n_1 \times n_2$ matrix.*

1. *If A is an $n_1 \times n_1$ matrix, then*

$$\|AB\|_b \le \|A\|_a \|B\|_b .$$

2. *If C is an $n_2 \times n_2$ matrix, then*

$$\|BC\|_b \le \|B\|_b \|C\|_c .$$

Proof. We will show the proof of (2). For it, note that

$$\|Bx\|_a \le \|B\|_b \|x\|_c$$

for all $n_2 \times 1$ vectors x. Thus,

$$\|BCx\|_a \le \|B\|_b \|Cx\|_c \le \|B\|_b \|C\|_c \|x\|_c .$$

Thus,

$$\|BC\|_b \le \|B\|_b \|C\|_c ,$$

which is what we need. ■

Using this lemma, we will show the convergence of a special infinite series which we need later.

Lemma 6.2 *In the infinite series*

$$L_2B_1 + L_3B_2C_1 + \cdots + L_kB_{k-1}C_{k-2}\cdots C_1 + \cdots$$

the matrices $L_2, L_3, \ldots$ are $n_1 \times n_1$, the matrices $B_1, B_2, \ldots$ are $n_1 \times n_2$, and the matrices $C_1, C_2, \ldots$ are $n_2 \times n_2$. The series converges if, for all k,

1. *$\|L_k\|_a \le K_1$ for some vector norm $\|\cdot\|_a$ and constant K_1,*
2. *$\|C_k\|_c \le \gamma$ for some vector norm $\|\cdot\|_c$ and constant γ, $\gamma < 1$, and*
3. *$\|B_k\|_b \le \beta$ for some constant β.*

Proof. We show that the series, given in the theorem, converges by showing the sequence $\langle L_2B_1 + \cdots + L_kB_{k-1}C_{k-2}\cdots C_1\rangle$ is Cauchy. To see this, observe that if $i > j$, the difference between the i-th and j-th terms of the sequence is

$$D_{ij} = L_{j+1}B_jC_{j-1}\cdots C_1 + \cdots + L_iB_{i-1}C_{i-2}\cdots C_1.$$

Thus, $\|D_{ij}\|_b =$

$$\begin{aligned}&\|L_{j+1}\|_a\|B_j\|_b\|C_{j-1}\cdots C_1\|_c + \cdots + \|L_i\|_a \|B_{i-1}\|_b \|C_{i-2}\cdots C_1\|_c\\ &\le K_1\beta\gamma^{j-1} + \cdots + K_1\beta\gamma^{i-2}.\end{aligned}$$

From this it is clear that the sequence is Cauchy, and thus converges. ■

The theorem about convergence of infinite products of reduced matrices follows.

Theorem 6.1 *Suppose each $n \times n$ matrix in the sequence $\langle M_k\rangle$ has the form*

$$M_k = \begin{bmatrix} A_1^{(k)} & B_{12}^{(k)} & B_{12}^{(k)} & \cdots & B_{1r}^{(k)} \\ 0 & C_1^{(k)} & B_{23}^{(k)} & \cdots & B_{2r}^{(k)} \\ & & & \cdots & \\ 0 & 0 & & \cdots & C_r^{(k)} \end{bmatrix}$$

where the $A_1^{(k)}$'s are $n_1 \times n_1$, $C_1^{(k)}$'s are $n_2 \times n_2, \ldots$, and the $C_r^{(k)}$'s are $n_{r+1} \times n_{r+1}$. We suppose there are vector norms $\|\cdot\|_a$ on F^{n_1} and $\|\cdot\|_{c_i}$ on $F^{n_{i+1}}$ such that

1. *$\left\|C_i^{(k)}\right\|_{c_i} \le \gamma$ for some constant $\gamma < 1$ and all i, k.*
2. *As given in (6.1), there is a positive constant K_3, such that*

$$\left\|B_{ij}^{(k)}\right\|_{b_{ij}} \le K_3$$

for all i, j, and k.

Finally, we suppose that for all s, the sequence $\langle A_k \cdots A_s\rangle$ converges to a matrix L_s and that

3. *$\|L_s\|_a \le K_2$ for all s and $\|L_s - A_k\cdots A_s\|_a \le K_1\alpha^{k-s+1}$ for some constants K_1 and $\alpha < 1$.*

Then the sequence $\langle M_k \cdots M_1\rangle$ converges.

Proof. We prove this result for $r = 1$. The general case is argued using Corollary 3.3. We use the notation

$$M_k = \begin{bmatrix} A_k & B_k \\ 0 & C_k \end{bmatrix}.$$

Then $M_k \cdots M_1 =$

$$\begin{bmatrix} A_k \cdots A_1 & A_k \cdots A_2 B_1 + A_k \cdots A_3 B_2 C_1 + \cdots + B_k C_{k-1} \cdots C_1 \\ 0 & C_k \cdots C_1 \end{bmatrix}.$$

By hypotheses, the sequence $\langle A_k \cdots A_s \rangle$ converges to L_s, and the sequence $\langle C_k \cdots C_1 \rangle$ converges to 0. We finish by showing that the sequence with k-th term

$$A_k \cdots A_2 B_1 + A_k \cdots A_3 B_2 C_1 + \cdots + B_k C_{k-1} \cdots C_1 \tag{6.2}$$

converges to

$$L_2 B_1 + L_3 B_2 C_1 + \cdots + L_{k+1} B_k C_{k-1} \cdots C_1 + \cdots . \tag{6.3}$$

Now letting D_k denote the difference between (6.3) and (6.2), and using Lemma 6.1, $\|D_k\|_{b_{12}} =$

$$\begin{aligned} &\| (L_2 - A_k \cdots A_2) B_1 + (L_3 - A_k \cdots A_3) B_2 C_1 + \cdots \\ &+ (L_{k+1} - I) B_k C_{k-1} \cdots C_1 + L_{k+2} B_{k+1} C_k \cdots C_1 + \cdots \|_{b_{12}} \\ &\leq \left(K_1 \alpha^{k-1} K_3 + K_1 \alpha^{k-2} K_3 \gamma + \cdots + K_1 K_3 \gamma^{k-1} \right) \\ &+ K_2 K_3 \gamma^k + \cdots + K_2 K_3 \gamma^{k+1} + \cdots \\ &\leq \left(K_1 K_3 \beta^{k-1} + \cdots + K_1 K_3 \beta^{k-1} \right) + K_2 K_3 \beta^k \frac{1}{1-\beta} \end{aligned}$$

where $\beta = \max\{\alpha, \gamma\}$. So

$$\|D_k\|_{b_{12}} \leq k K_1 K_3 \beta^{k-1} + K_2 K_3 \beta^k \frac{1}{1-\beta} \leq K k \beta^{k-1}$$

where $K = K_1 K_3 + K_2 K_3 \frac{1}{1-\beta}$. Thus, as $k \to \infty$, $D_k \to 0$ and so the sequence from (6.2) converges to the sum in (6.3). ■

Corollary 6.1 *Let $M_k = \begin{bmatrix} I & B_k \\ 0 & C_k \end{bmatrix}$ be an $n \times n$ matrix with I the $m \times m$ identity matrix. Suppose for some norm $\|\cdot\|_b$, as defined in (6.1), and constants β and γ, $\gamma < 1$, and a positive integer r*

1. $\|B_k\|_b \leq \beta$
2. $\|C_k\|_c \leq 1$ *and* $\|C_{k+r} \cdots C_{k+1}\|_c \leq \gamma$ *for all* k.

Then $\langle M_k \cdots M_1 \rangle$ converges at a geometrical rate.

Proof. Note that

$$M_k \cdots M_1 = \begin{bmatrix} I & B_1 + B_2 C_1 + \cdots + B_k C_{k-1} \cdots C_1 \\ 0 & C_k \cdots C_1 \end{bmatrix}.$$

By (2) of the theorem, $\|C_k \cdots C_1\| \leq \gamma^{\left[\frac{k}{r}\right]}$ and by (1) of the theorem,

$$\begin{aligned} &\|B_{k+1} C_k \cdots C_1 + B_{k+2} C_{k+1} \cdots C_1 + \cdots \|_b \\ &\leq r\beta\gamma^{\left[\frac{k}{r}\right]} + r\beta\gamma^{\left[\frac{k}{r}\right]+1} + \cdots \\ &\leq \frac{r\beta\gamma^{\left[\frac{k}{r}\right]}}{1-\gamma}. \end{aligned}$$

Thus, $\langle M_k \cdots M_1 \rangle$ converges to

$$\begin{bmatrix} I & B_1 + B_2 C_1 + B_3 C_2 C_1 + \cdots \\ 0 & 0 \end{bmatrix}$$

and at a geometric rate. ■

A special case of the theorem follows by applying (6.3).

Corollary 6.2 *Assume the hypothesis of the theorem and that each $L_s = 0$. Then the sequence $\langle M_k \rangle$ converges to 0.*

6.2 Convergence to 0

There are not many theorems on infinite products of matrices that converge to 0. In the last section, we saw one such result, namely Corollary 6.2. In the next section, we will see a few others. In this section, we show three direct results about convergence to 0.

The first result concerns nonnegative matrices and uses the measure U of full indecomposability as described in Chapter 2. In addition, we use that

$$r_i(A) = \sum_{k=1}^{n} a_{ik},$$

the i-th row sum of an $n \times n$ nonnegative matrix A.

Theorem 6.2 *Suppose that each matrix in the sequence $\langle A_k \rangle$ of $n \times n$ nonnegative matrices satisfies the following properties:*

1. $\max_i r_i(A_k) \leq r.$

2. $U(A_k) \geq u > 0.$

3. There is a number δ *such that* $\min_i r_i(A_k) \leq \delta.$

4. $\left(r^{n-1} - u^{n-1}\right) r^{r-1} + u^{n-1}\left(\delta r^{n-2}\right) = l < 1.$

Then $\prod_{k=1}^{\infty} A_k = 0.$

Proof. We first make a few observations.

Using properties of the measure of full indecomposability, Corollary 2.5, if for $s = 1, 2, \ldots$

$$B_s = \prod_{k=s(n-1)+1}^{(s+1)(n-1)} A_k$$

then the smallest entry in B_s,

$$\min_{i,j} b_{ij}^{(s)} \geq u^{n-1}.$$

And,

$$\max_i r_i(B_s) \leq r^{n-1}, \ \min_i r_i(B_s) \leq \delta r^{n-2}.$$

Then,

$$\begin{aligned} r_i(B_{s+1}B_s) &= \sum_{j=1}^{n}\sum_{k=1}^{n} b_{ik}^{(s+1)} b_{kj}^{(s)} \\ &= \sum_{k=1}^{n} b_{ik}^{(s+1)} r_k(B_s) \\ &\leq \sum_{\substack{k=1 \\ k\neq k_0}}^{n} b_{ik}^{(s+1)} r_k(B_s) + b_{ik_0}^{(s+1)}\left(\delta r^{n-2}\right) \end{aligned}$$

where we assume $r_{k_0}(B_s)$ is the smallest row sum. So

$$\begin{aligned} r_i(B_{s+1}B_s) &\leq \left(r^{n-1} - u^{n-1}\right) r^{n-1} + u^{n-1}\left(\delta r^{n-2}\right) \\ &= l < 1. \end{aligned}$$

Thus, since

$$\begin{aligned} \left\| \prod_{k=1}^{2m} B_k \right\|_1 &= \left\| \prod_{s=1}^{m} (B_{2s}B_{2s-1}) \right\|_1 \\ &\leq \prod_{s=1}^{m} \|(B_{2s}B_{2s-1})\|_1 \\ &\leq l^m \end{aligned}$$

it follows that $\prod_{k=1}^{\infty} A_k$ converges to 0. ■

This corollary is especially useful for substochastic matrices since in this case, we can take $r = 1$ and simplify (4).

The next result uses norms together with infinite series. To see this result, for any matrix norm $\|\cdot\|$, we let

$$\begin{aligned} \|A\|_+ &= \max\{\|A\|, 1\} \text{ and} \\ \|A\|_- &= \min\{\|A\|, 1\}. \end{aligned}$$

And, we state two conditions that an infinite sequence $\langle A_k \rangle$ of $n \times n$ matrices might have.

1. $\sum_{k=1}^{\infty}\left(\|A_k\|_+ - 1\right)$ converges.
2. $\sum_{k=1}^{\infty}\left(1 - \|A_k\|_-\right)$ diverges.

We now need a preliminary lemma.

Lemma 6.3 *Let* $A_1, A_2, \ldots$ *be a sequence of* $n \times n$ *matrices and* $\|\cdot\|$ *a matrix norm. If this sequence satisfies (1) and* $A_{i_1}, A_{i_2}, \ldots$ *is any rearrangement of it, then* $\|A_{i_1}\|_+, \|A_{i_2}\|_+ \|A_{i_1}\|_+, \ldots$ *and* $\|A_{i_1}\|, \|A_{i_2}A_{i_1}\|, \ldots$ *are bounded.*

Proof. First note that

$$\|A_{i_k} \cdots A_{i_1}\| \leq \|A_{i_k}\|_+ \cdots \|A_{i_1}\|_+$$

for all k. Now, using that

$$\sum_{k=1}^{\infty}\left(\|A_{i_k}\|_+ - 1\right)$$

converges, by Hyslop's Theorem 51 (See the Appendix.),

$$\prod_{k=1}^{\infty} \|A_{i_k}\|_+$$

converges. But this implies that $\|A_{i_1}\|_+, \|A_{i_2}\|_+ \|A_{i_1}\|_+, \ldots$ is bounded. ■

The following theorem says that if $\langle \|A_k\|_+ \rangle$ converges to 1 fast enough (condition 1) and $\langle \|A_k\|_- \rangle$ doesn't approach 1 or, if it does, it does so slowly (condition 2), then $\prod_{k=1}^{\infty} A_{i_k} = 0$.

Theorem 6.3 *Let $A_1, A_2, \ldots$ be a sequence of $n \times n$ matrices and $\|\cdot\|$ a matrix norm. If the sequence satisfies (1) and (2) and $A_{i_1}, A_{i_2}, \ldots$ is any rearrangement of the sequence, then we have $\prod_{k=1}^{\infty} A_{i_k} = 0$.*

Proof. Using that $\|A_{i_j}\| = \|A_{i_j}\|_- \|A_{i_j}\|_+$

$$\|A_{i_k} \cdots A_{i_1}\| \leq \|A_{i_k}\|_- \cdots \|A_{i_1}\|_- M,$$

where M is a bound on the sequence $\langle \|A_{i_k}\|_+ \cdots \|A_{i_1}\|_+ \rangle$. Since (2) is satisfied, by Hyslop's Theorem 52 (given in the Appendix), $\prod_{k=1}^{\infty} \|A_{i_k}\|_-$ converges to 0 or does not converge to any number. Since the sequence $\|A_{i_1}\|_-, \|A_{i_2}\|_- \|A_{i_1}\|_-, \cdots$ is decreasing, it must converge. Thus, this sequence converges to 0, and so

$$A_{i_1}, A_{i_2} A_{i_1}, \ldots$$

converge to 0. ■

The final result involves the generalized spectral radius $\hat{\rho}$ discussed in Chapter 2.

Theorem 6.4 *Let Σ be a compact matrix set. Then every infinite product, taken from Σ, converges to 0 iff $\hat{\rho}(\Sigma) < 1$.*

Proof. If $\hat{\rho}(\Sigma) < 1$, then by the characterization of $\hat{\rho}(\Sigma)$, Theorem 2.19, there is a norm $\|\cdot\|$ such that $\|A\| \leq \gamma, \gamma < 1$, for all $A \in \Sigma$. Thus for a product $A_{i_k} \ldots A_{i_1}$, from Σ, we have

$$\|A_{i_k} \ldots A_{i_1}\| \leq \gamma^k \to 0 \text{ as } k \to \infty.$$

Hence, all infinite products from Σ converge to 0.

Conversely, suppose that all infinite products taken from Σ converge to 0. Then by the norm-convergent result, Corollary 3.4, there is a norm $\|\cdot\|$ such that

$$\|A_i\| \leq 1$$

for all $A_i \in \Sigma$. We will prove that $\hat{\rho}(\Sigma) < 1$ by contradiction.

Suppose $\hat{\rho}(\Sigma) \geq 1$. Then $\hat{\rho}(\Sigma) = 1$. Since $\hat{\rho}_k$ is decreasing here, there exists a sequence $C_1, C_2, \ldots$ where C_k is a k-block taken from Σ, such that $\|C_k\| \geq 1$ for all k. Thus, $\|C_k\| = 1$ for all k. We use these C_k's to form an infinite product which does not converge to 0.

To do this, it is helpful to write

$$\begin{array}{lllll} C_1 = & & & & A_{11} \\ C_2 = & & & A_{22} & A_{21} \\ C_3 = & & A_{33} & A_{32} & A_{31} \\ & \cdots & & & \\ C_k = & A_{k4} & A_{k3} & A_{k2} & A_{k1} \\ & \cdots & & & \end{array} \tag{6.4}$$

where the $A_{i,j}$'s are taken from Σ. Now we know that Σ is product bounded, and so the sequence $A_{1,1}, A_{2,1}, \ldots$ has a subsequence $A_{l_1,1}, A_{l_2,1}, \ldots$ that converges to, say B_1 and $\|B_1\| = 1$. Thus, there is a constant L such that if $k \geq L$,

$$\|A_{l_k,1} - B_1\| < \frac{1}{8}.$$

Set $s_1(1) = l_L, s_1(2) = l_{L+1}, \ldots$ so $\|A_{s_1(k),1} - B_1\| < \frac{1}{8}$ for all k. Now, consider the subsequence $A_{s_1(1),2}, A_{s_1(2),2}, \ldots$. As before, we can find a subsequence of this sequence, say $A_{s_2(1),2}, A_{s_2(2),2}, \ldots$, which converge to B_2 and

$$\|A_{s_2(k),2} - B_2\| \leq \frac{1}{16} \text{ for all } k.$$

Continuing, we have

$$\|A_{s_j(k),j} - B_j\| \leq \frac{1}{2^{j+2}}$$

for all k. Using (6.4), a schematic showing how the sequences are chosen is given in Figure 6.1. We now construct the desired product by using

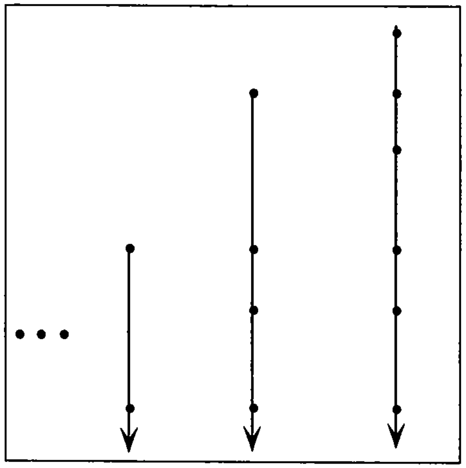

FIGURE 6.1. A diagram of the sequences.

$\pi_1 = A_{m_1}, \pi_2 = A_{m_2}A_{m_1}, \ldots$ where

$$\begin{aligned} A_{m_1} &= A_{s_1(1),1} \\ A_{m_2} &= A_{s_2(1),2} \\ &\cdots \\ A_{m_k} &= A_{s_k(1),k} \\ &\cdots \end{aligned}$$

which can be called a diagonal process.

We now make three important observations. Let $i > 1$ be given.

1. If $j \le i$,

$$\left\|A_{m_j} - A_{s_i(i),j}\right\| = \left\|A_{s_j(1),j} - A_{s_i(i),j}\right\| \le \frac{1}{2^{j+1}}.$$

(Note here that $s_i(k)$ is a subsequence of $s_j(k)$.)

2. Using (6.4),

$$\begin{aligned} 1 &= \left\|C_{s_i(i)}\right\| \\ &= \left\|A_{s_i(i),s_i(i)} \cdots A_{s_i(i),1}\right\| \\ &\le \left\|A_{s_i(i),s_i(i)} \cdots A_{s_i(i),i+1}\right\| \left\|A_{s_i(i),i} \cdots A_{s_i(i),1}\right\| \\ &\le \left\|A_{s_i(i),i} \cdots A_{s_i(i),1}\right\|. \end{aligned}$$

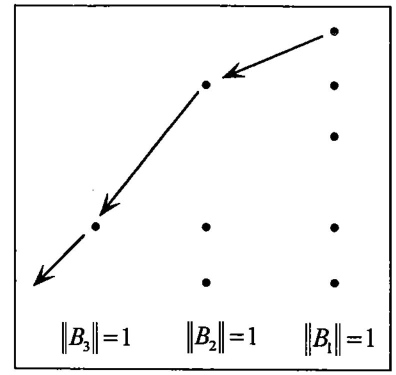

FIGURE 6.2. Diagonal product view.

3. Using a collapsing sum expression,

$$\begin{aligned} &\pi_i - A_{s_i(i),i}A_{s_i(i),i-1} \cdots A_{s_i(i),1} = \\ &A_{m_i} \cdots A_{m_2}\left(A_{m_1} - A_{s_i(i),1}\right) + \\ &A_{m_i} \cdots A_{m_3}\left(A_{m_2} - A_{s_i(i),2}\right)A_{s_i(i),1} + \cdots + \\ &\left(A_{m_i} - A_{s_i(i),i}\right)A_{s_i(i),i-1} \cdots A_{s_i(i),1} \end{aligned}$$

and taking the norm of both sides, we have from (1),

$$\begin{aligned} &\left\|\pi_i - A_{s_i(i),i}A_{s_i(i),i-1} \cdots A_{s_i(i),1}\right\| \\ &\le \left(\frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) \\ &= \frac{1}{2}. \end{aligned}$$

Putting together, by (2), $\left\|A_{s_i(i),i}A_{s_i(i),i-1} \cdots A_{s_i(i),1}\right\| \ge 1$ and by (3), $\left\|\pi_i - A_{s_i(i),i}A_{s_i(i),i-1} \cdots A_{s_i(i),1}\right\| \le \frac{1}{2}$. Thus $\|\pi_i\| \ge \frac{1}{2}$ for all i, which provides an infinite product from Σ that does not converge to 0. (See Figure 6.2.) This is a contradiction. So we have $\hat{\rho}(\Sigma) < 1$. ■

Concerning convergence rate, we have the following.

Corollary 6.3 *If Σ is a compact matrix set and $\hat{\rho}(\Sigma) < 1$, then all sequences $A_{i_1}, A_{i_2}A_{i_1}, A_{i_3}A_{i_2}A_{i_1}, \ldots$ converge uniformly to 0. And this convergence is at a geometric rate.*

Proof. By the characterization of $\hat{\rho}$, Theorem 2.19, there is a matrix norm $\|\cdot\|$ such that $\|A_k\| \leq \gamma$ where γ is a constant and $\gamma < 1$. Thus

$$\begin{aligned}\|A_{i_k} \cdots A_{i_1} - 0\| &= \|A_{i_k} \cdots A_{i_1}\| \\ &\leq \|A_{i_k}\| \cdots \|A_{i_1}\| \\ &\leq \gamma^k.\end{aligned}$$

Thus, any sequence $A_{i_1}, A_{i_2}A_{i_1}, \ldots$ converges to 0 at a geometric rate. And, since this rate is independent of the sequence chosen, the convergence is uniform. ■

Putting together two previous theorems, we obtain the following *norm-convergence to 0* result.

Corollary 6.4 *Let Σ be a compact matrix set. Then every infinite product, taken from Σ, converges to* 0 *iff there is a norm $\|\cdot\|$ such that $\|A\| \leq \gamma, \gamma < 1$, for all $A \in \Sigma$.*

Proof. If every infinite product taken from Σ converge to 0, then by the theorem, $\hat{p}(\Sigma) < 1$. Thus by Theorem 2.19, there is a norm $\|\cdot\|$ such that $\|A\| \leq \gamma$, $\gamma < 1$, for all $A \in \Sigma$. The converse is obvious. ■

6.3 Results on $\Pi\,(U_k + A_k)$

In this section, we look at convergence results for products of matrices of the form $U_k + A_k$. In this work, we will use that if $a_1, a_2 \ldots, a_k$ are nonnegative numbers, then

$$(1 + a_1) \cdots (1 + a_k) \leq e^{a_1 + \cdots + a_k}. \tag{6.5}$$

Wedderburn (1964) provides a result, given below, where each $U_k = I$.

Theorem 6.5 *Let $A_1, A_2, \ldots$ be a sequence of $n \times n$ matrices. If $\sum_{k=1}^{\infty} \|A_k\|$ converges, then $\prod_{k=1}^{\infty} \|I + A_k\|$ converges.*

Proof. Let

$$\begin{aligned}P_k &= (I + A_k) \cdots (I + A_1) \\ &= I + \sum A_{p_1} + \sum_{p_1 > p_2} A_{p_1} A_{p_2} + \cdots + A_k \cdots A_1.\end{aligned}$$

We show that the sequence $P_1, P_2, \ldots$ is Cauchy. For this, note that if $t > s$, then

$$\begin{aligned}\|P_t - P_s\| &= \left\| \sum_{p_1 > s} A_{p_1} + \sum_{\substack{p_1 > s \\ p_1 > p_2}} A_{p_1} A_{p_2} + \cdots + A_t \cdots A_1 \right\| \\ &\leq \sum_{p_1 > s} \|A_{p_1}\| + \sum_{\substack{p_1 > s \\ p_1 > p_2}} \|A_{p_1}\| \|A_{p_2}\| + \cdots + \|A_t\| \cdots \|A_1\| \\ &\leq \|A_{s+1}\| \left(1 + \sum_{i=1}^{t} \|A_i\| + \frac{\left(\sum_{i=1}^{t} \|A_i\| \right)^2}{2!} + \cdots \right) \\ &\quad + \|A_{s+2}\| \left(1 + \sum_{i=1}^{t} \|A_i\| + \frac{\left(\sum_{i=1}^{t} \|A_i\| \right)^2}{2!} + \cdots \right) + \cdots\end{aligned}$$

and using the power series expansion of e^x,

$$\begin{aligned}&\leq \|A_{s+1}\| \, e^{\sum_{i=1}^{t} \|A_i\|} + \|A_{s+2}\| \, e^{\sum_{i=1}^{t} \|A_i\|} + \cdots \\ &\leq \sum_{k=s+1}^{\infty} \|A_k\| \, e^{\sum_{i=1}^{\infty} \|A_i\|}.\end{aligned}$$

Now, given $\epsilon > 0$, since $\sum_{k=1}^{\infty} \|A_k\|$ converges, there is an N such that if $s > N$,

$$\sum_{k=s+1}^{\infty} \|A_k\| \, e^{\sum_{i=1}^{\infty} \|A_i\|} < \epsilon .$$

Thus, $P_1, P_2, \ldots$ is Cauchy and hence converges. ■

While Wedderburn's result dealt with products of the form $I + A_k$, Ostrowski (1973) considers products using $U + A_k$.

Theorem 6.6 *Let $U + A_1, U + A_2, \ldots$ be a sequence of $n \times n$ matrices. Given $\epsilon > 0$, there is a δ such that if $\|A_k\| < \delta$ for all k, then*

$$\|(U + A_k) \cdots (U + A_1)\| \leq \sigma\,(\rho + \epsilon)^k$$

for some constant σ and $\rho = \rho(U)$.

Proof. Using the upper triangular Jordan form, factor

$$U = PKP^{-1}$$

where K is the Jordan form with a super diagonal of 0's and $\frac{\epsilon}{2}$'s. Thus, $\|K\|_1 \le \rho + \frac{\epsilon}{2}$. Write $(U + A_k)\cdots(U + A_1)$

$$= \left(PKP^{-1} + A_k\right)\cdots\left(PKP^{-1} + A_1\right)$$
$$= P\left(K + P^{-1}A_kP\right)\cdots\left(K + P^{-1}A_1P\right)P^{-1}.$$

Let $\delta = \frac{\epsilon}{2\|P\|_1\|P^{-1}\|_1}$ so that

$$\left\|P^{-1}A_kP\right\|_1 \le \left\|P^{-1}\right\|_1 \|P\|_1 \|A_k\|_1 \le \left\|P^{-1}\right\|_1 \|P\|_1 \delta = \frac{\epsilon}{2}.$$

Then,

$$\|(U + A_k)\cdots(U + A_1)\|_1 \le \|P\|_1 \left\|P^{-1}\right\|_1 \left(\left(\rho + \frac{\epsilon}{2}\right) + \frac{\epsilon}{2}\right)^k$$
$$= \|P\|_1 \left\|P^{-1}\right\|_1 (\rho + \epsilon)^k.$$

Setting $\sigma = \|P\|_1 \left\|P^{-1}\right\|_1$, and noting that norms are equivalent, yields the theorem. ■

This theorem assures that if $\rho(U) < 1$ and

$$\|A_k\|_1 \le \frac{\epsilon}{2\|P\|_1 \|P^{-1}\|_1},$$

where $\rho + \epsilon < 1$, then $\prod_{k=1}^{\infty}(U + A_k) = 0$. So, slight perturbations of the entries of U, indicated by $A_1, A_2, \ldots$, will not change convergence to 0 of the infinite product.

We now consider infinite products $\prod_{k=1}^{\infty}(U_k + A_k)$ and $\prod_{k=1}^{\infty} U_k$. How far these products can differ is shown below.

Theorem 6.7 *Suppose $\|U_k\| \le 1$ for $k = 1, \ldots, r$. Then*

$$\|(U_r + A_r)\cdots(U_1 + A_1) - U_r\cdots U_1\| \le e^{\sum_{k=1}^{r}\|A_k\|} - 1.$$

Proof. Observe that by (6.5),

$$\|(U_r + A_r)\cdots(U_1 + A_1) - U_r\cdots U_1\|$$
$$\le \sum_i \|A_i\| + \sum_{i>j}\|A_i\|\|A_j\| + \cdots + \|A_r\|\cdots\|A_1\|$$
$$= (1 + \|A_r\|)\cdots(1 + \|A_1\|) - 1 \le e^{\sum_{k=1}^{r}\|A_k\|} - 1$$

the required inequality. ■

As a consequence, we have the following.

Corollary 6.5 *Suppose $\|U_k\| \le 1$ for $k = 1, 2, \ldots$ and that $\sum_{k=1}^{\infty}\|A_k\| < \infty$. Given $\epsilon > 0$, there is a constant N such that if $r \ge N$ and $t > r$, then*

$$\|(U_t + A_t)\cdots(U_r + A_r) - U_t\cdots U_r\| < \epsilon.$$

From these results, we might expect the following one.

Theorem 6.8 *Suppose $\|U_k\| \le 1$ for all k. Then the following are equivalent.*

1. *$\prod_{k=r}^{\infty} U_k$ converges for all r.*
2. *$\prod_{k=1}^{\infty}(U_k + A_k)$ converges for all sequences $\langle A_k\rangle$ when $\sum_{k=1}^{\infty}\|A_k\| < \infty$.*

Proof. That (2) implies (1) follows by using $A_1 = -U_1 + I$, then $A_2 = -U_2 + I, \ldots, A_{r-1} = -U_{r-1} + I$ and $A_r = \cdots = 0$.

Suppose (1), that $\prod_{k=r}^{\infty} U_k$ converges for all r and $\sum_{k=1}^{\infty}\|A_k\| < \infty$. Define, for $t > r$,

$$P_t = (U_t + A_t)\cdots(U_1 + A_1)$$
$$P_{t,r} = U_t\cdots U_{r+1}(U_r + A_r)\cdots(U_1 + A_1).$$

We show P_t converges by showing the sequence is Cauchy. For this, let $\epsilon > 0$ be given.

Using the triangular inequality, for $s, t > r$,

$$\|P_t - P_s\| \le \|P_t - P_{t,r}\| + \|P_{t,r} - P_{s,r}\| + \|P_{s,r} - P_s\|. \quad (6.6)$$

We now bound each of the three terms. We use (6.5) to observe that $\|P_r\| \le \beta$, where $\beta = e^{\sum_{k=1}^{\infty}\|A_k\|}$.

1. Using the previous corollary, there is an N_1 such that if $r = N_1$, then
$$\|P_t - P_{t,r}\| < \frac{\epsilon}{3} \text{ and } \|P_{s,r} - P_s\| < \frac{\epsilon}{3}.$$
2. $\|P_{t,r} - P_{s,r}\| \leq$
$$\|U_t \cdots U_{r+1} - U_s \cdots U_{r+1}\| \, \|(U_r + A_r) \cdots (U_1 + A_1)\|.$$
Since $\prod_{k=r+1}^{\infty} U_k$ converges, there is an $N_2, N_2 > r$, such that if $s, t > N_2$, then
$$\|P_{t,r} - P_{s,r}\| \leq \frac{\epsilon}{3}.$$

Putting (1) and (2) together in (6.6) yields that
$$\|P_t - P_s\| < \epsilon$$
for all $t, s \geq N_2$. Thus, P_t is Cauchy and the theorem is established. ■

From Theorems 6.7 and 6.8, we have something of a continuity result for infinite products of matrices. To see this, define $\|\langle A_k \rangle\| = \sum_{k=1}^{\infty} \|A_k\|$ for $\langle A_k \rangle$ such that $\|\langle A_k \rangle\| < \infty$. If $\prod_{k=r}^{\infty} U_k$ converges for all r, so does $\prod_{k=1}^{\infty} (U_k + A_k)$ and given $\epsilon > 0$, there is a $\delta > 0$ such that if $\|\langle A_k \rangle\| < \delta$, then
$$\left\| \prod_{k=1}^{\infty} U_k - \prod_{k=1}^{\infty} (U_k + A_k) \right\| < \epsilon.$$

Another corollary follows.

Corollary 6.6 *Let $\|U_k\| \leq 1$ for $k = 1, 2, \ldots$ and let $\langle A_k \rangle$ be such that $\sum_{k=1}^{\infty} \|A_k\| < \infty$. If $\prod_{k=r}^{\infty} U_k = 0$ for all r, then $\prod_{k=1}^{\infty} (U_k + A_k) = 0$.*

Proof. Theorem 6.8 assures us that $\prod_{k=1}^{\infty} (U_k + A_k)$ converges. Thus, using Corollary 6.5, given $\epsilon > 0$, there is an N_1 such that for $r > N_1$ and any $t > r$,
$$\|U_t \cdots U_{r+1} (U_r + A_r) \cdots (U_1 + A_1) - (U_t + A_t) \cdots (U_1 + A_1)\| < \frac{\epsilon}{2}.$$

Since $\prod_{k>r+1} U_k = 0$, there is an N_2 such that for $t > N_2 > r$,
$$\|U_t \cdots U_{r+1} (U_r + A_r) \cdots (U_1 + A_1) - 0\| < \frac{\epsilon}{2}.$$
Thus, by the triangular inequality,
$$\begin{aligned} &\|(U_t + A_t) \cdots (U_1 + A_1) - 0\| \leq \\ &\|(U_t + A_t) \cdots (U_1 + A_1) - U_t \cdots U_{r+1} (U_r + A_r) \cdots (U_1 + A_1)\| \\ &+ \|U_t \cdots U_{r+1} (U_r + A_r) \cdots (U_1 + A_1) - 0\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon . \end{aligned}$$

Hence, $\prod_{k=1}^{\infty} (U_k + A_k) = 0$. ■

6.4 Joint Eigenspaces

In this section we consider a set Σ of $n \times n$ matrices for which all left infinite products, say, $\prod_{k=1}^{\infty} A_k$, converge. Such sets Σ are said to have the *left convergence property* (LCP).

The eigenvalue properties of products taken from an LCP-set follows.

Lemma 6.4 *Let Σ be an LCP-set. If $A_1, \ldots, A_s \in \Sigma$ and λ is an eigenvalue of $B = A_s \cdots A_1$, then*

1. *$|\lambda| < 1$ or*
2. *$\lambda = 1$ and this eigenvalue is simple.*

Proof. Note that since Σ is an LCP-set, $\lim_{k \to \infty} B^k$ exists. Thus if λ is an eigenvalue of B, $|\lambda| \leq 1$ and if $|\lambda| = 1$, then $\lambda = 1$. Finally, that $\lambda = 1$ must be simple (on 1×1 Jordan blocks) is a consequence of the Jordan form of B. ■

For an eigenvector result, we need the following notation: Let A be an $n \times n$ matrix. The *1-eigenspace* of A is
$$E(A) = \{x : Ax = x\}.$$

Using this notion, we have the following.

Theorem 6.9 *Let $B = \prod_{k=1}^{\infty} A_i$ be taken from an LCP-set Σ. If $A \in \Sigma$ occurs infinitely often in the product $\prod_{k=1}^{\infty} A_i$, then every column of B is in $E(A)$.*

Proof. Since A occurs infinitely often in the product $\prod_{k=1}^{\infty} A_k$, there is a subsequence of $A_1, A_2A_1, \ldots$ with leftmost factor A, say,

$$AB_1, AB_2, \ldots$$

where the B_j's are products of A_k's. Since $A_1, A_2A_1, \ldots$ converges to B, so does $AB_1, AB_2, \ldots$ and $B_1, B_2, \ldots$. Thus,

$$\begin{aligned} AB &= \lim_{k\to\infty} AB_k \\ &= \lim_{k\to\infty} B_k \\ &= B. \end{aligned}$$

Hence, the columns of B are in $E(A)$. ■

As a consequence of this theorem, we see that for $B = \prod_{k=1}^{\infty} A_k$

$$\text{columns of } B \subseteq \cap E(A_i)$$

where the intersection is over all matrices A_i that occur infinitely often in $\prod_{k=1}^{\infty} A_k$. Thus, we have the following.

Corollary 6.7 *If $\prod_{k=1}^{\infty} A_k$ is convergent and $\cap E(A_i) = \{0\}$, where the intersection is over all $E(A_i)$ where A_i occurs infinitely often, then $\prod_{k=1}^{\infty} A_k = 0$.*

The sets $E(B), B = \prod_{k=1}^{\infty} A_k$, and $E(A_i)$'s are also related.

Corollary 6.8 *If $B = \prod_{k=1}^{\infty} A_k$ is convergent, then $E(B) \subseteq \cap E(A_i)$ where the intersection is over all A_i that occur in $\prod_{k=1}^{\infty} A_k$ infinitely often.*

In the next theorem we use the definition

$$E(\Sigma) = \cap E(A_i)$$

where the intersection is over all $A_i \in \Sigma$.

Theorem 6.10 *Let Σ be an LCP-set. If $E(A_i) = E(\Sigma)$ for all $A_i \in \Sigma$, then there is a nonsingular matrix P such that for all $A \in \Sigma$,*

$$P^{-1}AP = \begin{bmatrix} I & B \\ 0 & C \end{bmatrix}$$

where I is $s \times s$ and $\rho(C) < 1$.

Proof. Let $p_1, \ldots, p_s$ be a basis for $E(\Sigma)$. If $A \in \Sigma$, Lemma 6.4 assures that A has precisely s eigenvalues λ, where $\lambda = 1$, and all other eigenvalues λ satisfy $|\lambda| < 1$. Extend $p_1, \ldots, p_s$ to $p_1, \ldots, p_n$, a basis for F^n, and set $P = [p_1, \ldots, p_n]$. Then,

$$AP = P \begin{bmatrix} I & B \\ 0 & C \end{bmatrix}$$

for some matrices B and C. Thus,

$$P^{-1}AP = \begin{bmatrix} I & B \\ 0 & C \end{bmatrix}.$$

Finally, since $\rho(A) \leq 1$, $\rho(C) \leq 1$. If $\rho(C) = 1$, then A has $s+1$ eigenvalues equal to 1, a contradiction. Thus, we have $\rho(C) < 1$. ■

It is easily seen that Σ is an LCP-set if and only if

$$\Sigma_P = \{B : B = P^{-1}AP \text{ where } A \in \Sigma\}$$

is an LCP-set. Thus to obtain conditions that assure Σ is an LCP-set, we need only obtain such conditions on Σ_P. In case Σ satisfies the hypothesis, Corollary 6.1 can be of help.

6.5 Research Notes

In Section 2, Theorem 6.2 is due to Hartfiel (1974), Theorem 6.3 due to Neumann and Schneider (1999), and Theorem 6.4 due to Daubechies and Lagarias (1992). Also see Daubechies and Lagarias (2001) to get a view of the impact of this paper.

The results given in Section 3 are those of Wedderburn (1964) and Ostrowski (1973), as indicated there. Section 4 contains results given in Daubechies and Lagarias (1992).

In related work, Trench (1985 & 1999) provided results on when an infinite product, say $\prod_{k=1}^{\infty} A_k$, is invertible. Holtz (2000) gave conditions for an infinite right product, of the product form $\prod_{k=1}^{\infty} \begin{bmatrix} I & B_k \\ 0 & C_k \end{bmatrix}$, to converge.

Stanford and Urbano (1994) discussed matrix sets Σ, such that for a given vector x, matrices $A_1, A_2 \ldots$ can be chosen from Σ that assure $\prod_{k=1}^{\infty} A_k x = 0$.

Artzrouni (1986a) considered $f_U(A) = \prod_{k=1}^{\infty} (U_k + A_k)$, where he defined $U = \langle U_1, U_2, \ldots \rangle$ and $A = \langle A_1, A_2, \ldots \rangle$. He gave conditions that assure the functions form an equicontinuous family. He then applied this to perturbation in matrix products.

7 Continuous Convergence

In this chapter we look at LCP-sets in which the initial products essentially determine the infinite product; that is, whatever matrices are used, beyond some initial product, has little effect on the infinite product. This continuous convergence is a type of convergence seen in the construction of curves and fractals as we will see in Chapter 11.

7.1 Sequence Spaces and Convergence

Let $\Sigma = \{A_0, \ldots, A_{m-1}\}$, an LCP-set. The associated *sequence space* is

$$D = \{d : d = (d_1, d_2, \ldots)\}$$

where each $d_i \in \{0, \ldots, m-1\}$. On D, define

$$\partial\left(d, \hat{d}\right) = m^{-k}$$

where k is the first index such that $d_k \neq \hat{d}_k$. (So d and $\hat{d}_k$ agree on the first $k-1$ entries.) This ∂ is a metric on D.

Given $d = (d_1, d_2, \ldots)$, define the sequence

$$A_{d_1}, A_{d_2}A_{d_1}, A_{d_3}A_{d_2}A_{d_1}, \ldots.$$

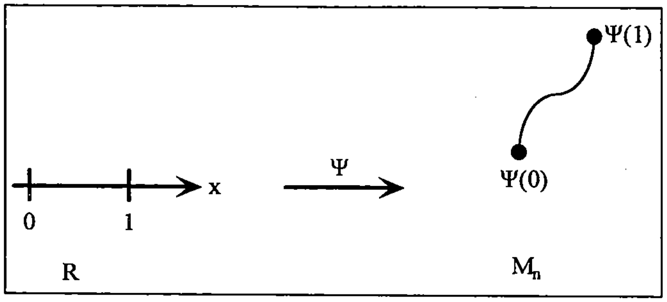

FIGURE 7.1. A view of Ψ.

Since Σ is an LCP-set, this sequence converges, i.e.

$$\prod_{i=1}^{\infty} A_{d_i} = A$$

for some matrix A.

Define $\varphi : D \to M_n$ by

$$\varphi(d) = A.$$

If this function is continuous using the metric ∂ on D and any norm on M_n, we say that Σ is a *continuous* LCP-set.

Continuity of φ can also be described as follows: φ is continuous at $d \in D$ if given any $\epsilon > 0$ there is an integer K such that if $k > K$, then $\left\|\varphi(d) - \varphi\left(\hat{d}\right)\right\| < \epsilon$ for all $\hat{d}$ that differ from d after the k-th digit. (The infinite product will not change much regardless of the choices of $A_{\hat{d}_{k+1}}, A_{\hat{d}_{k+2}}, \ldots$)

Not all LCP-sets are continuous. For example, if

$$\Sigma = \{I, P\}, P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

then φ is not continuous at $(0, 0, \ldots)$.

Now we use φ to define a function $\Psi : [0,1] \to M_n$. (See Figure 7.1.) As we will see in Chapter 11, such functions can be used to describe special curves in R^2.

If $x \in [0,1]$, we can write

$$x = d_1 m^{-1} + d_2 m^{-2} + \cdots$$

the p-adic expansion of x. Recall that if $0 < j < m$, then

$$\begin{aligned} & d_1 m^{-1} + \cdots + d_s m^{-s} + j m^{-s-1} \\ & + 0 m^{-s-2} + 0 m^{-s-3} + \cdots \\ & = d_1 m^{-1} + \cdots + d_s m^{-s} + (j-1) m^{-s-1} \\ & + (m-1) m^{-s-2} + (m-1) m^{-s-3} + \cdots \end{aligned}$$

give the same $x \in [0,1]$. Thus, to define

$$\Psi : [0,1] \to M_n$$

by $\Psi(x) = \varphi(d)$, we would need that

$$\varphi(d_1, \ldots, d_s, j, 0, \ldots) = \varphi(d_1, \ldots, d_s, j-1, m-1, \ldots).$$

When this occurs, we say that the continuous LCP-set Σ is *real definable.*

A theorem that describes when Σ is real definable follows.

Theorem 7.1 *A continuous LCP-set* $\Sigma = \{A_0, \ldots, A_{m-1}\}$ *is real definable iff*

$$A_0^{\infty} A_j = A_{m-1}^{\infty} A_{j-1}$$

for $j = 1, \ldots m-1$.

Proof. If $A_0^{\infty} A_j = A_{m-1}^{\infty} A_{j-1}$, then

$$\varphi(d_1, \ldots, d_s, j, 0, \ldots) = \varphi(d_1, \ldots, d_s, j-1, m-1, m-1 \ldots)$$

for any $s \geq 1$ and all $d_1, \ldots, d_s$. Thus, Σ is real definable.

Now suppose Σ is real definable. Then

$$\varphi(j, 0, 0, \ldots) = \varphi(j-1, m-1, m-1 \ldots)$$

for all $j \geq 1$. Thus,

$$A_0^{\infty} A_j = A_{m-1}^{\infty} A_{j-1},$$

as given in the theorem. ∎

An example may help.

Example 7.1 *Let*

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ .5 & .5 & 0 \\ .25 & .5 & .25 \end{bmatrix}, A_1 = \begin{bmatrix} .25 & .5 & .25 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\Sigma = \{A_0, A_1\}$. In Chapter 11 we will show that Σ is a continuous LCP-set. For now, note that

$$A_0^\infty = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_1^\infty = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since

$$A_0^\infty A_1 = A_1^\infty A_0,$$

Σ is real definable.

7.2 Canonical Forms

In this section we provide a canonical form for a continuous LCP-set $\Sigma = \{A_0, \ldots, A_{m-1}\}$. We again use the definition

$$E(\Sigma) = \bigcap_{i=0}^{m-1} E(A_i)$$

where $E(A_i)$ is the 1-eigenspace of A_i, for all i. We need a lemma.

Lemma 7.1 *If Σ is a continuous LCP-set, then $E(\Sigma) = E(A_i)$ for all i.*

Proof. Since $E(\Sigma) \subseteq E(A_i)$ for all i, it is clear that we only need to show that $E(A_i) \subseteq E(\Sigma)$, for all i.

For this, let $y \in E(A_i)$. Then $y = A_i^\infty y$. For any j, define

$$d^{(k)} \to (i, \ldots, i, j, j, \ldots)$$

where i occurs k times. Then

$$d^{(k)} \to (i, i, \ldots) \text{ as } k \to \infty.$$

Since Σ is continuous

$$\varphi\left(d^{(k)}\right) \to \varphi((i, i, \ldots)) \text{ as } k \to \infty,$$

so

$$A_j^\infty A_i^k \to A_i^\infty \text{ as } k \to \infty.$$

Hence

$$\lim_{k\to\infty} A_j^\infty A_i^k y = A_i^\infty y$$

or

$$A_j^\infty y = y.$$

By considering the Jordan form of A_j, we see that $A_j y = y$ also. Hence, $y \in E(A_j)$ from which it follows that $E(\Sigma) = \bigcap_{i=0}^{m-1} E(A_i)$ as required. ■

The canonical form follows.

Theorem 7.2 *Let $\Sigma = \{A_0, \ldots, A_{m-1}\}$. Then Σ is a continuous LCP-set iff there is a matrix P such that*

$$P^{-1}\Sigma P = \left\{ \begin{bmatrix} I & B_i \\ 0 & C_i \end{bmatrix} : \begin{bmatrix} I & B_i \\ 0 & C_i \end{bmatrix} = P^{-1}A_i P \right\}$$

where $\hat{\rho}(\Sigma_c) < 1$, $\Sigma_c = \{C_0, \ldots, C_{m-1}\}$.

Proof. Suppose Σ is a continuous LCP-set. Let $P = [p_1 \ldots p_s p_{s+1} \ldots p_n]$, a nonsingular matrix where $p_1, \ldots, p_s$ are in $E(\Sigma)$ and $\dim E(\Sigma) = s$. Then for any $A_i \in \Sigma$,

$$A_i P = P \begin{bmatrix} I & B_i \\ 0 & C_i \end{bmatrix}$$

for some B_i and C_i where I is the $s \times s$ identity matrix.

Now, since Σ is a continuous LCP-set, so is Σ_c. Thus for any infinite product $\prod_{k=1}^{\infty} C_k$ from Σ_c, by Theorem 6.9, its nonzero columns must be eigenvectors, belonging to 1, of every C_i that occurs infinitely often in the product. Since 1 is not an eigenvalue of any C_i, Lemma 7.1, the columns of $\prod_{k=1}^{\infty} C_k$ must be 0. Thus, by Theorem 6.4, $\hat{\rho}(\Sigma_c) < 1$.

Conversely, suppose $P^{-1}\Sigma P$ is as described in the theorem with $\hat{\rho}(\Sigma_c) < 1$. Since $\hat{\rho}(\Sigma_c) < 1$ by the definition of $\hat{\rho}(\Sigma_c)$, there is a positive integer r and a positive constant $\gamma < 1$ such that $\|\pi\|_1 \leq \gamma$ for all r-blocks π from Σ_c.

Now by Corollary 6.1, $P^{-1}\Sigma P$ is an LCP-set, and thus so is Σ. Hence, by Theorem 3.14, Σ is a product bounded set. We let β denote such a bound.

Let $d = (d_1, d_2, \ldots)$ be a sequence in D and $\epsilon > 0$. Let N be a positive number such that

$$2\beta \|P\|_1 \left\|P^{-1}\right\|_1 \gamma^N < \epsilon.$$

Let $\hat{d} = \left(\hat{d}_1, \hat{d}_2, \ldots\right)$ be a sequence such that $\delta\left(d, \hat{d}\right) < m^{-rN}$. Then

$$\begin{aligned}
&\left\|\varphi(d_1, d_2, \ldots) - \varphi\left(\hat{d}_1, \hat{d}_2, \ldots\right)\right\|_1 \\
&\le \left\|\left(\prod_{k=rN+1}^{\infty} A_{d_k} - \prod_{k=rN+1}^{\infty} A_{\hat{d}_k}\right)\prod_{k=1}^{rN} A_{d_k}\right\|_1 \\
&= \left\|\left(P\begin{bmatrix} I & S_1 \\ 0 & 0 \end{bmatrix}P^{-1} - P\begin{bmatrix} I & S_2 \\ 0 & 0 \end{bmatrix}P^{-1}\right)P\begin{bmatrix} I & S_3 \\ 0 & \pi \end{bmatrix}P^{-1}\right\|_1
\end{aligned}$$

where

$$\prod_{k=rN+1}^{\infty} A_{d_k} = P\begin{bmatrix} I & S_1 \\ 0 & 0 \end{bmatrix}P^{-1}, \quad \prod_{k=rN+1}^{\infty} A_{\hat{d}_k} = P\begin{bmatrix} I & S_2 \\ 0 & 0 \end{bmatrix}P^{-1}$$

and $\prod_{k=1}^{rN} A_{d_k} = P\begin{bmatrix} I & S_3 \\ 0 & \pi \end{bmatrix}P^{-1}$.

Continuing the calculation,

$$\begin{aligned}
&= \left\|P\begin{bmatrix} 0 & S_1 - S_2 \\ 0 & 0 \end{bmatrix}P^{-1}P\begin{bmatrix} I & S_3 \\ 0 & \pi \end{bmatrix}P^{-1}\right\|_1 \\
&= \left\|P\begin{bmatrix} 0 & S_1 - S_2 \\ 0 & 0 \end{bmatrix}P^{-1}P\begin{bmatrix} 0 & 0 \\ 0 & \pi \end{bmatrix}P^{-1}\right\|_1 \\
&\le \left\|P\begin{bmatrix} I & S_1 \\ 0 & 0 \end{bmatrix}P^{-1} - P\begin{bmatrix} I & S_2 \\ 0 & 0 \end{bmatrix}P^{-1}\right\|_1 \|\pi\|_1 \|P\|_1 \left\|P^{-1}\right\|_1 \\
&\le 2\beta \|\pi\|_1 \|P\|_1 \left\|P^{-1}\right\|_1 \\
&\le 2\beta\gamma^N \|P\|_1 \left\|P^{-1}\right\|_1 < \epsilon.
\end{aligned}$$

Thus, Σ is a continuous LCP-set. ■

As pointed out in the proof of the theorem, we have the following.

Corollary 7.1 *If $\Sigma = \{A_0, \ldots, A_{m-1}\}$ is a continuous LCP-set, then there is a nonsingular matrix P, such that for any sequence $(d_1, d_2, \ldots)$, there is a matrix S where*

$$\prod_{i=1}^{\infty} A_{d_i} = P\begin{bmatrix} I & S \\ 0 & 0 \end{bmatrix}P^{-1},$$

I the $s \times s$ identity matrix and $s = \dim E(\Sigma)$.

Also, we have the following.

Corollary 7.2 *If $\Sigma = \{A_0, \ldots, A_{m-1}\}$ is a continuous LCP-set, then infinite products from Σ converge uniformly and at a geometric rate.*

Proof. Note that in the proof of the theorem, β and γ do not depend on the infinite products considered. ■

As a final corollary, we show when the function Ψ, introduced in Section 1, is continuous.

Corollary 7.3 *Let $\Sigma = \{A_0, \ldots, A_{m-1}\}$ be a continuous LCP-set and suppose Σ is real definable. Then Ψ is continuous.*

Proof. We will only prove that Ψ is right side continuous at x, $x \in [0, 1)$. Left side continuous is handled in the same way.

Write

$$x = d_1 m^{-1} + d_2 m^{-2} + \cdots.$$

We will assume that this expansion does not terminate with repeated $m-1$'s. (Recall that any such expansion can be replaced by one with repeated 0's.)

Let $\epsilon > 0$. Using Corollary 7.2, choosing N, where we have $d_{N+1} \neq m-1$, and such that if any two infinite products, say B and $\hat{B}$ have their first k factors identical, $k \ge N$, then

$$\left\|B - \hat{B}\right\| < \epsilon. \tag{7.1}$$

Let $y \in (0, 1)$ where $y > x$ and $y - x < m^{-N-1}$. Thus, say,

$$y = d_1 m^{-1} + \cdots + d_N m^{-N} + \delta_{N+1} m^{-N-1} + \delta_{N+2} m^{-N-2} + \cdots.$$

Now, let

$$A = \prod_{k=1}^{\infty} A_{d_k} \text{ and } \hat{A} = \cdots A_{\delta_{N+2}} A_{\delta_{N+1}} A_{d_N} \cdots A_{d_1}.$$

Then, using (7.1),

$$\|\Psi(x) - \Psi(y)\| = \left\|A - \hat{A}\right\| < \epsilon$$

which shows that Ψ is right continuous at x. ■

7.3 Coefficients and Continuous Convergence

Again, in this section, we use a finite set $\Sigma = \{A_0, \ldots, A_{m-1}\}$ of $n \times n$ matrices. We will show how subspace contraction coefficients can be used to show that Σ is a continuous LCP-set.

To do this, we will assume that Σ is *τ-proper*, that is

$$E(\Sigma) = E(A_i)$$

for all i. If $p_1, \ldots, p_s$ is a basis for that eigenspace and $p_1, \ldots, p_n$ a basis for F^n, then $P = [p_1, \ldots, p_n]$ is nonsingular. Further,

$$A_k = P \begin{bmatrix} I & B_k \\ 0 & C_k \end{bmatrix} P^{-1}$$

where I is $s \times s$, for all $A_k \in \Sigma$.

Let

$$E = [p_1, \ldots, p_s]$$

and

$$W = \{x : xE = 0\}.$$

Recall from Chapter 2 that

$$\tau_W(A) = \max_{\substack{xE=0 \\ x \neq 0}} \frac{\|xA\|}{\|x\|}$$

is a subspace contraction coefficient. And, if $A_k \in \Sigma$ and we have $A_k = P \begin{bmatrix} I & B_k \\ 0 & C_k \end{bmatrix} P^{-1}$, then

$$\tau(A_k) = \|C_k\|_J$$

where the norm $\|\cdot\|_J$ is defined there. Recall that subspace contraction coefficients are all equivalent, so to prove convergence, it doesn't matter which norm is used.

Theorem 7.3 *Let $\Sigma = \{A_0, \ldots, A_{m-1}\}$ be τ-proper. The set Σ is a continuous LCP-set iff there is a subspace contraction coefficient τ_W and a positive integer r such that $\tau_W(\pi) < 1$ for all r-blocks π from Σ.*

Proof. If Σ is a continuous LCP-set, using any norm, a subspace contraction coefficient τ_W can be defined. Since by Theorem 7.2, using the notation there, $\hat{\rho}(\Sigma_c) < 1$, there is a positive integer r such that $\|C\|_J < 1$ for all r-blocks C from Σ_c. Thus, since $\tau_W\left(\prod_{k=1}^{r} A_{d_k}\right) = \left\|\prod_{k=1}^{r} C_{d_k}\right\|_J$, it follows that $\tau_W(\pi) < 1$ for all r-blocks from Σ.

Conversely, suppose τ_W is a contraction coefficient such that $\tau_W(\pi) < 1$ for all r-blocks π from Σ. Thus, $\|\hat{\pi}\|_J < 1$ for all r-blocks $\hat{\pi}$ from Σ_c. So, $\hat{\rho}_r(\Sigma_c) < 1$ which shows that $\hat{\rho}(\Sigma_c) < 1$. This shows, by using Theorem 7.2, that Σ is a continuous LCP-set. ■

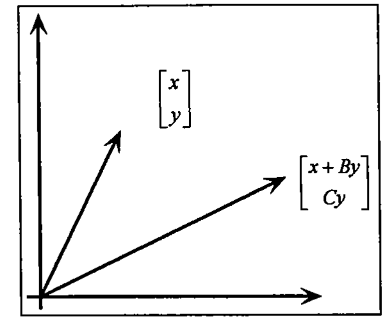

FIGURE 7.2. Convergence view of Corollary 7.4.

We can also prove the following.

Corollary 7.4 *If Σ is a τ-proper compact matrix set and we have $\tau_W(\pi) \leq \tau < 1$ for all r-blocks π in Σ, then Σ is an LCP-set.*

A view of the convergence here can be seen by observing that

$$\begin{bmatrix} I & B \\ 0 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + By \\ Cy \end{bmatrix},$$

where $\begin{bmatrix} x \\ y \end{bmatrix}$ is partitioned compatibly to $\begin{bmatrix} I & B \\ 0 & C \end{bmatrix}$. So the y vector contracts toward 0 while the x vector is changed by some (bounded) matrix constant of y, namely By. A picture is given in Figure 7.2.

We conclude with the following observation. In the definition

$$\tau_W(A) = \max_{\substack{x \in W \\ \|x\|=1}} \|xA\|,$$

matrix multiplication is on the right. And, we showed that $\tau_W(A_1A_2) \leq \tau_W(A_1)\,\tau_W(A_2)$, so we are talking about right products. Yet, τ_W defined in this way establishes LCP-sets.

Example 7.2 *Let* $\Sigma = \left\{ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \right\}$. *Set* $E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Then* $\tau_W(\Sigma) = 0$ *and* Σ *is an LCP-set. But,* Σ *is not an RCP-set.*

7.4 Research Notes

The sequence space and continuity results of Section 1 and the canonical form work in Section 2 are, basically, as in Daubechies and Lagarias (1992a). Section 3 used the subspace contraction coefficients results of Hartfiel and Rothblum (1998).

Applications and further results, especially concerning differentiability of Ψ, rather than continuity, can be found in Daubechies and Lagarias (1992b) and Micchelli and Prautzsch (1989).

8
Paracontracting

An $n \times n$ matrix A is *paracontracting* or PC with respect to a vector norm $\|\cdot\|$, if

$$\|Ax\| < \|x\| \text{ whenever } Ax \neq x$$

for all vectors x. Note that this implies that $\|A\| \leq 1$. We can view paracontracting by noting that $L(x) = Ax$ is the identity on $E(A)$ but contracts all other vectors. This is depicted, somewhat, in Figure 8.1.

If there is a positive constant γ such that

$$\|Ax\| \leq \|x\| - \gamma \|Ax - x\|$$

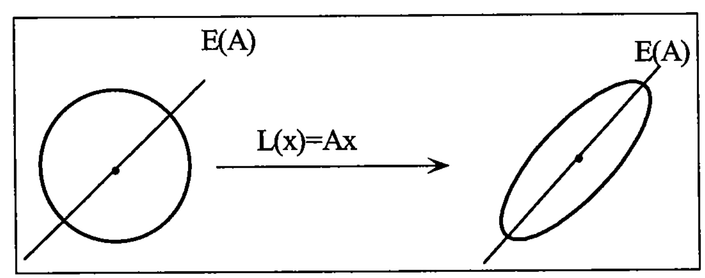

FIGURE 8.1. A view of paracontracting.

for all x, then A is called *γ-paracontracting*or γPC. It is clear that γPC implies PC. Paracontracting and γ-paracontracting sets are sets containing those kinds of matrices.

In this chapter we show that for a finite set $\Sigma$of $n \times n$matrices, paracontracting and γ-paracontracting are the same. In addition, both paracontracting and γ-paracontracting sets are LCP-sets.

8.1 Convergence

For any matrix set $\Sigma$and any vector x_1, the sequence

$$\begin{aligned} x_2 &= A_{i_1} x_1 \\ x_3 &= A_{i_2} x_2 \\ &\cdots \end{aligned}$$

where each $A_{i_k} \in \Sigma$, is called a *trajectory* of Σ. Any finite sequence, $x_1, \ldots, x_p$is called an *initial piece* of the trajectory.

Trajectories are linked to infinite products of matrices by the following lemma.

Lemma 8.1 *A matrix set Σ is an LCP-set iff all trajectories of Σ converge.*

Proof. The proof follows by noting that if $x_1 = e_i$, e_i the $(0,1)$-vector with a 1 only in the i-th position, then $A_{i_k} \cdots A_{i_1} e_i = i$-th column of $A_{i_k} \cdots A_{i_1}$. So convergence of trajectories implies column convergence of the infinite products and vice versa. And from this, the lemma follows. ■

Using this lemma, we show that for finite Σ, paracontracting sets are LCP-sets. The converse of this result will be given in Section 2.

Theorem 8.1 *If $\Sigma = \{A_1, \ldots, A_m\}$ is a paracontracting set with respect to $\|\cdot\|$, then Σ is an LCP-set.*

Proof. Let x_1 be a vector and set

$$x_2 = A_{i_1} x_1,\ x_3 = A_{i_2} A_{i_1} x_1, \ldots .$$

Since $\|Ay\| \leq \|y\|$ for all y and all $A \in \Sigma$, the sequence is bounded. Actually, if $p > q$, then $\|x_p\| \leq \|x_q\|$.

Suppose, reindexing if necessary, that $A_1, \ldots, A_s$ occur as factors infinitely often in $\prod_{k=1}^{\infty} A_{i_k}$. Let $y_1, y_2, \ldots$ be the subsequence of $x_1, x_2, \ldots$ such that $A_1 y_1, A_1 y_2, \ldots$ are in the sequence. And from this, take a convergent subsequence, say, $y_{j_1}, y_{j_2}, \ldots \to y$. Thus, $A_1 y_{j_1}, A_1 y_{j_2}, \ldots \to A_1 y$.

Since y is the limit of $y_{j_1}, y_{j_2}, \ldots$,

$$\|y\| \leq \|A_1 y_{j_k}\|$$

for all k so in the limit

$$\|y\| \leq \|A_1 y\| .$$

Thus, since A is paracontracting, $\|y\| = \|A_1 y\|$ and so $A_1 y = y$.

We show that $A_i y = y$ for $i = 1, \ldots, s$. For this, suppose $A_1, \ldots, A_w$ satisfies $A_i y = y$ for some w, $1 \leq w < s$. Consider a sequence, say without loss of generality $A_{w+1}\pi_1 y_{j_1}, A_{w+1}\pi_2 y_{j_2}, \ldots$ where the matrices in the products π_k are from $\{A_1, \ldots, A_w\}$. The sequence converges to $A_{w+1} y$ and thus,

$$\|y\| \leq \|A_{w+1} y\| .$$

So $\|y\| = \|A_{w+1} y\|$ and consequently $A_{w+1} y = y$. From this, it follows that

$$A_i y = y$$

for all $i \leq s$.

Finally,

$$\|x_k - y\| = \|\pi_k z_k - y\|$$

where $\pi_k z_k = x_k$, π_k a product of $A_1, \ldots, A_s$ and z_k the vector in $y_{j_1}, y_{j_2}, \ldots$ that immediately proceeds x_k. (Here k is large enough that no $A_{s+1}, \ldots, A_m$ reappears as a factor.) Then,

$$\begin{aligned} \|x_k - y\| &= \|\pi_k z_k - \pi_k y\| \\ &\leq \|z_k - y\| . \end{aligned}$$

So $\|x_k - y\| \to 0$ as $k \to \infty$. It follows that $\left\langle \prod_{j=1}^{k} A_{i_j} x_1 \right\rangle$ converges, and thus by the lemma, the result follows. ■

The example below shows that this result can be false when Σ is infinite.

Example 8.1 *Let*

$$\Sigma = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha_k \\ 0 & \alpha_k & 0 \end{bmatrix} : \alpha_k = \frac{2^k - 1}{2^k} \text{ for } k = 2, 3, \ldots \right\} .$$

Using the vector 2-norm, it is clear that Σ is a paracontracting set. Now, let $a > 0$. Then

$$\begin{aligned}\alpha_2 a &= \left(1-\frac{1}{2^2}\right)a = a - \frac{1}{4}a\\ \alpha_3\alpha_2 a &= \alpha_3\left(a-\frac{1}{4}a\right) = \left(1-\frac{1}{2^3}\right)\left(a-\frac{1}{4}a\right)\\ &> a - \frac{1}{4}a - \frac{1}{8}a\\ \alpha_4\alpha_3\alpha_2 a &> \alpha_4\left(a-\frac{1}{4}a-\frac{1}{8}a\right)\\ &= \left(1-\frac{1}{2^4}\right)\left(a-\frac{1}{4}a-\frac{1}{8}a\right)\\ &> a-\frac{1}{4}a-\frac{1}{8}a-\frac{1}{16}a.\end{aligned}$$

And, in general

$$\begin{aligned}\alpha_k\cdots\alpha_2 a &> a - \frac{1}{2^2}a - \frac{1}{2^3}a - \cdots - \frac{1}{2^k}a\\ &= a - \frac{1}{4}\left(\frac{1}{1-\frac{1}{2}}\right)a\\ &= \frac{1}{2}a.\end{aligned}$$

Thus, the sequence $\langle \alpha_k\cdots\alpha_1\rangle$ does not converge to 0. Hence, by observing entries, the sequence

$$\left\langle \begin{bmatrix}1&0&0\\0&0&\alpha_k\\0&\alpha_k&0\end{bmatrix}\cdots\begin{bmatrix}1&0&0\\0&0&\alpha_2\\0&\alpha_2&0\end{bmatrix}\right\rangle$$

does not converge, and so Σ is not an LCP-set.

In the next section, we will show that for finite sets, paracontracting sets, γ-paracontracting sets, and LCP-sets are equivalent.

Concerning continuous LCP-sets, we have the following.

Theorem 8.2 *Let $\Sigma = \{A_1, \ldots, A_m\}$. Then Σ is a continuous LCP-set iff Σ is a paracontracting set and $E(\Sigma) = E(A_i)$ for all i.*

Proof. Suppose Σ is a continuous LCP-set. By Lemma 7.1, $E(\Sigma) = E(A_i)$ for all i. Theorem 7.2 assures a P such that

$$A_i' = P^{-1}A_iP = \begin{bmatrix} I & B_i\\ 0 & C_i\end{bmatrix} \tag{8.1}$$

for all i, where $\hat\rho(\Sigma_c) < 1$. Thus by a characterization of $\hat\rho(\Sigma)$, Theorem 2.19, there is an induced norm $\|\cdot\|$ and an $\alpha < 1$ such that

$$\|C_i\| \le \alpha$$

for all i.

Now, partitioning $x = \begin{bmatrix} x_1\\ x_2\end{bmatrix}$ compatible to A_i', define, for $\epsilon > 0$, a vector norm

$$\|x\|_\epsilon = \left\|\begin{bmatrix} x_1\\ x_2\end{bmatrix}\right\|_\epsilon = \epsilon\|x_1\|_2 + \|x_2\|.$$

Thus,

$$\begin{aligned}\|A_i'x\|_\epsilon &= \left\|\begin{bmatrix} x_1 + B_ix_2\\ C_ix_2\end{bmatrix}\right\|_\epsilon\\ &= \epsilon\|x_1 + B_ix_2\|_2 + \|C_ix_2\|\\ &\le \epsilon\|x_1\|_2 + (\epsilon\|B_i\|_b + \alpha)\|x_2\|\end{aligned}$$

where

$$\|B_i\|_b = \max_{x_2\ne 0}\frac{\|B_ix_2\|_2}{\|x_2\|}.$$

Take ϵ such that

$$\gamma = \epsilon\|B_i\|_b + \alpha < 1$$

for all i.

Then

$$\begin{aligned}\|A_i'x\|_\epsilon &\le \epsilon\|x_1\|_2 + \gamma\|x_2\|\\ &\le \epsilon\|x_1\|_2 + \|x_2\|\\ &= \|x\|_\epsilon.\end{aligned}$$

And equality holds iff $\|x_2\| = 0$, i.e. $x_2 = 0$. Thus,

$$\|A_i'x\|_\epsilon \le \|x\|_\epsilon$$

with equality iff $A_i'x = x$. And it follows that $\Sigma' = \{A_1', \ldots, A_r'\}$, and thus Σ, is a paracontracting set. (Use $\|x\| = \left\|P^{-1}x\right\|_\epsilon$.)

Conversely, suppose Σ is a paracontracting set, which satisfies $E(\Sigma) = E(A_i)$ for all i. Then there is a matrix P such that $A_i' = P^{-1}A_iP$ has form given in (8.1). Since $E(C_i) = \{0\}$ for all i and since Σ is an LCP-set, by Theorem 6.9, $\prod_{i=1}^{\infty} C_{d_i} = 0$ for any sequence $(d_1, d_2, \ldots)$. Thus, by Theorem 6.4, $\hat\rho(\Sigma_c) < 1$. Hence, by Theorem 7.2, Σ is a continuous LCP-set. ■

8.2 Types of Trajectories

We describe three kinds of matrix sets Σ in terms of the behaviors of the trajectories determined by them.

Definition 8.1 *Let Σ be a matrix set.*

1. *The set Σ is* bounded variation stable *(BVS) if*

$$\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| < \infty$$

for all trajectories $x_1, x_2, \ldots$ of Σ. Here, $\sum_{i=1}^{\infty} \|x_{i+1} - x_i\|$ is called the variation of the trajectory $x_1, x_2, \ldots$.

2. *The set Σ is* uniformly bounded variation stable *(uniformly BVS) if there is a constant L such that*

$$\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| \leq L \|x_1\|$$

for all trajectories $x_1, x_2, \ldots$ of Σ.

3. *The set Σ has* vanishing steps *(VS) if*

$$\lim_{k \to \infty} \|x_{i+1} - x_i\| = 0$$

for all trajectories $x_1, x_2, \ldots$ of Σ.

An immediate consequence of BV follows.

Lemma 8.2 *If $\sum_{i=1}^{\infty} \|x_{i+1} - x_i\|$ converges, then $\langle x_i \rangle$ converges.*

Proof. Note that $\langle x_i \rangle$ is a Cauchy sequence, and thus it must converge. ∎

It should be noticed that deciding if Σ has one of the properties, 1 through 3, does not depend on the norm used (due to the equivalence of norms). And, in addition, if Σ has one of these properties, so does $P^{-1}\Sigma P$ for any nonsingular matrix P.

What we will show in this section is that, if Σ is finite, then all of properties, 1 through 3, are equivalent. And for finite Σ, we will show that LCP-sets, paracontracting sets, and γ-paracontracting sets are equivalent

to properties 1 through 3. To do this, notationally, throughout this section, we will assume that

$$\Sigma = \{A_1, \ldots, A_m\}.$$

We need a few preliminary results. The first of these allows us to trade our given problem for one with a special feature. Since

$$E(\Sigma) = \{x : A_i x = x \text{ for all } A_i \in \Sigma\},$$

there is a nonsingular matrix P such that, for all i,

$$A_i' = P^{-1} A_i P = \begin{bmatrix} I & B_i \\ 0 & C_i \end{bmatrix} \tag{8.2}$$

where I is $s \times s$, $s = \dim E(\Sigma)$. We let $\Sigma' = \{A_i' : A_i \in \Sigma\}$ and prove our first result for Σ'.

Lemma 8.3 *Suppose Σ' is VS. Then there exist positive constants α and β such that if $x = \begin{bmatrix} p \\ q \end{bmatrix}$, partitioned as in (8.2),*

$$\alpha \|C_i q - q\| \leq \|A_i' x - x\| \leq \beta \|C_i q - q\|$$

for all i.

Proof. By equivalence of norms, we can prove this result for $\|\cdot\|_2$. For this, note that

$$A_i' x = \begin{bmatrix} p + B_i q \\ C_i q \end{bmatrix}$$

so

$$\|A_i' x - x\|_2 = \left\| \begin{bmatrix} B_i q \\ C_i q - q \end{bmatrix} \right\|_2.$$

If $C_i q = q$ and $B_i q \neq 0$, then $\begin{bmatrix} p+B_i q \\ q \end{bmatrix}, \begin{bmatrix} p+2B_i q \\ q \end{bmatrix}, \begin{bmatrix} p+3B_i q \\ q \end{bmatrix}, \ldots$ is a trajectory of Σ'. This trajectory is not V.S. Thus, we must have that $B_i q = 0$. This implies that the null space of $C_i - I$ is a subset of the null space of B_i. And thus there is a matrix D_i such that $D_i (C_i - I) = B_i$.

From this we have that

$$\begin{aligned} \left\| \begin{bmatrix} B_i q \\ C_i q - q \end{bmatrix} \right\|_2 &= \left(\|B_i q\|_2^2 + \|(C_i - I) q\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \left(\|D_i\|_2^2 \|(C_i - I) q\|_2^2 + \|(C_i - I) q\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \left(\|D_i\|_2^2 + 1 \right)^{\frac{1}{2}} \|(C_i q - q)\|_2 . \end{aligned}$$

Setting $\beta = \max_i \left\{ \left(\|D_i\|_2^2 + 1 \right)^{\frac{1}{2}} \right\}$ yields the upper bound. A lower bound is obviously seen as $\alpha = 1$. ■

Since paracontracting, γ-paracontracting, LCP, BVS, and uniformly BVS all imply VS, this lemma allows us to change our problem of showing the equivalence of the properties, 1 through 3, to the set $\Sigma'' = \{C_1, \ldots, C_m\}$.

A useful tool in actually showing the equivalence results follows.

Lemma 8.4 *Suppose Σ'' is a uniformly BVS-set. Then there is an $\epsilon > 0$ such that*

$$\max_{1 \le i \le p-1} \|x_{i+1} - x_i\| \ge \epsilon \sum_{i=1}^{p-1} \|x_{i+1} - x_i\| \tag{8.3}$$

for all initial pieces $x_1, x_2, \ldots, x_p$ of trajectories $x_1, x_2, \ldots$ of Σ.

Proof. We prove the theorem using that $E(\Sigma'') = \{0\}$. We precede by induction on r, the number of matrices in Σ''. If $r = 1$, by Lemma 8.2, $\rho(C_1) < 1$. Thus, there is a vector norm $\|\cdot\|$ such that $\|C_1\| < 1$. From this, we have

$$\begin{aligned} \sum_{i=1}^{p-1} \left\| C_1^i x_1 - C_1^{i-1} x_1 \right\| &\le \sum_{i=1}^{p-1} \|C_1\|^{i-1} \|C_1 x_1 - x_1\| \\ &\le \frac{1}{1 - \|C_1\|} \|C_1 x_1 - x_1\|. \end{aligned}$$

Setting $\epsilon = 1 - \|C_1\|$ yields the result.

Now suppose the theorem holds for all Σ'' having $r - 1$ matrices. Let Σ'' be such that it has r matrices. Since Σ'' is a uniformly BVS-set, all trajectories converge, and thus, by Corollary 3.4, there is a vector norm $\|\cdot\|$ such that $\|C_i\| \le 1$ for all i. We now use this norm.

Note that (8.3) is true iff it is true for initial pieces of trajectories $x_1, x_2, \ldots, x_p$ such that $\max_{1 \le i \le p-1} \|x_{i+1} - x_i\| \le 1$. (This is a matter of scaling.) Thus we show that there is a number K such that

$$K \ge \sum_{i=1}^{p-1} \|x_{i+1} - x_i\|$$

where $\max_{1 \le i \le p-1} \|x_{i+1} - x_i\| \le 1$.

We argue by contradiction. Thus suppose there is a sequence of initial pieces $x_1^j, x_2^j, \ldots, x_{p_j}^j$ where $j = 1, 2, \ldots$ and such that

$$\sum_{i=1}^{p_j - 1} \left\| x_{i+1}^j - x_i^j \right\| \to \infty$$

as $j \to \infty$. By the uniformly BVS property, $\lim_{j \to \infty} \left\| x_1^j \right\| = \infty$.

Now let $\Sigma_k'' = \Sigma'' - \{C_k\}$ for $k = 1, \ldots, r$. Each Σ_k'' satisfies the induction hypothesis so there is a number M_k such that

$$M_k \ge \sum_{i=1}^{p-1} \|x_{i+1} - x_i\|$$

for initial pieces of trajectories of Σ_k'' with $\max_{1 \le i \le p-1} \|x_{i+1} - x_i\| \le 1$ for all i.

Let

$$M > \max_k M_k.$$

Now since $\sum_{i=1}^{p_j - 1} \left\| x_{i+1}^j - x_i^j \right\| \to \infty$ as $j \to \infty$, the initial pieces where $\sum_{i=1}^{p_j - 1} \left\| x_{i+1}^j - x_i^j \right\| > M$ must use all matrices in Σ''. Take all of these initial pieces $x_1^j, x_2^j, \ldots, x_{p_j}^j$ and from them take initial pieces $x_1^j, x_2^j, \ldots, x_{n_j}^j$ $(n_j \le p_j)$ such that

$$\sum_{i=1}^{n_j - 1} \left\| x_{i+1}^j - x_i^j \right\| \le M$$

and

$$\sum_{i=1}^{n_j} \left\| x_{i+1}^j - x_i^j \right\| > M.$$

Note that

$$\sum_{i=1}^{n_j} \left\| x_{i+1}^j - x_i^j \right\| \le M + 1. \tag{8.4}$$

Now let

$$y_1^j = \frac{x_1^j}{\left\| x_1^j \right\|}$$

and h the limit of a subsequence, say $y_1^{k_1}, y_1^{k_2}, \ldots$. We show that h is an eigenvector, belonging to the eigenvalue 1, of each matrix in Σ''.

Rewriting (8.4) yields

$$\sum_{i=1}^{n_j} \left\| y_{i+1}^{k_j} - y_i^{k_j} \right\| \leq \frac{M+1}{\left\| x_1^{k_j} \right\|}$$

so we have

$$\lim_{j\to\infty} \sum_{i=1}^{n_j} \left\| y_{i+1}^{k_j} - y_i^{k_j} \right\| = 0. \tag{8.5}$$

Suppose C_1 (reindexing if necessary) occurs as the leftmost matrix in $y_2^{l_1}, y_2^{l_2}, \ldots$. Then from (8.5), h is an eigenvector of C_1. Now, of those products $y_1^{l_1}, y_2^{l_1}, \ldots; y_1^{l_2}, y_2^{l_2}, \ldots; \ldots$, take the largest initial products that contain C_1, say $y_{m_1}^{l_1}, y_{m_2}^{l_2}, \ldots$. (So $y_{m_1}^{l_1} = C_1 \cdots C_1 y_1^{l_1}, y_{m_2}^{l_2} = C_1 \cdots C_1 y_1^{l_2}$, etc.) and such that C_2 (reindexing if necessary) occurs as the first factor in each of the iterates $y_{m_1+1}^{l_1}, y_{m_2+1}^{l_2}, \ldots$. Then by (8.5), h is an eigenvector of C_2. Continuing this procedure, we see that h is an eigenvector, belonging to the eigenvalue 1, of all matrices in Σ'', a contradiction. Thus, the lemma is true for Σ'' and the induction concluded. The result follows. ∎

We now establish the main result in this section.

Theorem 8.3 *If Σ'' is VS, then Σ'' is uniformly BVS.*

Proof. We prove the theorem by induction on r, the number of matrices in Σ''.

If Σ'' contains exactly one matrix, then $\rho(C_1) < 1$. Thus, there is a vector norm $\|\cdot\|$ such that $\|C_1\| < 1$ and so

$$\begin{aligned}\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| &\leq \sum_{i=1}^{\infty} \|C_1\|^{i-1} \|x_2 - x_1\| \\ &\leq \frac{1}{1 - \|C_1\|} \|C_1 - I\| \|x_1\|.\end{aligned}$$

So, using $L = \frac{1}{1-\|C_1\|} \|C_1 - I\|$, we see that Σ'' is uniformly BVS.

Suppose the theorem is true for all Σ'' containing $r-1$ matrices. Now suppose Σ'' has r matrices. Then, since every proper subset of Σ'' is VS, we have by the induction hypothesis that these proper subsets are uniformly BVS.

We now argue several needed smaller results.

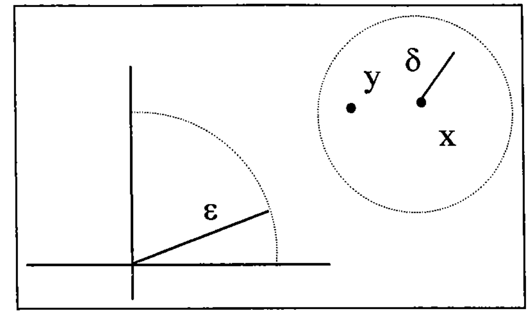

FIGURE 8.2. Sketch for ϵ, δ view.

1. We show that if $\epsilon > 0$, then there is a δ such that if $\|x\| \geq \epsilon$ and $\|y - x\| < \delta$, then $\|C_i y - y\| > \delta\,\epsilon$ for some C_i. (C_i depends on x.) Note that if this is false, then taking $\delta = \frac{1}{k}$, there is an $\|x_k\| \geq \epsilon$ and a y_k, $\|y_k - x_k\| < \frac{1}{k}$ such that $\|C_j y_k - y_k\| \leq \frac{1}{k}\,\epsilon$ for all j. (See Figure 8.2.) Thus, $\left\| C_j \frac{y_k}{\|x_k\|} - \frac{y_k}{\|x_k\|} \right\| \leq \frac{1}{k}$. Now there is a subsequence $\frac{x_{i_1}}{\|x_{i_1}\|}, \frac{x_{i_2}}{\|x_{i_2}\|}, \ldots$ of $\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \ldots$ that converges to, say x. Hence, $\frac{y_{i_1}}{\|x_{i_1}\|}, \frac{y_{i_2}}{\|x_{i_2}\|}, \ldots$ converges to x as well. Thus, since
$$\left\| C_j \frac{y_{i_k}}{\|x_{i_k}\|} - \frac{y_{i_k}}{\|x_{i_k}\|} \right\| \leq \frac{1}{k},$$
we have that $C_j x = x$ for all j. This implies $E(\Sigma'') \neq \{0\}$, a contradiction.

2. Let X be a trajectory of Σ''. We show that variations of segments determined by a partition on X converge to 0.

   We first need an observation. Let S be the set of all finite sequences from trajectories, determined from proper subsets of Σ''. Lemma 8.4 assures that there is a constant L such that if $z_1, \ldots, z_t$ is any such sequence, then
$$\sum_{i=1}^{t-1} \|z_{i+1} - z_i\| \leq L \max_{1\leq i\leq t-1} \|z_{i+1} - z_i\|. \tag{8.6}$$

Now partition X in segments $X_{i_1,s_1}, X_{i_2,s_2}, \ldots$ where

$$X_{i_k,s_k} \text{ is } x_{i_k}, \ldots, x_{s_k},\ i_k = s_{k-1} + 1$$

and each X_{i_k,s_k} is determined by a proper subset of Σ'' but $X_{i_k,i_{k+1}}$ is not. However, using (8.6), the variation $S(X_{i_k,s_k})$, of X_{i_k,s_k}, converges to 0 as $k \to \infty$. And using that Σ'' is VS on the last term of the expression, it follows that the variation $S\left(X_{i_k,i_{k+1}}\right)$ converges to 0 as $k \to \infty$ as well.

3. We show that X, as given in (2), converges to 0. We do this by contradiction; thus suppose X does not converge to 0. Then there is an $\epsilon > 0$ (We can take $\epsilon < 1$.) and a subsequence $x_{j_1}, x_{j_2}, \ldots$ of X such that $\|x_{j_k}\| \geq \epsilon$ for all k. But now, by (1), there is a $\delta > 0$ such that if $\|y - x_{j_k}\| < \delta$, then $\|C_j y - y\| \geq \delta\,\epsilon$ for some j. Since by (2) $S\left(X_{i_k,i_{k+1}}\right) \to 0$ as $k \to \infty$, we can take N sufficiently large so that if $k \geq N$, $S\left(X_{i_k,i_{k+1}}\right) < \delta$. Now take an interval i_k, i_{k+1} where $k \geq N$ and $i_k \leq j_k \leq i_{k+1}$. Then, every C_i occurs in the trajectory $x_{i_k}, C_{i_k}x_{i_k}, \ldots, C_{i_{k+1}} \cdots C_{i_k}x_{i_k}$, and $\|x_t - x_{j_k}\| < \delta$ for all t, $i_k \leq t \leq i_{k+1}$. Since all matrices in Σ'' are involved in $X_{i_k,i_{k+1}}$, there is a t, $i_k \leq t \leq i_{k+1}$ such that $C_j x_t = x_{t+1}$, and this C_j is as described in (1). But, $\|C_j x_t - x_t\| > \delta\,\epsilon$, which contradicts that X is VS. Hence, X must converge to 0. Since X was arbitrary, all trajectories converge to 0. Thus, there is a vector norm $\|\cdot\|$ such that $\|C_i\| < 1$ for all i (Corollary 6.4). Since all norms are equivalent, we can complete the proof using this norm, which we will do. Further, we use

$$q = \max_i \|C_i\|$$

in the remaining work.

Now, we show that Σ'' is uniformly BVS. Since each proper subset of Σ'' is uniformly BVS, there are individual L's, as given in the definition for these sets. We let L denote the largest of these numbers, and let $C = \max_j \|C_j - I\|$. Take any trajectory X of Σ'' and write X in terms of segments as in (2). Then using X_{i_1,i_2}

$$\begin{aligned}\sum_{i=i_1}^{i_2-1} \|x_{i+1} - x_i\| &= \sum_{i=i_1}^{i_2-2} \|x_{i+1} - x_i\| + \|x_{i_2} - x_{i_2-1}\| \\ &\leq L\|x_{i_1}\| + C\|x_{s_1}\| \\ &\leq L\|x_1\| + C\|x_1\| \\ &= (L+C)\|x_1\|\end{aligned}$$

since $\|C_i\| < 1$ for all i assures that $\|x_{k+1}\| \leq \|x_k\|$ for all k. And using X_{i_2,i_3}

$$\begin{aligned}\sum_{i=i_2}^{i_3-1} \|x_{i+1} - x_i\| &= \sum_{i=i_2}^{i_2-2} \|x_{i+1} - x_i\| + \|x_{i_3} - x_{i_3-1}\| \\ &\leq L\|x_{s_2}\| + C\|x_{s_2}\| \\ &\leq (L+C)\,q\,\|x_1\|.\end{aligned}$$

Continuing, we get

$$\sum_{i=i_k}^{i_{k+1}-1} \|x_{i+1} - x_i\| \leq (L+C)\,q^{k-1}\|x_1\|.$$

Finally, putting together

$$\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| = \frac{1}{1-q}(L+C)\|x_1\|$$

Thus Σ'' is uniformly BVS. ■

Theorem 8.4 *The properties γPC and uniformly BVS are equivalent.*

Proof. Suppose Σ is γPC. Then Σ is PC, and so Σ is an LCP-set. By the definition of γPC, there is a norm $\|\cdot\|$ and a $\gamma > 0$ such that

$$\|Ax\| \leq \|x\| - \gamma\|Ax - x\|$$

for all $A \in \Sigma$ and all x. Then, for any vector x_1, the trajectory $x_1, x_2, \ldots$ satisfies

$$\begin{aligned}\sum_{i=1}^{\infty} \|x_{i+1} - x_i\| &\leq \frac{1}{\gamma}\lim_{k\to\infty}\sum_{i=1}^{k}(\|x_i\| - \|x_{i+1}\|) \\ &= \frac{1}{\gamma}\lim_{k\to\infty}(\|x_1\| - \|x_{k+1}\|) \\ &\leq \frac{1}{\gamma}(2\|x_1\|) \\ &= \frac{2}{\gamma}\|x_1\|.\end{aligned}$$

Hence, Σ is uniformly BVS.

Now suppose Σ is uniformly BVS. Since any trajectory of bounded variation converges, Σ is an LCP-set. Thus, by Corollary 3.4, there is a vector norm $\|\cdot\|$, such that $\|A\| \leq 1$ for all $A \in \Sigma$. Set

$$\|x_1\|_\Sigma = \sup \sum_{i=1}^{\infty} \|x_{i+1} - x_i\|$$

where the sup is over all trajectories starting at x_1. This sup is finite since Σ is uniformly BVS. Furthermore, for any vectors y and z, we have $\|y+z\|_\Sigma \leq \|y\|_\Sigma + \|z\|_\Sigma$, and for every scalar α, $\|\alpha y\|_\Sigma \leq |\alpha| \, \|y\|_\Sigma$. Using the definition,

$$\|Ax\|_\Sigma \leq \|x\|_\Sigma - \|Ax - x\|$$

for any $A \in \Sigma$.

Now define a norm by

$$\|x\|_b = \frac{1}{2} \|x\| + \|x\|_\Sigma .$$

Then, for any $A \in \Sigma$

$$\begin{aligned} \|Ax\|_b &= \frac{1}{2} \|Ax\| + \|Ax\|_\Sigma \\ &\leq \frac{1}{2} \|x\| + (\|x\|_\Sigma - \|Ax - x\|) \\ &= \|x\|_b - \|Ax - x\| \\ &\leq \|x\|_b - \gamma \|Ax - x\|_b \end{aligned}$$

using the equivalence of norms to determine γ. Thus, Σ is an γPC-set, as required. ■

Implications between the various properties of Σ are given in Figure 8.3. The unlabeled implications are obvious.

1. This follows from Theorem 8.1.
2. This follows by Theorem 8.3.
3. This follows by Theorem 8.4.

8.3 Research Notes

The notion of paracontracting, as given in Section 1, appeared in Nelson and Neumann (1987) although Halperin (1962) and Amemiya and Ando

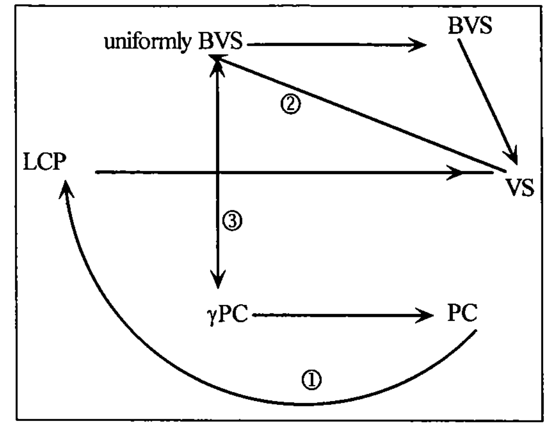

FIGURE 8.3. Relationships among the various properties.

(1965) used similar such notions in their work. Theorem 1 was given in Elsner, Koltracht, and Neumann (1990) while Theorem 2 was shown by Beyn and Elsner (1997). Beyn and Elsner also introduced the definition of γ-paracontracting.

The results of Section 2 occurred in Vladimirov, Elsner, and Beyn (2000). Gurvits (1995) provided similar such work.

9 Set Convergence

In this chapter we look at convergence, in the Hausdorff metric, of sequences of sets obtained from considering all possible outcomes in matrix products.

9.1 Bounded Semigroups

Recall, from Chapter 3, that if Σ is a product bounded subset of $n \times n$ matrices, then the limiting set for $\langle \Sigma^k \rangle$ is

$$\Sigma^\infty = \left\{ A : A \text{ is the limit of a matrix subsequence of } \langle \Sigma^k \rangle \right\}.$$

In this section, we give several results about how $\langle \Sigma^k \rangle \to \Sigma^\infty$ in the Hausdorff metric. A first such result, obtained by a standard argument, follows.

Theorem 9.1 *Let Σ be a compact subset of $n \times n$ matrices. If $\Sigma^2 \subseteq \Sigma$, then $\langle \Sigma^k \rangle$ converges to $\bigcap_{k=1}^{\infty} \Sigma^k$ in the Hausdorff metric.*

Now let

$$\hat{\Sigma}^\infty = \left\{ A \in M_n : A \text{ is the limit of a matrix sequence of } \langle \Sigma^k \rangle \right\}.$$

If $\Sigma^\infty = \hat{\Sigma}^\infty$, we call Σ^∞ the *strong limiting set* of $\langle \Sigma^k \rangle$. When Σ^∞ is a strong limiting set is given below.

Theorem 9.2 *Let Σ be a compact product bounded subset of M_n. Then Σ^∞ is the strong limiting set of $\langle \Sigma^k \rangle$ iff $\langle \Sigma^k \rangle$ converges to Σ^∞ in the Hausdorff metric.*

Proof. For the direct implication suppose that Σ^∞ is the strong limiting set for $\langle \Sigma^k \rangle$. We prove that $\langle \Sigma^k \rangle$ converges to Σ^∞ in the Hausdorff metric by contradiction. Thus, suppose there is an $\epsilon > 0$ such that $h\left(\Sigma^k, \Sigma^\infty\right) > \epsilon$ for infinitely many k's. We look at cases.

Case 1. Suppose $\partial\left(\Sigma^k, \Sigma^\infty\right) > \epsilon$ for infinitely many k's. From these Σ^k's, we can find matrix products $\pi_{k_1}, \pi_{k_2}, \ldots$ such that

$$d\left(\pi_{k_i}, \Sigma^\infty\right) > \epsilon$$

for all i. Since Σ is product bound, there is a subsequence $\pi_{j_1}, \pi_{j_2}, \ldots$ of π_{k_1}, π_{k_2} that converges, say to π. But by definition, $\pi \in \Sigma^\infty$, and yet we have that

$$d\left(\pi, \Sigma^\infty\right) \geq \epsilon,$$

a contradiction.

Case 2. Suppose $\partial\left(\Sigma^\infty, \Sigma^k\right) > \epsilon$ for infinitely many k's. In this case the Σ^k's yield a sequence $\pi_{k_1}, \pi_{k_2}, \ldots$ in Σ^∞ such that $d\left(\pi_{k_i}, \Sigma^{k_i}\right) > \epsilon$ for all i. Since Σ^∞ is bounded, $\pi_{k_1}, \pi_{k_2}, \ldots$ has subsequence which converges to, say, π. Thus $d\left(\pi, \Sigma^{k_i}\right) > \frac{\epsilon}{2}$ for all i sufficiently large. But this means that π is not the limit of a matrix sequence of $\langle \Sigma^k \rangle$, a contradiction.

Since both of these cases lead to contradictions, it follows that Σ^k converges to Σ^∞ in the Hausdorff metric.

Conversely, suppose that Σ^k converges to Σ^∞ in the Hausdorff metric. We need to show that Σ^∞ is the strong limiting set of $\langle \Sigma^k \rangle$.

Let $\pi \in \Sigma^\infty$. Since $h\left(\Sigma^k, \Sigma^\infty\right) \to 0$ as $k \to \infty$, we can find a sequence $\pi_1, \pi_2, \ldots$, taken from $\Sigma^1, \Sigma^2, \ldots$, such that $\langle \pi_k \rangle$ converges to π. Thus, π is the limit of a matrix sequence of $\langle \Sigma^k \rangle$ and thus $\pi \in \hat{\Sigma}^\infty$. Hence, $\Sigma^\infty \subseteq \hat{\Sigma}^\infty$.

Finally, it is clear that $\hat{\Sigma}^\infty \subseteq \Sigma^\infty$ and thus $\Sigma^\infty = \hat{\Sigma}^\infty$. It follows that Σ^∞ is the strong limiting set of $\langle \Sigma^k \rangle$. ■

For a stronger result, we define, for an LCP-set Σ, the set L which is the closure of all of its infinite products, that is,

$$L = \overline{\left\{ \prod_{k=1}^{\infty} A_{i_k} : A_{i_k} \in \Sigma \text{ for all } k \right\}}.$$

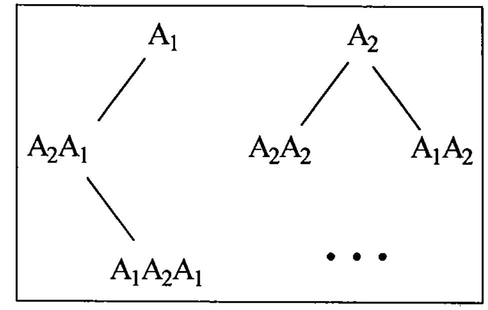

FIGURE 9.1. A possible tree graph of G.

Theorem 9.3 *Let $\Sigma = \{A_1, \ldots, A_m\}$ be an LCP-set. Then*

$$\Sigma^\infty = L.$$

Proof. By definition, $L \subseteq \Sigma^\infty$. To show $\Sigma^\infty \subseteq L$, we argue by contradiction.

Suppose $\pi \in \Sigma^\infty$ where $d(\pi, L) > \epsilon$, $\epsilon > 0$. Define a graph G with vertices all products $A_{i_k} \cdots A_{i_1}$ such that there are matrices $A_{i_t}, \ldots, A_{i_{k+1}}, t > k$, satisfying

$$d\left(A_{i_t} \ldots A_{i_k} \ldots A_{i_1}, \pi\right) < \epsilon.$$

Since $\pi \in \Sigma^\infty$, there are infinitely many such $A_{i_k} \cdots A_{i_1}$.

If $A_{i_{k+1}} \cdots A_{i_1}$ is in G, then so is $A_{i_k} \cdots A_{i_1}$, and we define an arc $\left(A_{i_k} \ldots A_{i_1}, A_{i_{k+1}} \ldots A_{i_1}\right)$ from $A_{i_k} \ldots A_{i_1}$ to $A_{i_{k+1}} \ldots A_{i_1}$. This defines a tree, e.g., Figure 9.1. Thus, $S_k = \{A_{i_k} \ldots A_{i_1} : A_{i_k} \ldots A_{i_1} \in G\}$ is the k-th strata of G. Since this tree satisfies the hypothesis of König's infinity lemma (See the Appendix.), there is a product $\prod_{k=1}^{\infty} A_{i_k}$ such that each $A_{i_k} \ldots A_{i_1} \in G$ for all k. Thus,

$$d\left(\prod_{k=1}^{\infty} A_{i_k}, \pi\right) \leq \epsilon$$

which contradicts that

$$d(\pi, L) > \epsilon$$

and from this, the theorem follows. ■

Corollary 9.1 *If Σ is a finite subset of M_n and Σ is an LCP-set, then $\langle \Sigma^k \rangle$ converges to Σ^∞ in the Hausdorff metric.*

Proof. Note that since $L \subseteq \hat{\Sigma}^\infty \subseteq \Sigma^\infty$, and by the theorem $L = \Sigma^\infty$, we have that $\Sigma^\infty = \hat{\Sigma}^\infty$, from which the result follows from Theorem 3.14 and Theorem 9.2. ∎

9.2 Contraction Coefficient Results

We break this section into two subsections.

9.2.1 *Birkhoff Coefficient Results*

Let Σ denote a set of $n \times n$ row allowable matrices. Let U be a subset of $n \times 1$ positive vectors. In this section, we see when the sequence $\langle \Sigma^k U \rangle$ converges, at least in the projective sense.

To do this, recall from Chapter 2 that S^+ denotes the set of $n \times 1$ positive stochastic vectors. And for each $A \in \Sigma$, recall that $w_A(x) = \frac{Ax}{\|Ax\|_1}$, $\Sigma_p = \{w_A : A \in \Sigma\}$, and for any $U \subseteq S^+$

$$\Sigma_p U = \{w_A(x) : w_A \in \Sigma_p \text{ and } x \in U\}.$$

We intend to look at the convergence of these projected sets. If U and Σ are compact, then so is $\Sigma_p U$. Thus, since p is a metric on S^+, we can use it to define the Hausdorff metric h, and thus measure the difference between two sets, say $\Sigma_p U$ and $\Sigma_p V$ of S^+.

A lemma in this regard follows.

Lemma 9.1 *Let Σ be a compact set of row allowable $n \times n$ matrices. If U and V are nonempty compact subsets of S^+, then*

$$h(\Sigma_p U, \Sigma_p V) \le \tau_B(\Sigma)\, h(U, V).$$

Proof. As in Theorem 2.9. ∎

For the remaining work, we will assume the following.

1. Σ is a compact set of row allowable matrices.
2. There is a positive number m such that for any $A \in \Sigma$

$$m \le \min_{a_{ij}>0} a_{ij} \le \max a_{ij} \le 1.$$

(Scaling A will not affect projected distances.)

3. There is a positive integer r such that all r-blocks from Σ are positive.

By (3) it is clear that each matrix in Σ has nonzero columns. Thus for any $A \in \Sigma$, w_A is defined on S. And

$$\Sigma_p S \subseteq S$$
$$\Sigma_p^2 S \subseteq \Sigma S \subseteq S.$$

Thus, we can define

$$L = \bigcap_{k=1}^{\infty} \Sigma_p^k S$$

a compact set of positive stochastic vectors. The following is a rather standard argument.

Lemma 9.2 *The sequence $\langle \Sigma_p^k S \rangle$ converges to L in the Hausdorff metric.*

Using this lemma, we have the following.

Lemma 9.3 $h(\Sigma_p L, L) = 0$.

Proof. Using Lemma 9.1, for any $k \ge 1$ we get

$$\begin{aligned} h(\Sigma_p L, L) &\le h(\Sigma_p L, \Sigma_p^k S) + h(\Sigma_p^k S, L) \\ &\le h(L, \Sigma_p^{k-1} S) + h(\Sigma_p^k S, L). \end{aligned}$$

Thus, by the previous lemma, taking the limit as $k \to \infty$, we have the equation

$$h(\Sigma_p L, L) = 0,$$

the desired result. ∎

Theorem 9.4 *Let U be a compact subset of S and Σ, as described in 1 through 3. If $\tau_B(\pi) \le \tau_r$ for all r-blocks π of Σ, then*

$$h(\Sigma_p^k U, L) \le \tau_r^{\left[\frac{k}{r}\right]} h(U, L).$$

Thus, if $\tau_r < 1$, $\Sigma_p^k U \to L$ with geometric rate.

Proof. Using Lemma 9.1 and Lemma 9.3,

$$\begin{aligned} h\left(\Sigma_p^k U, L\right) &= h\left(\Sigma_p^k U, \Sigma_p^k L\right) \\ &= h\left(\Sigma_p^r\left(\Sigma_p^{k-r} U\right), \Sigma_p^r\left(\Sigma_p^{k-r} L\right)\right) \\ &\leq \tau_r h\left(\Sigma_p^{k-r} U, \Sigma_p^{k-r} L\right) \\ &\cdots \\ &\leq \tau_r^{\left[\frac{k}{r}\right]} h\left(U, L\right). \end{aligned}$$

This proves the theorem. ∎

9.2.2 *Subspace Coefficient Results*

Let Σ be a compact, τ-proper product bounded set of $n \times n$ matrices. Since Σ is product bounded by Theorem 3.12, there is a vector norm $\|\cdot\|$ such that $\|A\| \leq 1$ for all $A \in \Sigma$. Let τ_W be the corresponding subspace contractive coefficient.

Let $x_0 \in F^n$ and $G = x_0 E$. Then

$$x_0 + W = \{x \in F^n : xE = G\}.$$

We suppose $S \subseteq x_0 + W$ such that

$$S\Sigma \subseteq S.$$

(For example, $S = \{x : \|x\| \leq 1\}$.) Then

$$S\Sigma^2 \subseteq S\Sigma \subseteq S$$

and we define

$$L = \cap S\Sigma^k.$$

Then mimicking the results of the previous section, we end with the following.

Theorem 9.5 *Let U be a compact subset of S. If $\tau_W(\pi) \leq \tau_r$ for all r-blocks π of Σ and $\tau_r < 1$, then $U\Sigma^k \to L$ at a geometric rate.*

A common situation in which this theorem arises is when we have $E = e$, Σ the set of stochastic matrices, and S the set of stochastic vectors.

We conclude this section with a result which is a bit stronger than the previous one.

Theorem 9.6 *If $\tau_W(\Sigma^r) < 1$ for some integer r, then $\Sigma^k \to \Sigma^\infty$ at a geometric rate.*

Proof. Define $W = \{B \in M_n : BE = 0\}$ and let

$$S = \{B \in I + W : \|B\| \leq 1\}.$$

Then $S\Sigma \subseteq S$ and L follows. Now use the 1-norm so that

$$\|BA\|_1 \leq \tau_W(A)\|B\|_1$$

for all $B \in S$, and mimic the previous results. Finally, use $U = \{I\}$ and the equivalence of norms. ∎

9.3 Convexity in Convergence

To compute $\Sigma^k U$ and $\Sigma_p^k U$, it is helpful to know when these sets are convex. In these cases, we can compute the sets by computing their vertices. Thus in this section, we discuss when $\Sigma^k U$ and $\Sigma_p^k U$ are convex.

A matrix set Σ is *column convex* if whenever $A, B \in \Sigma$, the matrix $[\alpha_1 a_1 + \beta_1 b_1, \ldots, \alpha_n a_n + \beta_n b_n]$ of convex sums of corresponding columns, is in Σ. Column convex sets can map convex sets to convex sets.

Theorem 9.7 *Let Σ be a column convex matrix set of row allowable matrices. If U is a convex set of nonnegative vectors, then ΣU is a convex set of nonnegative vectors.*

Proof. Let Ax, $By \in \Sigma U$ where $A, B \in \Sigma$ and $x, y \in U$. We show the convex sum $\alpha Ax + \beta By \in \Sigma U$.

Define

$$K = (\alpha AX + \beta BY)(\alpha X + \beta Y)^+ + R$$

where $X = \operatorname{diag}(x_1, \ldots, x_n)$, $Y = \operatorname{diag}(y_1, \ldots, y_n)$, $(\alpha X + \beta Y)^+$ the generalized inverse of $\alpha X + \beta Y$, and R such that

$$r_{ij} = \begin{cases} 0 \text{ if } \alpha x_j + \beta y_j > 0 \\ a_{ij} \text{ otherwise.} \end{cases}$$

Using that a_j, b_j denote the j-th columns of A and B, respectively, the j-th column of K is

$$\frac{\alpha a_j x_j + \beta b_j y_j}{\alpha x_j + \beta y_j} = \frac{\alpha x_j}{\alpha x_j + \beta y_j} a_j + \frac{\beta y_j}{\alpha x_j + \beta y_j} b_j$$

if $\alpha x_j + \beta y_j > 0$ and a_j if $\alpha x_j + \beta y_j = 0$. Thus, $K \in \Sigma$. Furthermore, for $e = (1, 1, \ldots, 1)^t$,

$$\begin{aligned} K(\alpha x + \beta y) &= K(\alpha X + \beta Y)e \\ &= (\alpha AX + \beta BY)e \\ &= \alpha Ax + \beta By \end{aligned}$$

which is in ΣU. From this, the result follows. ■

Applying the theorem more than once yields the following corollary.

Corollary 9.2 *Using the hypotheses of the theorem, $\Sigma^k U$ is convex for all $k \geq 1$.*

The companion result for Σ_p uses the following lemma.

Lemma 9.4 *If U is a convex subset of nonnegative vectors, none of which are zero, then $U_p = \left\{\frac{u}{\|u\|_1} : u \in U\right\}$ is a convex subset of stochastic vectors.*

Proof. Let $\frac{x}{\|x\|_1}, \frac{y}{\|y\|_1} \in U_p$ where $x, y \in U$. Then any convex sum $\alpha x + \beta y \in U$. Thus, $\frac{\alpha x + \beta y}{\|\alpha x + \beta y_1\|} \in U_p$ and

$$\frac{\alpha x + \beta y}{\|\alpha x + \beta y\|_1} = \frac{\alpha \|x\|_1}{\|\alpha x + \beta y\|_1} \frac{x}{\|x\|_1} + \frac{\beta \|y\|_1}{\|\alpha x + \beta y\|_1} \frac{y}{\|y\|_1}$$

is a convex sum of $\frac{x}{\|x\|_1}, \frac{y}{\|y\|_1} \in U_p$. And when $\alpha = 0$, the vector is $\frac{y}{\|y\|_1}$, while when $\beta = 0$, it is $\frac{x}{\|x\|_1}$. Thus, we see that all vectors between $\frac{x}{\|x\|_1}$ and $\frac{y}{\|y\|_1}$ are in U_p. So U_p is convex. ■

As a consequence, we have the following theorem.

Theorem 9.8 *Let Σ be a column convex matrix set of row allowable matrices. If U is a convex set of positive stochastic vectors, then $\Sigma_p^k U$ is a convex set.*

Proof. Using the previous corollary and lemma and that $\Sigma_p^k U$ is the projection of $\Sigma^k U$ to norm 1 vectors, since $\Sigma^k U$ is convex, so is $\Sigma_p^k U$. ■

It is known (Eggleston, 1969) and easily shown, that the limit of convex sets, assuming the limit exists, is itself convex. Thus, the previous two theorems can be extended to show $\Sigma_p^\infty U$ is convex.

Actually, we would like to know about the vertices of these sets. The following theorem is easily shown.

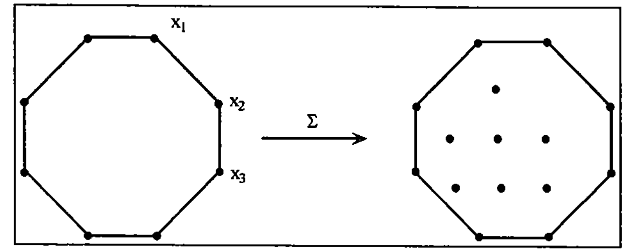

FIGURE 9.2. A view of ΣU.

Theorem 9.9 *If $\Sigma = \text{convex}\{A_1, \ldots, A_s\}$ is a column convex matrix set of nonnegative matrices and $U = \text{convex}\{x_1, \ldots, x_t\}$ a set of nonnegative vectors, then*

$$\Sigma U = \text{convex}\{A_i x_j : 1 \leq i \leq s, 1 \leq j \leq t\}.$$

Not all vectors $A_i x_j$ need be vertices of ΣU. The appearance may be as it appears in Figure 9.2. The more intricate theorem to prove uses the following lemma.

Lemma 9.5 *Let U be a subset of nonnegative, nonzero, vectors. If*

$$U = \text{convex}\{x_1, \ldots, x_t\}$$

then

$$U_p = \text{convex}\left\{\frac{x_1}{\|x_1\|_1}, \ldots, \frac{x_t}{\|x_t\|_1}\right\}.$$

Proof. Let $x = \alpha_1 x_1 + \cdots + \alpha_t x_t$ be a convex sum. Then

$$\begin{aligned} \frac{x}{\|x\|_1} &= \frac{\sum_{k=1}^t \alpha_k x_k}{\|x\|_1} \\ &= \sum_{k=1}^t \left(\frac{\alpha_k \|x_k\|_1}{\|x\|_1}\right) \frac{x_k}{\|x_k\|_1}. \end{aligned}$$

Since $\sum_{k=1}^t \left(\frac{\alpha_k \|x_k\|_1}{\|x\|_1}\right) = \frac{\|x\|_1}{\|x\|_1} = 1$, it follows that $\frac{x}{\|x\|_1}$ is a convex sum of vectors listed in U_p. That U_p is convex follows from Lemma 9.4. ■

The theorem follows.

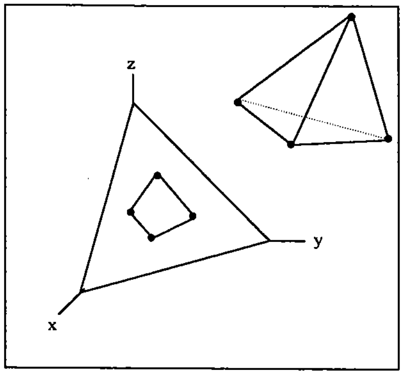

FIGURE 9.3. A projected symplex.

Theorem 9.10 *Let $\Sigma = \text{convex}\{A_1, \ldots, A_s\}$, a column convex matrix set of row allowable matrices, and $U = \text{convex}\{x_1, \ldots, x_t\}$, containing positive vectors. Then*

$$(\Sigma U)_p = \text{convex}\left\{ \frac{A_i x_j}{\|A_i x_j\|_1} : 1 \leq i \leq s,\ 1 \leq j \leq t \right\}.$$

Proof. The proof is an application of the previous theorem and lemma. ■

We give a view of this theorem in Figure 9.3.

9.4 Research Notes

Section 1 extends the work of Chapter 3 to sets. Section 2 is basically contained in Seneta (1984) which, in turn, used previously developed material from Seneta and Sheridan (1981).

Computing, or estimating, the limiting set can be a problem. In Chapters 11 and 13, we show how, in some cases, this can be done. In Hartfiel (1995,1996), iterative techniques for finding component bounds on the vectors in the limiting set are given. Both papers, however, are for special matrix sets. There is no known method for finding component bounds in general.

Much of the work in Section 4 generalizes that of Hartfiel (1998).

10
Perturbations in Matrix Sets

Let Σ and $\hat{\Sigma}$ be compact subsets of M_n. In this chapter we show conditions assuring that when Σ and $\hat{\Sigma}$ are close, so are $X\Sigma^\infty$ and $Y\hat{\Sigma}^\infty$.

10.1 Subspace Coefficient Results

Let Σ and $\hat{\Sigma}$ be product bounded compact subsets of M_n. We suppose that Σ and $\hat{\Sigma}$ are τ-proper, $E(\Sigma) = E\left(\hat{\Sigma}\right)$, and that τ_W is a corresponding contraction coefficient as described in Section 7.3. Also, we suppose that $S \subseteq F^n$ such that

1. $S \subseteq x_0 + W$ for some vector x_0,
2. $S\Sigma \subseteq S$, $S\hat{\Sigma} \subseteq S$.

Our perturbation result of this section uses the following lemma.

Lemma 10.1 *Let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be sequences of matrices taken from Σ and $\hat{\Sigma}$, respectively. Suppose $\tau_W(A_k) \leq \tau$ and $\|A_k - B_k\| \leq \epsilon$ for all k. Then*

$$\|xA_1 \cdots A_k - yB_1 \cdots B_k\| \leq \tau^k \|x - y\| + \left(\tau^{k-1} + \cdots + 1\right) \beta\epsilon$$

where $x, y \in S$ *and* $\beta = \sup_i \|yB_1 \cdots B_i\|$.

Proof. The proof is done by induction on k.
If $k = 1$,

$$\begin{aligned}\|xA_1 - yB_1\| &\le \|xA_1 - yA_1\| + \|yA_1 - yB_1\| \\ &\le \tau_W(A_1)\|x-y\| + \|y\|\|A_1 - B_1\| \\ &\le \tau\|x-y\| + \beta\epsilon.\end{aligned}$$

Assume the result holds for $k-1$ matrices. Then

$$\begin{aligned}&\|xA_1 \cdots A_k - yB_1 \cdots B_k\| \le \|xA_1 \cdots A_{k-1}A_k - yB_1 \cdots B_{k-1}A_k\| \\ &+ \|yB_1 \cdots B_{k-1}A_k - yB_1 \cdots B_{k-1}B_k\| \\ &\le \tau_W(A_k)\|xA_1 \cdots A_{k-1} - yB_1 \cdots B_{k-1}\| \\ &+ \|yB_1 \cdots B_{k-1}\|\|A_k - B_k\| \\ &\le \tau\|xA_1 \cdots A_{k-1} - yB_1 \cdots B_{k-1}\| + \beta\epsilon,\end{aligned}$$

and by the induction hypothesis, this leads to

$$\begin{aligned}&\le \tau\left(\tau^{k-1}\|x-y\| + \left(\tau^{k-2} + \cdots + 1\right)\beta\epsilon\right) + \beta\epsilon \\ &= \tau^k\|x-y\| + \left(\tau^{k-1} + \cdots + 1\right)\beta\epsilon.\end{aligned}$$

The perturbation result follows. ∎

Theorem 10.1 *Suppose* $\tau_W(\Sigma) \le \tau$ *and* $\tau_W\left(\hat{\Sigma}\right) \le \tau$. *Let* X *and* Y *be compact subsets of* F^n *such that* $X, Y \subseteq S$. *Then*

1. $h\left(X\Sigma^k, Y\hat{\Sigma}^k\right) \le \tau^k h(X,Y) + \left(\tau^{k-1} + \cdots + 1\right) h\left(\Sigma, \hat{\Sigma}\right)\beta$, *where* $\beta = \max\left(\sup\|xA_1 \cdots A_i\|, \sup\|yB_1 \cdots B_i\|\right)$ *and the sup is over all* $x \in X, y \in Y$, *and all* $A_1, \ldots, A_i \in \Sigma$ *and* $B_1, \ldots, B_i \in \hat{\Sigma}$.

If $\tau < 1$, *then*

2. $h\left(X\Sigma^\infty, Y\hat{\Sigma}^\infty\right) \le \frac{1}{1-\tau} h\left(\Sigma, \hat{\Sigma}\right)\beta$.

Proof. To prove (1), let $A_1 \cdots A_k \in \Sigma^k$ and $x \in X$. Take $y \in Y$ such that $\|x - y\| \le h(X,Y)$. Take $B_1, \ldots, B_k$ in $\hat{\Sigma}$ such that $\|A_i - B_i\| \le h\left(\Sigma, \hat{\Sigma}\right)$ for all i. Then, by the previous lemma,

$$\|xA_1 \cdots A_k - yB_1 \cdots B_k\| \le \tau^k\|x-y\| + \left(\tau^{k-1} + \cdots + 1\right) h\left(\Sigma, \hat{\Sigma}\right)\beta.$$

So,

$$\delta\left(X\Sigma^k, Y\hat{\Sigma}^k\right) \le \tau^k h(X,Y) + \left(\tau^{k-1} + \cdots + 1\right) h\left(\Sigma, \hat{\Sigma}\right)\beta.$$

Similarly,

$$\delta\left(Y\hat{\Sigma}^k, X\Sigma^k\right) \le \tau^k h(Y,X) + \left(\tau^{k-1} + \cdots + 1\right) h\left(\Sigma, \hat{\Sigma}\right)\beta.$$

Thus,

$$h\left(X\Sigma^k, Y\hat{\Sigma}^k\right) \le \tau^k h(X,Y) + \left(\tau^{k-1} + \cdots + 1\right) h\left(\Sigma, \hat{\Sigma}\right)\beta,$$

which yields (1).

For (2), Theorem 9.6 assures that Σ^∞ and $\hat{\Sigma}^\infty$ exist. Thus, (2) is obtained from (1) by calculating the limit as $k \to \infty$. ∎

For r-blocks, we have the following.

Corollary 10.1 *Suppose* $\tau_W(\Sigma^r) \le \tau_r$ *and* $\tau_W\left(\hat{\Sigma}^r\right) \le \tau_r$. *Let* X *and* Y *be compact subsets of* F^n *such that* $X, Y \subseteq S$. *Then*

$$h\left(X\Sigma^k, Y\hat{\Sigma}^k\right) \le \tau_r^{\left[\frac{k}{r}\right]} M_{XY} + \left(\tau_r^{\left[\frac{k}{r}\right]-1} + \cdots + 1\right) h\left(\Sigma^r, \hat{\Sigma}^r\right)\beta,$$

where $M_{XY} = \max_{0 \le t < r} h\left(X\Sigma^t, Y\hat{\Sigma}^t\right)$ *and* β *as given in the theorem.*

Proof. The proof mimics that of the theorem where we block the products. The blocking of the products can be done as in the example

$$A_1 \cdots A_k = A_1 \cdots A_s B_1 \cdots B_q$$

where $k = rq + s$. ∎

A consequence of this theorem is that we can approximate Σ^∞ by a $\hat{\Sigma}^\infty$, where $\hat{\Sigma}$ is finite. And, in doing this our finite results can be used on $\hat{\Sigma}$.

10.2 Birkhoff Coefficient Results

Let Σ be a matrix set of row allowable matrices. In this section, we develop some perturbation results for Σ_p. Before doing this, we show several basic results about projection maps.

Equality of two projective maps is given in the following lemma.

Lemma 10.2 *For projective maps, $w_A = w_B$ iff $p(Ax, Bx) = 0$ for all $x \in S^+$.*

Proof. Suppose

$$w_A(x) = w_B(x)$$

for all $x \in S^+$. Then

$$\frac{Ax}{\|Ax\|_1} = \frac{Bx}{\|Bx\|_1}$$

and

$$p(Ax, Bx) = 0$$

for all $x \in S^+$.

Conversely, suppose $p(Ax, Bx) = 0$ for all $x \in S^+$. Then

$$Ax = c(x) Bx$$

where $c(x)$ is a constant for each x. Thus

$$\frac{Ax}{\|Ax\|_1} = c(x) \frac{\|Bx\|_1}{\|Ax\|_1} \frac{Bx}{\|Bx\|_1}.$$

Since $\frac{Ax}{\|Ax\|_1}$ and $\frac{Bx}{\|Bx\|_1}$ are stochastic vectors,

$$e \frac{Ax}{\|Ax\|_1} = ec(x) \frac{\|Bx\|_1}{\|Ax\|_1} \frac{Bx}{\|Bx\|_1}$$

where $e = (1, 1, \ldots, 1)$, or

$$1 = c(x) \frac{\|Bx\|_1}{\|Ax\|_1}.$$

It follows that

$$\frac{Ax}{\|Ax\|_1} = \frac{Bx}{\|Bx\|_1}$$

or

$$w_A(x) = w_B(x).$$

Hence $w_A = w_B$. ■

An example follows.

Example 10.1 *We can show by direct calculation, if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, then $w_A = w_B$, while if $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$, then $w_A \neq w_B$.*

Let Σ be a compact set of row allowable $n \times n$ matrices such that if $A, B \in \Sigma$, then for corresponding signum matrices we have $A^* = B^*$, i.e. A and B have the same 0-pattern. Define

$$\Sigma_p = \{w_A : A \in \Sigma\}.$$

If $w_A, w_B \in \Sigma_p$, then

$$w_B \circ w_A(x) = \frac{B\left(\frac{Ax}{\|Ax\|_1}\right)}{\left\|B\left(\frac{Ax}{\|Ax\|_1}\right)\right\|_1} = \frac{BAx}{\|BAx\|_1},$$

a projective map. And in general,

$$w_{A_{i_k}} \circ \cdots \circ w_{A_{i_1}}(x) = \frac{A_{i_k} \cdots A_{i_1} x}{\|A_{i_k} \cdots A_{i_1} x\|_1}.$$

We define a metric on Σ_p as follows. If $w_A, w_B \in \Sigma_p$, then

$$p(w_A, w_B) = \sup_{x \in S^+} p(Ax, Bx).$$

A formula for p in terms of the entries of A and B follows.

Theorem 10.2 *For projective maps w_A and w_B, we have that $p(w_A, w_B) = \max_{i,j,r,s} \ln \frac{a_{ir} b_{js} + a_{is} b_{jr}}{b_{ir} a_{js} + b_{is} a_{jr}}$, where the quotient contains only positive entries.*

Proof. By definition,

$$p(w_A, w_B) = \sup_{x>0} p(Ax, Bx) = \sup_{x>0} \max_{i,j} \ln \frac{a_i x}{b_i x} \frac{b_j x}{a_j x}$$

where a_k, b_k are the k-th rows of A, B, respectively. Now

$$\begin{aligned}\frac{a_i x}{b_i x}\frac{b_j x}{a_j x} &= \frac{a_{i1}x_1 + \cdots + a_{in}x_n}{b_{i1}x_1 + \cdots + b_{in}x_n}\frac{b_{j1}x_1 + \cdots + b_{jn}x_n}{a_{j1}x_1 + \cdots + a_{jn}x_n} \\ &= \frac{\sum_{r,s}\left(a_{ir}b_{js} + a_{is}b_{jr}\right)x_r x_s}{\sum_{r,s}\left(b_{ir}a_{js} + b_{is}a_{jr}\right)x_r x_s} \\ &\le \max_{r,s}\frac{a_{ir}b_{js} + a_{is}b_{jr}}{b_{ir}a_{js} + b_{is}a_{jr}}.\end{aligned}$$

That equality holds is seen from this inequality by letting $x_r = x_s = t$ and $x_i = \frac{1}{t}$ for $i \neq r, s$ and letting $t \to \infty$. ■

For intervals of matrices in Σ, $p\left(w_A, w_B\right)$ can be bounded as follows.

Corollary 10.2 *If*

$$A - \epsilon A \le B \le A + \epsilon A$$

for some $\epsilon > 0$ *and* $A - \epsilon A, A + \epsilon A \in \Sigma$, *then*

$$p\left(w_A, w_B\right) \le \ln\frac{1+\epsilon}{1-\epsilon}.$$

Proof. By the theorem,

$$\begin{aligned}p\left(w_A, w_B\right) &= \max_{i,j,r,s}\ln\frac{a_{ir}b_{js} + a_{is}b_{jr}}{b_{ir}a_{js} + b_{is}a_{jr}} \\ &\le \max_{i,j,r,s}\ln\frac{a_{ir}\left(1+\epsilon\right)a_{js} + a_{is}\left(1+\epsilon\right)a_{jr}}{\left(1-\epsilon\right)a_{ir}a_{js} + \left(1-\epsilon\right)a_{is}a_{jr}} \\ &= \ln\frac{\left(1+\epsilon\right)}{\left(1-\epsilon\right)},\end{aligned}$$

the desired result. ■

Actually, (Σ_p, p) is a complete metric space which is also compact.

Theorem 10.3 *The metric space* (Σ_p, p) *is complete and compact.*

Proof. To show that Σ_p is complete, let $w_{A_1}, w_{A_2}, \ldots$ be a Cauchy sequence in Σ_p. Since $A_1, A_2, \ldots$ are in Σ, and Σ is compact, there is a subsequence $A_{i_1}, A_{i_2}, \ldots$ of this sequence that converges to, say $A \in \Sigma$. So by Theorem 10.2, $p\left(w_{A_{i_k}}, w_A\right) \to 0$ as $k \to \infty$.

We now show that $p\left(w_{A_i}, w_A\right) \to 0$ as $i \to \infty$, thus showing (Σ_p, p) is complete. For this, let $\epsilon > 0$. Then there is an $N > 0$, such that if $i, j > N$,

$$p\left(w_{A_i}, w_{A_j}\right) < \epsilon.$$

Thus, if $i_k > N$,

$$p\left(w_{A_{i_k}}, w_{A_j}\right) < \epsilon,$$

and so, letting $k \to \infty$, yields

$$p\left(w_A, w_{A_j}\right) \le \epsilon.$$

But, this says that $w_{A_j} \to w_A$ as $j \to \infty$ which is what we want to show.

To show that Σ_p is also compact is done in the same way. ■

We now give the perturbation result of this section. Mimicking the proof of Theorem 10.1, we can prove the following perturbation results.

Theorem 10.4 *Let* Σ *and* $\hat{\Sigma}$ *be compact subsets of positive matrices in* M_n. *Suppose* $\tau_B(\Sigma) \le \tau$ *and* $\tau_B\left(\hat{\Sigma}\right) \le \tau$. *Let* X *and* Y *be compact subsets of positive stochastic vectors. Then, using* p *as the metric for* h,

1. $h\left(\Sigma_p^k X, \hat{\Sigma}_p^k Y\right) \le \tau^k h(X, Y) + \left(\tau^{k-1} + \cdots + 1\right) h\left(\Sigma_p, \hat{\Sigma}_p\right)$.

And if $\tau < 1$,

2. $h\left(L, \hat{L}\right) \le \frac{1}{1-\tau} h\left(\Sigma_p, \hat{\Sigma}_p\right)$ *and by definition* $L = \lim_{k\to\infty}\Sigma_p^k X$ *and* $\hat{L} = \lim_{k\to\infty}\hat{\Sigma}_p^k Y$.

Converting to an r-block result, we have the following.

Corollary 10.3 *Let* Σ *and* $\hat{\Sigma}$ *be compact subsets of row allowable matrices in* M_n. *Suppose* $\tau_B(\Sigma^r) \le \tau_r$, $\tau_B\left(\hat{\Sigma}^r\right) \le \tau_r$. *Let* X *and* Y *be compact subsets of stochastic vectors. Then,* $h\left(\Sigma_p^k X, \hat{\Sigma}_p^k Y\right) \le \tau_r^{\left[\frac{k}{r}\right]} M_{XY} + \left(\tau^{\left[\frac{k}{r}\right]-1} + \cdots + 1\right) h\left(\Sigma_p^r, \hat{\Sigma}_p^r\right)$ *where* $M_{XY} = \max_{0\le t<r} h\left(\Sigma_p^t X, \hat{\Sigma}_p^t Y\right)$.

Computing $\tau_B\left(\Sigma^k\right)$, especially when k is large, can be a problem. If there is a $B \in \Sigma$ such that the matrices in Σ have pattern B and

$$B \le A$$

for all $A \in \Sigma$, then some bound on $\tau_B\left(\Sigma^k\right)$ can be found somewhat easily. To see this, let

$$RE = \max_{\substack{A\in\Sigma \\ b_{ij}>0}} \frac{a_{ij}-b_{ij}}{b_{ij}}$$

the largest relative error in the entries of B and the A's in Σ. Then we have the following.

Theorem 10.5 *If $B^k > 0$, then*

$$\tau_B\left(A_{i_k}\cdots A_{i_1}\right) \leq \frac{1-\frac{1}{(1+RE)^k}\sqrt{\varphi\left(B^k\right)}}{1+\frac{1}{(1+RE)^k}\sqrt{\varphi\left(B^k\right)}}.$$

Proof. Note that

$$B \leq A_i \leq B + (RE)\,B$$

for all i, and

$$B^k \leq A_{i_k}\cdots A_{i_1} \leq (1+RE)^k\,B^k$$

for all $i_1,\ldots,i_k$. Then if $A = A_{i_k}\cdots A_{i_1}$ and $\varphi(A) = \frac{a_{ij}a_{rs}}{a_{rj}a_{is}}$,

$$\varphi(A) \geq \frac{b_{ij}^{(k)}b_{rs}^{(k)}}{(1+RE)^{2k}\,b_{rj}^{(k)}b_{is}^{(k)}} \geq \frac{1}{(1+RE)^{2k}}\varphi\left(B^k\right).$$

So,

$$\tau_B(A) = \frac{1-\sqrt{\varphi(A)}}{1+\sqrt{\varphi(A)}} \leq \frac{1-\frac{1}{(1+RE)^k}\sqrt{\varphi\left(B^k\right)}}{1+\frac{1}{(1+RE)^k}\sqrt{\varphi\left(B^k\right)}},$$

the desired result. ■

10.3 Research Notes

The work in this chapter is new. To some extent, the chapter contains theoretical results which parallel those in Hartfiel (1998).

11 Graphics

This chapter shows how to use infinite products of matrices to draw curves and construct fractals. Before looking at some graphics, we provide a section developing the techniques we use.

11.1 Maps

In this section, we outline the general methods we use to obtain the graphics in this chapter.

Mathematically, we take an $n \times k$ matrix X (corresponding to points in R^2) and a finite set Σ of $n \times n$ matrices. To obtain the graphic, we need to compute $\Sigma^\infty X$ and plot the corresponding points in R^2.

We will use the subspace coefficient τ_W to show that the sequence $\left\langle\Sigma^k\right\rangle$ converges in the Hausdorff metric. To compute the limiting set, $\Sigma^\infty X$, it will be sufficient to compute $\Sigma^s X$ for a 'reasonable' s.

To compute $\Sigma^s X$, we could proceed directly, computing ΣX, then $\Sigma(\Sigma X)$, and $\Sigma\left(\Sigma^2 X\right),\ldots,\Sigma\left(\Sigma^{s-1}X\right)$. However, $\Sigma^k X$ can contain $|\Sigma|^k|X|$ matrices, and this number can become very large rapidly. Keeping a record of these matrices thus becomes a serious computational problem.

To overcome this problem, we need a method for computing $\Sigma^s X$ which doesn't require our keeping track of lots of matrices. A method for doing this is a Monte Carlo method, which we will describe below.

Monte-Carlo Method

1. Randomly (uniform distribution) choose a matrix in X, say X_j.
2. Randomly (uniform distribution) choose a matrix in Σ, say A_i. Compute $A_i X_j$.
3. If $A_{i_t} \cdots A_{i_1} X_j$ has been computed, randomly (uniform distribution) choose a matrix, say $A_{i_{t+1}}$ in Σ. Compute $A_{i_{t+1}} A_{i_t} \cdots A_{i_1} X_j$.
4. Continue until $A_{i_s} \cdots A_{i_1} X_j$ is found. Plot in R^2.
5. Return to (1) for the next run. Repeat sufficiently many times. (This may require some experimenting.)

11.2 Graphing Curves

In this section, we look at two examples of graphing curves.

Example 11.1 *We look at constructing a curve generated by a corner cutting method. This method replaces a corner as in Figure 11.1 by less sharp corners as shown in Figure 11.2. This is equivalent to replacing $\triangle ABC$*

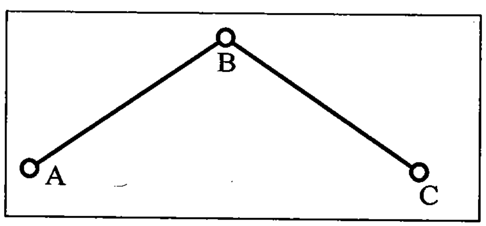

FIGURE 11.1. A corner.

with $\triangle ADE$ and $\triangle EFC$. This corner cutting can then be continued on polygonal lines (or triangles) ADE and EFC. In the limit, we have some curve as in Figure 11.3.

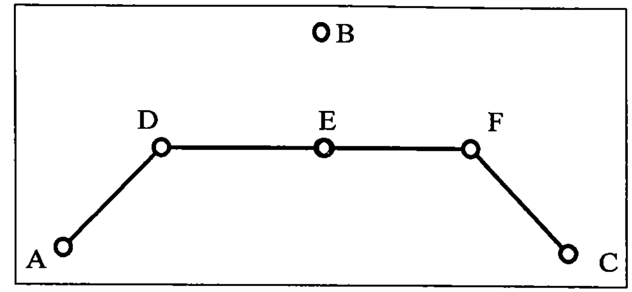

FIGURE 11.2. A corner cut into two corners.

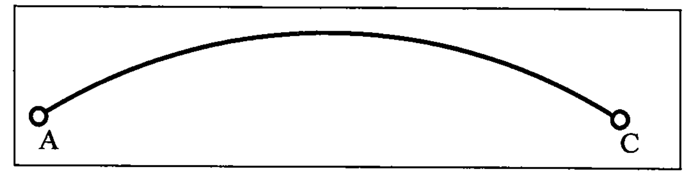

FIGURE 11.3. A curve generated by corner cutting.

Mathematically, this amounts to taking points $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ and generating

$$A = A$$
$$D = .5A + .5B$$
$$E = .25A + .5B + .25C$$

and

$$E = .25A + .5B + .25C$$
$$F = .5B + .5C$$
$$C = C.$$

This can be achieved by matrix multiplication

$$A_1 \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}, \quad A_2 \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ .5 & .5 & 0 \\ .25 & .5 & .25 \end{bmatrix}, \quad A_2 = \begin{bmatrix} .25 & .5 & .25 \\ 0 & .5 & .5 \\ 0 & 0 & 1 \end{bmatrix}.$$

And, continuing we have

$$A_1A_1P, A_2A_1P, A_1A_2P, A_2A_2P \tag{11.1}$$

where $P = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$, *etc. The products* $\prod_{k=1}^{\infty} A_{i_k}P$ *are plotted to give the points on the curve.*

Calculating, we can see that $\Sigma = \{A_1, A_2\}$ *is a* τ*-proper set with*

$$E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The corresponding subspace contraction coefficient satisfies

$$\tau_W(A_1) = .75 \text{ and } \tau_W(A_2) = .75,$$

using the 1*-norm. Thus by Theorem 9.6,* $\langle \Sigma^k P \rangle$ *converges.*

Given a corner $P = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$, *we apply the corner cutting technique 10 times. Thus, we compute* $\Sigma^{10}P$ *by Monte-Carlo where the number of runs (Step 5) is 5,000. The graph shown in Figure 11.4 was the result.*

Example 11.2 *In this example, we look at replacing a segment with a polygonal line introducing corners. If the segment is* $\overline{AB}$, *as shown in Figure 11.5 we partition it into three equal parts and replace the center segment by a corner labeled* CDE, *with sides congruent to the replaced segment. See Figure 11.6.*

Given $A(x_1, y_1)$ *and* $B(x_2, y_2)$, *we see that*

$$C\left(\frac{2}{3}A + \frac{1}{3}B\right) = C\left(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{2}{3}y_1 + \frac{1}{3}y_2\right).$$

Thus, listing coordinates columnwise, if

$$A_1 \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix},$$

then

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{bmatrix}.$$

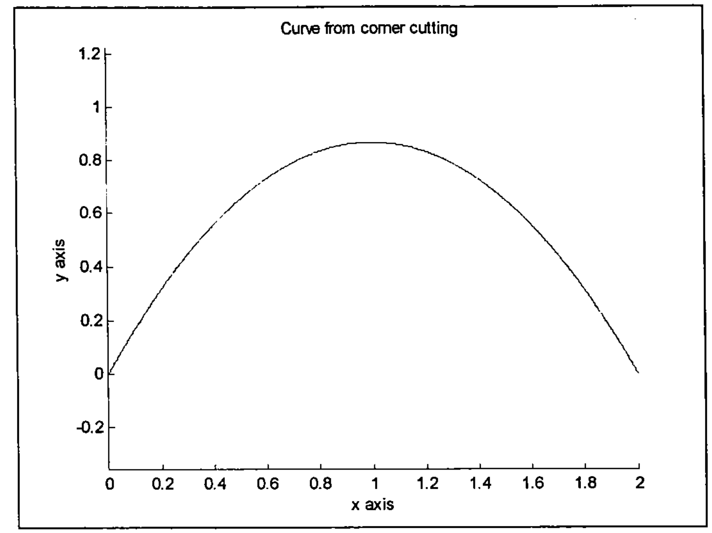

FIGURE 11.4. Curve from corner cutting.

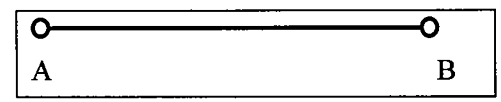

FIGURE 11.5. A segment.

To find $\overline{CD}$, *we use a vector approach to get*

$$\begin{aligned} D &= A + \frac{1}{3}(B-A) + \frac{1}{3}(B-A)\begin{bmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} \\ &= D\left(\frac{1}{2}x_1 + \frac{\sqrt{3}}{6}y_1 + \frac{1}{2}x_2 - \frac{\sqrt{3}}{6}y_2, -\frac{\sqrt{3}}{6}x_1 + \frac{1}{2}y_1 + \frac{\sqrt{3}}{6}x_2 + \frac{1}{2}y_2\right). \end{aligned}$$

Thus,

$$A_2 \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix}$$

where

$$A_2 = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{1}{2} \end{bmatrix}.$$

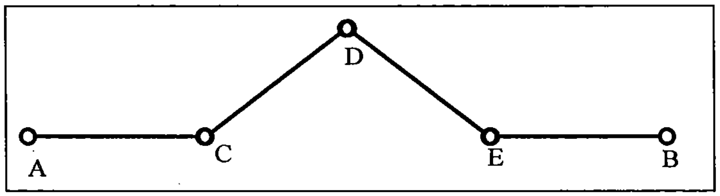

FIGURE 11.6. A corner induced by the segment.

Continuing,

$$A_3 \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} D \\ E \end{bmatrix}$$

for

$$A_3 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{3}}{6} & \frac{1}{2} & \frac{\sqrt{3}}{6} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

and

$$A_4 \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} E \\ B \end{bmatrix}$$

for

$$A_4 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The set $\Sigma = \{A_1, A_2, A_3, A_4\}$ is a τ-proper set and

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus,

$$W = \text{span}\{u_1, u_2\}$$

where $u_1 = (1, 0, -1, 0)$ and $u_2 = (0, 1, 0, -1)$. The unit sphere in the 1-norm is

$$\text{convex}\left\{\pm\frac{1}{2}u_1, \pm\frac{1}{2}u_2\right\}.$$

Hence, using Theorem 2.12,

$$\tau_W(A) = \max\left\{\left\|\pm\frac{1}{2}u_1 A\right\|_1, \left\|\pm\frac{1}{2}u_2 A\right\|_1\right\} = \max\left\{\frac{1}{2}\|a_1 - a_3\|_1, \frac{1}{2}\|a_2 - a_3\|_1\right\}$$

where a_k is the k-th row of A.

Applying our formula to Σ, we get

$$\tau_W(\Sigma) = \frac{2}{3}.$$

Thus, by Theorem 9.6, $\Sigma^\infty P$ exists. To compute and graph this set, we use $\Sigma^s P$ where $P = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, and we take $s = 6$. The result of applying the Monte-Carlo techniques, with 3,000 runs, is shown in Figure 11.7.

Of course, other polygonal lines can be used to replace a segment, e.g., see Figure 11.8.

11.3 Graphing Fractals

In this section, we use products of matrices to produce fractals. We look at two examples.

Example 11.3 *To construct a Cantor set, we can note that if $\begin{bmatrix} a \\ b \end{bmatrix}$ is an interval on the real line, then for*

$$A_1 = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad A_1 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ \frac{2}{3}a + \frac{1}{3}b \end{bmatrix}$$

gives the first $\frac{1}{3}$ of the interval and for

$$A_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & 1 \end{bmatrix}, \quad A_2 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{3}a + \frac{2}{3}b \\ b \end{bmatrix}$$

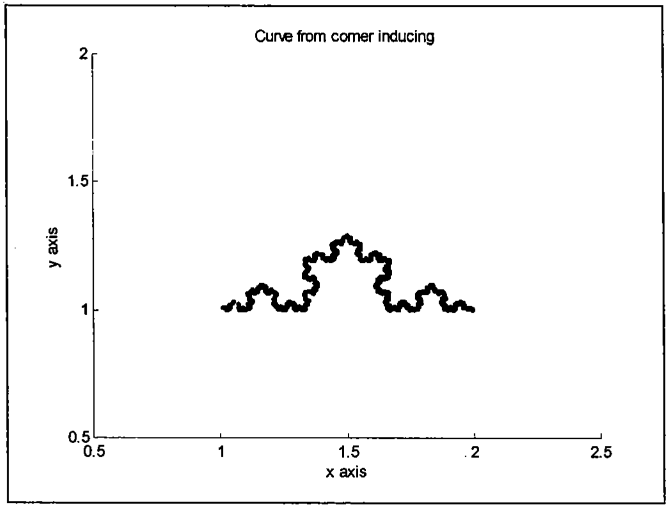

FIGURE 11.7. Curve from corner inducing.

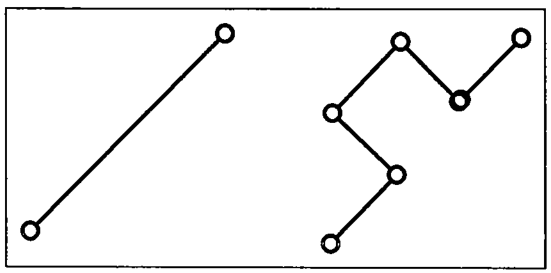

FIGURE 11.8. Square induced by segment.

gives the second third of the interval. See Figure 11.9. Thus, graphing all products $\Sigma^\infty \begin{bmatrix} a \\ b \end{bmatrix}$ *gives the* $\frac{1}{3}$ *Cantor set.*

Calculation shows $\Sigma = \{A_1, A_2\}$ *is a* τ*-proper set where,*

$$E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So

$$\tau_W(A) = \frac{1}{2} \max_{i,j} \|a_i - a_j\|_1$$

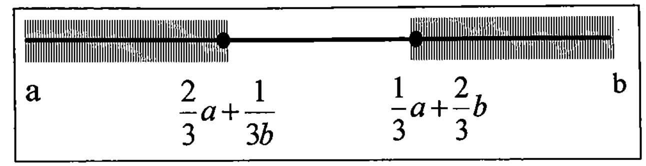

FIGURE 11.9. One third of segment removed.

where a_k *is the k-th row of A. Thus*

$$\tau_W(\Sigma) = \frac{1}{3},$$

and so $\Sigma^\infty \begin{bmatrix} a \\ b \end{bmatrix}$ *exists.*

To see a picture, we computed $\Sigma^s \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *where* $s = 4$. *With 1,000 runs, we obtained the graph in Figure 11.10.*

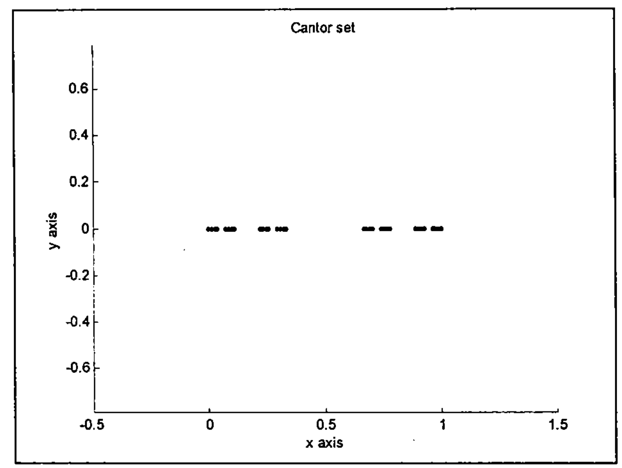

FIGURE 11.10. The beginning of the Cantor set.

The $\frac{1}{4}$ *Canter set, etc. can also be obtained in this manner.*

Example 11.4 *To obtain a Sierpenski triangle, we take three points which form a triangle. See Figure 11.11.*

We replace this triangle (See Figure 11.12.) with three smaller ones, $\triangle ADF$, $\triangle DBE$, $\triangle FEC$.

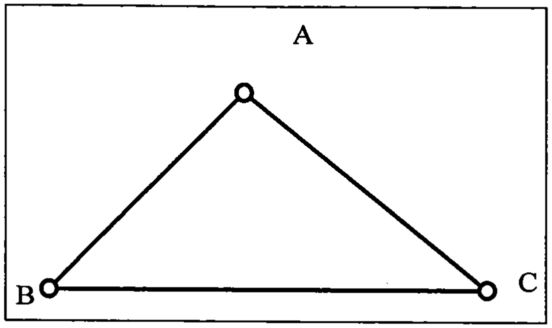

FIGURE 11.11. A triangle.

If $A(x_U, y_U)$, $B(x_L, y_L)$, $C(x_R, y_R)$ are given, we obtain the coordinates of A, D, F as

$$A_1 \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} A \\ D \\ F \end{bmatrix}$$

or numerically,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_U \\ y_U \\ x_L \\ y_L \\ x_R \\ y_R \end{bmatrix} = \begin{bmatrix} x_U \\ y_U \\ \frac{1}{2}x_U + \frac{1}{2}x_L \\ \frac{1}{2}y_U + \frac{1}{2}y_L \\ \frac{1}{2}x_U + \frac{1}{2}x_R \\ \frac{1}{2}y_U + \frac{1}{2}y_R \end{bmatrix}.$$

And

$$A_2 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

provides $\triangle DBE$, while

$$A_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

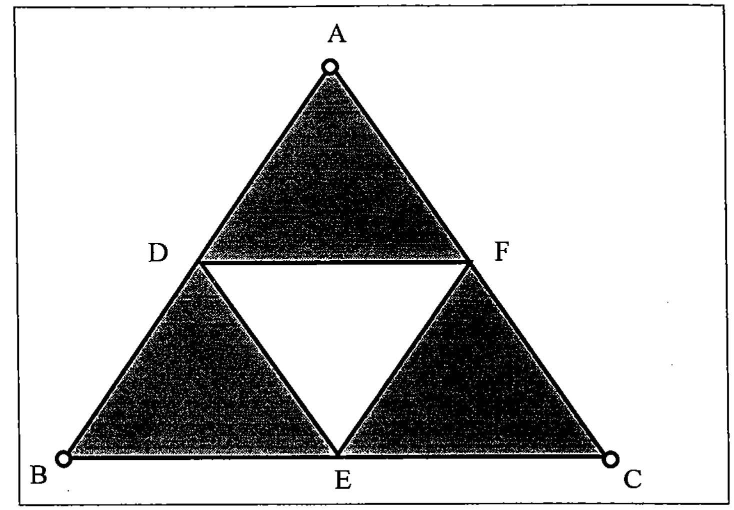

FIGURE 11.12. One triangle removed.

provides $\triangle FEC$.

The set $\Sigma = \{A_1, A_2, A_3\}$ is a τ-proper set with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This yields that

$$\tau_W(A) = \frac{1}{2} \max \left\{ \begin{matrix} \|a_1 - a_3\|_1, \|a_1 - a_5\|_1, \|a_3 - a_5\|_1 \\ \|a_2 - a_4\|_1, \|a_2 - a_6\|_1, \|a_4 - a_6\|_1 \end{matrix} \right\}.$$

Thus,

$$\tau_W(\Sigma) = \frac{1}{2}.$$

This assures us that $\langle \Sigma^k P \rangle$ converges.

Using $P = \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, $s = 6$, *and 3,000 runs, we obtained the picture given in Figure 11.13. Other triangles or polygonal shapes can also be used*

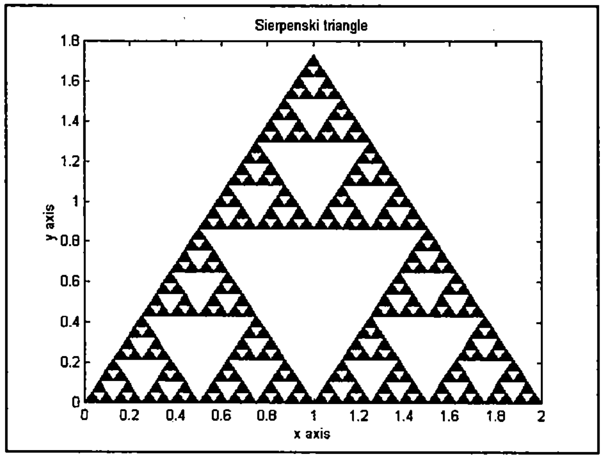

FIGURE 11.13. Sierpenski triangle.

in this setting.

11.4 Research Notes

The graphing method outlined in Section 1 is well known, although not particularly used in the past in this setting. Barnsley (1988) provided a different iterative method for graphing fractals; however, some of that work, on designing fractals, is patented.

Additional work for Section 2 can be found in Micchelli and Prautzsch (1989) and Daubechies and Lagarias (1992). References there are also helpful.

For Section 3, Diaconis and Shahshani (1986), and the references there, are useful.

11.5 MATLAB Codes

Curve from Corner Cutting

```
A1=[1  0  0;.5  .5  0;.25  .5  .25];
A2=[.25 .5 .25;0 .5 .5;0 0 1];
axis equal
xlabel('x axis')
ylabel('y axis')
title('Curve from corner cutting')
hold on
for k=1:5000
    P=[0,0;1,sqrt(3);2,0];
    for i=1:10
        G=rand;
        if G<=1/2
            P=A1*P;
        else
            P=A2*P;
        end
    end
    w=[P(1,1)  P(2,1)  P(3,1)  P(1,1)];
    z=[P(1,2)  P(2,2)  P(3,2)  P(1,2)];
    plot(w,z)
end
```

Contraction Coefficient

```
A=[1  0  0;.5  .5  0;.25  .5  .25];
T=0;
for i=1:3
    for j=1:3
        M=.5*norm(A(i,:)'-A(j,:)',1);
        T=max([T,M]);
    end
end
T
```

Curve from Corner Inducing

```
A1=[1  0  0 0;0  1  0 0;2/3  0  1/3 0;0 2/3 0 1/3];
A2=[2/3  0  1/3 0;0  2/3  0 1/3;1/2  sqrt(3)/6  1/2
```

```
    -sqrt(3)/6;-sqrt(3)/6 1/2 sqrt(3)/6 1/2];
A3=[1/2  sqrt(3)/6  1/2  -sqrt(3)/6;-sqrt(3)/6 1/2
    sqrt(3)/6  1/2;1/3 0 2/3 0;0 1/3 0 2/3];
A4=[1/3  0  2/3 0;0  1/3  0 2/3;0  0  1 0;0 0 0 1];
hold on
axis [.5 2.5 0.5 2.5]
axis equal
xlabel('x axis')
ylabel('y axis')
title('Curve from corner inducing')
for k=1:3000
    P=[1;1;2;1];
    for i=1:10
        G=rand;
        if G<1/4
            P=A1*P;
        end
        if G<=1/4&G<1/2
            P=A2*P;
        end
        if G>=3/4
            P=A3*P;
        end
        if G>=3/4
            P=A4*P;
        end
    end
    plot(P(1),P(2))
    plot(P(3),P(4))
end
```

Cantor Set

```
A1=[1  0;2/3 1/3];
A2=[1/3 2/3;0 1];
axis [-.5 1.5 -.5 .5]
axis equal
xlabel('x axis')
ylabel('y axis')
title('Cantor set')
hold on
```



```
for k=1:1000
    x=[1;0];
    for i=1:4
        G=rand;
        if G<.5, B=A1;
        else, B=A2;
        end
        x=B*x;
    end
    plot(x(1),0,'.')
    plot(x(2),0,'.')
end
```

Serpenski Triangle

```
A1=[1 0 0 0 0 0;0 1 0 0 0 0;.5 0 .5 0 0 0;
    0 .5 0 .5 0 0;.5 0 0 0 .5 0;0 .5 0 0 0 .5];
A2=[.5  0  .5 0 0  0;0 .5 0 .5 0 0;0 0 1 0  0 0;
    0 0 0 1 0  0;0 0 .5 0 .5 0;0 0 0 .5 0 .5];
A3=[.5 0 0 0 .5 0;0 .5 0 0 0 .5;0 0 .5 0 .5 0;
    0 0 0 .5 0 .5;0 0 0 0 1 0;0 0 0 0 0 1];
axis equal
xlabel('x axis')
ylabel('y axis')
title('Serpenski triangle')
hold on
plot(0,0)
plot(1,sqrt(3))
plot(2,0)
for k=1:3000
    x=[0;0;1;sqrt(3);2;0];
    for i=1:5
        G=rand;
        if G<1/3
            x=A1*x;
        elseif G>=1/3&G<2/3
            x=A2*x;
        else
            x=A3*x;
        end
    end
```

```
    w=(x(1),x(3),x(5),x(1));
    z=(x(2),x(4),x(6),x(2));
    fill(w,z,'k')
end
```

12
Slowly Varying Products

When finite products $A_1, A_2A_1, \ldots, A_k \ldots A_2A_1$ vary slowly, some terms in the trajectory $\left\langle \prod_{i=1}^{k} A_i x \right\rangle$ can sometimes be estimated by using the current matrix, or recently past matrices. This chapter provides results of this type.

12.1 Convergence to 0

In this section, we give conditions on matrices that assure slowly varying products converge to 0. The theorem will require several preliminary results.

We consider the equation

$$A^*SA - S = -I, \tag{12.1}$$

where A is an $n \times n$ matrix and I the $n \times n$ identity matrix.

Lemma 12.1 *If $\rho(A) < 1$, then a solution S to (12.1) exists.*

Proof. Define

$$S = \frac{1}{2\pi i} \oint \left(A^* - z^{-1}I\right)^{-1} (A - zI)^{-1} z^{-1} dz$$

where the integration is over the unit circle.

To show that S satisfies (12.1), we use the identities

$$\begin{aligned}(A-zI)^{-1}A &= I+z(A-zI)^{-1}\\ A^*\left(A^*-z^{-1}I\right)^{-1} &= I+z^{-1}\left(A^*-z^{-1}I\right)^{-1}.\end{aligned}$$

Then

$$\begin{aligned}A^*SA &= \frac{1}{2\pi i}\oint A^*\left(A^*-z^{-1}I\right)^{-1}(A-zI)^{-1}Az^{-1}dz\\ &= \frac{1}{2\pi i}\oint\left[I+z^{-1}\left(A^*-z^{-1}I\right)^{-1}\right]\left[I+z(A-zI)^{-1}\right]z^{-1}dz\\ &= \frac{1}{2\pi i}\oint\left(z^{-1}I+(A-zI)^{-1}+(A^*z-I)^{-1}z^{-1}\right)dz+S\\ &= I+\frac{1}{2\pi i}\oint(A-zI)^{-1}dz+\frac{1}{2\pi i}\oint(A^*z-I)^{-1}z^{-1}dz+S.\end{aligned}$$

Now, since $f(A)=\frac{1}{2\pi i}\oint f(z)(Iz-A)^{-1}dz$ for any analytic function f, taking $f(z)=1$, we have

$$\frac{1}{2\pi i}\oint(A-zI)^{-1}dz=-I.$$

Changing the variable z to z^{-1} and replacing A by A^* yields

$$\frac{1}{2\pi i}\oint(A^*z-I)^{-1}z^{-1}dz=-I.$$

Plugging these in, we get

$$A^*SA=S-I$$

or

$$A^*SA-S=-I,$$

which proves the lemma. ■

We now need a few bounds on the eigenvalues of S. To get these bounds, we note that for the parametrization

$$z=e^{i\theta},\quad -\pi\le\theta\le\pi,$$

$$S=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(A^*-e^{-i\theta}I\right)^{-1}\left(A-e^{i\theta}I\right)^{-1}d\theta$$

or setting $G=A-e^{i\theta}I$,

$$S=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(G^{-1}\right)^*G^{-1}d\theta,$$

which is Hermitian.

To show that S is also positive definite, note that $\left(G^{-1}\right)^*G^{-1}$ is positive definite. Thus, if $x\neq 0$, $x^*\left(G^{-1}\right)^*G^{-1}x>0$ for all θ and so

$$x^*Sx=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^*\left(G^{-1}\right)^*G^{-1}x\,d\theta>0.$$

Hence, S is positive definite.

The bounds are on the largest eigenvalue $\rho(S)$ and the smallest eigenvalue $\sigma(S)$ of S follow.

Lemma 12.2 *If $\rho(A)<1$, then*

1. $\rho(S)\le(\|A\|_2+1)^{2n-2}/(1-\rho(A))^{2n}$
2. $\sigma(S)\ge 1$.

Proof. For (1), since $G=A-e^{i\theta}I$, then, using a singular value decomposition of G, we see that $\|G\|_2$ is the largest singular value of G, $\left\|G^{-1}\right\|_2$ is the reciprocal of the smallest singular value of G, and $|\det G|$ is the product of the singular values. Thus,

$$\left\|G^{-1}\right\|_2\le\frac{\|G\|_2^{n-1}}{|\det G|}.$$

And, since $|\det G|=\left|\lambda_1-e^{i\theta}\right|\cdots\left|\lambda_n-e^{i\theta}\right|$, where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A,

$$\begin{aligned}\left\|G^{-1}\right\|_2 &\le\left\|A-e^{i\theta}I\right\|_2^{n-1}/\left|\lambda_1-e^{i\theta}\right|\cdots\left|\lambda_n-e^{i\theta}\right|\\ &\le(\|A\|_2+1)^{n-1}/(1-\rho(A))^n.\end{aligned}$$

Since S is Hermitian,

$$\begin{aligned}\rho(S) &= \|S\|_2\\ &\le\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\|G^{-1}\right\|_2^2d\theta\\ &= \frac{(\|A\|_2+1)^{2n-2}}{(1-\rho(A))^{2n}}.\end{aligned}$$

For (2), we use Hermitian forms. Of course,

$$\sigma(S)\, x^* x \le x^* S x$$

for all x. Setting $x = Ay$, we have

$$\sigma(S)\, y^* A^* A y \le y^* A^* S A y.$$

Using that $A^* S A = S - I$, we get

$$\sigma(S)\, \sigma(A^* A)\, y^* y \le y^* S y - y^* y$$

or

$$(1 + \sigma(S)\, \sigma(A^* A))\, y^* y \le y^* S y.$$

Since this inequality holds for all y, and thus for all y such that $Sy = \sigma(S)\, y$,

$$1 + \sigma(S)\, \sigma(A^* A) \le \sigma(S).$$

Hence,

$$1 \le \sigma(S),$$

the required inequality. ■

We now consider the system

$$x_{k+1} = B_k x_k, \tag{12.2}$$

where

1. $\|B_k\| \le K$
2. $\rho(B_k) \le \beta < 1$

   for positive constants K, β and all $k \ge 1$.

Theorem 12.1 *Using the system 12.2 and conditions (1) and (2), there is an $\epsilon > 0$ such that if*

$$\|B_{k+1} - B_k\| \le \epsilon$$

for all k, then $\langle x_k \rangle \to 0$ as $k \to \infty$.

Proof. Let $q > 1$, q an integer to be determined. We consider the interval $0 \le k \le 2q$,

$$x_{k+1} = B_q x_k + [B_k - B_q]\, x_k.$$

Since $\|B_{s+1} - B_s\| \le \epsilon$ for all s, then

$$\|[B_k - B_q]\, x_k\| \le |q - k|\, \|x_k\|\, \epsilon.$$

To shorten notation, let

$$A = B_q$$
$$f(k) = [B_k - B_q]\, x_k.$$

So we have

$$x_{k+1} = A x_k + f(k).$$

By hypothesis $\rho(A) \le \beta$, so $\rho(\beta^{-1} A) < 1$. Thus, there is a positive definite Hermitian matrix S such that

$$(\beta^{-1} A)^* S (\beta^{-1} A) - S = -I$$

or

$$A^* S A = \beta^2 S - \beta^2 I.$$

Let $V(x_k) = x_k^* S x_k$. Then

$$V(x_{k+1}) = x_k^* A^* S A x_k + f(k)^* S f(k) + x_k^* A^* S f(k) + f(k)^* S A x_k. \tag{12.3}$$

Now we need a few bounds. For these, let $\alpha > 0$. (A particular α will be chosen later.) Since

$$(\alpha f(k)^* - x_k^* A^*)\, S\, (\alpha f(k) - A x_k) \ge 0,$$
$$\alpha f(k)^* S f(k) + \alpha^{-1} x_k^* A^* S A x_k \ge f(k)^* S A x_k + x_k^* A^* S f(k).$$

Plugging this into (12.3), we have

$$V(x_{k+1}) \le (1 + \alpha)\, f(k)^* S f(k) + (1 + \alpha^{-1})\, x_k^* A^* S A x_k$$
$$= (1 + \alpha^{-1})\left[\alpha f(k)^* S f(k) + x_k^* A^* S A x_k\right].$$

Continuing the calculation

$$\begin{aligned} V(x_{k+1}) &\le \left(1+\alpha^{-1}\right)\left[\alpha f(k)^* Sf(k) + \beta^2 x_k^*(S-I)x_k\right] \\ &= \left(1+\alpha^{-1}\right)\left(\alpha f(k)^* Sf(k) + \beta^2 V(x_k) - \beta^2 \|x_k\|^2\right). \end{aligned}$$

Now,

$$f(k)^* Sf(k) \le \rho(S)\|f(k)\|^2 \le \rho(S)(q-k)^2 \epsilon^2 \|x_k\|^2.$$

So, by substitution,

$$V(x_{k+1}) \le \left(1+\alpha^{-1}\right)\left[\beta^2 V(x_k) + \left(\alpha\rho(S)(q-k)^2\epsilon^2 - \beta^2\right)\|x_k\|^2\right].$$

Let $\alpha = \frac{\beta^2}{\rho(S)(q-k)^2\epsilon^2}$ to get

$$\begin{aligned} V(x_{k+1}) &\le \left(1+\alpha^{-1}\right)\beta^2 V(x_k) \\ &= \left(\beta^2 + \rho(S)(q-k)^2\epsilon^2\right)V(x_k). \end{aligned}$$

By Lemma 12.2,

$$\begin{aligned} \rho(S) &\le \frac{\left(\|A\|_2+1\right)^{2n-2}}{\left(1-\rho(A)\right)^{2n}} \\ &\le \frac{(K+1)^{2n-2}}{(1-\beta)^{2n}}. \end{aligned}$$

Set

$$\rho = \frac{(K+1)^{2n-2}}{(1-\beta)^{2n}}.$$

By continuing the calculation

$$V(x_{k+1}) \le \left(\beta^2 + \rho(q-k)^2\epsilon^2\right)V(x_k).$$

Thus,

$$V(x_1) \le \left(\beta^2 + \rho q^2\epsilon^2\right)V(x_0),$$

and by iteration,

$$V(x_{2q}) \le \left(\beta^2 + \rho q^2\epsilon^2\right)^{2q} V(x_0).$$

Finally, we have

$$\sigma(S)\|x_{2q}\|^2 \le \rho\left(\beta^2 + \rho q^2\epsilon^2\right)^{2q}\|x_0\|^2.$$

So, by Lemma 12.2

$$\|x_{2q}\| \le \sqrt{\rho\left(\beta^2 + \rho q^2\epsilon^2\right)^{2q}}\,\|x_0\|.$$

Now, choose ϵ and q such that

$$\rho\left(\beta^2 + \rho q^2\epsilon^2\right)^{2q} < 1$$

and set

$$T = \sqrt{\rho\left(\beta^2 + \rho q^2\epsilon^2\right)^{2q}}$$

so

$$\|x_{2q}\| \le T\|x_0\|.$$

This inequality can be achieved for the interval $[2q, 4q]$ to obtain

$$\|x_{4q}\| \le T\|x_{2q}\|;$$

continuing,

$$\left\|x_{2(m+1)q}\right\| \le T\|x_{2mq}\|$$

which shows that $x_{mq} \to 0$ as $m \to \infty$.

Repeating the argument, for intervals $[2mq + r, 2(m+1)q + r]$ yields

$$\left\|x_{2(m+1)q+r}\right\| \le T\|x_{2mq+r}\|,$$

so $x_{2mq+r} \to 0$ as $m \to \infty$. Thus, putting together $x_k \to 0$ as $k \to \infty$. ■

Corollary 12.1 *Using the hypotheses of the theorem*

$$\lim_{k\to\infty} B_k \cdots B_1 = 0.$$

Proof. By the theorem

$$\lim_{k\to\infty} B_k \cdots B_1 x_1 = 0$$

for all x_1. Thus,

$$\lim_{k\to\infty} B_k \cdots B_1 = 0,$$

the desired result. ■

12.2 Estimates of x_k from Current Matrices

Let $A_1, A_2, \ldots$ be $n \times n$ primitive nonnegative matrices and x_1 an $n \times 1$ positive vector. Define the system

$$x_{k+1} = A_k x_k.$$

The Perron-Frobenius theory (Gantmacher, 1964) assures that each A_k has a simple positive eigenvalue λ_k such that $\lambda_k > |\lambda|$ for all other eigenvalues λ of A_k. And there is a positive stochastic eigenvector v_k belonging to λ_k. If the stochastic eigenvectors $v_k, \ldots, v_s$ of the last few matrices, $A_k, \ldots, A_s$ vary slowly with k, then v_k may be a good estimate of $\frac{A_k \cdots A_s x_s}{\|A_k \cdots A_s x_s\|}$. In this section we show when this can happen.

We make the following assumptions.

1. There are positive constants m, and M such that
$$m \leq \inf_{a_{ij}^{(k)}>0} a_{ij}^{(k)}$$
$$M \geq \sup a_{ij}^{(k)},$$
where inf and sup are over all i, j, and k.
2. There is a positive constant r such that
$$A_{t+r} \cdots A_{t+1} > 0$$
for all positive integers t.

These two conditions assure that

$$m^r \leq (A_{t+r} \cdots A_{t+1})_{ij} \leq n^{r-1} M^r$$

for all i, j, and t. Thus,

$$\phi(A_{t+r} \cdots A_{t+1}) \geq \left(\frac{m^r}{n^{r-1}M^r}\right)^2.$$

Set $\phi = \left[\frac{m^r}{n^{r-1}M^r}\right]^2$ and $\tau_r = \frac{1-\sqrt{\phi}}{1+\sqrt{\phi}} < 1$. We see that

$$\tau_B(\Sigma^r) \leq \tau_r.$$

Theorem 12.2 *Assuming conditions 1 and 2,*

$$p(x_{k+1}, v_k) \leq \tau_r^{\left[\frac{k}{r}\right]} p(x_2, v_1) + \sum_{j=1}^{k-1} \tau_r^{\left[\frac{j}{r}\right]} p(v_{k-j}, v_{k-j+1}). \qquad (12.4)$$

Proof. By the triangle inequality $p(x_{k+1}, v_k) \leq$

$$p(A_k \cdots A_1 x_1, A_k \cdots A_1 v_1) + p(A_k \cdots A_2 A_1 v_1, A_k \cdots A_2 v_2) + \cdots + p(A_k A_{k-1} v_{k-1}, A_k v_k)$$

and using that $A_s v_s = \lambda_s v_s$ for all s,

$$= p(A_k \cdots A_2 A_1 x_1, A_k \cdots A_2 A_1 v_1) + p(A_k \cdots A_2 v_1, A_k \cdots A_2 v_2) + \cdots + p(A_k v_{k-1}, A_k v_k)$$
$$\leq \tau_r^{\left[\frac{k}{r}\right]} p(x_2, v_1) + \tau_r^{\left[\frac{k-1}{r}\right]} p(v_1, v_2) + \cdots + \tau_r^{\left[\frac{1}{r}\right]} p(v_{k-1}, v_k)$$

which gives (12.4). ■

If

$$\delta = \sup_j p(v_j, v_{j+1}),$$

then the theorem yields

$$p(x_{k+1}, v_k) \leq \tau_r^{\left[\frac{k}{r}\right]} p(x_2, v_1) + \sum_{j=1}^{k-1} \tau_r^{\left[\frac{j}{r}\right]} \delta.$$

Thus, if in the recent past, starting the system at a recent vector so k is small, we see that if δ and τ_r are small, v_k gives a good approximation of x_{k+1}.

This gives us some insight into the behavior of a system. For example, suppose we have only the latest transition matrix, say A. We know

$$x_{k+1} = A x_k$$

but don't know x_k and thus neither do we know x_{k+1}. However, we need to estimate $\frac{x_{k+1}}{\|x_{k+1}\|_1}$.

Let v be the stochastic eigenvector of A belonging to $\rho(A)$. The theorem tells us that if we feel that in recent past the eigenvectors, say $v_1, \ldots, v_k$, didn't vary much and τ_r is small, then v is an estimate of $\frac{x_{k+1}}{\|x_{k+1}\|_1}$. (We might add here that some estimate, reasonably obtained, is often better than nothing at all.)

Some numerical work is given in the following example.

Example 12.1 *Let*

$$A = \begin{bmatrix} .2 & .4 & .4 \\ .9 & 0 & 0 \\ 0 & .9 & 0 \end{bmatrix}$$

and let Σ be the set of matrices C, such that

$$A - .02A \leq C \leq A + .02A.$$

Thus, we allow a 2% variation in the entries of A. Now, we start with $x_1 = \begin{bmatrix} 0.3434 \\ 0.3333 \\ 0.3232 \end{bmatrix}$ *and randomly generate* $A_1 = \left[a_{ij}^{(1)}\right]$ *where*

$$a_{ij}^{(1)} = b_{ij} + \text{rand}\,(.04)\, a_{ij}$$

for all i, j, where rand *is a randomly generated number between* 0 *and* 1 *and $B = A - .02A$. Then,*

$$x_2 = A_1 x_1$$

and

$$\bar{x}_2 = \frac{x_2}{\|x_2\|_1},$$

etc. We now apply this technique to demonstrate the theoretical bounds given in Theorem 12.2.

Let v_k denote the stochastic eigenvector for A_k; we obtain for a run of 50 iterates the data in the table.

k	48	49	50
$\bar{x}_{k+1}$	$\begin{bmatrix} 0.3430 \\ 0.3299 \\ 0.3262 \end{bmatrix}$	$\begin{bmatrix} 0.3599 \\ 0.3269 \\ 0.3132 \end{bmatrix}$	$\begin{bmatrix} 0.3445 \\ 0.3470 \\ 0.3085 \end{bmatrix}$
v_k	$\begin{bmatrix} 0.3474 \\ 0.3330 \\ 0.3196 \end{bmatrix}$	$\begin{bmatrix} 0.3541 \\ 0.3330 \\ 0.3129 \end{bmatrix}$	$\begin{bmatrix} 0.3476 \\ 0.3354 \\ 0.3170 \end{bmatrix}$
$p(\bar{x}_{k+1}, v_k)$	0.0301	0.0351	0.0609
$p(v_k, v_{k-1})$	0.0224	0.0403	0.0314

Using Theorem 12.2 and three iterates, we have

$$x_{49} = A_{48} x_{48}$$
$$x_{50} = A_{49} x_{49}$$
$$x_{51} = A_{50} x_{50}.$$

So

$$p(x_{51}, v_{50}) \leq \tau_B(\Sigma^3)\, p(x_{49}, v_{48}) + \tau_B(\Sigma^2)\, p(v_{48}, v_{49}) + \tau_B(\Sigma)\, p(v_{49}, v_{50}).$$

And, using Theorem 10.5, $\tau_B(\Sigma^3) \leq 0.5195$, so

$$p(x_{51}, v_{50}) \leq 0.0899.$$

The actual calculation is $p(x_{51}, v_{50}) = 0.0609$, and the error is

$$error = 0.029.$$

To see that δ is always finite, we can proceed as follows. Let

$$Av = \lambda v$$

where $A = A_k, v = v_k, \lambda = \lambda_k$ for some k. Suppose

$$x_q = \max_k x_k$$

where $v = (x_1, \ldots, x_n)^t$. Then, since v is stochastic,

$$x_q \geq \frac{1}{n}.$$

Now, since A is primitive, $A^r > 0$ for some r. Thus $a_{i,j}^{(r)} \geq m^r$ for all i, j. Now

$$x_i = \frac{(A^r v)_i}{\|A^r v\|_1} = \frac{\sum_{k=1}^{n} a_{ik}^{(r)} x_k}{\sum_{s=1}^{n}\sum_{k=1}^{n} a_{sk}^{(r)} x_k},$$

and since there are no more than n^{r-1} distinct paths from any v_i to any v_j, of length r,

$$x_i \geq \frac{a_{iq}^{(r)} x_q}{\sum_{s=1}^{n}\sum_{k=1}^{n} n^{r-1} M^r x_k} \geq \frac{m^r x_q}{n^2 n^{r-1} M^r x_q} = \frac{m^r}{n^2 n^{r-1} M^r}.$$

Since this inequality holds for all i,

$$\min_i x_i \geq \frac{m^r}{n^2 n^{r-1} M^r} > 0.$$

And, from this it follows that $\rho(v_j, v_{j+1})$ is bounded.

We add to our assumptions.

3. There are positive constants β and λ such that

$$\|A_k\|_\infty \leq \beta \text{ and } \lambda_k \leq \lambda$$

for all k. And $\gamma > 0$ is a lower bound on the entries of v_k for all k.

An estimate of λ_k can be obtained as follows.

Theorem 12.3 *Assuming the conditions 1 through 3,*

$$\left| \frac{(x_{k+1})_i}{(x_k)_i} - \lambda_k \right| \leq \left(\frac{\beta}{\gamma} + \lambda \right) e^{\rho(x_k, v_k)} \rho(x_k, v_k).$$

Proof. Using Theorem 2.2, for fixed k, there is a positive constant r and a diagonal matrix M, with positive main diagonal, such that the vectors x_k and v_k satisfy

$$x_k = r(v_k + Mv_k).$$

Thus,

$$\begin{aligned} x_{k+1} &= A_k x_k \\ &= r(\lambda_k v_k + A_k M v_k), \end{aligned}$$

and so

$$\frac{(x_{k+1})_i}{(x_k)_i} = \frac{r(\lambda_k v_k + A_k M v_k)_i}{r(v_k + Mv_k)_i}.$$

Hence,

$$\begin{aligned} \left| \frac{(x_{k+1})_i}{(x_k)_i} - \lambda_k \right| &= \left| \frac{(\lambda_k v_k + A_k M v_k)_i - \lambda_k (v_k + Mv_k)_i}{(v_k + Mv_k)_i} \right| \\ &= \left| \frac{(A_k M v_k)_i - \lambda_k (Mv_k)_i}{(v_k + Mv_k)_i} \right| \\ &\leq \left| \frac{(A_k M v_k)_i - \lambda_k (Mv_k)_i}{(v_k)_i} \right| \\ &= \left| \frac{(A_k M v_k)_i}{(v_k)_i} - \lambda_k m_i \right| \\ &\leq \frac{\|A_k M v_k\|_\infty}{(v_k)_i} + \lambda_k m_i \\ &\leq \frac{\|A_k\|_\infty \|M\|_\infty \|v_k\|_\infty}{\gamma} + \lambda_k m_i \\ &\leq \frac{\beta \|M\|_\infty}{\gamma} + \lambda_k \|M\|_\infty. \end{aligned}$$

And since by Theorem 2.2, $\|M\|_\infty \leq e^{\rho(x_k, v_k)} - 1$, we get the bound

$$\leq \left(\frac{\beta}{\gamma} + \lambda_k \right) \left(e^{\rho(x_k, v_k)} - 1 \right).$$

Now we can write

$$e^u - 1 \leq ue^u$$

for any $u \geq 0$, so we have

$$\left| \frac{(x_{k+1})_i}{(x_k)_i} - \lambda_k \right| \leq \left(\frac{\beta}{\gamma} + \lambda \right) e^{\rho(x_k, v_k)} \rho(x_k, v_k)$$

for all i. ■

From this theorem, we see that if the v_k's and x_k's are near, then $\frac{(x_{k+1})_i}{(x_k)_i}$ is an estimate of λ_k. Of course, if the v_i's vary slowly then the x_{i+1}'s are close to the v_i's and are thus themselves close, so the v_i's and x_i's are close.

12.3 State Estimates from Fluctuating Matrices

Let A be an $n \times n$ primitive nonnegative matrix and y_1 an $n \times 1$ positive vector. Define

$$y_{k+1} = Ay_k.$$

It can be shown that $\left\langle \frac{y_k}{\|y_k\|_1} \right\rangle$ converges to π, the stochastic eigenvector belonging to the eigenvalue $\rho(A)$ of A.

Let $A_1, A_2, \ldots$ be fluctuations of A and consider the system

$$x_{k+1} = A_k x_k \tag{12.5}$$

where $x_1 > 0$. In this section, we see how well π approximates $\frac{x_k}{\|x_k\|_1}$, especially for large k.

It is helpful to divide this section into subsections.

12.3.1 Fluctuations

By a fluctuation of A, we mean a matrix $A + E \geq 0$ where the entries in $E = [e_{ij}]$ are small compared with those of A. More particularly, we suppose the entries of E are bounded, say

$$|e_{ij}| \leq \mathcal{E}_{ij}$$

where

$$a_{ij} - \mathcal{E}_{ij} > 0$$

when $a_{ij} > 0$ and $\mathcal{E}_{ij} = 0$ when $a_{ij} = 0$. Thus,

$$A - \mathcal{E} \leq A + E \leq A + \mathcal{E}.$$

Define

$$\delta = \sup p\,(Ax, (A+E)\,x)$$

where the *sup* is over all positive vectors x and fluctuations $A + E$. To show δ is finite, we use the notation,

$$RE = \max \frac{\mathcal{E}_{ij}}{a_{ij}}$$

where the maximum is over all $a_{ij} > 0$.

Theorem 12.4 *Using that $RE < 1$,*

$$\delta \leq \ln \frac{1+RE}{1-RE}.$$

Proof. Let $x > 0$. For simplicity, set

$$z_k = (Ax)_k$$
$$e_k = (Ex)_k\,.$$

Then

$$\sup_{x>0} \frac{|(Ex)_i|}{(Ax)_i} = \sup_{x>0} \frac{|e_{i1}x_1 + \cdots + e_{in}x_n|}{a_{i1}x_1 + \cdots + a_{in}x_n} \leq \sup_{x>0} \frac{\mathcal{E}_{i1}x_1 + \cdots + \mathcal{E}_{in}x_n}{a_{i1}x_1 + \cdots + a_{in}x_n}$$

and by using the quotient bound result (2.3),

$$\leq \max_{a_{ij}>0} \frac{\mathcal{E}_{ij}}{a_{ij}} = RE.$$

Furthermore, using that

$$\frac{z_j + e_j}{z_j} = 1 + \frac{e_j}{z_j}$$

and that

$$\frac{z_i}{z_i + e_i} = \frac{1}{1 + \frac{e_i}{z_i}},$$

we have

$$\frac{(Ax)_i}{((A+E)\,x)_i} \frac{((A+E)\,x)_j}{(Ax)_j} = \frac{z_i}{z_i + e_i} \frac{z_j + e_j}{z_j} = \frac{1 + \frac{e_j}{z_j}}{1 + \frac{e_i}{z_i}} \leq \frac{1+RE}{1-RE}.$$

Thus,

$$p\,(Ax, (A+E)\,x) \leq \ln \frac{1+RE}{1-RE},$$

the inequality we need. ■

12.3.2 Normalized Trajectories

Suppose the trajectory for (12.5) is $x_1, x_2, \ldots$. We normalize to stochastic vectors and set

$$\bar{x}_k = \frac{x_k}{\|x_k\|_1}.$$

We will assume that $\tau_B(A^r) = \tau_r < 1$. How far $\bar{x}_k$ is from π is given in the following theorem.

Theorem 12.5 *For all k and $0 < t \le r$,*

$$p(\pi, \bar{x}_{kr+t}) \le \tau_r^k p(\pi, \bar{x}_1) + \left(\tau_r^{k-1} + \cdots + \tau_r + 1\right) r\delta + (t-1)\delta.$$

Proof. We first make two observations.

1. For a positive vector x and a positive integer t,

$$\begin{aligned}
&p\left(A^t x, A_t \cdots A_1 x\right) \\
&\le p\left(A^t x, AA_{t-1} \cdots A_1 x\right) + p\left(AA_{t-1} \cdots A_1 x, A_t \cdots A_1 x\right) \\
&\le p\left(A^{t-1} x, A_{t-1} \cdots A_1 x\right) + \delta,
\end{aligned}$$

and by continuing,

$$\le (t-1)\delta + \delta = t\delta.$$

2. For all $k > 1$,

$$\begin{aligned}
p(\pi, \bar{x}_{kr+1}) &= p\left(A^r \pi, A^r \bar{x}_{(k-1)r+1}\right) \\
&\quad + p\left(A^r \bar{x}_{(k-1)r+1}, A_{kr} \cdots A_{(k-1)r+1} \bar{x}_{(k-1)r+1}\right) \\
&\le \tau_r p\left(\pi, \bar{x}_{(k-1)r+1}\right) + r\delta
\end{aligned}$$

and by continuing

$$p(\pi, \bar{x}_{kr+1}) \le \tau_r^k p(\pi, \bar{x}_1) + \left(\tau_r^{k-1} + \cdots + \tau_r + 1\right) r\delta.$$

Now, putting (1) and (2) together, we have for $0 < t \le r$,

$$\begin{aligned}
&p(\pi, \bar{x}_{kr+t}) \\
&= p\left(A^{t-1}\pi, A_{kr+t-1} \cdots A_{kr+1} \bar{x}_{kr+1}\right) \\
&\le p\left(A^{t-1}\pi, A^{t-1}\bar{x}_{kr+1}\right) + p\left(A^{t-1}\bar{x}_{kr+1}, A_{kr+t-1} \cdots A_{kr+1}\bar{x}_{kr+1}\right) \\
&\le p(\pi, \bar{x}_{kr+1}) + (t-1)\delta \\
&\le \tau_r^k p(\pi, \bar{x}_1) + \left(\tau_r^{k-1} + \cdots + \tau_r + 1\right) r\delta + (t-1)\delta,
\end{aligned}$$

the desired inequality. ■

12.3.3 Fluctuation Set

The vectors $\bar{x}_k$ need not converge to π. What we can expect, however, is that, in the long run, the $\bar{x}_k$'s fluctuate toward a set about π. By using Theorem 12.5, we take this set as

$$C = \left\{ x \in S^+ : p(x, \pi) \le \frac{r\delta}{1 - \tau_r} + (r-1)\delta \right\}.$$

Using

$$d(x, C) = \min_{c \in C} p(x, c),$$

we show that

$$d(\bar{x}_k, C) \to 0 \text{ as } k \to \infty.$$

We need a lemma.

Lemma 12.3 *If $x \in S^+$ and $0 \le \alpha \le 1$, then*

$$p(\pi, x) = p(\pi, \alpha\pi + (1-\alpha)x) + p(\alpha\pi + (1-\alpha)x, x).$$

Proof. We assume without loss of generality (We can reindex.) that

$$\frac{\pi_1}{x_1} \ge \cdots \ge \frac{\pi_n}{x_n}.$$

Thus, if $i < j$, then

$$\frac{\pi_i}{x_i} \ge \frac{\pi_j}{x_j}$$

or

$$\pi_i x_j \ge \pi_j x_i.$$

So

$$\alpha\pi_i\pi_j + (1-\alpha)\pi_i x_j \ge \alpha\pi_i\pi_j + (1-\alpha)\pi_j x_i$$

and

$$\frac{\pi_i}{\alpha\pi_i + (1-\alpha)x_i} \ge \frac{\pi_j}{\alpha\pi_j + (1-\alpha)x_j}.$$

From this, we have that

$$\begin{aligned}
&p(\pi, x) \\
&= \ln \frac{\pi_1}{x_1} \frac{x_n}{\pi_n} \\
&= \ln \left(\frac{\pi_1}{\alpha\pi_1 + (1-\alpha) x_1} \frac{\alpha\pi_n + (1-\alpha) x_n}{\pi_n} \right) \\
&+ \ln \left(\frac{\alpha\pi_1 + (1-\alpha) x_1}{x_1} \frac{x_n}{\alpha\pi_n + (1-\alpha) x_n} \right) \\
&= p(\pi, \alpha\pi + (1-\alpha) x) + p(\alpha\pi + (1-\alpha) x, x),
\end{aligned}$$

as required. ■

The theorem follows.

Theorem 12.6 *For any k and $0 < t \le r$,*

$$d(\bar{x}_{kr+t}, C) \le \tau_r^k p(\bar{x}_1, \pi).$$

Proof. If $\bar{x}_{kr+t} \in C$, the result is obvious. Suppose $\bar{x}_{kr+t} \notin C$. Then choose α, $0 < \alpha < 1$, such that

$$x(\alpha) = \alpha\pi + (1-\alpha)\bar{x}_{kr+t}$$

satisfies

$$p(x(\alpha), \pi) = \frac{r\delta}{1 - \tau_r} + (r-1)\delta.$$

Thus, $x(\alpha) \in C$.

By the lemma

$$\begin{aligned}
p(\bar{x}_{kr+t}, \pi) &= p(\bar{x}_{kr+t}, x(\alpha)) + p(x(\alpha), \pi) \\
&= p(\bar{x}_{kr+t}, x(\alpha)) + \frac{r\delta}{1 - \tau_r} + (r-1)\delta.
\end{aligned}$$

Now, by Theorem 12.5, we have

$$p(\bar{x}_{kr+t}, \pi) \le \tau_r^k p(\bar{x}_1, \pi) + \frac{r\delta}{1 - \tau_r} + (r-1)\delta,$$

so

$$p(\bar{x}_{kr+t}, x(\alpha)) \le \tau_r^k p(\bar{x}_1, \pi).$$

Thus,

$$d(\bar{x}_{kr+t}, C) \le \tau_r^k p(\bar{x}_1, \pi),$$

which is what we need. ■

Using the notation in Chapter 9, we show how C is related to limiting sets.

Corollary 12.2 *If U is a compact subset of S^+ and $\Sigma_p^k U \to L$ as $k \to \infty$, then $L \subseteq C$.*

Using this corollary, Theorem 12.4, and work of Chapter 10, we can compute a bound on L, even when L itself cannot be computed.

We conclude this section by showing how a pair $\bar{x}_i, \bar{x}_{i+1}$ from a trajectory indicates the closeness of $\bar{x}_i$ to C.

Theorem 12.7 *If $\tau_r < 1$, then*

$$d(\bar{x}_i, C) \le \frac{r}{1 - \tau_r} p(\bar{x}_i, \bar{x}_{i+1}).$$

Proof. Throughout the proof, i will be fixed. Generate a new sequence

$$x_1, \ldots, x_i, A_i x_i, A_i^2 x_i, \ldots.$$

Since A_i is primitive, using the Perron-Frobenius theory

$$\lim_{k \to \infty} \frac{A_i^k x_i}{\|A_i^k x_i\|_1} = \pi_i,$$

the stochastic eigenvector for the eigenvalue $\rho(A_i)$ of A_i. Thus,

$$\lim_{k \to \infty} p(A_i^k x_i, \pi_i) = 0.$$

Hence, using the sequence

$$A_1, A_2, \ldots, A_i, A_i, A_i, \ldots$$

since Theorem 12.6 still holds,

$$d(\pi_i, C) = 0$$

and so $\pi_i \in C$.

Now, using the triangle inequality

$$\begin{aligned} p\left(\bar{x}_i, A_i^{kr+t}\bar{x}_{i+1}\right) &\leq p\left(\bar{x}_i, \bar{x}_{i+1}\right) + p\left(A_i\bar{x}_i, A_i\bar{x}_{i+1}\right) \\ &+ \cdots + p\left(A_i^{kr+t}\bar{x}_i, A_i^{kr+t}\bar{x}_{i+1}\right) \\ &\leq rp\left(\bar{x}_i, \bar{x}_{i+1}\right) + r\tau_r p\left(\bar{x}_i, \bar{x}_{i+1}\right) \\ &+ \cdots + r\tau_r^k p\left(\bar{x}_i, \bar{x}_{i+1}\right) \\ &\leq \frac{r}{1-\tau_r} p\left(\bar{x}_i, \bar{x}_{i+1}\right). \end{aligned}$$

Now, letting $k \to \infty$,

$$p\left(\bar{x}_i, \pi_i\right) \leq \frac{r}{1-\tau_r} p\left(\bar{x}_i, \bar{x}_{i+1}\right)$$

and since $\pi_i \in C$,

$$d\left(\bar{x}_i, C\right) \leq \frac{r}{1-\tau_r} p\left(\bar{x}_i, \bar{x}_{i+1}\right),$$

as desired. ■

The intuition given by these theorems is that in fluctuating systems, the iterates need not converge. However, they do converge to a sphere about π. See Figure 12.1 for a picture.

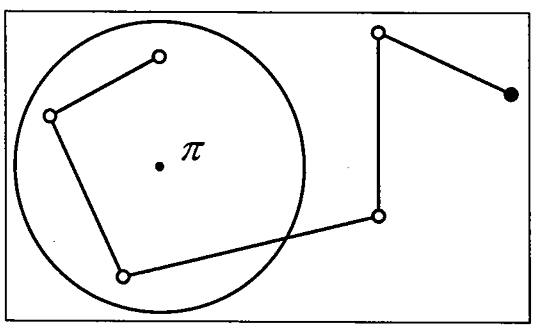

FIGURE 12.1. A view of iterations toward C.

An example providing numerical data follows.

Example 12.2 *Let*

$$A = \begin{bmatrix} .2 & .4 & .4 \\ .9 & 0 & 0 \\ 0 & .9 & 0 \end{bmatrix}.$$

Let Σ be the set of matrices C such that

$$A - .02A \leq C \leq A + .02A,$$

thus allowing for a 2% variation in the entries of A.

The stochastic eigenvector for A is $\pi = \begin{bmatrix} 0.3495 \\ 0.3331 \\ 0.3174 \end{bmatrix}$. *Starting with* $x_1 = \begin{bmatrix} 0.3434 \\ 0.3333 \\ 0.3232 \end{bmatrix}$ *and randomly generating the A_k's, we get*

$$x_{k+1} = A_k x_k \text{ and } \bar{x}_{k+1} = \frac{x_{k+1}}{\|x_{k+1}\|_1}.$$

Iterations 48 to 50 are shown in the table.

k	48	49	50
$\bar{x}_{k+1}$	$\begin{bmatrix} 0.3440 \\ 0.3299 \\ 0.3262 \end{bmatrix}$	$\begin{bmatrix} 0.3599 \\ 0.3269 \\ 0.3132 \end{bmatrix}$	$\begin{bmatrix} 0.3445 \\ 0.3470 \\ 0.3085 \end{bmatrix}$
$p\left(\bar{x}_{k+1}, \pi\right)$	0.0432	0.0483	0.0692

Using three iterates, we have

$$\begin{aligned} p\left(\bar{x}_{51}, \pi\right) &\leq \tau_B\left(A^3\right) p\left(\pi, \bar{x}_{48}\right) + 2\delta \\ &= 0.1044. \end{aligned}$$

This compares to the actual difference

$$p\left(\bar{x}_{51}, \pi\right) = 0.0692.$$

The error is

$$error = 0.0352.$$

12.4 Quotient Spaces

The trajectory $x_1, x_2, \ldots$ determined from an $n \times n$ matrix A, namely,

$$x_{k+1} = Ax_k$$

may tend to infinity. When this occurs, we can still discern something about the behavior of the trajectory. For example, if A and x_1 are positive, then we can look at the projected vectors $\frac{x_k}{\|x_k\|_1}$. In this section we develop an additional approach for studying such trajectories. We do this work in R^n so that formulas can be computed for the various norms.

Let e be the $n \times 1$ vector of 1's. Using R^n, define

$$W = \operatorname{span}\{e\}.$$

The quotient space R^n/W is the set

$$\{x + W : x \in R^n\}$$

where addition and scalar multiplication are done the obvious way, that is, $(x+W)+(y+W) = (x+y)+W$, $\alpha(x+W) = \alpha x + W$. This arithmetic is well defined, and the quotient space is a vector space.

12.4.1 Norm on R^n/W

Let

$$C = \{c : c^t w = 0 \text{ for all } w \in W\}$$

and

$$C_1 = \{c \in C : \|c\| = 1\}.$$

(C_1 depends on the norm used.) Then for any $x \in R^n/W$,

$$\|x + W\|_C = \max_{c \in C_1} |c^t x|.$$

It is easily seen that $\|\cdot\|_C$ is well defined and a norm on R^n/W.

Example 12.3 *Let $\|\cdot\|_1$ denote the 1-norm. It is known that, in this case, C_1 is the convex set with vertices those vectors with precisely two nonzero entries, $\frac{1}{2}$ and $-\frac{1}{2}$. Thus,*

$$\|x + W\|_C = \max_{c \in C_1} |c^t x|,$$

and if $c = \sum_{k=1}^{s} \alpha_k c_k$, a convex sum of the vertices $c_1, \ldots, c_s$ of C_1, then

$$\begin{aligned}\|x + W\|_C &\le \max_{c \in C_1} \sum_{k=1}^{s} \alpha_k |c_k^t x| \\ &= \max_k |c_k^t x| \\ &= \frac{1}{2} \max_{p,q} |x_p - x_q|.\end{aligned}$$

Since this max is achieved for some c_k, it follows that equality holds.

We conclude this subsection by showing how close a vector is to a coset. To do this, we define the distance from a vector y to a coset $x + W$ by

$$d(y, x+W) = \min_{w \in W} \|y - (x+w)\|_1.$$

We need a lemma.

Lemma 12.4 *Let $c \in C$ where $c = (c_1, \ldots, c_n)$. Then*

$$\frac{1}{2} \max c_i - \frac{1}{2} \min_i c_i \ge \frac{1}{n} \|c\|_1.$$

Proof. Without loss of generality, suppose that

$$c = (p_1, \ldots, p_r, q_1, \ldots, q_s)$$

where $r + s = n$ and

$$p_1 \ge \cdots \ge p_r \ge 0 \ge q_1 \ge \cdots \ge q_s.$$

Note that $\sum_{k=1}^{r} p_k = -\sum_{k=1}^{s} q_k$.

Now,

$$\begin{aligned}\frac{1}{2}p_1 - \frac{1}{2}q_s &\ge \frac{1}{2}\left(\frac{p_1 + \cdots + p_r}{r}\right) - \frac{1}{2}\left(\frac{q_1 + \cdots + q_s}{s}\right) \\ &= \frac{1}{2r}(p_1 + \cdots + p_r) - \frac{1}{2s}(q_1 + \cdots + q_s) \\ &= \left(\frac{1}{2r} + \frac{1}{2s}\right)(p_1 + \cdots + p_r) \\ &\ge \frac{2}{r+s}(p_1 + \cdots + p_r) \\ &= \frac{1}{r+s}(p_1 + \cdots + p_r) - \frac{1}{r+s}(q_1 + \cdots + q_s) \\ &= \frac{1}{r+s}\|c\|_1,\end{aligned}$$

the desired result. ■

Using the 1-norm to determine C_1, we have the following.

Theorem 12.8 *Suppose $\|(y+W) - (x+W)\|_C \le \epsilon$. Then we have that $d(y, x+W) \le n\epsilon$.*

Proof. First suppose that $y, x \in C$. Then, using the example and lemma, we have $\|(y+W)-(x+W)\|_C$

$$= \left(\frac{1}{2}\max_i (y_i - x_i) - \frac{1}{2}\min_j (y_j - x_j)\right)$$
$$\geq \left(\frac{1}{n}\sum_{i=1}^{n} |y_i - x_i|\right)$$
$$= \frac{1}{n}\|y-x\|_1 .$$

Now, let $x, y \in R^n$. Write $x = \hat{x} + w_1$, $y = \hat{y} + w_2$ where $\hat{x}, \hat{y} \in C$ and $w_1, w_2 \in W$. Then, using the first part of the proof,

$$\begin{aligned}\|(y+W)-(x+W)\|_C &= \|(\hat{y}+W)-(\hat{x}+W)\|_C \\ &\geq \frac{1}{n}\|\hat{y}-\hat{x}\|_1 \\ &= \frac{1}{n}\|(y-w_2)-(x-w_1)\|_1 \\ &= \frac{1}{n}\|y-(x+w)\|_1\end{aligned}$$

where $w = w_2 - w_1$. And, from this it follows that

$$d(y, x+W) \leq n\epsilon,$$

which was required. ■

12.4.2 Matrices in R^n/W

Let A be an $n \times n$ matrix such that

$$A : W \to W.$$

Thus $Ae = \rho e$ for some real eigenvalue ρ. (In applications, we will have $\rho = \rho(A)$.)
Define

$$A : R^n/W \to R^n/W$$

by

$$A(x+W) = Ax + W.$$

It can be observed that this is a coset map, not a set map. In terms of sets, $A(x+W) \subseteq Ax + W$ with equality not necessarily holding. The map $A : R^n/W \to R^n/W$ is well defined and linear on the vector space R^n/W.
Inverses, when they exist, for maps can be found as follows. Using the Schur decomposition

$$A = P\begin{bmatrix} \rho & y \\ 0 & B \end{bmatrix} P^t$$

where P is orthogonal and has $\frac{e}{\sqrt{n}}$ as its first column. If B is nonsingular, set

$$A^+ = P\begin{bmatrix} \rho & y \\ 0 & B^{-1} \end{bmatrix} P^t.$$

(Other choices for A^+ are also possible.)

Lemma 12.5 *A^+ is the inverse of A on R^n/W.*

Proof. To show that $A^+ : W \to W$, let $w \in W$. Then $w = \alpha e$ for some scalar α. Thus,

$$\begin{aligned}A^+ w &= A^+(\alpha e) \\ &= \alpha P\begin{bmatrix} \rho & y \\ 0 & B^{-1} \end{bmatrix} P^t e \\ &= \alpha P\begin{bmatrix} \rho & y \\ 0 & B^{-1} \end{bmatrix}\begin{bmatrix} \frac{n}{\sqrt{n}} \\ 0 \end{bmatrix} \\ &= \alpha P\begin{bmatrix} \rho\frac{n}{\sqrt{n}} \\ 0 \end{bmatrix} \\ &= \alpha\rho e \in W.\end{aligned}$$

Now,

$$AA^+ = P\begin{bmatrix} \rho^2 & \rho y + yB^{-1} \\ 0 & I \end{bmatrix} P^t.$$

If $x \in C$, then

$$\begin{aligned}AA^+ x &= P\begin{bmatrix} \rho^2 & \rho y + yB^{-1} \\ 0 & I \end{bmatrix} P^t x \\ &= P\begin{bmatrix} \rho^2 & \rho y + yB^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 \\ P_2 x \end{bmatrix}\end{aligned}$$

where $P^t = \begin{bmatrix} \frac{e^t}{\sqrt{n}} \\ P_2 \end{bmatrix}$,

$$= P \begin{bmatrix} \left(\rho y + yB^{-1}\right) P_2 x \\ P_2 x \end{bmatrix}$$
$$= P \begin{bmatrix} \beta \\ P_2 x \end{bmatrix}$$

where $\beta = \left(\rho y + yB^{-1}\right) P_2 x$,

$$= \beta \frac{e}{\sqrt{n}} + P_2^t P_2 x$$
$$= \beta \frac{e}{\sqrt{n}} + x \in x + W.$$

Thus, $AA^+ (x + W) = x + W$ and since x was arbitrary AA^+ is the identity on R^n/W. Similarly, so is A^+A. So A^+ is the inverse of A on R^n/W. ■

When $A : W \to W$, we can define the norm on the matrix $A : R^n/W \to R^n/W$ as

$$\|A\|_C = \max_{x \notin W} \frac{\|Ax + W\|_C}{\|x + W\|_C}$$
$$= \max_{\|x+W\|_C = 1} \|Ax + W\|_C .$$

For special norms, expressions for norms on A can be found. An example for the 1-norm follows.

Example 12.4 *For the* 1*-norm on* R^n*: Define*

$$\|A\|_C = \max_{x \notin W} \frac{\|Ax + W\|_C}{\|x + W\|_C}$$
$$= \max_{\|x+W\|_C = 1} \|Ax + W\|_C .$$

Now

$$\|Ax + W\|_C = \max_{c \in C_1} \left|c^t Ax\right|$$

which by Example 12.3,

$$= \frac{1}{2} \max_{i,j} |a_i x - a_j x|$$
$$= \frac{1}{2} \max_{i,j} |(a_i - a_j)\, x|$$

where a_k *is the k-th row of A. Since* $\|x + W\|_C = 1$,

$$\frac{1}{2} \max_{i,j} |x_i - x_j| = 1.$$

And, since $x + \alpha e \in x + W$ *for all* α*, we can assume that* x_i *is nonnegative for all i and* 0 *for some i. Thus, the largest entry* x_i *in* x *is* 2*. It follows that*

$$\max_{i,j} |(a_i - a_j)\, x|$$

over all x, $0 \le x_i \le 2$ *is achieved by setting* $x_k = 2$ *when the k-th entry in* $a_i - a_j$ *is positive and* 0 *otherwise. Hence,*

$$\max_{i,j} |(a_i - a_j)\, x| = \max_{i,j} \|a_i - a_j\|_1 .$$

Thus,

$$\|Ax + W\|_C = \frac{1}{2} \max_{i,j} \|a_i - a_j\|_1$$

and so

$$\|A\|_C = \frac{1}{2} \max_{i,j} \|a_i - a_j\|_1 .$$

To obtain the usual notation, set

$$\tau_1(A) = \|A\|_C .$$

This gives the following result.

Theorem 12.9 *Using the* 1*-norm on* R^n*,*

$$\|A\|_C = \tau_1(A) .$$

12.4.3 Behavior of Trajectories

To see how to use quotient spaces to analyze the behavior of trajectories, let

$$x_{k+1} = Ax_k + b$$

where $A : W \to W$. In terms of quotient spaces, we convert the previous equation into

$$x_{k+1} + W = A(x_k + W) + (b + W) . \tag{12.6}$$

To solve this system, we subtract the k-th equation from the $(k+1)$-st one. Thus, if we let

$$z_{k+1} + W = (x_{k+1} + W) - (x_k + W),$$

then

$$z_{k+1} + W = A(z_k + W)$$

so

$$z_{k+1} + W = A^k(z_0 + W).$$

Using norms,

$$\|z_{k+1} + W\|_C \le \tau_1(A)^k \|z_1 + W\|_C. \qquad (12.7)$$

If $\tau_1(A) < 1$, then $z_k + W$ converges to W geometrically.

As a consequence, $\langle x_k + W \rangle$ is Cauchy and thus converges to say $x + W$. (It is known that the quotient space is complete.) So we have

$$x + W = A(x + W) + (b + W). \qquad (12.8)$$

Now, subtracting (12.8) from (12.6), we have

$$x_{k+1} - x + W = A(x_k - x) + W.$$

Thus, by (12.7),

$$\|x_{k-1} - x + W\|_C \le \tau_1(A)^k \|x_1 - x + W\|_C.$$

And, using that $d(x_k - x, W) = d(x_k, x + W)$, as well as Theorem 12.8,

$$d(x_k, x + W) \le n\tau_1(A)^k \|x_1 - x + W\|_C. \qquad (12.9)$$

It follows that x_k converges to $x + W$ at a geometric rate.

Example 12.5 *Let* $A = \begin{bmatrix} .8 & .3 \\ .3 & .8 \end{bmatrix}$ *and*

$$x_{k+1} = Ax_k.$$

Note that $\langle A^k \rangle$ *tends componentwise to* ∞. *The eigenvalues for A are 1.1 and .5, with 1.1 having eigenvector* $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *Let*

$$W = span\{e\}.$$

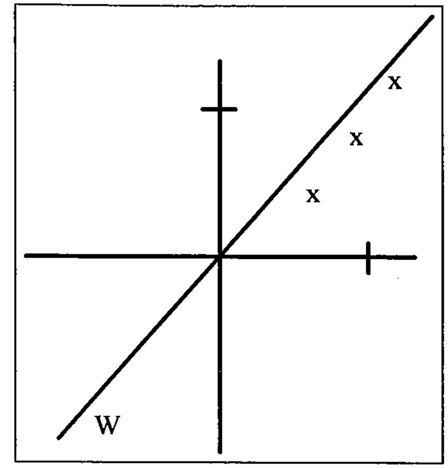

FIGURE 12.2. Iterates converging to W.

Then, using (12.9) with $x = 0$, *we have*

$$d(x_k, W) \le 2\tau_1(A)^k \|x_1 + W\|_C.$$

Then, since $\tau_1(A) = .5$,

$$d(x_k, W) \le 2(.5)^k \|x_1 + W\|_C.$$

So, x_k *converges to W at a geometric rate.*

A sample of iterates, letting $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, *follows in the table below. By direct calculation,* $\|x_1 + W\|_C = .5$.

k	4	8	12
x_k	$(0.763, 0.3700)^t$	$(1.073, 1.069)^t$	$(1.569, 1.569)^t$
$d(x_k, W)$	0.125	0.0078	0.0005

After 12 iterations, no change was seen in the first four digits of the x_k*'s. However, growth in the direction of e still occurred, as seen in Figure 12.2.*

Extending a bit, let

$$A_1 = \begin{bmatrix} .8 & .3 \\ .3 & .8 \end{bmatrix}, A_2 = \begin{bmatrix} .2 & .5 \\ .5 & .2 \end{bmatrix}, A_3 = \begin{bmatrix} .7 & .9 \\ .9 & .7 \end{bmatrix}$$

and

$$\Sigma = \{A_1, A_2, A_3\}.$$

Since W, for each matrix in Σ, is span$\{e\}$, *where* $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, *and* $\tau_1(\Sigma) = \max_i \tau(A_i) = .5$, *we have the same situation for the equation*

$$x_{k+1} = A_{i_k} x_k$$

where $\langle A_{i_k} \rangle$ *is any sequence from* Σ.

The following examples show various adjustments that can be made in applying the results given in this section.

Example 12.6 *Consider*

$$x_{k+1} = Ax_k$$

where

$$A = \begin{bmatrix} .604 & .203 & 0 \\ 2.02 & 0 & 0 \\ 0 & 2.02 & 0 \end{bmatrix}.$$

Here the eigenvalues of A are 1.01, −.4060, *and* 0 *with* $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ *an eigenvector belonging to* 1.01. *Let* $D = \operatorname{diag}(1, 2, 4)$. *Then*

$$Dx_{k+1} = DAD^{-1}Dx_k$$

or

$$y_{k+1} = By_k$$

where $y_k = Dx_k$ *and* $B = DAD^{-1}$. *Here*

$$B = \begin{bmatrix} .604 & .406 & 0 \\ 1.01 & 0 & 0 \\ 0 & 1.01 & 0 \end{bmatrix}.$$

and an eigenvector of B for 1.01 *is* $e = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. *So we have* $W = \operatorname{span}\{e\}$.

Now, $\tau_1(B) = 1.01$; *however,* $\tau_1(B^3) \approx .1664$. *Thus, from (12.9),*

$$d(y_k, W) \text{ converges to } 0$$

at a geometric rate. Noting that

$$d\left(D^{-1}y_k, D^{-1}W\right) \le \max_i \left|d_i^{-1}\right| d(y_k, W),$$

it follows that

$$d\left(x_k, D^{-1}W\right) \text{ converges to } 0$$

at a geometric rate.

Example 12.7 *Consider*

$$x_{k+1} = Ax_k + b$$

where $A = \begin{bmatrix} .7 & .4 \\ .4 & .7 \end{bmatrix}$ *and* $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. *Note that* $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *is an eigenvector of A belonging to the eigenvalue* 1.1 *and so we have* $W = \operatorname{span}\{e\}$. *Further,* $\tau_1(A) = .3$. *The corresponding quotient space equation is*

$$x_{k+1} + W = A(x_k + W) + (b + W).$$

The sequence $\langle x_k + W \rangle$ *converges to, say,* $x + W$. *Thus,*

$$x + W = A(x + W) + (b + W).$$

Solving for $x + W$ *yields*

$$(I - A)(x + W) = b + W,$$

so

$$(x + W) = (I - A)^+ (b + W),$$

where

$$(I - A)^+ = \begin{bmatrix} .9 & .2 \\ .2 & .9 \end{bmatrix},$$

and thus

$$(I - A)^+ b = \begin{bmatrix} 1.3 \\ 2 \end{bmatrix}.$$

It follows by (12.8) that

$$x_k + W \to \begin{bmatrix} 1.3 \\ 2 \end{bmatrix} + W,$$

or using $x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$d(x_k, x+W) \leq 2 \times .3^k \times \|x+W\|_C$$
$$\leq 2 \times .3^k \times .35 \leq .7 \times .3^k.$$

A few iterates follow.

k	2	4	8
x_k	$(2.5, 3.8)^t$	$(6.253, 7.70)^t$	$(16.44, 17.87)^t$
$d(x_k, x+W)$	0.063	0.00567	0.000046

12.5 Research Notes

The results in Section 1, slowly varying products and convergence to 0, are basically due to Smith (1966a, 1966b). Some alterations were done to obtain a simpler result. Section 2 contains a result of Artzrouni (1996). For other such results, see Artznouni (1991). Application work can be found in Artzrouni (1986a, 1986b).

Section 3 is due to Hartfiel (2001) and Section 4, Hartfiel (1997). Rhodius (1998) also used material of this type.

Often bound work is not exact, but when not exact, the work can still give some insight into a system's behavior.

In related research, Johnson and Bru (1990) showed for slowly varying positive eigenvectors, $\rho(A_1 \cdots A_k) \approx \rho(A_1) \cdots \rho(A_k)$. Bounds are also provided there.

13 Systems

This chapter looks at how infinite products of matrices can be used in studying the behavior of systems. To do this, we include a first section to outline techniques.

13.1 Projective Maps

Let Σ be a set of $n \times n$ row allowable matrices and X a set of positive $n \times 1$ vectors. In this section, we outline the general idea of finding bounds on the components of the vectors in a set, say $\Sigma^s X$ or $\Sigma_p^s X$. Basically we use that for a convex polytope, smallest and largest component bounds occur at vertices as depicted in Figure 13.1.

Let Σ also be a column convex and U a convex subset of positive vectors. If

$$\Sigma = convex\{A_1, \ldots, A_p\}$$

and

$$U = convex\{x_1, \ldots, x_q\},$$

then, as shown in 9.3, ΣU is a convex polytope of positive vectors, whose vertices are among the vectors in $V = \{A_i x_j : A_i$ and x_j are vertices in Σ

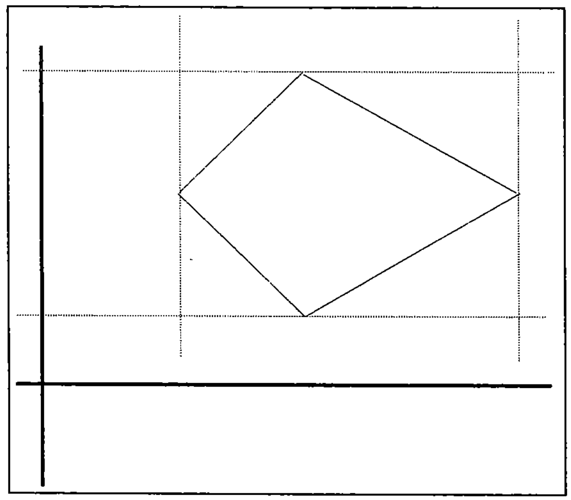

FIGURE 13.1. Component bounds for a convex polytope.

and U, respectively}. Thus, if we want component bounds on $\Sigma^s X$, we need only compute them on $A_{i_s} \cdots A_{i_1} x_j$ over all choices of $i_s, \ldots, i_1$, and j. For Σ_p, component bounds are found by finding them on $w_{A_{i_s}} \circ \cdots \circ w_{A_{i_1}}(x_j)$ over all choices of $i_s, \ldots, i_1$, and j.

To compute component bounds on these vectors, we use the Monte-Carlo method.

Component Bounds

1. Randomly (uniform distribution) generating the vertex matrices involved, compute

$$x = A_{i_1} x_j \text{ or } x = w_{A_{i_1}}(x_j).$$

2. Suppose $x = A_{i_t} \cdots A_{i_1} x_j$ or $x = w_{A_{i_t}} \circ \cdots \circ w_{A_{i_1}}(x_j)$ have been found. If $t < s$, randomly (uniform distribution) generating the vertex matrix involved, compute

$$x = A_{i_{t+1}} \cdots A_{i_1} x_j \text{ or } x = w_{A_{i_{t+1}}} \circ \cdots \circ w_{A_{i_1}}(x_j).$$

Continue until $x = A_{i_s} \cdots A_{i_1} x_j$ or $x = w_{A_{i_s}} \circ \cdots \circ w_{A_{i_1}}(x_j)$ is found.

3. Set $L_1 = H_1 = x$.

4. Repeat (1) and (2). After the $k+1$-st run, set

$$l_i^{(k+1)} = \min\left\{l_i^{(k)}, x_i\right\}, h_i^{(k+1)} = \max\left\{h_i^{(k)}, x_i\right\}$$

and form $L_{k+1} = \left(l_i^{(k+1)}\right), H_{k+1} = \left(h_i^{(k+1)}\right)$.

5. Continue for sufficiently many runs. (Some experimenting may be required here.)

13.2 Demographic Problems

This section provides two problems involving populations, partitioned into various categories, at discrete intervals of time.

Taking a small problem, suppose a population is divided into three groups: 1=young, 2=middle, and 3=old, where young is aged 0 to 5, middle 5 to 10, and old 10 to 15.

Let $x_k^{(1)}$ denote the population of group k for $k = 1, 2, 3$. After 5 years, suppose this population has changed to

$$\begin{aligned} x_1^{(2)} &= b_{11}x_1^{(1)} + b_{12}x_2^{(1)} + b_{13}x_3^{(1)} \\ x_2^{(2)} &= s_{21}x_1^{(1)} \\ x_3^{(2)} &= s_{32}x_2^{(1)} \end{aligned}$$

or

$$x_2 = Ax_1$$

where $x_k = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\right)$ for $k = 1, 2$, and

$$A = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ s_{21} & 0 & 0 \\ 0 & s_{32} & 0 \end{bmatrix}.$$

The matrix A is called a *Leslie matrix.*

Continuing, after 10 years, we have

$$x_3 = Ax_2 = A^2 x_1,$$

etc.

Data indicates that birth (the b_{ij}'s) and survival (the s_{ij}'s) rates change during time periods, and thus we will consider the situation

$$x_{k+1} = A_k \cdots A_1 x_1$$

where $A_1, \ldots, A_k$ are the Leslie matrices for time periods $1, \ldots, k$. In this section, we look at component bounds on $\bar{x}_{k+1}$.

Example 13.1 *Let the Leslie matrix be*

$$A = \begin{bmatrix} .2 & .4 & .4 \\ .9 & 0 & 0 \\ 0 & .9 & 0 \end{bmatrix}.$$

Allowing for a 2% variation in the entries of A, we assume the transition matrices satisfy

$$A - .02A \leq A_k \leq A + .02A$$

for all k. Thus, Σ is the convex polytope with vertices

$$C = [a_{ij} \pm .02a_{ij}].$$

We start the system at $x = \begin{bmatrix} 0.3434 \\ 0.3333 \\ 0.3232 \end{bmatrix}$, *and estimate the component bounds on $\Sigma_p^{10}x$, the 50-year distribution vectors of the system by Monte Carlo. We did this for 1000 to 200,000 runs to compare the results. The results are given in the table below.*

k	L_k	H_k
1000	(0.3364, 0.3242, 0.2886)	(0.3669, 0.3608, 0.3252)
5000	(0.3363, 0.3232, 0.2859)	(0.3680, 0.3632, 0.3267)
10,000	(0.3351, 0.3209, 0.2841)	(0.3682, 0.3635, 0.3286)
100,000	(0.3349, 0.3201, 0.2835)	(0.3683, 0.3641, 0.3313)
200,000	(0.3340, 0.3195, 0.2819)	(0.3690, 0.3658, 0.3318)

Of course, the accuracy of our results is not known. Still, using experimental probability, we feel the distribution vector, after 50 years, will be bounded by our L and H, with high probability.

A picture of the outcome of 1000 runs is shown in Figure 13.2. We used

$$T = \begin{bmatrix} 0 & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix}$$

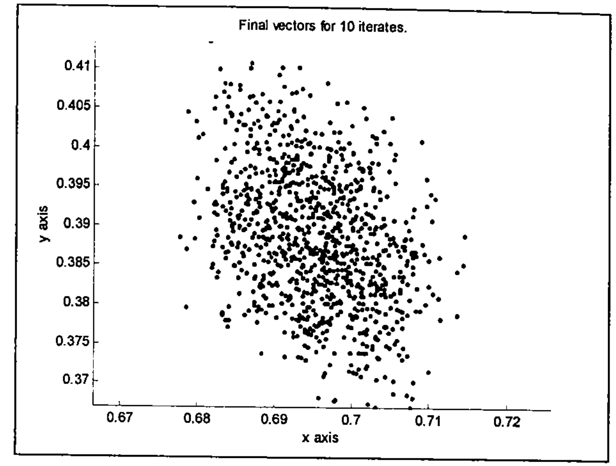

FIGURE 13.2. Final vectors of 10 iterates.

to map S^+ into R^2.

Recall that $\Sigma_p^{10}x$ is the convex hull of vertices. Yet as can be seen, many, many calculations $A_{k_{10}} \cdots A_{k_1}x$ do not yield vertices of $\Sigma_p^{10}x$. In fact, they end up far in the interior of the convex hull.

Taking some point in the interior, say the projection of the average of the lower and upper bounds after 200,000 runs, namely

$$ave = (0.3514, 0.3425, 0.3062),$$

we can empirically estimate the probability that the system is within some specified distance δ of average. Letting c denote the number of times a run ends with a vector x, $\|x - ave\|_\infty < \delta$, we have the results shown in the table below. We used 10,000 runs.

δ	c
.005	1873
.009	5614
.01	6373
.02	9902

Interpreting $\delta = .01$, we see that in 6373 runs, out of 10,000 runs, the result x agreed with those of ave. to within .01.

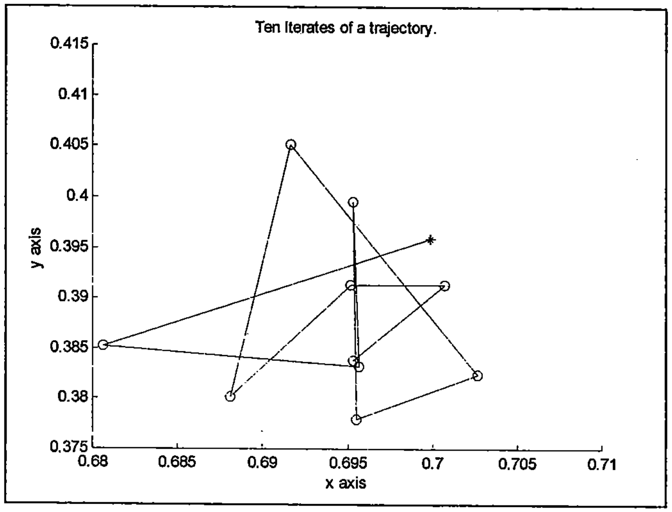

FIGURE 13.3. Ten iterates of a trajectory.

Finally, it may be of some interest to see the movement of $x_1, x_2, \ldots, x_{10}$ *for some run. To see these vectors in* R^2*, we again use the matrix*

$$T = \begin{bmatrix} 0 & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix}$$

and plotted $Tx_1, Tx_2, \ldots, Tx_{10}$*. A picture of a trajectory is shown in Figure 13.3.*

In this figure, the starting vector is shown with a $*$ *and other vectors with a* $\circ$*. The vectors are linked sequentually by segments to show where vectors go in proceeding steps.*

13.3 Business, Man Power, Production Systems

Three additional examples using infinite products of matrices are given in this section.

Example 13.2 *A taxi driver takes fares within and between two towns, say* T_1 *and* T_2*. When without a fare the driver can*

1. *cruise for a fare, or*
2. *go to a taxi stand.*

The probabilities of the drivers' actions on (1) and (2) are given by the matrices

$$A = \begin{bmatrix} .55 & .45 \\ .6 & .4 \end{bmatrix}, \quad B = \begin{bmatrix} .5 & .5 \\ .4 & .6 \end{bmatrix}$$

for each of the towns, as seen from the diagram in Figure 13.4.

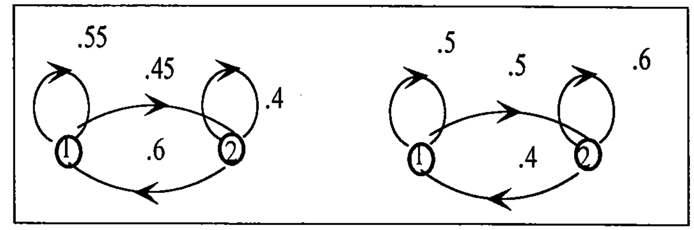

FIGURE 13.4. Diagram of taxi options.

We can get the possible probabilities of the cab driver being in (1) or (2) by finding Σ^∞ *where* $\Sigma = \{A, B\}$*, a* τ*-proper set. Here* $E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and the corresponding subspace coefficient is* $\tau_W(\Sigma) = .2$*. Three products should be enough for about 2 decimal place accuracy. Here,*

$$A^3 = \begin{bmatrix} 0.5450 & 0.4550 \\ 0.5460 & 0.4540 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 0.4550 & 0.5450 \\ 0.4540 & 0.5460 \end{bmatrix}$$

$$AB^2 = \begin{bmatrix} 0.4450 & 0.5550 \\ 0.4460 & 0.5540 \end{bmatrix}, \quad ABA = \begin{bmatrix} 0.5550 & 0.4450 \\ 0.5540 & 0.4460 \end{bmatrix}$$

$$BAB = \begin{bmatrix} 0.4550 & 0.5450 \\ 0.4560 & 0.5440 \end{bmatrix}, \quad B^2A = \begin{bmatrix} 0.5550 & 0.4450 \\ 0.5560 & 0.4440 \end{bmatrix}$$

$$BA^2 = \begin{bmatrix} 0.5450 & 0.4550 \\ 0.5440 & 0.4560 \end{bmatrix}, \quad BA^2 = \begin{bmatrix} 0.5450 & 0.4550 \\ 0.5440 & 0.4560 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0.4450 & 0.5550 \\ 0.4440 & 0.5560 \end{bmatrix}.$$

If

$p_{ij} =$ *probability that if the taxi driver is in i initially, he eventually (in the long run) ends in j,* *then*

$$.44 \le p_{11} \le .56 \qquad .44 \le p_{12} \le .56$$
$$.44 \le p_{21} \le .56 \qquad .44 \le p_{22} \le .56.$$

Note the probabilities vary depending on the sequence of A's and B's. However, regardless of the sequence, the bounds above hold.

In the next example, we estimate component bounds on a limiting set.

Example 13.3 *A two-phase production process has input β at phase 1. In both phase 1 and phase 2 there is a certain percentage of waste and a certain percentage of the product at phase 2 is returned to phase 1 for a repeat of that process. These percentages are shown in the diagram in Figure 13.5.*

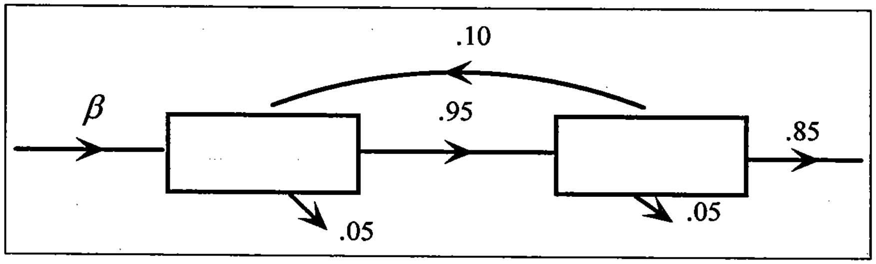

FIGURE 13.5. A two-phase production process.

For the mathematical model, let

$$a_{ij} = \text{percentage of the product in process } i \text{ that goes to process } j.$$

Then

$$A = \begin{bmatrix} 0 & .95 \\ .10 & 0 \end{bmatrix}.$$

And if we assume there is a fluctuation of at most 5% in the entries of A at time k, then

$$A - .05A \le A_k \le A + .05A.$$

Thus, if $x_k = \left(x_1^{(k)}, x_2^{(k)}\right)$ where

$$x_1^{(k)} = \text{amount of product at phase 1}$$
$$x_2^{(k)} = \text{amount of product at phase 2},$$

then our model would be

$$x_{k+1} = x_k A_k + b \tag{13.1}$$

where $b = (\beta, 0)$ and numerically,

$$\begin{bmatrix} 0 & 0.9025 \\ 0.095 & 0 \end{bmatrix} \le A_k \le \begin{bmatrix} 0 & 0.9975 \\ 0.105 & 0 \end{bmatrix}.$$

If we put this into our matrix equation form, we have

$$(1, x_{k+1}) = (1, x_k) \begin{bmatrix} 1 & b \\ 0 & A_k \end{bmatrix}.$$

(We can ignore the first entries in these vectors to obtain (13.1).) Then Σ is convex and has vertices

$$A_1 = \begin{bmatrix} 1 & \beta & 0 \\ 0 & 0 & .9025 \\ 0 & .095 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & \beta & 0 \\ 0 & 0 & .9025 \\ 0 & .105 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 1 & \beta & 0 \\ 0 & 0 & .9975 \\ 0 & .095 & 0 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 1 & \beta & 0 \\ 0 & 0 & .9975 \\ 0 & .105 & 0 \end{bmatrix}.$$

As in the previous example, we estimate component bounds on $y\Sigma^{10}$. Using $\beta = 1$, $y = (1, 0.5, 0.5)$, and doing 10,000 and 20,000 runs, we have the data in the table below.

no. of runs	L	H
10,000	(1, 1.0938, 0.9871)	(1, 1.1170, 1.1142)
20,000	(1, 1.0938, 0.9871)	(1, 1.1170, 1.1142)

We can actually calculate exact component bounds on $y\Sigma^{\infty}$ by using

$$L = yA_1^{10} = (1,\ 1.0938,\ 0.9871)$$
$$H = yA_4^{10} = (1,\ 1.1170,\ 1.1142).$$

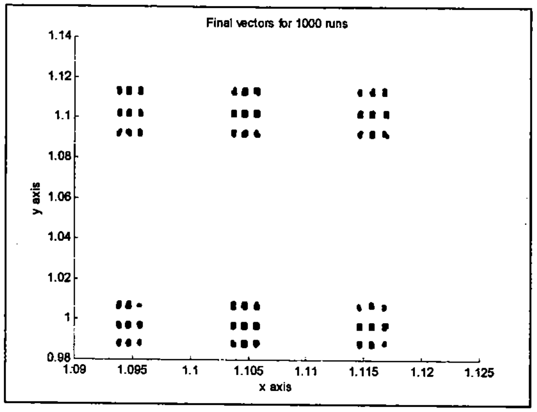

FIGURE 13.6. Final vectors for 1000 runs.

Thus, our estimated bounds are correct. To see why, it may be helpful to plot the points obtained in these runs. Projected into the yz-plane (All x coordinates are 1.), we have the picture shown in Figure 13.6. Observe that in this picture, many of the points are near vertices. The picture, to some extent, tells why the estimates were exact.

To estimate convergence rates to $y\Sigma^\infty$*, note that* Σ *is* τ*-proper where*

$$E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus,

$$\tau_W(B) = \max\{\|b_2\|_1, \|b_3\|_1\},$$

where b_k *is the k-th row of* $B \in \Sigma$*. So,*

$$\tau_W(\Sigma) = .9975.$$

The value of $\tau_W(\Sigma)$ *can be made smaller by simultaneously scaling rows and columns to get each row sum the same. To do this let*

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{0.525}{0.9975}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2294 \end{bmatrix}.$$

Then, using the norm

$$\|(1,x)\|_d = \|(1,x)D\|_1,$$

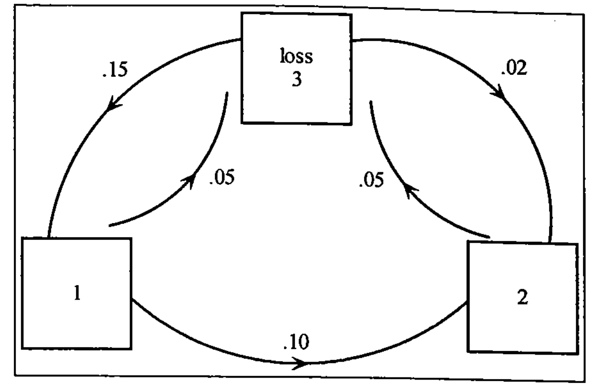

FIGURE 13.7. Management structure.

we have $\|B\|_d = \|D^{-1}BD\|_1$ *for any* 3×3 *matrix B, and so*

$$\tau_W(\Sigma) = 0.2294.$$

By Theorem 9.5,

$$h\left(y\Sigma^k, y\Sigma^\infty\right) \le \tau_W(\Sigma)^k\, h(y, y\Sigma^\infty) \tag{13.2}$$

which shows more rapid convergence.

The last example concerns management structures.

Example 13.4 *We analyze the management structure of a business by partitioning all managing personal into categories: 1 = staff and 2 = executive. We also have 3 = loss (due to change of job, retirement, etc.) state. And, we assume that we hire a percentage of the number of people lost.*

Suppose the 1 year flow in the structure is as given in the diagram in Figure 13.7.

If $p_k = (x_k, y_k, z_k)$ *gives the number of employees in 1,2,3, respectively, at time k, then 1 year later we would have*

$$\begin{aligned} x_{k+1} &= .85x_k + .10y_k + .05z_k \\ y_{k+1} &= + .95y_k + .05z_k \\ z_{k+1} &= .15x_k + .02y_k + .83z_k \end{aligned}$$

or

$$p_{k+1} = Ap_k$$

where $A = \begin{bmatrix} .85 & .10 & .05 \\ 0 & .95 & .05 \\ .15 & .02 & .83 \end{bmatrix}$.

Of course, we would expect retirements, new jobs, etc. to fluctuate some, and thus we suppose that matrix A fluctuates, yielding

$$p_{k+1} = A_k p_k.$$

For this example, we will suppose that each A_k *has no more than 2% fluctuation from A, so*

$$A - .02A \leq A_k \leq A + .02A$$

for all k. Let Σ *denote the set of all of these matrices.*

The set Σ *is* τ*-proper with* $E = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. *Then*

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

whose unit circle in the 1-norm is

$$convex\{c_1, c_2, c_3\}$$

where $c_1 = \pm\frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, c_2 = \pm\frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, c_3 = \pm\frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. *Thus, using Theorem 2.12*

$$\tau_W(A) = \max\left\{\frac{1}{2}\|a_1 - a_2\|_1, \frac{1}{2}\|a_1 - a_3\|_1, \frac{1}{2}\|a_2 - a_3\|_1\right\}$$

where a_k *is the k-th row of A. Using the formula, we get*

$$\tau_W(\Sigma) \leq 0.9166.$$

So Σ^∞ *exists; however, convergence may be very slow. Thus, we only show what can occur to this system in 10 years by finding component bounds* $\Sigma^{10}x$ *for* $x = \begin{bmatrix} 300 \\ 50 \\ 15 \end{bmatrix}$. *Using k runs, we find the following.*

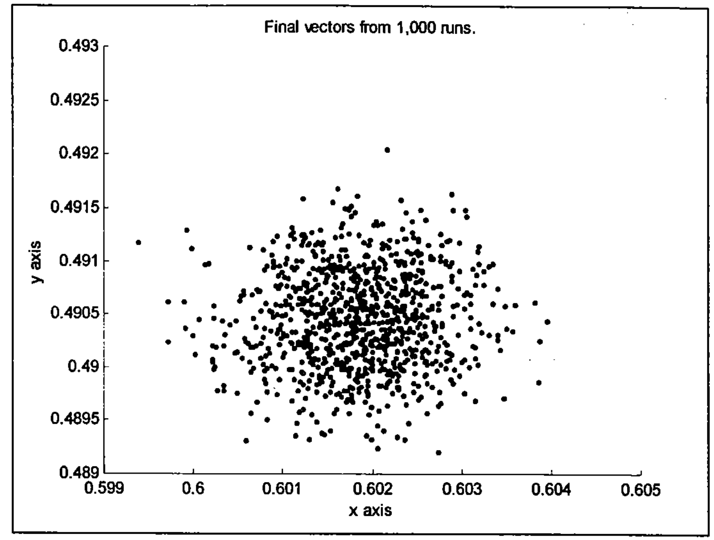

FIGURE 13.8. Final vectors of 1000 runs.

k	L	H
500	(122.54, 73.63, 130.90)	(125.42, 75.33, 134.01)
1000	(122.16, 73.73, 130.87)	(125.32, 75.33, 134.05)
10,000	(121.98, 73.51, 130.53)	(125.42, 75.47, 134.20)

A picture, showing where the system might be, depending on the run, can be seen, for 1,000 runs, in Figure 13.8.

The points occurring at the end of the runs were mapped into R^2 *using the matrix* $T = \begin{bmatrix} 0 & \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix}$.

13.4 Research Notes

The work in this chapter extends that in Chapter 11. The taxi problem can be found in Howard (1960).

Hartfiel (1998) showed how to obtain precise component bounds for those Σ which are intervals of stochastic matrices. However, no general such technique is known.

13.5 MATLAB Codes

Component Bounds for Demographic Problem

```
A=[.2  .4  .4;  .9  0  0;  0  .9  0];
B=[.196  .392  .392; .882  0  0;0 .882 0]
L=ones(1,3);
H=zeros(1,3);
for m=1:5000
    x=[.3434; .3333; .3232];
    for k=1:10
        for i=1:3
            for j=1:3
                G=rand;
                if G<=.5
                    d=1;
                else
                    d=0;
                end
            C(i,j)=B(i,j)+d*.04*A(i,j);
            end
        end
        x=C*x/norm(C*x,1);
    end
    for i=1: 3
    L(i)=min([L(i), x(i)]);
    H(i)=max([H(i), x(i)]);
    end
end
L
H
```

Final Vectors Graph for Demographics Problem

```
A=[.2  .4  .4;  .9  0  0;  0  .9  0];
B=[.196  .392  .392; .882  0  0;0 .882 0];
T=[0 sqrt(2) 1/sqrt(2);0 0 sqrt(6)/2];
hold on
axis equal
xlabel('x axis')
ylabel('y axis')
title('Final vectors for 10 iterates')
```



```
hold on
for r=1:1000
    x=[.3434; .3333; .3232];
    for k=1:10
        for i=1:3
            for j=1:3
                G=rand;
                if G<=.5
                    d=1;
                else
                    d=0;
                end
                C(i,j)=B(i,j)+d*.04*A(i,j);
            end
        end
        x=C*x/norm(C*x,1);
    end
    y=T*x
    plot(y(1),y(2))
end
```

Trajectory for Demographics Problem

```
A=[.2  .4  .4;  .9  0  0;  0  .9  0];
B=[.196  .392  .392; .882  0  0;0 .882 0];
T=[0 sqrt(2) 1/sqrt(2);0 0 sqrt(6)/2];
y=[.3434, .3333, .3232];
z=T*y;
x=[.3434; .3333; .3232];
xlabel('x axis')
ylabel('y axis')
title('Ten iterates of a trajectory')
hold on
plot(z(1),z(2),'k:*')
for k=1:10
    for i=1:3
        for j=1:3
            G=rand;
            if G<=.5
                d=1;
            else
```

```
            d=0;
         end
         C(i,j)=B(i,j)+(d*.04)*A(i,j);
      end
   end
   x=C*x/norm(C*x,1);
   p=T*x;
   q=T*y;
   plot(p(1),p(2),'o')
   plot([p(1),q(1)],[p(2),q(2)])
   y=x
end
```

Appendix

We give a few results used in the book.

Perron-Frobenius Theory:

Let A be an $n \times n$ nonnegative matrix. If $A^k > 0$ for some positive integer k, then A is primitive. If A isn't primitive, but is irreducible, there is an integer r called A's index of imprimitivity. For this r, there is a permutation matrix P such that

$$PAP^t = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ & & & \cdots & \\ A_r & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where the r main diagonal 0-blocks are square and r the largest integer producing this canonical form.

If A is nonnegative, A has an eigenvalue $\rho = \rho(A)$ where

$$Ay = \rho y$$

and y is a nonnegative eigenvector. If A is primitive, it has exactly one eigenvalue ρ where

$$\rho > |\lambda|$$

for all eigenvalues $\lambda \neq \rho$ of A. If A is irreducible, with index r, then A has eigenvalues

$$\rho, \rho e^{i\frac{2\pi}{r}}, \rho e^{i\frac{4\pi}{r}}, \ldots$$

all of multiplicity one with all other eigenvalues λ satisfying

$$|\lambda| < \rho.$$

For the eigenvalue ρ, when A is irreducible (includes primitive), A has a unique positive stochastic eigenvector y, so that

$$Ay = \rho y.$$

Hyslop's Theorems:

We give two of these theorems. In the last two theorems, divergence includes convergence to 0.

Theorem 14, Hyslop *Let $a_k \geq 0$ for all positive integers k. Let $a_{i_1}, a_{i_2}, \ldots$ be a rearrangement of $a_1, a_2, \ldots$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} a_{i_k}$ converges.*

Theorem 51, Hyslop *Let $a_k \geq 0$ for all positivc integers k. Then $\sum_{k=1}^{\infty} a_k$ and $\prod_{k=1}^{\infty} (1 + a_k)$ converge or diverge together.*

Theorem 52, Hyslop If $-1 < a_k \leq 0$, then $\sum_{k=1}^{\infty} a_k$ and $\prod_{k=1}^{\infty} (1 + a_k)$ converge or diverge together.

König's Infinity Lemma:

The statement of this lemma follows.

Lemma Let $S_1, S_2, \ldots$ be a sequence of finite nonempty sets and suppose that $S = \cup S_k$ is infinite. Let $\triangle \subseteq S \times S$ be such that for each k, and each $x \in S_{k+1}$, there is a $y \in S_k$ such that $(y, x) \in \triangle$. Then there exist elements $x_1, x_2, \ldots$ of S such that $x_k \in S_k$ and $(x_k, x_{k+1}) \in \triangle$ for all k.

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