Tensor Rank is NP-complete

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Abstract: We prove that computing the rank of a three-dimensional tensor over any finite field is NP-complete. Over the rational numbers the problem is NP-hard.

Key words: NP-completeness, tensor rank, bilinear complexity, multiplicative complexity

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1. Introduction.

One of the most fundamental quantities in linear algebra is the rank of a matrix. This is a well understood, easy to compute number. The purpose of this paper is to study a higher dimensional analogue, namely the rank of a three-dimensional tensor.

Let us define this number before we continue. For comparison we first give a slightly unusual definition of matrix rank. A matrix is a two-dimensional array of numbers. It has rank 1 iff it can be written as the outer product of two vectors. By this we mean that there are vectors x and y such that $m_{ij} = x_i y_j$. The rank of a general matrix M is now the minimal number of rank 1 matrices M_i such that $M = \sum M_i$. In the same way, a three-dimensional tensor is a three-dimensional array of numbers. It has rank 1 iff it can be written as the outer product of three vectors and the rank of a general tensor T is the minimal number of rank 1 tensors T_i such that $T = \sum T_i$.

Despite the fact that the rank of a tensor is a very natural object, our knowledge of its properties is surprisingly limited. For instance, it does not seem to be known in any field what is the maximal rank of an $n \times n \times n$ tensor. In this paper we prove that over most fields it is NP-hard to compute the rank of a tensor. Thus unless $NP = P$ there will be no easily computable characterization of rank and furthermore if $NP \neq coNP$ there will be no easy to verify characterization of the property "having rank at least r". These facts might explain at least partly the lack of progress in the study of tensor rank. One can here draw a parallel with graph theory where the NP-complete problem of Hamiltonian circuit has been much more elusive than many other properties of graphs.

In spite of the interesting and natural questions above, our main motivation to study tensor rank is its connection with the multiplicative complexity of collections of bilinear forms. It is well known (see for instance [S1]) that the rank of the corresponding tensor is exactly equal to minimal number of multiplications needed to compute a collection of bilinear forms by a bilinear noncommutative algorithm. Our interest in tensor rank was initiated by an effort to prove lower bounds on this measure of complexity. With this in mind, our present result has some negative implications. Unless $NP = coNP$ there will not be any optimal, easy to verify, lower bound proof techniques for the complexity of general bilinear forms. It is very amusing to observe how complexity theory bites its own tail in this argument. On the other hand one should not be too pessimistic. It is still possible that it is easier to prove close to optimal lower bounds or that the bilinear forms we are interested in will be easier to handle than general bilinear forms. In particular, in view of the enormous efforts spent on obtaining upper bounds on the complexity of matrix multiplication (the current champion is [CW]) it would be very interesting to improve the lower bounds beyond $2n^2-1$ in the general case [AS] and beyond $2.5n^2-o(n^2)$ in the $GF(2)$ case [B].

The fact that estimating the number of multiplications when computing bilinear forms was NP-complete had been proved before in some restricted cases. In particular when no subtraction is allowed and only the constants 0 and 1 may be used the result was proved by Gonzalez and Ja'Ja' [GoJ]. These restrictions, however, makes the nature of the problem much more combinatorial and NP-completeness comes easier. Our result was conjectured in their paper.

This is the journal version of the conference paper [H]. The conference paper contains a longer, more selfcontained proof of the main theorem and hence that might be easier to read for the non-expert.

2. Main Result

We will be working with three-dimensional tensors and we will use the notation $T=(t_{ijk})$ where i will range from 1 to n_1, j will range from 1 to n_2, and k will range from 1 to n_3. The matrix obtained by fixing the eth coordinate to a given value will be called an *e-slice* of T.
Let us now make a formal definition. Let F be a field.

TENSOR RANK OVER F. Given numbers in F, t_{ijk} where $1\le i\le n_1$, $1\le j\le n_2$ and $1\le k\le n_3$ and an integer r. Are there vectors $v_e^{(l)}$ $1\le l\le r, 1\le e\le 3$ where $v_e^{(l)}\in F^{n_e}$ such that $t_{ijk}=\sum_{l=1}^{r} v_1^{(l)}(i)v_2^{(l)}(j)v_3^{(l)}(k)$ for all i,j,k?

We will sometimes write the last equation as

$$T=\sum_{l=1}^{r} v_1^{(l)}v_2^{(l)}v_3^{(l)},$$

dropping the indices i,j and k. We will use the phrase "M appears in the expansion of T," if M is a rank 1 matrix and M is a scalar multiple of the outer product of $v_{e_1}^{(l)}$ and $v_{e_2}^{(l)}$ for some l The pair of indices e_1 and e_2 will be clear from the context. The rank of T will be denoted by $r(T)$.

Now we can state our main theorem.

Theorem 1: *TENSOR RANK is NP-complete for any finite field and NP-hard for the rational numbers.*

Proof: First observe that it is easy to verify that the problem is in NP for a finite field since we have no trouble guessing the vectors $v_e^{(l)}$. Over the rational numbers there might be some problem that the number of bits needed to specify these vectors might be large.

We now reduce 3SAT which is known to be NP-complete [C] (cf [GaJ]) to tensor rank. 3SAT is the problem of given a Boolean formula of n variables in CNF-form with at most 3 variables in each of m clauses, is it possible to find a satisfying assignment for the formula. We transform this to the problem of computing the rank of a tensor T of size $(2+n+2m)\times 3n\times(3n+m)$.

T has the following 3-slices:

1. n variable matrices V_i
2. n help matrices S_i
3. n help matrices M_i
4. m clause matrices C_l

Let us describe these matrices in detail.

V_i. The matrix V_i has a 1 in positions $(1,2i-1)$ and $(2,2i)$ while all other elements are 0.

S_i. The matrix S_i has a 1 in position $(1, 2n+i)$ and is otherwise 0.

M_i. The matrix M_i has a 1 in positions $(1, 2i-1)$, $(2+i, 2i)$ and $(2+i, 2n+i)$ and is 0 otherwise.

C_l. Let x_i be a vector with only a 1 in position $2i-1$ and let $\bar{x}_i$ be a vector with 1 in positions $2i-1$ and $2i$. Now we can identify literals with vectors. Suppose the clause c_l contains the literals u_1, u_2 and u_3. Then we define the matrix C_l as follows.

Row 1 is the vector u_1.
Row $2+n+2l-1$ is the vector $u_1 - u_2$.
Row $2+n+2l$ is the vector $u_1 - u_3$.

Before we continue let us give an example and the intuition behind the construction. Let us construct the tensor corresponding to three variables and the two clauses $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$.

$$V_1 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}, V_2 = \begin{pmatrix} 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}, S_1 = \begin{pmatrix} 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}, S_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Let us explain the idea behind the construction. As can be seen from the example the 3-slices are fairly independent in the sense that they have very few common nonzero elements. Now use the characterization that the rank is the minimal number of rank 1 matrices N_i such that any of the above 3-slices can be written as a linear combination of the N_i. By the above mentioned independence the same rank 1 matrix cannot be useful in too many places. In particular, the matrices M_i and S_i make sure that the matrices V_i are written as a sum of two matrices using one of the two equations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

We get a matrix whose only nonzero row is the first and that takes value x_i or $\bar{x}_i$. Then one only needs the fact that one of these is helpful for obtaining C_l iff the literal appears in the corresponding clause. Let us now make this formal. We have:

Lemma 2: *The constructed tensor has rank $4n + 2m$ iff the formula is satisfiable. Otherwise the rank is larger.*

Remark: Clearly Lemma 2 implies Theorem 1.

Proof: Let us first prove that if the formula is satisfiable the rank is at most $4n + 2m$. Let $x_i = \alpha_i$ be a satisfying assignment. We now construct $4n + 2m$ rank 1 matrices such that the V_i, S_i, M_i and C_l can be written as linear combinations of these matrices.

Matrices $V_i^{(1)}$ and $V_i^{(2)}$ where $V_i^{(1)}$ has first row equal to x_i iff $\alpha_i = 1$ and otherwise $\bar{x}_i$. All the other rows are 0. We set $V_i^{(2)} = V_i - V_i^{(1)}$.

Matrices S_i.

Matrices $M_i^{(1)}$ where $M_i^{(1)} = M_i - V_i^{(1)}$ if $\alpha_i = 1$ and $M_i^{(1)} = M_i - V_i^{(1)} - S_i$ if $\alpha_i = 0$.

Matrices $C_l^{(1)}$ and $C_l^{(2)}$. Let $x_i = \alpha_i$ be the assignment that makes the clause c_l true. Then $C_l - V_i^{(1)}$ has rank 2, since either it has just two nonzero rows (in the case where x_i is the first variable in the clause) or it has three nonzero rows of which two are equal. In both cases we just need two additional rank 1 matrices.

The total number of rank 1 matrices sufficient is $2n + n + n + 2m = 4n + 2m$ and thus the rank of the constructed tensor is at most $4n + 2m$ when the formula is satisfiable. That the rank is exactly $4n + 2m$ is not needed for the NP-completeness proof but will follow from the argument below showing that the rank is greater than $4n + 2m$ when the formula is not satisfiable. Let us now turn to proving the lower bound.

In T the 1-slices corresponding to $i = 3, 4, \ldots n + 2m + 2$ are all of rank 1 and are linearly independent, hence by [HK], Lemma 2, these slices can all be made to appear in a minimal expansion of T. We do not know what multiples of these matrices to be subtracted from the first two 1-slices and we hence leave this as variables for the moment. We get

$$r(T) = n + 2m + \min r(\tilde{T})$$

where $\tilde{T}$ is a $2 \times 3n \times (3n + m)$ tensor described by the following 3-slices:

The matrices V_i and S_i truncated to two rows.

Matrices $\tilde{M}_i$. The first row of $\tilde{M}_i$ is $e_{2i-1} + k_i^1(e_{2i} + e_{2n+i})$, while the second row is $k_i^2(e_{2i} + e_{2n+i})$.

Matrices $\tilde{C}_l$. The first row of $\tilde{C}_l$ is $(1 + c_l^1 + c_l^2)u_1 - c_l^1 u_2 - c_l^2 u_3$ and the second is $(d_l^1 + d_l^2)u_1 - d_l^1 u_2 - d_l^2 u_3$.

Here $k_i^1, k_i^2, c_l^1, c_l^2, d_l^1$ and d_l^2 are independent scalar variables, and the minimum is taken over these variables.

Now we observe that the 3-slices S_i are of rank 1 and can hence be made to appear in the expansion of $\tilde{T}$ (by [HK]). The question is only in what multiples of S_i will be subtracted for the other matrices. To determine these coefficients let us prove a lemma.

Lemma 3: *If the second row of any $\tilde{M}_i$ is nonzero then the rank of $\tilde{T}$ is at least 3n+1.*

Proof: Suppose without loss of generality that the second row of $\tilde{M}_1$ is nonzero. After we have subtracted suitable multiples of S_i from all other 3-slices we have 3-slices $\bar{V}_i, i = 1, \ldots n$ and $\bar{M}_1$ (and some other matrices). We claim that the tensor, $\tilde{T}'$, given by these $n + 1$ 3-slices already have rank at least $2n + 1$ and this obviously implies the lemma. We know that in the first $2n$ columns the $\bar{V}_i$ look like the original V_i and furthermore in position $(2, 2n + 1)$ they all have a 0. On the other hand $\bar{M}_1$ has a nonzero element in this position by the assumption that the second row of $\tilde{M}_1$ was nonzero. Consider the first $2n + 1$ 2-slices, $B_j, j = 1, \ldots 2n + 1$, of $\tilde{T}'$ (these are now matrices of size $2 \times (n + 1)$).

B_1 has a 1 in positions $(1, 1)$ and $(1, n + 1)$ and is otherwise zero.

B_2 has k_1^1 in position $(1, n + 1)$, 1 in position $(2, 1)$, k_1^2 in position (2,n+1) and is zero otherwise.

$B_{2i-1}, 2 \leq i \leq n$ has a 1 in position $(1, i)$ and is zero otherwise.

$B_{2i}, 2 \leq i \leq n$ has a 1 in position $(2, i)$ and is zero otherwise.

B_{2n+1} has unknown first row, while the second row is 0 except that in position $n+1$ it has the entry k_1^2 which by assumption is nonzero.

We claim that these matrices are linearly independent. It is clear that B_i, $1 \leq i \leq 2n$ are linearly independent since they each have precisely one 1 in the first n columns and these ones are placed in different positions. Our only problem is that B_{2n+1} might be a linear combination of the other B_j. But since the second row of B_{2n+1} has zeros in the first n positions this would have to be a linear combination of the odd indexed matrices. But all these matrices have a zero in position $(2, n+1)$ where B_{2n+1} has a nonzero element and hence the B_j are linearly independent. This implies that the rank of $\tilde{T}'$ is at least $2n+1$ since if the rank is r we can only get r linearly independent 2-slices in the tensor. The proof of the lemma is complete. ∎

Since for T to have rank $n + 2m$, $\tilde{T}$ must have rank $3n$, we can assume that $k_i^2 = 0$ for all i. Now if we subtract k_i^1 times S_i from $\tilde{M}_i$ and leave the other 3-slices as they are we make all 2-slices for $j > 2n$ identically 0. All other choices would not change the first $2n$ 2-slices and make some other 2-slice nonzero. Such a choice could clearly only increase the rank. Thus, we get

$$r(T) = 2n + 2m + \min r(\bar{T})$$

where $\bar{T}$ is a tensor of rank $2 \times 2n \times (2m + 2n)$ given by the following 3-slices.

V_i (the original matrices truncated).

$\bar{M}_i$. The first row of $\bar{M}_i$ is $e_{2i-1} + k_i^1 e_{2i}$. The second row is 0.

Matrices $\tilde{C}_l$. The first row of $\tilde{C}_l$ is $(1 + c_l^1 + c_l^2)u_1 - c_l^1 u_2 - c_l^2 u_3$ and the second is $(d_l^1 + d_l^2)u_1 - d_l^1 u_2 - d_l^2 u_3$.

where the minimum is taken over the constants c_l^e, d_l^e and k_i^1. The entire question is reduced to the question whether the tensor $\bar{T}$ can have rank as low as $2n$.

Since the $\bar{M}_i$ have rank 1 and are linearly independent they can all be made to appear in the expansion of $\bar{T}$. Next we have

Lemma 4: *For any k we can assume that $V_k - \bar{M}_k$ as well as all the $\bar{M}_i$ appear in the expansion of $\bar{T}$.*

Proof: Observe first that the matrix $V_k - \bar{M}_k$ have rank 1. Now assume that it does not appear in the expansion. Then V_k is written as a linear combination of the occurring rank 1 matrices $V_k = \sum_{j=1}^{r} a_j N_j$. We already know that $\bar{M}_k$ appears in the expansion of $\bar{T}$. Thus $V_k - \bar{M}_k$ is also a linear combination of the chosen N_j. Furthermore this linear combination does not only contain matrices $\bar{M}_i$ since $V_k - \bar{M}_k$ is linearly independent of these matrices. Hence we can eliminate one of the N_j which is not equal to $\bar{M}_i$ for any i and introduce $V_i - \bar{M}_i$. The lemma follows. ∎

We need a slight extension of Lemma 4.

Lemma 5: *We can assume that all the matrices $V_i - \bar{M}_i$ as well as all the $\bar{M}_i$ appear in the expansion of $\bar{T}$.*

Proof: This follows by basically the same proof as Lemma 4. Only observe that since the matrices are linearly independent, we can introduce them one by one in the expansion without eliminating previously inserted matrices. ∎

Thus the question whether $\bar{T}$ has rank $2n$ is equivalent to whether $\tilde{C}_l$ can be written as a sum of the matrices $\bar{M}_i$ and $V_i - \bar{M}_i$. We have the following claim

Claim: If $\tilde{C}_l$ can be written as a linear combination of $\bar{M}_i$ and $V_i - \bar{M}_i$ then the second row of C_l is 0 and the first row of one of the $\bar{M}_i$ is u_i where u_i is one of the literals appearing in c_l.

To see the first part of the claim observe that if the second row of a $\tilde{C}_l$ is nonzero then it contains a nonzero element in an odd position. On the other hand both $\bar{M}_i$ and $V_i - \bar{M}_i$ have zeros in all odd positions on the second row. This proves the first part of the claim. Observe that this implies in particular that only $\bar{M}_i$'s appear in the sum giving C_l.

To establish the second part let u_j be a literal belonging to the variable x_i which appears in the first row of C_l with a nonzero coefficient. Since only $\bar{M}_i$ of all the $\bar{M}$ matrices has nonzero elements in either of the positions $(1, 2i-1)$ or $(1, 2i)$, $\bar{M}_i$ must be used to cancel these elements. Thus the first row of $\bar{M}_i$ must be a multiple of u_j and since the element in position $(1, 2i-1)$ of $\bar{M}_i$ is 1 this multiple must be 1. We have established the claim.

To complete the proof of Lemma 2 we just have to observe that if all the $\tilde{C}_l$ can be written as a sum of the $\bar{M}_i$ and the $V_i - \bar{M}_i$ then we get a satisfying assignment for the original formula by setting $x_i = 1$ if $\bar{M}_i$ has first row x_i and $x_i = 0$ otherwise. This completes the proof of Lemma 2 and hence of Theorem 1. ∎

3. Directions for further research.

The fact that tensor rank is NP-complete should not deter us from trying to prove lower bounds for the number of multiplications needed to compute collections of bilinear forms. In particular it would be very interesting to obtain nonlinear lower bounds for any natural problem, in particular for a well studied problem like matrix multiplication.

Maybe in the quest for lower bounds it would be helpful to study the concept of tensor rank as a mathematical subject rather than just pushing at the lower bound problem. Here a fundamental question is to determine the maximal rank of an $n \times n \times n$ tensor. It is known to be between roughly $n^2/2$ and $n^2/3$. For further information see [S2] and the references therein.

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