# A Rewriting Approach to Graph Invariants

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**Abstract**

The Generic Diamond Lemma of the author is applied to the problem of classifying all graph invariants satisfying a contract–delete recursion (like that of the chromatic polynomial). As expected, the recursion for the Tutte polynomial is found, along with some more degenerate invariants. The purpose of this exercise is to demonstrate techniques for applying the diamond lemma to diagrammatic calculations in general.

In addition, a concept of 'semigraph' is defined and some related elementary constructions of interest for algebraic applications are given.

## 1 Introduction

Diagrammatic calculations — some different flavours of which can be found in the works of for example Baez–Lauda [2], Cvitanović [5], Majid [8], and Major [9] — is a powerful tool that gets near indispensable when one tries to manage some of the newer algebraic structures that have been popping up in the last couple of decades. A key feature is that algebraic expressions, which in more classical mathematics has a linear or at worst tree-like structure, start to look much more like an electric circuit where components (functions, operations, interactions, or whatever) can be connected to form networks with pretty arbitrary structure as long as some basic syntactical conditions are fulfilled. While there often are more traditionally-looking ways of writing such expressions (tensors employing the Einstein convention for summing over indices in [5, 9], the Sweedler notation in [8]), these have the distinct disadvantage that it is hard to tell whether a given subexpression occurs in an expression; different parts of the subexpression may appear far from each other and the links between them are typically hard to discern. Diagrammatic expressions alleviates this by letting the mathematician take more advantage of the human capacity to find patterns in graphical data.

I would have loved to speak about this in the AGMF conference, and in particular about how my work on a generic diamond lemma [7] can be applied to the rewriting theory of such diagrams, but unfortunately there was a 20 minutes limit on the talks. Since it can not yet be assumed that the audience knows which graphs may be considered well-formed diagrams and how to interpret diagrams as expressions, this would have had to be explained, and then there wouldn't have been enough time to actually do something using the diagrams; proper diagrammatical calculations were thus out of the question. What I *could* do within the given time was to drop the aspiration of working with diagrams that correspond to actual expressions, and instead just do rewriting on ordinary graphs.

From a pedagogical perspective, graph rewriting is actually a rather good place to start, because in this one encounters most of the complications arising from applying the diamond lemma, whereas the object of study does not produce any particular complications of its own. My intention is that this paper should serve as a prototype for applications of the diamond lemma to diagrammatic calculations.

## 2 Graph theory

This section is an informal review of the graph-theoretical concepts that are relevant in this paper. The methods used in Sections 4–5 do not logically depend on this material, but it should help to explain why the given problem is relevant and how it relates to known results. For graph-theoretical concepts not defined in this text, I refer to Diestel [6]. I will mostly follow his terminology, but prefer to reserve the term *multigraph* for multigraphs without loops; if loops (and multiple edges) are allowed then the object will be called a *pseudograph.* $K_n$ is the complete graph on $n$ vertices. $K_n^{\mathrm{C}}$ is the complement of $K_n$, i.e., the graph with $n$ vertices but no edges.

The *chromatic polynomial* $P_\chi(G)$ for a graph $G$ is defined by the property that $P_\chi(G)(k)$ is the number of vertex-$k$-colourings of $G$, for any natural number $k$. That this function is always a polynomial is at first sight surprising, but an easy proof can be based on the delete–contract recursion for $P_\chi$:

$$P_\chi(G)(k) = P_\chi(G - e)(k) - P_\chi(G/e)(k) \quad \text{for all } k \in \mathbb{N} \text{ and } e \in \mathrm{E}(G). \tag{1}$$

Here the notation $G - e$ means "the graph $G$ with the edge $e$ deleted", whereas $G/e$ means "the graph $G$ with the edge $e$ contracted", i.e., the two endpoints of $e$ are identified; see Figure 1 for an example. The proof of this recursion is almost trivial: a $k$-colouring of $G - e$ either assigns different colours to the endpoints of $e$, and in that case it is a $k$-colouring of $G$, or assigns the same colour to the endpoints of $e$, and in that case it defines a $k$-colouring of $G/e$; subtract the latter, and you get the expression for the former. Since each step of the recursion decreases the size (number of edges) of the graphs involved by at least 1, one arrives after a finite number of steps at a linear combination of chromatic polynomials for size 0 graphs, and these are easily found to be polynomials in $k$; since any assignment of colours to the vertices in $K_n^{\mathrm{C}}$ is a colouring, there are exactly $k^n$ $k$-colourings of this graph, and hence $P_\chi(K_n^{\mathrm{C}})(x) = x^n$. The recursion (1) is the foremost tool for computing the chromatic polynomial, and by extension even for computing the chromatic number, of a general graph (even though relying solely on this recursion often makes the task much more laborious than it has to be).

A function of graphs which does not depend on which the vertices and edges are, but only on how they are connected, is called an *invariant*; formally a function $Q$ is an invariant if $Q(G) = Q(H)$ whenever $G$ and $H$ are isomorphic. The chromatic polynomial is an invariant, and interestingly enough there are also several other graph invariants which sport similar delete–contract recursions. Hence it becomes an interesting problem to classify these invariants and perhaps find new ones. In order to do so, one must however first clarify exactly

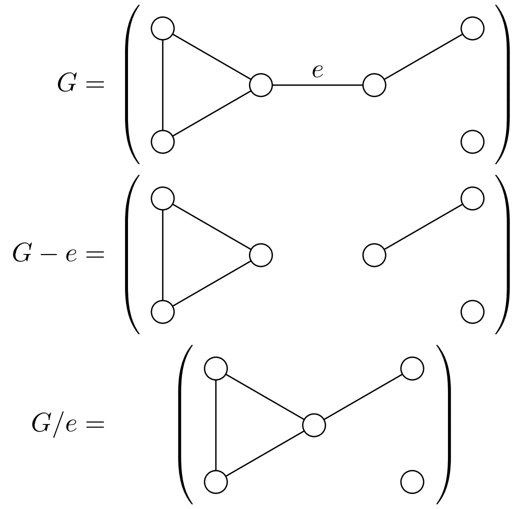



Figure 1: Deletion and contraction of an edge

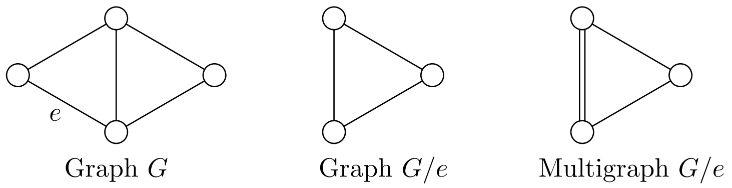



Figure 2: Contracting an edge in a triangle

what the deletion and contraction operations should do. There is a certain amount of hindsight here, in that the choices I make are primarily dictated by the method I want to apply, but there are also more generic reasons for making these choices, and it is certainly worth while to explain them.

The difficult operation is edge contraction. The first choice (Figure 2) one has to make when defining it is to decide what happens when one contracts an edge in a triangle (3-cycle): should the two remaining edges count as one edge or two? The disadvantage of keeping two parallel edges in this case is that it means edge contraction can transform a graph into a multigraph. For the chromatic polynomial one or two edges make no difference — the endpoints are not allowed to have the same colour in either case — but for some other invariants (e.g. the flow polynomial) it is crucial to distinguish these cases, and as it happens it is then the multigraph contraction that is the right one for the recursion. Hence it is natural to let the problem concern invariants of multigraphs.

Opening up for multigraphs leads to another problem, namely what should happen when one contracts a double (or even higher multiplicity) edge, as in Figure 3. Contracting one of two parallel edges will turn the other one into a loop — must one therefore extend the argument to consider also pseudographs? Although the tradition indeed is to do so, there is no compelling computational reason to take this route; loops just sit on particular vertices, so they behave as

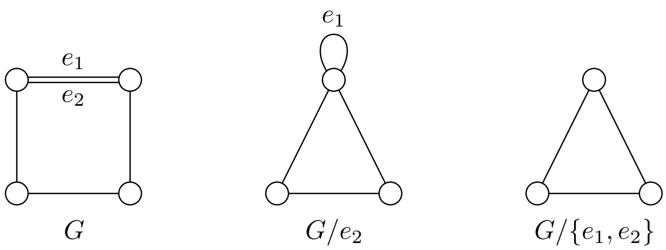



Figure 3: Contracting a double edge

an extra set of weights that for most part can be ignored. It is possible to stay within the realm of multigraphs if one decides to always delete or contract all edges between the two endpoints in a single step, rather than removing them one at a time; this way the multigraph recursion for the chromatic polynomial proceeds exactly as with graph contraction of edges.

The tradition to consider also pseudographs has mostly historical reasons: a lot of the theory for colourings and nowhere-zero flows grew out of attempts to prove the Four Colour Theorem (every plane graph has a vertex-4-colouring), and in that context the operation of forming the plane dual of a graph is of great importance. The dual of a plane graph will however in general be a pseudograph, and hence it is then more natural to work in the generality of plane pseudographs. In this paper, planarity is not an issue and hence a restriction to multigraphs is not problematic.

# 3 The problem

Formally, the problem studied here is to classify all multigraph invariants $Q$, with values in a vector space $W$ over some field $\mathcal{R}$, that for all multigraphs $G$ satisfy an identity on the form

$$Q(G) = \alpha_{|\bar{e}|} Q(G - \bar{e}) - \beta_{|\bar{e}|} Q(G/\bar{e}) \quad \text{for all } e \in \mathrm{E}(G). \tag{2}$$

By $\bar{e}$ is meant the set of all edges with the same endpoints as the edge $e$; that this set can be regarded as the closed hull $\bar{e}$ of $e$ is well established in matroid theory. The coefficients $\{\alpha_m\}_{m=1}^{\infty}$ and $\{\beta_m\}_{m=1}^{\infty}$ may be arbitrary elements of $\mathcal{R}$; to a great extent a class of invariants is determined by a particular parametrisation of these coefficients.

Most of the rewriting arguments in subsequent sections could just as well be carried out for coefficients from a commutative ring with unit, but the subsequent parametrisation of these coefficients for particular invariant classes becomes much easier if one can invert nonzero ring elements. As this is meant to be a pedagogical presentation, one shouldn't introduce unnecessary difficulties; those interested in the greater generality are referred to the classification of Bollobás and Riordan [4] instead.

For an algebraic treatment of graph invariants satisfying the prescribed kind of recursion, it is convenient to first fold away the condition that $Q$ should be an invariant. A formal solution to this would be to regard $Q$ as a function of isomorphism classes of multigraphs, as this is an alternative definition of invariant. In graph theory, such isomorphism classes are sometimes known as *unlabelled multigraphs* (multigraphs where no names have been assigned to the vertices and edges), whereas an ordinary multigraph $G$ for which one may speak of distinct sets of vertices $\mathrm{V}(G)$ and edges $\mathrm{E}(G)$ is a *labelled multigraph*.[1] Graph theorists rarely bother about upholding this distinction, but we will have to be careful about it when we get to do algebra with graphs.

Let $\mathcal{Y}$ denote the set of all unlabelled multigraphs, which will be interpreted as a set of equivalence classes of labelled multigraphs — hence its elements will be written as $[G]$ for some labelled multigraph $G$. The sought class of invariants is then just the set of those functions $Q\colon \mathcal{Y} \longrightarrow W$ which satisfy

$$Q\bigl([G]\bigr) = \alpha_{|\bar{e}|} Q\bigl([G - \bar{e}]\bigr) - \beta_{|\bar{e}|} Q\bigl([G/\bar{e}]\bigr) \tag{3}$$

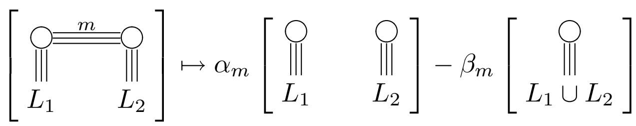
for all $e \in \mathrm{E}(G)$ and labelled multigraphs $G$. Next let $\mathcal{M}$ be the set of all formal $\mathcal{R}$-linear-combinations of elements of $\mathcal{Y}$, i.e., the $\mathcal{R}$-vector-space with basis $\mathcal{Y}$. Any invariant $Q\colon \mathcal{Y} \longrightarrow W$ extends to a linear map $Q\colon \mathcal{M} \longrightarrow W$, and those of interest here are those that satisfy (3), or equivalently

$$Q\Bigl([G] - \alpha_{|\bar{e}|}[G - \bar{e}] + \beta_{|\bar{e}|}[G/\bar{e}]\Bigr) = 0 \tag{4}$$

for all $e \in \mathrm{E}(G)$ and labelled multigraphs $G$! Let $\mathcal{I} \subseteq \mathcal{M}$ be the subspace spanned by all $[G] - \alpha_{|\bar{e}|}[G - \bar{e}] + \beta_{|\bar{e}|}[G/\bar{e}]$ for $e \in \mathrm{E}(G)$ and $[G] \in \mathcal{Y}$. The invariants which satisfy the wanted recursion are exactly the linear maps that factor over $\mathcal{M}/\mathcal{I}$, so they are completely determined by their values on a basis for this quotient. Determining bases for a quotient is a standard application for the diamond lemma, and the key to start using it is to express the identity $[G] \equiv \alpha_{|\bar{e}|}[G - \bar{e}] - \beta_{|\bar{e}|}[G/\bar{e}] \pmod{\mathcal{I}}$ as a rewrite rule.

# 4 Semigraphs

The recursion identity $[G] \equiv \alpha_{|\bar{e}|}[G - \bar{e}] - \beta_{|\bar{e}|}[G/\bar{e}] \pmod{\mathcal{I}}$ has a straightforward interpretation as a rewrite rule: two vertices connected by $m$ edges may be replaced by $\alpha_m$ times the same two vertices without those edges, minus $\beta_m$ times the contraction of those two vertices. More symbolically, this may be written as



(5)

where the $L_1$ and $L_2$ should be taken as the sets of edges between these two vertices and other vertices in $G$; it is important in the recursion that these are unaffected by the deletion and contraction operations.

In order to turn this into a rewrite rule for use with the diamond lemma of [7], the first step is to turn the diagrams in (5) into proper mathematical objects with which one may calculate. To that end, I now propose a generalisation of the ordinary multigraph concept, which I call *semigraph*. Intuitively, a semigraph

[1]In diagrammatic calculations, 'label' is often a term used for extra information attached to an edge or vertex, but in graph theory it is just the unique identifier of that particular item. If you need both concepts, then try using 'annotation' for the extra information kind of label.

may be thought of as an ordinary graph which someone has been placed on a hard surface and chopped in half with a cleaver. (Semi- = half.) Any edge between a vertex in one part and a vertex in the other must be split in half by such an operation, so each half-a-graph one is left with will in general have half-edges sticking out that are attached only to one vertex.

**Definition 4.1.** A *labelled semigraph* $G$ is a triplet $\bigl(\mathrm{V}(G), \mathrm{E}(G), \phi_G\bigr)$, where $\mathrm{V}(G)$ and $\mathrm{E}(G)$ are finite sets and

$$\phi_G \colon \mathrm{E}(G) \longrightarrow \bigl\{\, A \subseteq \mathrm{V}(G) \bigm| 1 \leqslant |A| \leqslant 2 \,\bigr\}$$

is a map. The elements of $\mathrm{V}(G)$ are called *vertices*, the elements of $\mathrm{E}(G)$ are called *edges*. The elements of $\phi_G(e)$ are called *endpoints* of the edge $e$. An edge is said to be *internal* if it has two endpoints and *external* otherwise. External edges are also called *legs*. The set of legs of $G$ is called the *boundary* of $G$ and is denoted $\partial G$.

While the formal definition of semigraph may look very similar to a formal definition of pseudograph, the two concepts are quite different: an external edge is an edge with one endpoint, whereas a loop would be an edge with two ends, although both of them happen to be attached to the same vertex. A careful formalisation of pseudograph should really have $\phi_G$ map edges to 2-element multisets of vertices, but substituting the set $\{v\}$ for the multiset $\{v, v\}$ is usually a reasonable simplification. A better characterisation would be that semigraphs are "$\{1, 2\}$-uniform hypergraphs" (hypergraphs where all edges are incident with 1 or 2 vertices), but semigraphs occur often enough in practice that they deserve a simple name. In particular, the 'gadgets' that graph theorists sometimes use to transform graphs or build graphs with particular properties are precisely semigraphs.

**Definition 4.2.** An *isomorphism* of two semigraphs $G$ and $H$ is a pair $(\alpha, \beta)$ of bijections $\alpha \colon \mathrm{V}(G) \longrightarrow \mathrm{V}(H)$ and $\beta \colon \mathrm{E}(G) \longrightarrow \mathrm{E}(H)$ such that

$$\phi_H\bigl(\beta(e)\bigr) = \bigl\{\, \alpha(v) \bigm| v \in \phi_G(e) \,\bigr\} \qquad \text{for all } e \in \mathrm{E}(G).$$

Two semigraphs $G$ and $H$ are said to be *isomorphic*, written $G \cong H$, if there exists an isomorphism from $G$ to $H$. The isomorphism $(\alpha, \beta)$ is said to be *internal* if $\beta(e) = e$ for all $e \in \partial G$. Two semigraphs $G$ and $H$ are said to be *internally isomorphic*, written $G \simeq H$, if there exists an internal isomorphism from $G$ to $H$.

Let $\mathcal{D}$ be the set of labelled semigraphs $G$ with $\mathrm{V}(G), \mathrm{E}(G) \subset \mathbb{Z}_{>0}$ and define $\mathcal{D}(L)$ to be the set of those $G \in \mathcal{D}$ for which $\partial G = L$. Define $\mathcal{Y}(L) = \mathcal{D}(L)/{\simeq}$; this is the set of *unlabelled semigraphs* with boundary $L$ and if $G \in \mathcal{D}(L)$ then $[G]$ denotes the element of $\mathcal{Y}(L)$ containing $G$. Finally define $\mathcal{M}(L)$ to be the $\mathcal{R}$-vector-space with basis $\mathcal{Y}(L)$.

In this more general notation, the set of unlabelled multigraphs $\mathcal{Y}$ is the set $\mathcal{Y}(\varnothing)$ of unlabelled semigraphs with empty boundary, and similarly the vector space $\mathcal{M}$ of the previous section is $\mathcal{M}(\varnothing)$. For multigraphs the internal isomorphism concept is the same as ordinary isomorphism of multigraphs, but for more general semigraphs they are not, and consequently even unlabelled semigraphs have their external edges labelled. A natural analogy is with the names of variables in a function definition: it makes no difference which names are used, but one must use the same names in both sides of the defining equation. The rewriting will primarily operate on semigraphs with boundary. Note for example that the left hand side of (5) is an element of $\mathcal{Y}(L_1 \cup L_2)$ and the right hand side is an element of $\mathcal{M}(L_1 \cup L_2)$.

**Lemma 4.3.** *Let $L_1, L_2 \subset \mathbb{Z}_{>0}$ be given. If $\eta\colon L_1 \longrightarrow L_2$ is a bijection then it defines a bijection $\mathring{\eta}\colon \mathcal{Y}(L_1) \longrightarrow \mathcal{Y}(L_2)$ by*

$$\mathring{\eta}\bigl([G]\bigr) = \Bigl[\bigl(\mathrm{V}(G), E, \phi_G \circ \beta\bigr)\Bigr] \tag{6}$$

*where $\beta\colon E \longrightarrow \mathrm{E}(G)$ is a bijection such that $\beta\bigl(\eta(e)\bigr) = e$ for all $e \in L_1$, and $E \subset \mathbb{Z}_{>0}$ is arbitrary such that $|E| = \bigl|\mathrm{E}(G)\bigr|$ and $L_2 \subseteq E$.*

*Proof.* First observe that $(\mathrm{id}, \beta^{-1})$, where id denotes the identity map, is an isomorphism from $G$ to $H = \bigl(\mathrm{V}(G), E, \phi_G \circ \beta\bigr)$. Hence if $G' \in [G]$ is some other semigraph of the same class then $H' = \bigl(\mathrm{V}(G), E', \phi_{G'} \circ \beta'\bigr)$, where $E' \subset \mathbb{Z}_{>0}$ is such that $|E'| = \bigl|\mathrm{E}(G)\bigr|$ and $L_2 \subseteq E'$, and $\beta'\colon E' \longrightarrow \mathrm{E}(G')$ is some bijection such that $\beta'\bigl(\eta(e)\bigr) = e$ for all $e \in L_1$, satisfies $H' \cong G' \simeq G \cong H$. Furthermore that isomorphism from $H'$ to $H$ composed in the obvious way from $(\mathrm{id}, \beta')$, an internal isomorphism from $G'$ to $G$, and $(\mathrm{id}, \beta^{-1})$ will map elements of $L_2$ back to themselves, and thus be an internal isomorphism. Hence $\mathring{\eta}$ is well-defined. In order to see that it is a bijection, it suffices to observe that $(\eta^{-1})^{\circ}\colon \mathcal{Y}(L_2) \longrightarrow \mathcal{Y}(L_1)$ is the inverse of $\mathring{\eta}$. □

This lemma illustrates a common awkwardness in formalising operations on labelled semigraphs (or just graphs): although the operation may be very easy to define, one usually has to make some arbitrary choice of new labels in it. This means the operations do not really have a canonical definition; a different author (or perhaps more noticeably: a different programmer) will probably make a different choice. The different choices will however usually produce isomorphic results, so in general the operation is canonical as an operation on *unlabelled* semigraphs.

**Definition 4.4.** Let $G$ and $H$ be labelled semigraphs. $H$ is an *induced subsemigraph* of $G$, written $H \sqsubseteq G$, if

$$\begin{aligned}
&\mathrm{V}(H) \subseteq \mathrm{V}(G),\\
&\phi_H(e) = \phi_G(e) \cap \mathrm{V}(H) \qquad \text{for all } e \in \mathrm{E}(H),\\
&\mathrm{E}(H) = \Bigl\{\, e \in \mathrm{E}(G) \Bigm| 1 \leqslant \bigl|\phi_G(e) \cap \mathrm{V}(H)\bigr| \leqslant 2 \,\Bigr\}.
\end{aligned}$$

If $X \subseteq \mathrm{V}(G)$ then $G\{X\}$ denotes the induced subsemigraph of $G$ whose set of vertices is $X$.

For any $H \sqsubseteq G$, the *splice map* $G \div H\colon \mathcal{D}(\partial H) \longrightarrow \mathcal{D}(\partial G)$ is defined by

$$\begin{aligned}
n &= \max\bigl(\{0\} \cup \mathrm{V}(G) \setminus \mathrm{V}(H)\bigr),\\
m &= \max\Bigl(\{0\} \cup \mathrm{E}(G) \setminus \bigl(\mathrm{E}(H) \setminus \partial H\bigr)\Bigr),\\
\mathrm{V}\bigl((G \div H)(K)\bigr) &= \bigl(\mathrm{V}(G) \setminus \mathrm{V}(H)\bigr) \cup \{\, v + n \mid v \in \mathrm{V}(K) \,\},\\
\mathrm{E}\bigl((G \div H)(K)\bigr) &= \Bigl(\mathrm{E}(G) \setminus \bigl(\mathrm{E}(H) \setminus \partial H\bigr)\Bigr) \cup \{\, e + m \mid e \in \mathrm{E}(K) \setminus \partial K \,\},
\end{aligned}$$

$$\phi_{(G\div H)(K)}(e) = \begin{cases} \bigl(\phi_G(e) \setminus \mathrm{V}(H)\bigr) \cup \{\, v+n \mid v \in \phi_K(e) \,\} & \text{if } e \in \partial H, \\ \{\, v+n \mid v \in \phi_K(e-m) \,\} & \text{if } e > m, \\ \phi_G(e) & \text{otherwise;} \end{cases}$$

the idea is to replace in $G$ the internal parts of $H$ by the corresponding parts of $K$. The integers $n$ and $m$ are offsets added to labels from $K$ to avoid collisions with labels from $G$.

The sense in which $(G \div H)(K)$ is "$G$, but with the $H$ part replaced by $K$" is formalised in the next lemma.

**Lemma 4.5.** *Let $G \in \mathcal{D}$, $H \sqsubseteq G$, and $K \in \mathcal{D}(\partial H)$ be given. If $G' = (G \div H)(K)$ and $H' = G'\bigl\{\mathrm{V}(G') \setminus \mathrm{V}(G)\bigr\}$ then $H' \simeq K$ and $G' \div H' = G \div H$. Furthermore if $F \sqsubseteq G$ is such that $\mathrm{V}(F) \cap \mathrm{V}(H) = \varnothing$ then $F \sqsubseteq G'$.*

*Proof.* Let $n = \max\bigl(\{0\} \cup \mathrm{V}(G) \setminus \mathrm{V}(H)\bigr)$ and $m = \max\bigl(\{0\} \cup (\mathrm{E}(G) \setminus \mathrm{E}(H)) \cup \partial H\bigr)$ be the offsets that the splice map $G \div H$ adds to vertices and internal edges of $K$ when duplicating them in $G'$. Let $\alpha(v) = v + n$ for all $v \in \mathrm{V}(K)$. Let $\beta \colon \mathrm{E}(K) \longrightarrow \mathbb{Z}_{>0}$ be defined by $\beta(e) = e$ for $e \in \partial K = \partial H$ and $\beta(e) = e + m$ for $e \in \mathrm{E}(K) \setminus \partial K$. Let $X = \mathrm{V}(G') \setminus \mathrm{V}(G)$. By the definition of $G \div H$,

$$X = \{\, v \in \mathrm{V}(G') \mid v > n \,\} = \{\, v + n \mid v \in \mathrm{V}(K) \,\},$$

hence $\alpha$ is a bijection from $\mathrm{V}(K)$ to $X = \mathrm{V}(H')$. Furthermore $\beta$ is a bijection from $\mathrm{E}(K)$ to $\mathrm{E}(H')$ and $\phi_{G'}\bigl(\beta(e)\bigr) \cap X = \bigl\{\, \alpha(v) \bigm| v \in \phi_K(e) \,\bigr\}$ for all $e \in \mathrm{E}(K)$, thus $H' \simeq K$ as claimed. That $G' \div H' = G \div H$ is immediate from the definition, since $\partial H' = \partial H$, $\mathrm{E}(G') \setminus \mathrm{E}(H') = \mathrm{E}(G) \setminus \mathrm{E}(H)$, and $\mathrm{V}(G') \setminus \mathrm{V}(H') = \mathrm{V}(G) \setminus \mathrm{V}(H)$; in particular the two offsets $n$ and $m$ will be exactly the same in the definition of $G' \div H'$ as in the definition of $G \div H$.

For the last claim, it is clear that $\mathrm{V}(F) \subseteq \mathrm{V}(G')$. That $\mathrm{E}(F) \subseteq \mathrm{E}(G')$ follows from the observation that no edge of $F$ may be an internal edge of $H$. Finally, $\phi_F(e) = \phi_G(e) \cap \mathrm{V}(F) = \phi_{G'}(e) \cap \mathrm{V}(F)$ for all $e \in \mathrm{E}(F)$ since $\phi_G(e) \setminus \phi_{G'}(e) \subseteq \mathrm{V}(H)$ and $\phi_{G'}(e) \setminus \phi_G(e) \subseteq X$, both of which are disjoint from $\mathrm{V}(F)$. □

The definition of the splice map admittedly contains a bit of arbitrariness when it comes to the assignment of vertex and edge labels — one could just as well have used $n+1$ and $m+1$ instead of $n$ and $m$, since the labels are mostly irrelevant. The canonical object is instead the corresponding splice map for unlabelled semigraphs.

**Lemma 4.6.** *Let $G \in \mathcal{D}$ and $H \sqsubseteq G$. If $K_1, K_2 \in \mathcal{D}(\partial H)$ are such that $K_1 \simeq K_2$ then $(G \div H)(K_1) \simeq (G \div H)(K_2)$. Hence $G \div H$ is well-defined as a map $\mathcal{Y}(\partial H) \longrightarrow \mathcal{Y}(\partial G)$ and extends to a linear map $\mathcal{M}(\partial H) \longrightarrow \mathcal{M}(\partial G)$.*

*Proof.* Let $n$ and $m$ be the vertex and edge respectively offsets added by $G \div H$. Let $G_i = (G \div H)(K_i)$ and $X_i = \bigl\{\, v \in \mathrm{V}(G_i) \bigm| v > n \,\bigr\}$ for $i = 1, 2$. By Lemma 4.5, $G_1\{X_1\} \simeq K_1 \simeq K_2 \simeq G_2\{X_2\}$. Let $(\alpha, \beta)$ be such an internal isomorphism from $G_1\{X_1\}$ to $G_2\{X_2\}$. By extending $\alpha$ to the whole of $\mathrm{V}(G_1)$ by $\alpha(v) = v$ for $v \in \mathrm{V}(G_1) \setminus X_1$ and $\beta$ to the whole of $\mathrm{E}(G_1)$ by $\beta(e) = e$ for $e \leqslant m$, one gets an internal isomorphism from $G_1$ to $G_2$. Hence $G_1 \simeq G_2$, as claimed. □

The splice maps themselves do however require some degree of semigraph labelling, as it is necessary to specify *which* induced subsemigraph is being replaced by a particular splice map. The next lemma demonstrates how different labellings of a semigraph may be used to express the same splice map.

**Lemma 4.7.** *Let $G_1, G_2, H_1 \in \mathcal{D}$ such that $G_1 \simeq G_2$ and $H_1 \sqsubseteq G_1$ be given. Then there exists some $H_2 \sqsubseteq G_2$ and bijection $\mathring{\eta}\colon \mathcal{Y}(\partial H_1) \longrightarrow \mathcal{Y}(\partial H_2)$ such that $H_2 \cong H_1$ and $G_1 \div H_1 = (G_2 \div H_2) \circ \mathring{\eta}$ as maps $\mathcal{Y}(\partial H_1) \longrightarrow \mathcal{Y}(\partial G_1)$.*

*Proof.* Let $(\alpha, \beta)$ be the internal isomorphism from $G_1$ to $G_2$. For $H_2 = G_2\bigl\{\alpha(\mathrm{V}(H_1))\bigr\}$, the restrictions of $\alpha$ and $\beta$ to $\mathrm{V}(H_1)$ and $\mathrm{E}(H_1)$ provide an isomorphism from $H_1$ to $H_2$. Let $\eta$ be the restriction of $\beta$ to $\partial H_1$ and let $\mathring{\eta}$ be the corresponding bijection as defined in Lemma 4.3. The task is now to show that for any $K_1 \in \mathcal{D}(\partial H_1)$ there is some $K_2 \in \mathring{\eta}\bigl([K_1]\bigr)$ such that $(G_1 \div H_1)(K_1) \simeq (G_2 \div H_2)(K_2)$.

Let $n_1$ and $m_1$ be the offsets $G_1 \div H_1$ adds to vertex and edge respectively labels. Let $n_2$ and $m_2$ be the offsets $G_2 \div H_2$ adds to vertex and edge respectively labels. Let $E = \partial H_2 \cup \bigl\{\, e + m_2 \bigm| e \in \mathrm{E}(K_1) \setminus \partial K_1 \,\bigr\}$ and define $\gamma\colon E \longrightarrow \mathrm{E}(K_1)$ by $\gamma\bigl(\eta(e)\bigr) = e$ for all $e \in \partial K_1$ and $\gamma(e + m_2) = e$ for all $e \in \mathrm{E}(K_1) \setminus \partial K_1$. Then $\gamma$ is a bijection and $(\mathrm{id}, \gamma^{-1})$ is an isomorphism from $K_1$ to $K_2 := \bigl(\mathrm{V}(K_1), E, \phi_{K_1} \circ \gamma\bigr)$, which implies $K_2 \in \mathring{\eta}\bigl([K_1]\bigr)$. The needed isomorphism $(\alpha', \beta')$ from $(G_1 \div H_1)(K_1)$ to $(G_2 \div H_2)(K_2)$ is defined by

$$\alpha'(v) = \begin{cases} \alpha(v) & \text{if } v \leqslant n_1, \\ v - n_1 + n_2 & \text{if } v > n_1, \end{cases}$$

$$\beta'(e) = \begin{cases} \beta(e) & \text{if } e \leqslant m_1, \\ \gamma^{-1}(e - m_1) + m_2 = e - m_1 + 2m_2 & \text{if } e > m_1 \end{cases}$$

for all $v \in \mathrm{V}\bigl((G_1 \div H_1)(K_1)\bigr)$ and $e \in \mathrm{E}\bigl((G_1 \div H_1)(K_1)\bigr)$. □

Another useful property of the splice maps is that the order of replacing disjoint parts of a semigraph is not important.

**Lemma 4.8.** *Let $G \in \mathcal{D}$ and $H_1, H_2 \sqsubseteq G$ be such that $\mathrm{V}(H_1) \cap \mathrm{V}(H_2) = \varnothing$. Then for any $K_1 \in \mathcal{D}(\partial H_1)$ and $K_2 \in \mathcal{D}(\partial H_2)$,*

$$\bigl((G \div H_1)(K_1) \div H_2\bigr)(K_2) \simeq \bigl((G \div H_2)(K_2) \div H_1\bigr)(K_1). \tag{7}$$

*It follows that there is a map $G \div H_1 \div H_2 : \mathcal{Y}(\partial H_1) \times \mathcal{Y}(\partial H_2) \longrightarrow \mathcal{Y}(\partial G)$ defined by*

$$(G \div H_1 \div H_2)\bigl([K_1], [K_2]\bigr) = \Bigl[\bigl((G \div H_1)(K_1) \div H_2\bigr)(K_2)\Bigr] \tag{8}$$

*that extends to a bilinear map $\mathcal{M}(\partial H_1) \times \mathcal{M}(\partial H_2) \longrightarrow \mathcal{M}(\partial G)$.*

*Proof.* By Lemma 4.5, $H_2 \sqsubseteq (G \div H_1)(K_1)$ and $H_1 \sqsubseteq (G \div H_2)(K_2)$, hence both sides of (7) are well-defined. Let $n_i$ and $m_i$ be the offsets added to vertex and edge respectively labels of $K_i$ in the left hand side of (7), and let $n_i'$ and $m_i'$ be the offsets added to vertex and edge respectively labels of $K_i$ in the right hand side of (7), for $i = 1, 2$. The wanted isomorphism $(\alpha, \beta)$ from left hand side to right hand side is then given by

$$\alpha(v) = \begin{cases} v - n_2 + n_2' & \text{if } v > n_2, \\ v - n_1 + n_1' & \text{if } n_2 \geqslant v > n_1, \\ v & \text{if } n_1 \geqslant v, \end{cases}$$

$$\beta(e) = \begin{cases} e - m_2 + m_2' & \text{if } e > m_2, \\ e - m_1 + m_1' & \text{if } m_2 \geqslant e > m_1, \\ e & \text{if } m_1 \geqslant e. \end{cases}$$

It follows from Lemma 4.6 that the right hand side of (8) depends only on the internal isomorphism class $[K_2]$ of $K_2$, and not on its exact labelling. (8) is furthermore by (7) equivalent to

$$(G \div H_1 \div H_2)\bigl([K_1], [K_2]\bigr) = \Bigl[\bigl((G \div H_2)(K_2) \div H_1\bigr)(K_1)\Bigr]$$

and hence the values of $G \div H_1 \div H_2$ is indeed determined only by the internal isomorphism classes of its arguments. □

The next lemma treats the opposite case of composition of splice maps: what happens when one part is contained in another. The proof is left as an exercise to the reader.

**Lemma 4.9.** *Let $F \in \mathcal{D}$ and $H \sqsubseteq G \sqsubseteq F$ be given. For any $K \in \mathcal{D}(\partial H)$,*

$$(F \div H)(K) \simeq (F \div G)\bigl((G \div H)(K)\bigr). \tag{9}$$

*More generally, if $G_1 \in \mathcal{D}(\partial G)$ and $H_1 \sqsubseteq G_1$ then there exist $F_2 \in \mathcal{D}(\partial F)$ and $H_2 \in \mathcal{D}(\partial H_1)$ such that $H_2 \sqsubseteq F_2$ and for all $K \in \mathcal{D}(\partial H_1)$ it holds that*

$$(F_2 \div H_2)(K) \simeq (F \div G)\bigl((G_1 \div H_1)(K)\bigr). \tag{10}$$

The previous lemmas have all concerned fairly generic properties of isomorphism and splicing in diagrams, in the sense that some variant on these properties should hold for diagrams no matter what how they are formalised, and they are part of a general toolbox for defining reductions on $\mathcal{M}(L)$. The utility of the next lemma is more specific to the concrete rewriting system studied.

**Lemma 4.10.** *For any $G \in \mathcal{D}$, $H \sqsubseteq G$, and $K \in \mathcal{D}(\partial H)$,*

$$\Bigl|\mathrm{E}\bigl((G \div H)(K)\bigr)\Bigr| = \bigl|\mathrm{E}(G)\bigr| - \bigl|\mathrm{E}(H)\bigr| + \bigl|\mathrm{E}(K)\bigr|. \tag{11}$$

*Proof.* By the construction of the edge offset factor $m$ in Definition 4.4, the union in the definition of $\mathrm{E}\bigl((G \div H)(K)\bigr)$ is disjoint. Hence

$$\Bigl|\mathrm{E}\bigl((G \div H)(K)\bigr)\Bigr| = \bigl|\mathrm{E}(G)\bigr| - \Bigl(\bigl|\mathrm{E}(H)\bigr| - |\partial H|\Bigr) + \Bigl(\bigl|\mathrm{E}(K)\bigr| - |\partial K|\Bigr) \tag{12}$$

and the claim follows from $\partial K = \partial H$. □

The splice maps for unlabelled graphs may be compared to maps multiplying by a constant monomial, in the sense that they play the same role in the next section as multiplication by a monomial does in Gröbner basis theory or Bergman's diamond lemma [3]. This may seem an overly modest foundation to build an algebraic theory on, but it does suffice, and it handles many issues — in particular the construction of simple reductions — quite elegantly. It is however not the only possibility for a "multiplication structure" on semigraphs.

A natural product of unlabelled semigraphs is to join up common external edges; formally one may define $G \cdot H$ for $G, H \in \mathcal{D}$ by

$$\begin{aligned} n &= \max\bigl(\{0\} \cup \mathrm{V}(G)\bigr), \\ m &= \max\bigl(\{0\} \cup \mathrm{E}(G) \cup \partial H\bigr), \\ \mathrm{V}(G \cdot H) &= \mathrm{V}(G) \cup \left\{\, v+n \,\middle|\, v \in \mathrm{V}(H) \,\right\}, \\ \mathrm{E}(G \cdot H) &= \mathrm{E}(G) \cup \partial H \cup \left\{\, e+m \,\middle|\, e \in \mathrm{E}(H) \setminus \partial H \,\right\}, \\ \phi_{G \cdot H}(e) &= \begin{cases} \left\{\, v+n \,\middle|\, n \in \phi_H(e-m) \,\right\} & \text{if } e > m, \\ \left\{\, v+n \,\middle|\, n \in \phi_H(e) \,\right\} & \text{if } e \in \partial H \setminus \partial G, \\ \phi_G(e) \cup \left\{\, v+n \,\middle|\, n \in \phi_H(e) \,\right\} & \text{if } e \in \partial H \cap \partial G, \\ \phi_G(e) & \text{otherwise.} \end{cases} \end{aligned}$$

If $G$ and $H$ are multigraphs, then this $G \cdot H$ is simply the disjoint union of $G$ and $H$. It should be no surprise that $G_1 \cdot H_1 \simeq G_2 \cdot H_2$ whenever $G_1 \simeq G_2$ and $H_1 \simeq H_2$, so this multiplication is well-defined also for unlabelled semigraphs. Slightly more surprising is perhaps that

$$G\bigl\{\mathrm{V}(G) \setminus \mathrm{V}(H)\bigr\} \cdot K \simeq (G \div H)(K), \tag{13}$$

meaning the "quotients" $G \div H$ can actually themselves be identified with semigraphs. Defining $\mathcal{A} = \bigoplus_{\text{finite } L \subset \mathbb{Z}_{>0}} \mathcal{M}(L)$, one has even produced an "$\mathcal{R}$-algebra of unlabelled semigraphs", which turns out to be graded by the group of finite subsets of $\mathbb{Z}_{>0}$ under symmetric difference, since $\partial(G \cdot H) = (\partial G \setminus \partial H) \cup (\partial H \setminus \partial G)$! This view has many nice features, but there is also a downside.

One problem with the semigraph algebra $\mathcal{A}$ is that its multiplicative structure is rather far from what is common in rewriting theories: there is no unique factorisation, not even cancellation (the definition of splice map $G \div H$ requires $G$ and $H$ to be labelled), and consequently $\mathcal{A}$ is rich with zero divisors. A more important problem is however that the semigraph multiplication doesn't generalise well to diagrams with more structure. In a directed semigraph, it wouldn't be possible to join an external edge $e$ of $G$ with the external edge $e$ of $H$ if both ends are heads or both ends are tails. In a directed acyclic semigraph, which is something found underneath a PROP (or symmetric monoidal category), it need not be allowed to join an output of $G$ to an input of $H$ at the same time as one joins an output of $H$ to an input of $G$, as this could create a cycle. If one is working with plane diagrams, then joining external edges with equal labels is likely to produce a non-planar diagram. And so on.

The key concept is "replacing part of a diagram with another diagram of the same sort", and that is what the splice maps do. Multiplication of diagrams assumes that "a diagram with a part removed" (which is the essential meaning of some $G \div H$) can be identified with another diagram, but that is often not possible. A plane diagram has to be a plane semigraph with all external edges in the outer facet, but a plane diagram with a piece missing may have external edges also in some inner facet, where the extra piece would be plugged in. Similarly a tree with a piece missing is generally not a tree, but rather a forest. Inserting any tree (connected acyclic semigraph) with the right boundary will turn the forest back into a tree however, so splicing on trees is a well-behaved operation in that it avoids going outside the set of trees. The same holds for the other types of diagrams mentioned above.

# 5 Applying the diamond lemma

This section is all about applying the machinery from [7] to the present semigraph rewriting problem. Not only results, but also definitions and notation from that paper will be used extensively and without restraint. The conclusion can be understood without first having read [7], but the proof of it probably cannot.

Parts of the basic framework for the diamond lemma has already been set up, but a review may be in order:

- The index set $I$ — the set of sorts — will be the set of all finite subsets of $\mathbb{Z}_{>0}$, i.e., the set of all boundaries of semigraphs in $\mathcal{D}$.
- The abelian group of all expressions of sort $L \in I$ will be the vector space $\mathcal{M}(L)$ of Definition 4.2.
- Similarly, the set of all "monomials" of sort $L \in I$ will be the set $\mathcal{Y}(L)$ of unlabelled semigraphs with boundary $L$ from that same definition.

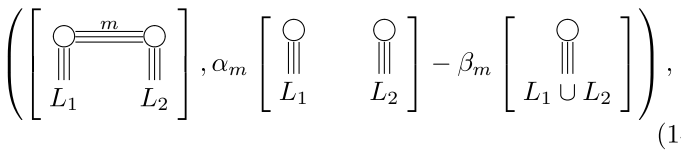
In addition, there are some advanced aspects of the framework which we won't have to worry about:

- The set $R(L)$ of endomorphisms of $\mathcal{M}(L)$ is simply going to be the set of actions of $\mathcal{R}$ on $\mathcal{M}(L)$, so an $R(L)$-module (or $R$-module, in the single-sorted notation) is just an $\mathcal{R}$-vector-space.
- The topology used will be the discrete topology, so $B_n(L) = \{0\}$ for all $n \geqslant 1$ and $L \in I$. This also implies that $\overline{\mathcal{M}}(L) = \mathcal{M}(L)$ for all $L \in I$, that all maps $\mathcal{M}(L) \longrightarrow \mathcal{M}(L)$ are equicontinuous, and that the topological descending chain condition is the same thing as the ordinary descending chain condition (any subset contains a minimal element).

The above should be considered the default settings for these two aspects of the generic diamond lemma framework, as it is often what one wants for diagrammatical calculations, and those that want something more advanced generally know that they do.

Even the main step of defining the simple reductions has been hinted at, but now is the time to do it properly. For all $m \in \mathbb{Z}_{>0}$ and $L_1, L_2 \in I$ such that $L_1 \cap L_2 = \varnothing$ define

$$s(L_1, m, L_2) :=$$




(14)

which is an element of $\mathcal{Y}(L_1 \cup L_2) \times \mathcal{M}(L_1 \cup L_2)$. For all $L \in I$, let $S(L)$ be the set of all $s(L_1, m, L \setminus L_1)$ for $m \in \mathbb{Z}_{>0}$ and $L_1 \subseteq L$, and let $S = \bigcup_{L \in I} S(L)$.

It is convenient to introduce a less spacious notation for unlabelled semigraphs without internal edges, as the results once rewriting is complete will involve a lot of these. Therefore let $[L_1, \dots, L_n]$ denote the element $[G] \in \mathcal{Y}(L_1 \cup \cdots \cup L_n)$ where $\mathrm{V}(G) = \{1, \dots, n\}$, $\mathrm{E}(G) = \bigcup_{k=1}^{n} L_k$, and $\phi_G(e) = \{k\}$ for all $e \in L_k$, i.e., $L_1$ through $L_n$ are the sets of external edges incident with vertices 1 through $n$ respectively. (The only aspect of this notation that is not uniquely determined by the underlying element of $\mathcal{Y}(L)$ is the order of the $L_k$ sets.) For $(\mu, a) = s(L_1, m, L_2)$ this means $a$ can be expressed as $\alpha_m[L_1, L_2] - \beta_m[L_1 \cup L_2]$.

For any finite $L, L' \in I$ define

$$V(L, L') = \left\{ G \div H \colon \mathcal{M}(L') \longrightarrow \mathcal{M}(L) \,\middle|\, G \in \mathcal{D}(L), H \sqsubseteq G, \partial H = L' \right\}. \quad (15)$$

Note that by Lemma 4.9, if $v \in V(L, L')$ and $w \in V(L', L'')$, then $v \circ w \in V(L, L'')$. For any $v \in V(L, L')$ and $s = (\mu_s, a_s) \in S(L')$ define $t_{v,s} \colon \mathcal{M}(L) \longrightarrow \mathcal{M}(L)$ to be the linear map which satisfies

$$t_{v,s}(\lambda) = \begin{cases} v(a_s) & \text{if } \lambda = v(\mu_s), \\ \lambda & \text{otherwise,} \end{cases} \qquad \text{for all } \lambda \in \mathcal{Y}(L). \quad (16)$$

Finally let

$$T_1(S)(L) = \left\{ t_{v,s} \,\middle|\, v \in V(L, L'), s \in S(L'), L' \in I \right\} \quad (17)$$

be the set of simple reductions on $\mathcal{M}(L)$, for every $L \in I$. With this in place, the derived sets $\mathcal{I}(S)(L)$, $\mathrm{Irr}(S)(L)$, etc. are defined for all $L \in I$. Also note that the maps in $V(L, L')$ are all advanceable with respect to $T(S)(L')$ and $T(S)(L)$, since $w\bigl(t_{v,s}(\lambda)\bigr) = t_{w \circ v, s}\bigl(w(\lambda)\bigr)$ for all $\lambda \in \mathcal{Y}(L')$, $w \in V(L, L')$, and $t_{v,s} \in T_1(S)(L')$.

In order to verify that this has anything do with the problem that was posed in Section 3, it must be established that $\mathcal{I} = \mathcal{I}(S)(\varnothing)$, or in more elementary language that the rules replace two adjacent vertices by the wanted linear combination of their delete–contract counterparts. By [7, Lemma 3.7], $\mathcal{I}(S)(\varnothing)$ is the set spanned by all $\lambda - t(\lambda)$ such that $\lambda \in \mathcal{Y}(\varnothing)$ and $t \in T_1(S)(\varnothing)$, i.e., the set spanned by all $\lambda - t_{v,s}(\lambda)$ such that $\lambda \in \mathcal{Y}(\varnothing)$, $v \in V(\varnothing, L)$, and $s \in S(L)$ for some $L \in I$. By the definition of $t_{v,s}$, $\lambda - t_{v,s}(\lambda) = 0$ unless $\lambda = v(\mu_s)$, and in that case $\lambda - t_{v,s}(\lambda) = v(\mu_s) - v(a_s)$.

Every $v \in V(\varnothing, L)$ is of the form $G \div H$ for some $G \in \mathcal{D}(\varnothing)$ and $H \sqsubseteq G$ such that $\partial H = L$. Similarly every $s = (\mu_s, a_s) \in S(L)$ is such that $v(\mu_s) = \bigl[(G \div H)(K)\bigr]$ for some $K \in \mathcal{D}(L)$ such that $\mathrm{V}(K) = \{v_1, v_2\}$, and if one furthermore defines $L_i = \left\{ e \in L \,\middle|\, \phi_K(e) = \{v_i\} \right\}$ for $i = 1, 2$ and $m = \bigl|\mathrm{E}(K) \setminus L\bigr|$ then $a_s = \alpha_m[L_1, L_2] - \beta_m[L]$. Letting $G' = (G \div H)(K)$ and $e \in \mathrm{E}(G') \setminus \mathrm{E}(G)$, it follows that $|\bar{e}| = m$, $[G' - \bar{e}] = (G \div H)\bigl([L_1, L_2]\bigr)$, and $[G'/\bar{e}] = (G \div H)\bigl([L]\bigr)$, whence $v(\mu_s) - v(a_s) = [G'] - \alpha_m[G' - \bar{e}] + \beta_m[G'/\bar{e}] \in \mathcal{I}$. Hence $\mathcal{I}(S)(\varnothing) \subseteq \mathcal{I}$.

Conversely, for any labelled multigraph $G \in \mathcal{D}(\varnothing)$ and every edge $e \in \mathrm{E}(G)$, there are two endpoints $\{v_1, v_2\} = \phi_G(e)$. Let $H$ be the induced subsemigraph $G\bigl\{\phi_G(e)\bigr\}$, let $L_i = \left\{ e \in \partial H \,\middle|\, \phi_H(e) = \{v_i\} \right\}$ for $i = 1, 2$, and let $m = |\bar{e}|$. Then $s(L_1, m, L_2) \in S(\partial H)$ satisfies $\mu_{s(L_1,m,L_2)} = [H]$ and $G \div H \in V(\varnothing, \partial H)$.

Furthermore $[G-\bar{e}] = (G \div H)\bigl([L_1, L_2]\bigr)$ and $[G/\bar{e}] = (G \div H)\bigl([L_1 \cup L_2]\bigr)$, hence

$$\begin{aligned}
&[G] - \alpha_m[G-\bar{e}] + \beta_m[G/\bar{e}] = \\
&\quad = (G \div H)\bigl([H]\bigr) - \alpha_m(G \div H)\bigl([L_1, L_2]\bigr) + \beta_m(G \div H)\bigl([L_1 \cup L_2]\bigr) = \\
&\qquad\qquad = (G \div H)(\mu_{s(L_1,m,L_2)}) - (G \div H)(a_{s(L_1,m,L_2)}) \in \mathcal{I}(S)(\varnothing)
\end{aligned}$$

and thus $\mathcal{I} \subseteq \mathcal{I}(S)(\varnothing)$.

The next necessary step is to define a suitable partial order on the unlabelled multigraphs, but for resolving ambiguities it is convenient to have corresponding partial orders defined for unlabelled semigraphs of all sorts. Constructing these things can be quite complicated for some diagrammatical calculation problems, but in the present case it is sufficient to compare elements by size (number of edges). Thus for every sort $L \in I$, let the partial order $P(L)$ on $\mathcal{Y}(L)$ be defined by $[G] < [H]$ in $P(L)$ iff $\bigl|\mathrm{E}(G)\bigr| < \bigl|\mathrm{E}(H)\bigr|$. Since there are only finitely many possible values for $\bigl|\mathrm{E}(G)\bigr|$ if $[G] < [H]$ in $P(L)$ for some given $[H]$, it follows that all $P(L)$ satisfy the descending chain condition.

For the issue of whether the simple reductions are compatible with these partial orders, one should first observe that for all $v \in V(L, L')$ and $\mu, \nu \in \mathcal{Y}(L')$, it follows from Lemma 4.10 that $v(\mu) < v(\nu)$ in $P(L)$ if and only if $\mu < \nu$ in $P(L')$, and hence

$$v\Bigl(\mathrm{DSM}\bigl(\nu, P(L')\bigr)\Bigr) \subseteq \mathrm{DSM}\bigl(v(\nu), P(L)\bigr). \tag{18}$$

Thus the issue of whether $t_{v,s}$ is compatible with $P(L)$ reduces to the issue of whether $a_s \in \mathrm{DSM}\bigl(\mu_s, P(L')\bigr)$, and that is easily verified by considering the explicit form $s(L_1, m, L_2)$ for the rule $s$: any $H \in \mu_{s(L_1,m,L_2)}$ has $\bigl|\mathrm{E}(H)\bigr| = |L_1| + m + |L_2|$ whereas $H \in [L_1, L_2]$ or $H \in [L_1 \cup L_2]$ has $\bigl|\mathrm{E}(H)\bigr| = |L_1| + |L_2|$. The general conditions for the diamond lemma are thus fulfilled, and one may conclude that:

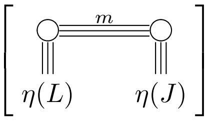
- $\mathrm{Irr}(S)(\varnothing)$ is the subspace of $\mathcal{M}(\varnothing)$ spanned by the unlabelled graphs without edges [7, Lemma 5.6].
- $\mathcal{M}(\varnothing) = \mathrm{Irr}(S)(\varnothing) \oplus \mathcal{I}(S)(\varnothing)$ if and only if all ambiguities of $T_1(S)(\varnothing)$ are resolvable relative to $P(\varnothing)$ [7, Theorem 5.11].

It is when verifying that the ambiguities are resolvable that one ends up making diagrammatic calculations, but most of them are about semigraphs of other sorts than $\varnothing$, as the critical part of an ambiguity of $T_1(S)(\varnothing)$ is typically a much smaller ambiguity of some other $T_1(S)(L)$. For the purpose of analysing ambiguities, the following lemma is very convenient.

**Lemma 5.1.** *Let $L \in I$, $t \in T_1(S)(L)$, and $\lambda \in \mathcal{Y}(L)$ such that $t$ acts nontrivially on $\lambda$ be given. For every $G \in \lambda$ there exists some $H \sqsubseteq G$ and $s \in S(\partial H)$ such that $H \in \mu_s$ and $t = t_{G \div H, s}$.*

*Proof.* Since $t$ acts nontrivially on $\lambda$ it must be the case that $t = t_{v,s'}$, where $v(\mu_{s'}) = \lambda$, for some $v \in V(L, L')$, $s' \in S(L')$, and $L' \in I$. By Lemma 4.5, this means $v = G' \div K$ for some $G' \in \lambda$ and $K \in \mu_{s'}$ such that $K \sqsubseteq G'$. By Lemma 4.7 and since $G' \simeq G$, there exists some $H \sqsubseteq G$ and $\mathring{\eta}\colon \mathcal{Y}(\partial K) \longrightarrow \mathcal{Y}(\partial H)$ such that $H \cong K$ and $v = G' \div K = (G \div H) \circ \mathring{\eta}$. Letting $\mathring{\eta}$ act on both parts of $s'$, which must be on the form $s(L,m,J)$ for some $L, J \in I$ and $m \in \mathbb{Z}_{>0}$, one finds that

$$\mathring{\eta}(\mu_{s(L,m,J)}) =$$



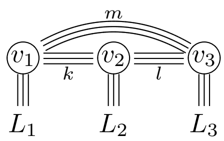

$$= \mu_{s(\eta(L),m,\eta(J))},$$

$$\begin{aligned} \mathring{\eta}(a_{s(L,m,J)}) &= \mathring{\eta}\big(\alpha_m[L,J] - \beta_m[L \cup J]\big) = \\ &= \alpha_m \mathring{\eta}\big([L,J]\big) - \beta_m \mathring{\eta}\big([L \cup J]\big) = \\ &= \alpha_m \big[\eta(L), \eta(J)\big] - \beta_m \big[\eta(L) \cup \eta(J)\big] = \\ &= a_{s(\eta(L),m,\eta(J))}. \end{aligned}$$

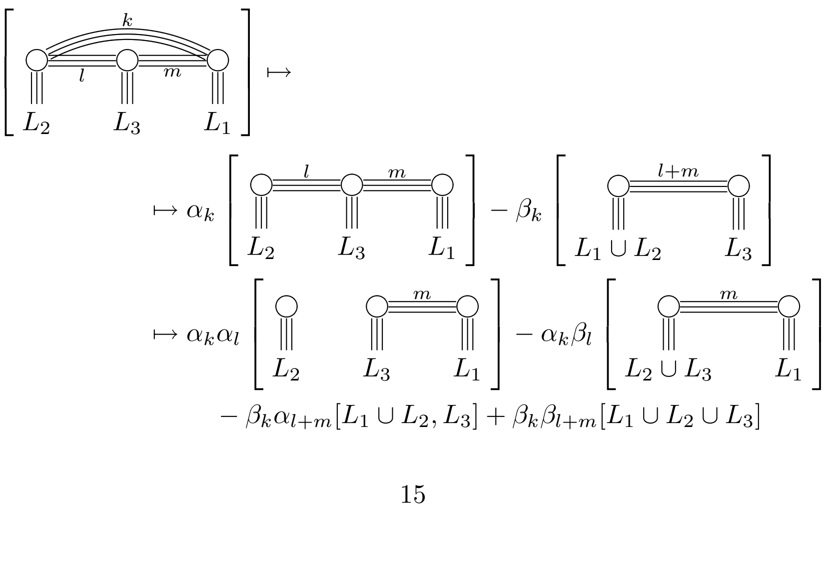
Hence for $s = s(\eta(L), m, \eta(J))$, the wanted result is obtained. $\square$

Let an ambiguity $(t_1, \mu, t_2)$ of $T_1(S)(\varnothing)$ be given, and fix some $G \in \mu$. The lemma then implies that $(t_1, \mu, t_2)$ is of the form $(t_{G \div H_1, s_1}, [G], t_{G \div H_2, s_2})$, where $H_i \sqsubseteq G$ satisfies $H_i \in \mu_{s_i}$ for $i = 1, 2$. This description of ambiguities is concrete enough that a resolution can be computed. There are essentially three ambiguity cases in this system, and these may be distinguished by the number of vertices that $H_1$ and $H_2$ have in common. Since $\big|\mathrm{V}(H_1)\big| = \big|\mathrm{V}(H_2)\big| = 2$, the first case is that $\mathrm{V}(H_1) = \mathrm{V}(H_2)$, so that both reductions act on exactly the same part of $G$. Due to the symmetry of the rules ($s(L,m,J) = s(J,m,L)$ for all rules $s(L,m,J)$), all such cases have $t_1 = t_2$ and are thus trivially resolvable.

A not at all trivial case occurs if $\big|\mathrm{V}(H_1) \cap \mathrm{V}(H_2)\big| = 1$. Consider first what happens in $H = G\big\{\mathrm{V}(H_1) \cup \mathrm{V}(H_2)\big\}$. This semigraph has the form



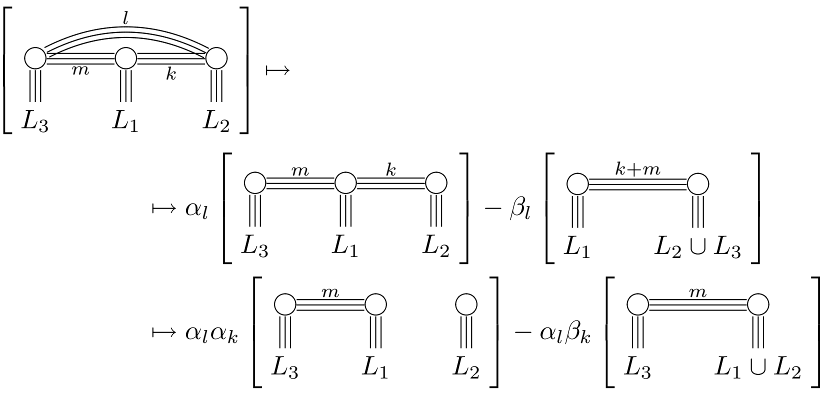

where $v_1 \in \mathrm{V}(H_1) \setminus \mathrm{V}(H_2)$, $v_2 \in \mathrm{V}(H_1) \cap \mathrm{V}(H_2)$, $v_3 \in \mathrm{V}(H_2) \setminus \mathrm{V}(H_1)$, and $L_i = \big\{ e \in \partial H \,\big|\, \phi_H(e) = \{v_i\} \big\}$ for $i = 1, 2, 3$. $k \geqslant 1$ is the number of edges between $v_1$ and $v_2$, $l \geqslant 1$ is the number of edges between $v_2$ and $v_3$, whereas $m \geqslant 0$ is the number of edges between $v_3$ and $v_1$. This is the site of the ambiguity $\big(t_{H \div H_1, s_1}, [H], t_{H \div H_2, s_2}\big)$ of $T_1(S)(\partial H)$, which is resolved as follows. On the $t_{H \div H_1, s_1}$ side,




$$- \beta_k \alpha_{l+m}[L_1 \cup L_2, L_3] + \beta_k \beta_{l+m}[L_1 \cup L_2 \cup L_3]$$

$$
\begin{aligned}
&\mapsto \alpha_k\alpha_l\alpha_m[L_1, L_2, L_3] - \alpha_k\alpha_l\beta_m[L_1 \cup L_3, L_2] \\
&\qquad - \alpha_k\beta_l\alpha_m[L_1, L_2 \cup L_3] - \beta_k\alpha_{l+m}[L_1 \cup L_2, L_3] \\
&\qquad + (\alpha_k\beta_l\beta_m + \beta_k\beta_{l+m})[L_1 \cup L_2 \cup L_3],
\end{aligned} \tag{19}
$$

and on the $t_{H \div H_2, s_2}$ side,




$$
\begin{aligned}
&\qquad - \beta_l\alpha_{k+m}[L_1, L_2 \cup L_3] + \beta_l\beta_{k+m}[L_1 \cup L_2 \cup L_3] \\
&\mapsto \alpha_l\alpha_k\alpha_m[L_3, L_1, L_2] - \alpha_l\alpha_k\beta_m[L_1 \cup L_3, L_2] \\
&\qquad - \alpha_l\beta_k\alpha_m[L_1 \cup L_2, L_3] - \beta_l\alpha_{k+m}[L_1, L_2 \cup L_3] \\
&\qquad + (\alpha_l\beta_k\beta_m + \beta_l\beta_{k+m})[L_1 \cup L_2 \cup L_3].
\end{aligned} \tag{20}
$$

The $[L_1, L_2, L_3]$ and $[L_1 \cup L_3, L_2]$ terms of these reductions are always equal, but the $[L_1 \cup L_2, L_3]$ terms are only equal if $\beta_k\alpha_{l+m} = \beta_k\alpha_l\alpha_m$, the $[L_1, L_2 \cup L_3]$ terms are only equal if $\alpha_k\beta_l\alpha_m = \beta_l\alpha_{k+m}$, and the $[L_1 \cup L_2 \cup L_3]$ terms are only equal if $\alpha_k\beta_l\beta_m + \beta_k\beta_{l+m} = \alpha_l\beta_k\beta_m + \beta_l\beta_{k+m}$. These are thus the conditions under which this ambiguity is resolvable.

There is a slight formal complication in that the reduction steps removing edges between $v_1$ and $v_3$ are not really allowed if $m = 0$, as that means there aren't any edges there to remove, but if one sets $\alpha_0 = 1$ and $\beta_0 = 0$ then the reduction step carried out is equivalent to applying the identity reduction id. Hence the $m = 0$ cases can be resolved using a calculation of the same form as the $m > 0$ cases, although a more direct approach would be to make a separate calculation for the $m = 0$. The given equations furthermore turn out to be trivially fulfilled for $m = 0$ if $\alpha_0 = 1$ and $\beta_0 = 0$, so it can be argued that $m = 0$ really represents a different type of ambiguity than the $m > 0$ cases.

All of this has been about an ambiguity of $T_1(S)(\partial H)$, however — what about the ambiguity $(t_1, \mu, t_2)$ that we began with? By [7, Lemma 6.3] this ambiguity is resolvable relative to $P(\varnothing)$, since it is a shadow of the ambiguity at $[H]$ that was resolved above. The key to making this claim is Lemma 4.9, which implies that $(G \div H)\bigl(t_{H \div H_i, s_i}(\lambda)\bigr) = t_i\bigl((G \div H)(\lambda)\bigr)$ for all $\lambda \in \mathcal{Y}(\partial H_i)$ and $i = 1, 2$, and as explained above more generally implies that $G \div H \in V(\partial G, \partial H)$ is an advanceable map.

What remains is therefore the case $\mathrm{V}(H_1) \cap \mathrm{V}(H_2) = \varnothing$, which again is practically trivial (although the theory is somewhat involved). What happens in this case is that $(t_1, \mu, t_2)$ is a montage of the two pieces $\bigl([H_1], t_{\mathrm{id}, s_1}\bigr)$ and $\bigl([H_2], t_{\mathrm{id}, s_2}\bigr)$ under the composition map $w = G \div H_1 \div H_2$ as detailed in Lemma 4.8; the biadvanceability of $w$ is immediate from $w\bigl([K_1], \cdot\bigr) = (G \div H_1)(K_1) \div H_2 \in$

$V(\partial G, \partial H_2)$ and $w\bigl(\cdot, [K_2]\bigr) = (G \div H_2)(K_2) \div H_1 \in V(\partial G, \partial H_1)$. Since these expressions via (18) also demonstrate that the conditions in [7, Lemma 6.7] are fulfilled, it follows that all ambiguities of this last class are resolvable relative to $P(\varnothing)$.

Since $\mathcal{M}(\varnothing) = \mathrm{Irr}(S)(\varnothing) \oplus \mathcal{I}(S)(\varnothing)$ implies $\mathcal{M}(\varnothing)/\mathcal{I}(S)(\varnothing) \cong \mathrm{Irr}(S)(\varnothing)$, it has thus been shown that:

**Lemma 5.2.** *If the coefficients* $\{\alpha_m\}_{m=1}^{\infty}, \{\beta_m\}_{m=1}^{\infty} \subseteq \mathcal{R}$ *satisfy*

$$(\alpha_k \alpha_m - \alpha_{k+m})\beta_l = 0, \tag{21a}$$

$$\alpha_k \beta_l \beta_m + \beta_k \beta_{l+m} = \alpha_l \beta_k \beta_m + \beta_l \beta_{k+m} \tag{21b}$$

*for all* $k, l, m \in \mathbb{Z}_{>0}$ *then a* $W$*-valued multigraph invariant* $Q$ *which satisfies* (2) *is uniquely determined by its values for graphs with no edges, and every assignment of values to the graphs with no edges extends to an invariant* $Q$ *for all multigraphs.*

The part about the invariant being uniquely determined by its values on graphs with no edges is pretty easy to arrive at by elementary methods, but the part that every possible assignment of values to these graphs is allowed for such invariants is not. Likewise, it is fairly easy to show that (21) are necessary conditions for such an invariant, but harder to show that they are also sufficient.

# 6 Classification of invariants

The two main cases in the classification are (i) invariants for which the conditions of Lemma 5.2 are fulfilled and (ii) invariants for which these conditions are not fulfilled. In the latter case, it is possible to derive additional relations which lead to simpler recursions or classifications, which means these on the whole tend to have fewer degrees of freedom. The former case is more interesting, so let us begin with that.

Denote by $q^n$ the value of $Q(K_n^{\mathrm{C}})$. (Even if $W$ is not formally required to be a space of polynomials, it turns out to be a very natural identification, and there is no loss of information as long as one preserves the coefficients of the polynomial.) Also let $|G|$ denote the order (number of vertices) of the multigraph $G$, let $\|G\|$ denote the size (number of edges) of the multigraph $G$, and let $c(G)$ denote the number of components (connected nonempty induced subsemigraphs without external edges) in $G$. Some invariants turn out to be explainable in terms of these elementary invariants alone.

**Invariant class 1.** $\beta_m = 0$ for all $m \in \mathbb{Z}_{>0}$. In this case (21) is fulfilled for all $\{\alpha_m\}_{m=1}^{\infty} \subseteq \mathcal{R}$, and so these may be chosen arbitrarily.

Since $Q(G) = \alpha_m Q(G - \bar{e})$ if $|\bar{e}| = m$, the coefficient $\alpha_m$ is essentially a weight attributed to edges of multiplicity $m$, and apart from that $Q$ only keeps track of the number of vertices.

If $\beta_l \neq 0$ for some $l$ then (21a) implies $\alpha_{k+m} = \alpha_k \alpha_m$ for all $k$ and $m$. This has the unique solution $\alpha_m = \alpha_1^m$, and so the values of $\alpha$'s in the remaining cases are determined by the values of $\alpha_1$.

**Invariant class 2.** $\beta_m \neq 0$ for some $m$ and $\alpha_m = 0$ for all $m$. (21b) simplifies to $\beta_k \beta_{l+m} = \beta_l \beta_{k+m}$, which for $l = 1$ and $k = m + 1$ reads $\beta_{m+1}^2 = \beta_1 \beta_{2m+1}$.

Since $\beta_m \neq 0$ for some $m$, it follows that $\beta_1 \neq 0$. Setting $k = 2$ and $l = 1$ one gets $\beta_2\beta_{m+1} = \beta_1\beta_{m+2}$ for all $m \in \mathbb{Z}_{>0}$, from which follows that

$$\beta_k = \beta_1^{2-k}\beta_2^{k-1} = \beta_1(\beta_1^{-1}\beta_2)^{k-1} \qquad \text{for all } k \in \mathbb{Z}_{>0}. \tag{22}$$

Hence the parameters of this invariant class are $\beta_1$, $\beta_2/\beta_1$, and $\{q^n\}_{n=0}^{\infty}$.

As with the previous invariant class, $Q(G)$ is always a single $q^n$ times some weight factors, but in this case $n$ will be the number of components $c(G)$. The exponent on $\beta_1$ will be the number of contractions made, i.e., $|G| - c(G)$ and the exponent on $\beta_2/\beta_1$ will be the number of edges minus the number of contractions. Hence the value of this invariant is completely determined by $c(G)$, $|G|$, and $\|G\|$.

Now assume $\beta_l \neq 0$ for some $l$ and $\alpha_k = \alpha_1^k \neq 0$ for all $k$. Setting $l = 1$ and $k = m+1$ in (21b) leads to $\alpha_1^{m+1}\beta_1\beta_m + \beta_{m+1}^2 = \alpha_1\beta_{m+1}\beta_m + \beta_1\beta_{2m+1}$, or from collecting terms $\beta_1(\alpha_1^{m+1}\beta_m - \beta_{2m+1}) = \beta_{m+1}(\alpha_1\beta_m - \beta_{m+1})$. If $\beta_1 = \beta_m = 0$ then this implies $\beta_{m+1} = 0$ too, and in particular $\beta_1 = 0$ implies $\beta_2 = 0$. As it was assumed some $\beta_m \neq 0$, it follows that $\beta_1 \neq 0$.

Having established that, it is possible to fix

$$\beta_m = \alpha_1^m \cdot (\beta_1/\alpha_1) \cdot \gamma_m \tag{23}$$

for some new family of parameters $\{\gamma_m\}_{m=1}^{\infty} \subseteq \mathcal{R}$ where $\gamma_1 = 1$. Inserting this into (21b) yields the homogeneous equation system

$$\gamma_l\gamma_m + \gamma_k\gamma_{l+m} = \gamma_k\gamma_m + \gamma_l\gamma_{k+m} \qquad \text{for all } k, l, m \in \mathbb{Z}_{>0}. \tag{24}$$

Let $z = \gamma_2 - 1$, $l = 1$, and $k = 2$. This equation then becomes $\gamma_m + (1+z)\gamma_{m+1} = (1+z)\gamma_m + \gamma_{m+2}$, or equivalently $z(\gamma_{m+1} - \gamma_m) = \gamma_{m+2} - \gamma_{m+1}$. Hence $\gamma_{m+1} - \gamma_m = z^m$ and $\gamma_m = \sum_{k=0}^{m-1} z^k$ — the so-called $z$-natural numbers.

**Invariant class 3** (The Tutte polynomial)**.** If some $\beta_k$ and $\alpha_k$ are both nonzero then the parameters of the invariant are $\alpha_1$, $\beta_1/\alpha_1$, $z$, and $\{q^n\}_{n=0}^{\infty}$, where

$$\alpha_m = \alpha_1^m, \tag{25a}$$

$$\beta_m = \alpha_1^m \cdot (\beta_1/\alpha_1) \cdot \sum_{k=0}^{m-1} z^k. \tag{25b}$$

It should be observed that $\alpha_1$ is essentially a weight on edges and $\beta_1/\alpha_1$ is a weight on vertices, where the latter however has some interaction with the $q^n$.

If one denotes by $Q'$ an invariant of this class with values in $\mathcal{R}[q]$ and $\alpha_1 = \beta_1 = 1$ but the same value of $z$ as a particular $\mathcal{R}[q]$-valued $Q$, then these two are related by

$$Q(G)(q) = \alpha_1^{\|G\|} \cdot (\beta_1/\alpha_1)^{|G|} \cdot Q'(G)(q\alpha_1/\beta_1),$$

so any nontrivial information from $Q$ has been encoded into this simpler $Q'$. What is this invariant $Q'$?

For $z = 0$ one recovers the chromatic polynomial, but $Q'(G)$ seen as a polynomial in the two variables $q$ and $z$ also happens to be a known graph invariant: it is the *Tutte polynomial* — in a variant of $q$-state Potts model variables; see [10] for a nice overview of different forms of the Tutte polynomial — and a closed form expression for it is

$$Q'(G) = (-1)^{|G|} \sum_{F \subseteq \mathrm{E}(G)} (-q)^{c(G-F)} (z-1)^{\|G-F\| + c(G-F) - |G|}. \tag{26}$$

In order to verify that this satisfies the recursion (2), one may collect the terms of the sum depending on whether $F \supseteq \bar{e}$ or $F \not\supseteq \bar{e}$, where the former group turns out to sum to $Q'(G - \bar{e})$ and the latter group sums to $-\gamma_{|\bar{e}|} Q'(G/\bar{e})$.

For the invariants whose coefficients do not satisfy (21), one may return to the identity

$$\begin{aligned}(\beta_k \alpha_{l+m} - \beta_k \alpha_l \alpha_m)[L_1 \cup L_2, L_3] + (\alpha_k \beta_l \alpha_m - \beta_l \alpha_{k+m})[L_1, L_2 \cup L_3] + \\ + (\alpha_k \beta_l \beta_m + \beta_k \beta_{l+m} - \alpha_l \beta_k \beta_m - \beta_l \beta_{k+m})[L_1 \cup L_2 \cup L_3] \in \\ \in \mathcal{I}(S)(L_1 \cup L_2 \cup L_3)\end{aligned} \tag{27}$$

that in (19) and (20) was derived for all $k, l, m \in \mathbb{Z}_{>0}$ and disjoint $L_1, L_2, L_3 \in I$. It is convenient to consider the special case $L_2 = \varnothing$ (which arises when the common vertex $v_2$ have no neighbours other than $v_1$ and $v_3$), as the identity then simplifies to the two terms

$$\begin{aligned}(\beta_k \alpha_{l+m} - \beta_k \alpha_l \alpha_m + \alpha_k \beta_l \alpha_m - \beta_l \alpha_{k+m})[L_1, L_3] + \\ + (\alpha_k \beta_l \beta_m + \beta_k \beta_{l+m} - \alpha_l \beta_k \beta_m - \beta_l \beta_{k+m})[L_1 \cup L_3] \in \mathcal{I}(S)(L_1 \cup L_3).\end{aligned}$$

**Invariant class 4.** If $\beta_k \alpha_{l+m} - \beta_k \alpha_l \alpha_m + \alpha_k \beta_l \alpha_m - \beta_l \alpha_{k+m} \neq 0$ for some $k, l, m \in \mathbb{Z}_{>0}$ then the invariant $Q$ besides (2) also satisfies a recursion on the form

$$Q(G) = \gamma_0 Q(G/xy) \qquad \text{for all nonadjacent } x, y \in \mathrm{V}(G), \tag{28}$$

and from applying this to (2) they can both be combined into

$$Q(G) = \gamma_{|\overline{xy}|} Q(G/xy) \qquad \text{for all } x, y \in \mathrm{V}(G), \tag{29}$$

where $\gamma_m = \gamma_0 \alpha_m - \beta_m$ and $\overline{xy} = \{\, e \in \mathrm{E}(G) \,|\, \phi_G(e) = \{x, y\} \,\}$ denotes the set of edges between $x$ and $y$.

Invariants of class 4 are subject to the same uncertainties as those satisfying (2) in general as to whether these really are the simplest possible recursions, or whether there are some still simpler identities that can be derived. Since the formal process of checking this exactly mirrors what was done in Section 5, except that the calculations are a bit simpler, it seems appropriate to leave this as an exercise for the reader. It may be noted, however, that $q^n = \gamma_0^{n-1} q$, which indeed limits the degrees of freedom quite considerably.

**Invariant class 5.** If $\beta_k \alpha_{l+m} - \beta_k \alpha_l \alpha_m + \alpha_k \beta_l \alpha_m - \beta_l \alpha_{k+m} = 0$ for all $k, l, m \in \mathbb{Z}_{>0}$ but $\alpha_k \beta_l \beta_m + \beta_k \beta_{l+m} - \alpha_l \beta_k \beta_m - \beta_l \beta_{k+m} \neq 0$ for some $k, l, m \in \mathbb{Z}_{>0}$ then the invariant $Q$ becomes really trivial, since there is then an identity stating that

$$(\alpha_k \beta_l \beta_m + \beta_k \beta_{l+m} - \alpha_l \beta_k \beta_m - \beta_l \beta_{k+m}) Q(G) = 0 \tag{30}$$

if $G$ has at least one vertex. In other words, $Q$ may at most distinguish between the empty graph (no vertices or edges) and all other graphs.

For a generic choice of parameters for an invariant of class 4, it is very likely that one ends up with a parameter behaving as the ones in class 5. The final class of invariants is a bit more interesting.

**Invariant class 6.** The only case remaining has $\beta_k\alpha_{l+m} - \beta_k\alpha_l\alpha_m + \alpha_k\beta_l\alpha_m - \beta_l\alpha_{k+m} = 0$ and $\alpha_k\beta_l\beta_m + \beta_k\beta_{l+m} - \alpha_l\beta_k\beta_m - \beta_l\beta_{k+m} = 0$ for all $k, l, m \in \mathbb{Z}_{>0}$, but $\beta_k\alpha_{l+m} \neq \beta_k\alpha_l\alpha_m$ for some $k, l, m \in \mathbb{Z}_{>0}$. (27) then takes on the form

$$(\beta_k\alpha_{l+m} - \beta_k\alpha_l\alpha_m)\bigl([L_1 \cup L_2, L_3] - [L_1, L_2 \cup L_3]\bigr) \in \mathcal{I}(S)(L_1 \cup L_2 \cup L_3), \quad (31)$$

which in plain English means $Q$ cannot tell the difference between $G$ and $G'$ if they differ only in that $G$ has some set $L_2$ of edges attached to a vertex $v_1$ whereas $G'$ has them attached to the vertex $v_3$, where $v_1$ and $v_3$ are non-adjacent. It is furthermore easy to get rid of the non-adjacency condition by going via some additional vertex — if necessary, such a vertex can be manufactured by running the recursion (2) backwards: $G = G''/e'$, where one endpoint of $e'$ is a new leaf, so that $\beta_1 Q(G) = \alpha_1 Q(G'' - e) - Q(G'')$.

In other words, this kind of invariant does not care which vertices an edge is incident with, so it can at most keep track of the size and order of a graph.

# 7 Concluding remarks

In [4], Bollobás and Riordan perform a similar classification of delete–contract invariants, which arrives at the much nicer conclusion that the Tutte polynomial is universal. How does this not contradict the results in the previous section, which arrived at a much larger number of invariants? Probably by considering slightly different problems. Bollobás and Riordan delete and contract edges one at a time, which quite probably should eliminate invariant class 1. There are some remarks in [4, Remark 4] that allowing coefficients from a ring may unify distinct invariant classes of coefficients from a field, but this seems unlikely for the invariant classes above. A thorough analysis is however beyond the scope of this paper.

One problem that could be worth studying is whether the value of an invariant of class 1 can be computed from the (ordinary) Tutte polynomial for the same multigraph. Since both are determined by the cycle matroid and order of the multigraph — the multiplicity of an edge is 1 more then the number of 2-circuits it is contained in — and since the Tutte polynomial is often celebrated as "the strongest matroid invariant there is", one would expect that the answer is "yes", but at the same time there doesn't seem to be any obvious substitution that turns the Tutte polynomial into a generic class 1 invariant. In the case of class 2 invariants there is no such uncertainty, as the three graph parameters these may encode can all be determined from the Tutte polynomial.

It may seem curious, in the case of invariant classes 2 and 3, that $\alpha_1$, $\beta_1$, and $\beta_2$ determine the values for all other coefficients: why should nothing new happen with edges of higher multiplicities? A bit of hands-on manipulation of the multigraphs will however reveal the answer: by running the recursion backwards, it is possible to subdivide any edge and thus express the value of the invariant for a multigraph using values of that invariant for some ordinary graphs. When computing $Q(G)$ for a graph $G$, it is possible to process the edges in such an order that no edge ever receives a multiplicity higher than 2, and hence there is a way of calculating $Q(G)$ that never uses any coefficients other than $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$! Subdivision of edges is however not possible unless $\beta_1 \neq 0$, which is why invariant class 1 can have degrees of freedom corresponding to higher multiplicities.

A more traditional formalism for graph rewriting, in that it avoids semi-graphs and half-edges, can be found in [1]. The difference is mostly that the roles of vertices and edges are swapped, so that vertices act as connectors between edges rather than vice versa, but of course either way will work. It is a bit curious, though, that mathematicians (myself included) should be so reluctant towards equipping vertices with extra structure and rather seek to impose it on edges, when the intuitive setting for many applications is to have it the other way around.

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