DOUBLY STOCHASTIC PROCESSING ON JACKET MATRICES

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ABSTRACT

We describe so- called doubly stochastic processing by using a simple factorization scheme on Jacket matrices..

1. INTRODUCTION

The doubly stochastic matrix is with entrywise nonnegative with all rows and columns sum one, and it is a special process in combinatorial theory, and probability [1]. Otherwise, Hadamard matrices are used widely in communication, and signal processing. Motivated by the Hadamard, a generalized form called Jacket has been reported and its applications in image processing and communications have been pointed out [2],[3]. The basic idea of Jacket was motivated by the cloths of Jacket. As our two sided Jacket is inside and outside compatible, at least two positions of a Jacket matrix are replaced by their inverse; these elements are changed in their position and are moved, for example, from inside of the middle circle to outside or from to inside without loss of sign. Recently, [4] gives contributions on mixed-radix representation for Jacket transform, which unifies all Hadamard transforms, and Jacket transforms, and also applicable for any even length vectors. In this paper, we investigate a simple Jacket factorization scheme for doubly stochastic processing, which includes a special *orthostochastic* case for any even length.

2. JACKET MATRICES AND THEIR PROPERTIES

In [3], a basic two by two kernel Jacket matrix is defined as

$$[J]_2 = \begin{bmatrix} a & b \\ b^T & -c \end{bmatrix}, \tag{1}$$

where T denotes the transpose, and a, b, c are all real nonzero element. By considering a $[J]_2$ is a unitary or orthogonal, we should have

$$[J]_2 \cdot [J]_2^T = 2[I]_2$$
$$= \begin{bmatrix} a & b \\ b^T & -c \end{bmatrix}\begin{bmatrix} a & b^T \\ b & -c \end{bmatrix}$$
$$= \begin{bmatrix} a^2+b^2 & ab^T - bc \\ ab^T - bc & (b^T)^2 + a^2 \end{bmatrix}, \tag{2}$$

thus we have the solution $b^T = b$, $a = c$, and the orthogonal $[J]_2$ can be written by

$$[J]_2 = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \tag{3}$$

where a, b are real nonzero elements. Additionally, an inverse property should be hold, which is

$$[J]_N^{-1} = \frac{1}{N}\left[(J_{ij})^{-1}\right]^T, \tag{4}$$

where J_{ij} is the i th row and j th column element in $[J]_N$. It implies that the inverse of the Jacket matrix is the entrywise inverse and transpose of itself. Therefore, (3) should be rewritten by

$$[J]_2^{-1} = \frac{1}{2}\begin{bmatrix} 1/a & 1/b \\ 1/b & -1/a \end{bmatrix}, \tag{5}$$

where we should force $a = b$. Clearly the result is a classical two by two Hadamard matrix

$$[J]_2 = a \cdot [H]_2 = \begin{bmatrix} a & a \\ a & -a \end{bmatrix}, \tag{6}$$

where $[H]_2$ is the size two Hadamard matrix. The result shows that the two by two orthogonal Jacket matrix is exist and only is the Hadamard case. Therefore, in several works [4][5], Jacket matrices are from four by four form, which is defined as

$$[J]_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\omega & \omega & -1 \\ 1 & \omega & -\omega & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \tag{7}$$

and its inverse is

$$[J]_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1/\omega & 1/\omega & -1 \\ 1 & 1/\omega & -1/\omega & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \tag{8}$$

where ω denotes a weight factor, that can be 2^n, j, and other complex numbers. The higher order

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Jacket matrices can be obtained by using the recursive function with $N = 2^n, n \in \{2,3,4,...\}$ as

$$[J]_N = [J]_{N/2} \otimes [H]_2, \quad (9)$$

where $\otimes$ is the kronecker product, and $[H]_2$ is defined by

$$[H]_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \quad (10)$$

Thus, the unitary Jacket matrices only exist when $\frac{1}{\omega} = \omega *$. Since the unitary Jacket matrix needs

$$[J]_N \cdot [J]_N^H = N[I]_N, \quad (11)$$

that implies

$$[J]_N^H = N[J]_N^{-1}, \quad (12)$$

where H denotes the Hermitian of a matrix, and * is the conjugate of the element.

3. DOUBLY STOCHASTIC PROCESSING ON JACKET MATRICES

An *N* by *N* matrix is said to be doubly stochastic if $[P]_N = \lfloor P_{ij} \rfloor$, $\sum_j P_{ij} = 1$ and $\sum_i P_{ij} = 1$ for all row i, and column j. Now, we can write a set of doubly stochastic matrices according to a simple factorization scheme as follows.

Theorem 1: Assuming a nonnegative probability matrix $[P]_N$ is a nonnegative diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2 \cdots \lambda_N$, it can be a doubly stochastic matrix if

$$[P]_N = [J]_N[diag(\lambda_1, \lambda_2 \cdots, \lambda_N)][J]_N^{-1}, \quad (13)$$

where $[J]_N$ should be unitary, $\lambda_1 = 1$, and $\lambda_2, \lambda_3, ...$ are any values which can guarantee that $[P]_N$ is nonnegative. The inverse form can be easily written by

$$[P]_N^{-1} = [J]_N[diag(1/\lambda_1, 1/\lambda_2 \cdots, 1/\lambda_N)][J]_N^{-1}. \quad (14)$$

Proof: Based on the basic matrix (7), we have

$$[P]_4 = [J]_4[diag(\lambda_1, ..., \lambda_4)][J]_4^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & -\omega & -1 \\ 1 & -\omega & \omega & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \lambda_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & -\omega & -1 \\ 1 & -\omega & \omega & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} a & b & c & d \\ b\_pair & a & d & c\_pair \\ c\_pair & d & a & b\_pair \\ d & c & b & a \end{bmatrix}, \quad (15)$$

where

$$a = 1/4 \times (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$$
$$b = 1/4 \times (\lambda_1 + (1/\omega)\lambda_2 - (1/\omega)\lambda_3 - \lambda_4)$$
$$c = 1/4 \times (\lambda_1 - (1/\omega)\lambda_2 + (1/\omega)\lambda_3 - \lambda_4)$$
$$d = 1/4 \times (\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4)$$
$$b\_pair = 1/4 \times (\lambda_1 + \omega\lambda_2 - \omega\lambda_3 - \lambda_4)$$
$$c\_pair = 1/4 \times (\lambda_1 - \omega\lambda_2 + \omega\lambda_3 - \lambda_4).$$

The sum of rows and that of columns in $[P]_4$ are listed by

Rows: $a + b\_pair + c\_pair + d = \lambda_1$;

$$b + a + d + c = \lambda_1; \quad (16)$$

Columns: $a + b + c + d = \lambda_1$;

$$b\_pair + a + d + c\_pair = \lambda_1. \quad (17)$$

It is clearly that the $[P]_4$ is doubly stochastic if $\lambda_1 = 1$. By using the recursive function as (9), the higher order probability matrix $[P]_N$ also is the doubly stochastic. And we call the set of these doubly stochastic matrices as doubly stochastic Jacket matrices (DSJM), several cases are listed in Table 1.

Theorem 2: A square doubly stochastic matrix of the form $[P]_N = [U]_N \circ [U]_N *$ ***for some unitary U*** is said to be *orthostochastic* [1][7][8]. If $[U]_N = [J]_N$, the resulted matrix is *orthostochastic*, and we always can find a special matrix is *orthostochastic* from the matrices set according to the **Theorem 1**, when $\lambda_1 = 1, \lambda_2 = ... = \lambda_N = 0$.

Proof: Let $[U]_4 = [J]_4$, we obtain

$$[P]_4 = [U]_4 \circ [U]_4 *$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\omega & \omega & -1 \\ 1 & \omega & -\omega & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\omega* & \omega* & -1 \\ 1 & \omega* & -\omega* & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega\omega* & \omega\omega* & 1 \\ 1 & \omega\omega* & \omega\omega* & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad (18)$$

where $\frac{1}{\omega} = \omega *$, $\omega\omega* = \omega \cdot \frac{1}{\omega} = 1$, and $\circ$ denotes the Hadamard product. Clearly, it is *orthostochastic* matrix, and its eigenvalues matrix is like as shown in Table 1 (b), where $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \lambda_4 = 0$, then

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (19)$$

Similarly the higher order *orthostochastic* probability matrix according to the recursive function as

$$[P]_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

with

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$N = 2^n, n \geq 2. \quad (20)$$

For the special two by two unitary Jacket matrix, i.e. Hadamard, the probability matrix generated according to **Theorem 1** has the same properties as that of the four by four unitary Jacket matrix. In general, the Theorem 1 can not only be applied for $N = 2^n$, but also be used for $N = 2n$, with $n \geq 3$. Since the unitary $2n \times 2n$ Jacket matrix is based on [4]

$$[J]_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}, \quad (21)$$

where $\alpha^6 = 1$. Thus we give factorization as

$$[P]_6 = [J]_6 [diag(\lambda_1,...,\lambda_6)][J]_6^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_4 & & \\ & & & & \lambda_5 & \\ & & & & & \lambda_6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \mathrm{a} & b & c & d & e & \mathrm{f} \\ d & \mathrm{a} & b & e & \mathrm{f} & c \\ e & d & \mathrm{a} & \mathrm{f} & c & b \\ b & c & \mathrm{f} & \mathrm{a} & d & e \\ c & \mathrm{f} & e & b & \mathrm{a} & d \\ \mathrm{f} & e & d & c & b & \mathrm{a} \end{bmatrix}, \quad (22)$$

where

$$\begin{aligned} a &= \frac{1}{6}(\lambda_1 + \lambda_2 + \cdots + \lambda_6) \\ b &= \frac{1}{6}(\lambda_1 + \alpha^5\lambda_2 + \alpha^4\lambda_3 + \alpha\lambda_4 + \alpha^2\lambda_5 - \lambda_6) \\ c &= \frac{1}{6}(\lambda_1 + \alpha^4\lambda_2 + \alpha^2\lambda_3 + \alpha^2\lambda_4 + \alpha^4\lambda_5 + \lambda_6) \\ d &= \frac{1}{6}(\lambda_1 + \alpha\lambda_2 + \alpha^2\lambda_3 + \alpha^5\lambda_4 + \alpha^4\lambda_5 - \lambda_6) \\ e &= \frac{1}{6}(\lambda_1 + \alpha^2\lambda_2 + \alpha^4\lambda_3 + \alpha^4\lambda_4 + \alpha^2\lambda_5 + \lambda_6), \\ f &= \frac{1}{6}(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 - \lambda_6) \end{aligned} \quad (23)$$

and $\alpha = e^{j\pi/3}$ which is the complex roots of $\alpha^6 = 1$. It is clear that $\alpha^2 + \alpha^4 = -1$, and $\alpha^1 + \alpha^5 = 1$. Hence, we can find if diagonal value $\lambda_1 = 1$, other diagonal values can chose any values that *a,b,c,d,e,f* , are nonnegative ,then the probability matrix is a double stochastic process. Similar to **Theorem 2**, the $2n \times 2n$ with $n \geq 3$ Jacket matrix is a unitary matrix, thus we have $[J]_{2n\times 2n}^H = (2n \times 2n) \cdot [J]_{2n\times 2n}^{-1}$. We always can find a generalized doubly stochastic matrix is *orthostochastic*, if the first eigenvalue $\lambda_1 = 1$, and the other eigenvalues equal zero.

Proof: Let $[U]_6 = [J]_6$, we obtain

$$[P]_6 = [U]_6 \circ [U]_6{}^*$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad (24)$$

Clearly, it is *orthostochastic* matrix, and its eigenvalues matrix is likely as shown in **Theorem 2**,

$$\begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_3 & & & \\ & & & \lambda_4 & & \\ & & & & \lambda_5 & \\ & & & & & \lambda_6 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}. \quad (25)$$

Similarly the higher order *orthostochastic* probability matrix according to the recursive function is

$$[P]_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

with

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$N = 2n, \; n \geq 3. \tag{26}$$

Table.1: Different cases of 4×4 doubly stochastic Jacket matrices

Case \ DSJM	4×4 Doubly Stochastic Jacket Matrices
(a) Eigenvalues $\lambda_1=\lambda_2=\lambda_3=\lambda_4=1$	$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-2&2&-1\\1&2&-2&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-\frac{1}{2}&\frac{1}{2}&-1\\1&\frac{1}{2}&-\frac{1}{2}&-1\\1&-1&-1&1\end{bmatrix}$
(b) $\lambda_1=1 \neq \lambda_2=\lambda_3=\lambda_4$	$\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-2&2&-1\\1&2&-2&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-\frac{1}{2}&\frac{1}{2}&-1\\1&\frac{1}{2}&-\frac{1}{2}&-1\\1&-1&-1&1\end{bmatrix}$
(c) $\lambda_1=1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$	$\begin{bmatrix}\frac{3}{5}&\frac{11}{80}&\frac{17}{80}&\frac{1}{20}\\\frac{1}{40}&\frac{3}{5}&\frac{1}{20}&\frac{13}{40}\\\frac{13}{40}&\frac{1}{20}&\frac{3}{5}&\frac{1}{40}\\\frac{1}{20}&\frac{17}{80}&\frac{11}{80}&\frac{3}{5}\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-2&2&-1\\1&2&-2&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{7}{10}&0&0\\0&0&\frac{4}{10}&0\\0&0&0&\frac{3}{10}\end{bmatrix}\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-\frac{1}{2}&\frac{1}{2}&-1\\1&\frac{1}{2}&-\frac{1}{2}&-1\\1&-1&-1&1\end{bmatrix}$
(d) $\lambda_1=-\lambda_4$, $\lambda_2=-\lambda_3$	$\begin{bmatrix}0&\frac{3}{8}&\frac{5}{8}&0\\0&0&0&1\\1&0&0&0\\0&\frac{5}{8}&\frac{3}{8}&0\end{bmatrix}=\begin{bmatrix}1&1&1&1\\1&-2&2&-1\\1&2&-2&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&-\frac{1}{2}&0\\0&0&0&-1\end{bmatrix}\frac{1}{4}\begin{bmatrix}1&1&1&1\\1&-\frac{1}{2}&\frac{1}{2}&-1\\1&\frac{1}{2}&-\frac{1}{2}&-1\\1&-1&-1&1\end{bmatrix}$

4. CONCLUSION

We proposed a novel method to generalize a set of $2^n \times 2^n$ and $2n \times 2n$ matrices named generalized doubly stochastic Jacket matrices, also orthostochastic cases are included. The derivation shows that $2^n \times 2^n$ and $2n \times 2n$, $n \geq 2$, Jacket matrices always have the *orthostochastic* case if the eigenvalues $\lambda_1 = 1$, the others are zeros, however, for the doubly stochastic case they may be any values which could approach the elements in the probability matrix are nonnegative. Generally, the proposed scheme uses a simple matrix factorization method to represent the doubly stochastic, Markov vectors and eigenvalues, and it can be easily applied for stochastic signal processing, Markov random process, Miller coding system [7], and orthogonal design for signal processing such as space time codes pattern design [6], [9], [10],[11].

5. ACKNOWLEDGMENTS

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