Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Linear Algebra and its Applications 379 (2004) 77–83
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

The inverse eigenvalue problem for symmetric doubly stochastic matrices

Suk-Geun Hwang [a,*,1], Sung-Soo Pyo [b,2]

[a] *Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Republic of Korea*
[b] *School of Electrical Engineering and Computer Science, Kyungpook National University, Taegu 702-701, Republic of Korea*

Received 22 July 2002; accepted 24 December 2002

Submitted by F. Uhlig

Abstract

For a positive integer n and for a real number s, let Γ_n^s denote the set of all $n \times n$ real matrices whose rows and columns have sum s. In this note, by an explicit constructive method, we prove the following.

(i) Given any real n-tuple $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$, there exists a symmetric matrix in $\Gamma_n^{\lambda_1}$ whose spectrum is Λ.
(ii) For a real n-tuple $\Lambda = (1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ with $1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, if

$$\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{n(n-2)} + \cdots + \frac{\lambda_n}{2 \cdot 1} \geqslant 0,$$

then there exists a symmetric doubly stochastic matrix whose spectrum is Λ.

The second assertion enables us to show that for any $\lambda_2, \ldots, \lambda_n \in [-1/(n-1), 1]$, there is a symmetric doubly stochastic matrix whose spectrum is $(1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ and also that any number $\beta \in (-1, 1]$ is an eigenvalue of a symmetric positive doubly stochastic matrix of any order.
© 2003 Elsevier Inc. All rights reserved.

* Corresponding author. Tel.: +82-53-950-5887; fax: +82-53-950-6811.
E-mail addresses: sghwang@knu.ac.kr (S.-G. Hwang), sspyo@gauss.knu.ac.kr (S.-S. Pyo).
[1] Supported by Com^{2}MaC-KOSEF.
[2] This work was done while the second author had a BK21 Post-Doc. position at Kyungpook National University.

0024-3795/$ - see front matter © 2003 Elsevier Inc. All rights reserved.
doi:10.1016/S0024-3795(03)00366-5

AMS classification: 15A51; 15A18

Keywords: Doubly stochastic matrix; Inverse eigenvalue problem; Spectrum

1. Introduction

A real matrix A is called *nonnegative* (resp. *positive*), written $A \geqslant O$ (resp. $A > O$), if all of its entries are nonnegative (resp. positive). A square nonnegative matrix is called *doubly stochastic* if all of its rows and columns have sum 1. The set of all $n \times n$ doubly stochastic matrices is denoted by Ω_n.

For a square matrix A, let $\sigma(A)$ denote the spectrum of A. Given an n-tuple $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ of numbers, real or complex, deciding the existence of a matrix A with some specific properties such that $\sigma(A) = \Lambda$ has long time been one of the problems of main interest in the theory of matrices.

Given $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$, in order that $\Lambda = \sigma(A)$ for some positive matrix A, it is necessary that $\lambda_1 + \lambda_2 + \cdots + \lambda_n > 0$. Sufficient conditions for the existence of a positive matrix A with $\sigma(A) = \Lambda$ have been investigated by several authors such as Borobia [1], Fiedler [2], Kellog [4], and Salzmann [5].

In this paper we deal with the existence of certain real symmetric matrices and that of symmetric doubly stochastic matrices with prescribed spectrum under certain conditions.

Let $A \in \Omega_n$. Then the well known Gershgorin's theorem (see [3], for instance) implies that $\sigma(A)$ is contained in the closed unit disc centered at the origin in the complex plane. Thus if, in addition, A is symmetric, then $\sigma(A)$ lies in the real closed interval $[-1, 1]$.

Given a real n-tuple $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, in order that there exists $A \in \Omega_n$ with $\sigma(A) = \Lambda$, it is necessary that

$$\lambda_1 = 1, \quad \lambda_n \geqslant -1, \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_n \geqslant 0. \tag{1}$$

Thus, for instance, there exists no 3×3 doubly stochastic matrix with spectrum $(1, -0.5, -0.6)^{\mathrm{T}}$ or $(1, 1, -1.1)^{\mathrm{T}}$.

Certainly the condition (1) is not sufficient for $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ to be the spectrum of a doubly stochastic matrix.

In this paper, we find a sufficient condition on $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ satisfying (1) for the existence of a symmetric doubly stochastic matrix having Λ as its spectrum by a constructive method. For an n-tuple $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, let $\Delta\mathbf{x}$ and $\overleftarrow{\mathbf{x}}$ be defined by

$$\Delta\mathbf{x} = (x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n, x_n)^{\mathrm{T}},$$
$$\overleftarrow{\mathbf{x}} = (x_n, x_{n-1}, \ldots, x_2, x_1)^{\mathrm{T}}.$$

Note that $\overleftarrow{\Delta\mathbf{x}}$ is the difference sequence of the finite sequence $0, x_n, x_{n-1}, \dots, x_2, x_1$. Let

$$\mathbf{h}_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\right)^{\mathrm{T}}.$$

Then

$$\Delta\mathbf{h}_n = \left(\frac{1}{1\cdot 2}, \frac{1}{2\cdot 3}, \dots, \frac{1}{(n-1)\cdot n}, \frac{1}{n}\right)^{\mathrm{T}}.$$

We see that the ith component of $\Delta\mathbf{h}_n$ is the length of the ith subinterval of the division of the unit interval [0, 1] by inserting the $n-1$ points $\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$.

For a real number s, let Γ_n^s denote the set of all $n \times n$ real matrices all of whose rows and columns have sum s. The set Ω_n consists of all nonnegative matrices in Γ_n^1.

In what follows we show that for any real n-tuple $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)^{\mathrm{T}}$, there exists a symmetric matrix in $\Gamma_n^{\lambda_1}$ whose spectrum is Λ, and also that if a real n-tuple $\Lambda = (1, \lambda_2, \dots, \lambda_n)^{\mathrm{T}}$ with $1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ satisfies (1) and one of the two equivalent conditions (a) and (b) in the following Lemma 1, then there exists a symmetric doubly stochastic matrix of order n with Λ as its spectrum.

Lemma 1. *For a real n-tuple* $\Lambda = (1, \lambda_2, \dots, \lambda_n)^{\mathrm{T}}$. *Let* $\Delta\Lambda = (\delta_1, \delta_2, \dots, \delta_n)^{\mathrm{T}}$. *Then the following are equivalent.*

(a) $$\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \cdots + \frac{\lambda_n}{2\cdot 1} \geqslant 0, \tag{2}$$

(b) $$\frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_{n-1}}{2} + \delta_n \geqslant 0. \tag{3}$$

Proof. We see that

$$\begin{aligned}
&\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \cdots + \frac{\lambda_n}{2\cdot 1} \\
&\quad = 1\left(\frac{1}{n} - 0\right) + \lambda_2\left(\frac{1}{n-1} - \frac{1}{n}\right) + \cdots + \lambda_n\left(\frac{1}{1} - \frac{1}{2}\right) \\
&\quad = (1-\lambda_2)\frac{1}{n} + (\lambda_2 - \lambda_3)\frac{1}{n-1} + \cdots + (\lambda_{n-1} - \lambda_n)\frac{1}{2} + \lambda_n \\
&\quad = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_{n-1}}{2} + \delta_n.
\end{aligned}$$

Thus the equivalence of (a) and (b) follows. □

Notice that

$$\frac{1}{n} + \frac{\lambda_2}{n(n-1)} + \frac{\lambda_3}{(n-1)(n-2)} + \cdots + \frac{1}{2\cdot 1} = \Lambda^{\mathrm{T}}\overleftarrow{\Delta\mathbf{h}_n},$$

$$\frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_{n-1}}{2} + \delta_n = (\Delta\Lambda)^{\mathrm{T}}\overleftarrow{\mathbf{h}_n}.$$

Thus the inequalities (2) and (3) can be expressed as

$$\Lambda \cdot \overleftarrow{\Delta \mathbf{h}_n} \geqslant 0, \tag{4}$$

$$\Delta\Lambda \cdot \overleftarrow{\mathbf{h}_n} \geqslant 0 \tag{5}$$

respectively, where '·' stands for the Euclidean inner product. In the sequel we denote by I_n, J_n, O_n the identity matrix, the all 1's matrix and the zero matrix of order n respectively.

We let $\mathbf{e}_n$ denote a column of J_n. We sometimes write $I, J, O, \mathbf{e}$ in place of $I_n, J_n, O_n, \mathbf{e}_n$ in case that the size of the matrix or vector is clear within the context.

2. Main result

For $n \geqslant 2$, let

$$Q_n = \begin{bmatrix} \frac{1}{n} & \frac{1}{n}\mathbf{e}^{\mathrm{T}} \\ -\mathbf{e} & I_{n-1} \end{bmatrix}. \tag{6}$$

Then clearly Q_n is nonsingular.

We first observe the effect of the similarity transformation $X \to QXQ^{-1}$ on certain sets of matrices on which our discussion relies.

Lemma 2. *Let Q_n be the matrix defined in* (6), *then*

(a) $Q_n J_n Q_n^{-1} = nI_1 \oplus O_{n-1}$,
(b) $Q_n(I_1 \oplus A)Q_n^{-1} = I_1 \oplus A$, *for any* $A \in \Gamma_{n-1}^1$.

Proof. Clearly

$$Q_n J_n = \begin{bmatrix} 1 & \mathbf{e}^{\mathrm{T}} \\ \mathbf{0} & O \end{bmatrix} = \begin{bmatrix} n & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & O \end{bmatrix} Q_n,$$

and (a) follows. To show (b), observe that

$$Q_n \begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & A \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n}\mathbf{e}^{\mathrm{T}}A \\ -\mathbf{e} & A \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n}\mathbf{e}^{\mathrm{T}} \\ -\mathbf{e} & A \end{bmatrix},$$

$$\begin{bmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & A \end{bmatrix} Q_n = \begin{bmatrix} \frac{1}{n} & \frac{1}{n}\mathbf{e}^{\mathrm{T}} \\ -A\mathbf{e} & A \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n}\mathbf{e}^{\mathrm{T}} \\ -\mathbf{e} & A \end{bmatrix},$$

for any $A \in \Gamma_{n-1}^1$. Thus (b) holds. □

We first find a matrix with a prescribed spectrum in the set Γ_{n-1}^s.

Theorem 3. *Let $n \geqslant 2$. Then for any real n-tuple $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$, there exists a symmetric matrix $A \in \Gamma_n^{\lambda_1}$ with $\sigma(A) = \Lambda$.*

Proof. Let $(x_1, x_2, \ldots, x_n)$ be an n-tuple of indeterminates and let

$$A = x_1\left(\frac{1}{n}J_n\right) + x_2\left(I_1 \oplus \frac{1}{n-1}J_{n-1}\right) + \cdots + x_n\left(I_{n-1} \oplus \frac{1}{1}J_1\right). \tag{7}$$

We show that A is similar to $\operatorname{diag}(y_1, y_2, \ldots, y_n)$ where $y_i = x_i + x_{i+1} + \cdots + x_n$ $(i = 1, 2, \ldots, n)$. We proceed by induction on n.

If $n = 2$, then

$$Q_2 A Q_2^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{x_1}{2} + x_2 & \frac{x_1}{2} \\ \frac{x_1}{2} & \frac{x_1}{2} + x_2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} x_1 + x_2 & 0 \\ 0 & x_2 \end{bmatrix},$$

and the induction starts.

Suppose that $n > 2$. We have, by Lemma 1, that

$$\begin{aligned} Q_n A Q_n^{-1} &= x_1\,(I_1 \oplus O_{n-1}) + x_2\left(I_1 \oplus \frac{1}{n-1}J_{n-1}\right) \\ &\quad + x_3\left(I_2 \oplus \frac{1}{n-2}J_{n-2}\right) + \cdots + x_n\left(I_{n-1} \oplus \frac{1}{1}J_1\right) \\ &= y_1 I_1 \oplus B, \end{aligned}$$

where

$$B = x_2\left(\frac{1}{n-1}J_{n-1}\right) + x_3\left(I_1 \oplus \frac{1}{n-2}J_{n-2}\right) + \cdots + x_n\left(I_{n-2} \oplus \frac{1}{1}J_1\right).$$

By the induction hypothesis, there exists a nonsingular matrix Q of order $n-1$ such that $QBQ^{-1} = \operatorname{diag}(y_2, y_3, \ldots, y_n)$. Then

$$(I_1 \oplus Q)Q_n A Q_n^{-1}(I_1 \oplus Q^{-1}) = \operatorname{diag}(y_1, y_2, \ldots, y_n).$$

Now, let $(x_1, x_2, \ldots, x_n)^{\mathrm{T}} = \Delta\Lambda$. Then $(y_1, y_2, \ldots, y_n)^{\mathrm{T}} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$, and the theorem is proved. □

We are now ready to prove the following.

Theorem 4. *Let $\Lambda = (1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ be a real n-tuple with $1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. If Λ satisfies one of the inequalities* (2)–(5), *then there exists a symmetric $A \in \Omega_n$ with $\sigma(A) = \Lambda$. Moreover, if $1 > \lambda_2$, then the matrix A can be taken to be positive.*

Proof. Let $\Delta\Lambda = (\delta_1, \delta_2, \ldots, \delta_n)^{\mathrm{T}}$, and let $A = [a_{ij}]$ be the matrix defined as (7) with $x_i = \delta_i$ $(i = 1, 2, \ldots, n)$. Then $A \in \Lambda_n^1$. All we need to show is that $A \geqslant O$. Since $\delta_i \geqslant 0$ for $i = 1, 2, \ldots, n-1$, we see that all of the off-diagonal entries of A are nonnegative. We also see, for each $i = 1, 2, \ldots, n-1$, that

$$a_{ii} = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_i}{n-i+1} + \delta_{i+1} + \cdots + \delta_n,$$

and

$$a_{nn} = \frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_{n-1}}{2} + \delta_n$$

from which it follows that $a_{11} \geqslant a_{22} \geqslant \cdots \geqslant a_{nn}$ because $\delta_i \geqslant 0$ for $i = 1, 2, \ldots, n-1$. Now that $a_{nn} \geqslant 0$ follows from Lemma 1.

If $1 > \lambda_2$, then $\delta_1 > 0$, and it must be that $A > O$. □

Theorem 4 can be interpreted geometrically as

Corollary 5. *Let $\Lambda = (1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ be a real n-tuple with $1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. If Λ forms an acute or right angle with $\overleftarrow{\Delta \mathbf{h}_n}$, then there exists a symmetric $A \in \Omega_n$ with $\sigma(A) = \Lambda$.*

Let n be fixed and let Π_0 denote the hyperplane

$$\{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x}^{\mathrm{T}} \overleftarrow{\Delta \mathbf{h}_n} = 0\},$$

where $\mathbf{R}^n$ denotes the real Euclidean n-space. Then $\mathbf{R}^n - \Pi_0$ is divided into two parts

$$\Pi_+ = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x}^{\mathrm{T}} \overleftarrow{\Delta \mathbf{h}_n} > 0\}, \quad \Pi_- = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{x}^{\mathrm{T}} \overleftarrow{\Delta \mathbf{h}_n} < 0\}.$$

Let $S = \{(1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n \mid 1 \geqslant x_2 \geqslant \cdots \geqslant x_n\}$. Then Corollary 5 can be restated as

Corollary 6. *Every vector in $S \cap (\Pi_+ \cup \Pi_0)$ is the spectrum of a symmetric doubly stochastic matrix.*

An $n \times n$ matrix is *irreducible* if there does not exist a permutation matrix P such that $P^{\mathrm{T}}AP$ has the form

$$P^{\mathrm{T}}AP = \begin{bmatrix} B & O \\ * & C \end{bmatrix},$$

where B, C are nonvacuous square matrices. If A is a doubly stochastic, then there are irreducible doubly stochastic matrices $A_1, A_2, \ldots, A_k$, called the irreducible components of A, such that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$. The algebraic multiplicity of the eigenvalue 1 of A equals the number of irreducible components of A. For $\Lambda = (1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$, with $1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, if $1 = \lambda_2$, then there is no irreducible doubly stochastic matrix whose spectrum is Λ. But if $1 > \lambda_2$, then the matrix defined by (7) with $x_i = \delta_i$, $i = 1, 2, \ldots, n$, is a positive matrix.

From the proof of Theorem 3, we get

$$a_{nn} \geqslant \frac{\delta_1 + \delta_2 + \cdots + \delta_{n-1}}{n} + \lambda_n = \frac{1 + (n-1)\lambda_n}{n}.$$

Thus we have

Corollary 7. *If* $\lambda_2, \ldots, \lambda_n \in [-1/(n-1), 1]$, *then there exists a symmetric* $A \in \Omega_n$ *with* $\sigma(A) = (1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$.

Note that the lower bound $-1/(n-1)$ of the interval is the best possible for the statement of Corollary 7. For, if α is a number such that $\alpha < -1/(n-1)$, then there does not exist an $A \in \Omega_n$ such that $\sigma(A) = (1, \alpha, \ldots, \alpha)$ because A must have a nonnegative trace.

Let β be any number such that $-1 < \beta < 1$. Take any real number ε such that $0 < \varepsilon < \min\{1 + \beta, 1 - \beta\}$. Let

$$\lambda_i = \begin{cases} 1, & \text{if } i = 1, \\ 1 - \varepsilon, & \text{if } 2 \leqslant i \leqslant n - 1, \\ \beta, & \text{if } i = n \end{cases}$$

and let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$. Then $\Delta\Lambda = (\delta_1, \delta_2, \ldots, \delta_n)^{\mathrm{T}} = (\varepsilon, 0, \ldots, 0, 1 - \varepsilon - \beta, \beta)^{\mathrm{T}}$ so that

$$\frac{\delta_1}{n} + \frac{\delta_2}{n-1} + \cdots + \frac{\delta_{n-1}}{2} + \delta_n = \frac{\varepsilon}{n} + \frac{1 - \varepsilon - \beta}{2} + \beta > \frac{1 + \beta - \varepsilon}{2} > 0.$$

Thus we have the following.

Corollary 8. *Let* $n \geqslant 2$, *then for any* $\beta \in (-1, 1]$, *there exists a symmetric positive* $A \in \Omega_n$ *of which* β *is an eigenvalue.*

Certainly -1 is an eigenvalue of some $A \in \Omega_n$ for every $n \geqslant 2$, for example, of

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus I_{n-2}.$$

But -1 cannot be an eigenvalue of a positive doubly stochastic matrix. This fact follows directly from the Perron–Frobenius theory. We give here another simple proof. Suppose that $A\mathbf{x} = -\mathbf{x}$ where $A = [a_{ij}] \in \Omega_n$, $A > O$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. Without loss of generality we may assume that $x_1 = 1$ and $|x_i| \leqslant 1$ for all $i = 2, 3, \ldots, n$. Then $a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = -1 - a_{11}$ from which we have $1 < 1 + a_{11} < a_{12}|x_2| + \cdots + a_{1n}|x_n| \leqslant a_{12} + \cdots + a_{1n} = 1 - a_{11} < 1$, a contradiction.

References

[1] A. Borobia, On the nonnegative eigenvalue problem, Linear Algebra Appl. 223/224 (1995) 131–140.
[2] M. Fiedler, Eigenvalues of nonnegative symmetric matrices, Linear Algebra Appl. 9 (1974) 119–142.
[3] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[4] R.B. Kellog, Matrices similar to a positive or essentially positive matix, Linear Algebra Appl. 4 (1971) 191–204.
[5] F.L. Salzmann, A note on eigenvalues of nonnegative matrices, Linear Algebra Appl. 5 (1972) 329–338.