Weak Ergodicity of Population Evolution Processes

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ABSTRACT

The weak ergodic theorems of mathematical demography state that the age distribution of a closed population is asymptotically independent of the initial distribution. In this paper, we provide a new proof of the weak ergodic theorem of the multistate population model with continuous time. The main tool to attain this purpose is a theory of multiplicative processes, which was mainly developed by Garrett Birkhoff, who showed that ergodic properties generally hold for an appropriate class of multiplicative processes. First, we construct a general theory of multiplicative processes on a Banach lattice. Next, we formulate a dynamical model of a multistate population and show that its evolution operator forms a multiplicative process on the state space of the population. Subsequently, we investigate a sufficient condition that guarantees the weak ergodicity of the multiplicative process. Finally, we prove the weak and strong ergodic theorems for the multistate population and resolve the consistency problem.

1. INTRODUCTION

One of the main problems of mathematical demography is to investigate the long-run behavior of age-structured populations. The first successful attack on this problem was achieved by Sharpe and Lotka [30] when they showed that a closed population submitted to a time-invariant fertility and mortality schedule will converge to a fixed form, which is called a stable age distribution of the population. The stable age distribution and the eventual rate of increase of the population are completely determined by the fixed vital rates and are independent of the initial data. This is the *strong ergodic theorem* of demography. In other words, this Sharpe-Lotka theorem tells us that there is only a time-invariant stable age structure consisting of unchanged vital rates in the long run and other age structures are transient. Since the paper of Feller [15], there have been many satisfactory proofs of the strong ergodic theorem (Inaba [18, 19]; Metz and Diekmann [26]; Webb [33, 34]).

If the vital rates of the population change through time, we cannot expect a time-invariant age structure to exist. However, we can still ask whether the age distribution at a particular moment in time is transient, that is, whether the long-run behavior of the age distribution is independent of the initial data. This phenomenon is called *weak ergodicity* of the population (Lopez [22], p. 42). The history of investigation of weak ergodicity is slightly complicated. Weak ergodicity of human populations was first conjectured by Coale [11]. Lopez [22] first proved the weak ergodic theorem for the deterministic discrete-time model. For the discrete-time model, the weak ergodic theorem has been widely established under various assumptions (Golubitsky et al. [16]; Cohen [12, 13]; Kim [20]). On the other hand, the circumstances are quite different for the continuous-time model, for which there has been no successful proof. Lopez [23] and Le Bras [21] gave explanations for the weak ergodicity of the continuous-time model described by the Lotka renewal equation, but their argument was incomplete as a rigorous proof. Tuljapurkar [32] attempted to provide an operator-theoretic proof for the weak ergodicity of the Lotka equation by using the Hilbert projective metric, but his assumption for the net maternity function was unrealistic for human populations. However, as Tuljapurkar [32] pointed out, it must be stressed that the remarkable paper of Norton [27] had already attempted to establish weak ergodicity for the continuous-time model (Charlesworth [8, 9]). Although, from the modern mathematical point of view, Norton's proof seems to be loose, his idea will become a cornerstone of our proof of the weak ergodicity of the continuous-time multistate population model.

From the lattice-theoretic perspective, Birkhoff [2–5] constructed a theory of *multiplicative processes* and showed that ergodic properties are commonly found in some appropriate class of multiplicative processes, although he did not use the term "ergodicity." Thus, in what follows, we first construct a general theory of multiplicative processes using Birkhoff's results. After that, we introduce a linear multistate model of a one-sex age-structured population and define the population evolution operator. Making use of the fact that the evolution operator forms a multiplicative process on the state space of the population, we can prove a sufficient condition that guarantees the weak ergodicity of the multistate population.

Although usually the term "ergodic" refers to asymptotic behavior for future time that is independent of initial conditions in demography, there exists an alternative point of view for the ergodicity of population processes. Suppose that past time-series data of vital rates are given. Then we can ask whether there uniquely exists a time series of age distributions consistent with given time series of vital rates. This *consistency problem* also appeared ambiguously in Lotka [24, 25] but the meaning of his papers has been ignored for a long time. We will see that the consistency problem is naturally proposed and can be resolved in the context of the theory of multiplicative processes.

2. POSITIVE LINEAR OPERATORS AND THE HILBERT PROJECTIVE PSEUDOMETRIC

In this section we introduce the Hilbert projective pseudometric in partially ordered linear space (Bushell [6, 7]).

Let V be a real linear space containing a positive cone C of nonnegative elements: that is, a subset with properties (1) $C + C \subset C$, (2) $\alpha C \subset C$ for all real $\alpha \geqslant 0$, and (3) $C \cap -C = \{0\}$. The relation $\leqslant$ is defined in V by saying that $x \leqslant y$ if and only if $y - x \in C$. Then the linear space V is partially ordered by the relation $\leqslant$. Furthermore, we suppose that the partially ordered linear space $\{V, \leqslant\}$ is *Archimedean* (or *integrally closed*); that is, if $nx \leqslant y$ for $n = 1, 2, \ldots$, then $x \leqslant 0$. For $(x, y) \in V \times C^+$, we define

$$\sup\left(\frac{x}{y}\right) = \inf\{\lambda; x \leqslant \lambda y\}, \qquad \inf\left(\frac{x}{y}\right) = \sup\{\mu; \mu y \leqslant x\}, \tag{2.1}$$

where $C^+ = C - \{0\}$, and we adopt the convention $\inf \phi = \infty$ and $\sup \phi = -\infty$.

The *oscillation* of x and y is defined by

$$\operatorname{osc}\left(\frac{x}{y}\right) = \sup\left(\frac{x}{y}\right) - \inf\left(\frac{x}{y}\right). \tag{2.2}$$

The *Hilbert projective pseudometric* is defined in C^+ by

$$d(x, y) = \log\left[\frac{\sup(x/y)}{\inf(x/y)}\right]. \tag{2.3}$$

Then the following holds, although we omit the proof.

LEMMA 2.1

If $x, y \in C^+$, then

(1) *$d(x, y)$ is a pseudometric, that is, $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leqslant d(x, y) + d(y, z)$.*

(2) *$d(x, y) = 0$ if and only if $x = \lambda y$ for some $\lambda > 0$.*

(3) *$d(\lambda x, \mu y) = d(x, y)$ for $\lambda, \mu > 0$.*

Thus $\{C^+, d\}$ is a pseudometric space. A *connected component* of the pseudometric space $\{C^+, d\}$ is an equivalence class of elements for which $d(x, y) < \infty$. It is easy to see that x and y are in the same component if and only if x and y are strongly comparable, that is, there exist positive real numbers λ and μ such that $\mu y \leqslant x \leqslant \lambda y$. The *ray* is an equivalence class of elements for which $d(x, y) = 0$. In particular, if (V, C) forms a *Banach lattice*, the following proposition holds (Birkhoff [2,3]).

PROPOSITION 2.2

If (V, C) is a Banach lattice, the set of rays in a component forms a complete metric space in the projective metric.

In other words, any connected component of the unit sphere of any Banach lattice is a complete metric space in the projective metric. On the unit sphere of any Banach lattice, the relation between the norm of the Banach space and the projective distance is given as

$$\|f - g\| \leqslant \exp\{d(f, g)\} - 1, \tag{2.4}$$

where $\|\cdot\|$ is the norm of the Banach space V (Birkhoff [2]).

A linear operator A on V is called *nonnegative* when $A(C) \subset C$ and *positive* when $A(C^+) \subset C^+$. The relation of comparability is preserved under any positive linear operator A, and it follows immediately that

$$d(Ax, Ay) \leqslant d(x, y) \qquad \text{if } d(x, y) < \infty. \tag{2.5}$$

Then any positive linear operator A is continuous on a component of C^+ in the projective metric. The *projective diameter* of a positive operator A is defined by

$$\Delta(A) = \sup\{d(Ax, Ay); (x, y) \in C^+ \times C^+\}. \tag{2.6}$$

The *projective norm* of a positive operator A is defined as

$$\|A\|_p = \sup \frac{d(Ax, Ay)}{d(x, y)} \qquad \text{for } 0 < d(x, y) < \infty, \tag{2.7}$$

which is also called the *contraction ratio* of A and is denoted by $k(A)$. The *oscillation ratio* $N(A)$ of A is defined by

$$N(A) = \inf\left\{\lambda; \operatorname{osc}\left(\frac{Ax}{Ay}\right) \leqslant \lambda \operatorname{osc}\left(\frac{x}{y}\right), \quad x, y \in C^+\right\}. \tag{2.8}$$

Bushell [6] and Ostrowski [28] showed that $k(A) = N(A)$ if A is a positive linear mapping in V. A positive linear operator A is called *uniformly positive* when the projective diameter $\Delta(A)$ is finite. Then any element of C^+ is mapped into a connected component of C^+ by a uniformly positive operator. If some power of A is uniformly positive, it is called *uniformly primitive*. From (2.5), we know that any positive linear operator is contractive in the projective norm. In addition, we can prove the following sharper result, although the proof is too complicated to present here. The reader may find an elementary analytical proof in Bauer [1] and Bushell [7]. The original proof is given in Birkhoff [2, 3, 5].

PROPOSITION 2.3

Let A be a positive linear operator in the Archimedean partially ordered real space $\{V, C, \leqslant\}$. Then,

$$\|A\|_p \leqslant \tanh\left[\frac{\Delta(A)}{4}\right]. \tag{2.9}$$

From (2.9) we know that any uniformly positive linear operator on V is strictly contractive operator in the projective metric. Uniform positivity of a positive linear operator is characterized as follows.

LEMMA 2.4

A positive linear operator A is uniformly positive if and only if there exist $e \in C^+$, $\alpha \geqslant 1$ and a positive functional $\lambda(x)$ [that is, $\lambda(x) > 0$ for $x \in C^+$] such that

$$\lambda(x)e \leqslant Ax \leqslant \alpha\lambda(x)e. \tag{2.10}$$

Proof. If (2.10) holds, it follows that $d(Ax, e) \leqslant \log\alpha$. Then

$$d(Ax, Ay) \leqslant d(Ax, e) + d(e, Ay) \leqslant 2\log\alpha \qquad \text{for all } (x, y) \in C^+ \times C^+.$$

This shows that $\Delta(A) \leqslant 2\log\alpha$, that is, A is uniformly positive. Conversely, if A is uniformly positive, then for any fixed $y \in C^+$, $d(Ax, Ay) \leqslant \Delta(A) < \infty$ for all $x \in C^+$. Then we obtain

$$\inf\left(\frac{Ax}{Ay}\right)Ay \leqslant Ax \leqslant \exp[\Delta(A)]\inf\left(\frac{Ax}{Ay}\right)Ay.$$

If we let $e = Ay$, $\lambda(x) = \inf(Ax/e)$, and $\alpha = \exp[\Delta(A)]$, then (2.10) follows. ∎

Although we mainly treat linear problems in this paper, the projective metric technique can also be applied to nonlinear problems (Thieme [31]).

3. MULTIPLICATIVE PROCESSES

A *time-inhomogeneous multiplicative process* for positive time (or negative time) on (V, C) is a two-parameter family of nonnegative linear operators $U(t, s)$ on (V, C), defined for all $s \leqslant t$ in an infinite interval $S = [s_0, +\infty)$

$[S=(-\infty, s_0]$ for negative time], and satisfying the *multiplicative property*

$$U(u,t)U(t,s)=U(u,s) \qquad \text{for all } s \leqslant t \leqslant u \text{ in } S, \tag{3.1}$$

and $U(s,s)=I$, where I is the identity operator. $U(t,s)$ is called *positive* if $U(t,s)(C^+) \subset C^+$ for all $s \leqslant t$ in S. It is *uniformly primitive for positive time* when for some $\alpha < \infty$ there exists for any $K > s_0$ some $K < s < t$ such that $\Delta(U(t,s)) \leqslant \alpha$. It is *uniformly primitive for negative time* when for some $\alpha < \infty$ there exists for any $K < s_0$ for some $s < t < K$ such that $\Delta(U(t,s)) \leqslant \alpha$.

A function $f(t)$, defined for all $t \in S$ and with values $f(t) \in C$, is *consistent* with the multiplicative process $U(t,s)$, $s,t \in S$, when for all $s \leqslant t$ in S, $f(t)=U(t,s)f(s)$.

Let C^* be the set of linear nonnegative functionals on V. Then C^* is a convex cone in the set V^* of all differences f^*-g^*, $f^*, g^* \in C^*$. The directed vector space (V^*, C^*) is the *dual space* of (V,C). In particular, the dual of any Banach lattice is again a Banach lattice (Birkhoff [5], p. 368). We can define the *dual multiplicative process* $U^*(s,t)$, $s \leqslant t$, as

$$\langle U^*(s,t)v^*, \phi \rangle = \langle v^*, U(t,s)\phi \rangle,$$

where $\langle v, \phi \rangle$ denotes the value of v at ϕ. A function $v^*(t)$ defined for all $t \in S$ and with values $v^*(t) \in C^*$ is consistent with $U^*(s,t)$ when for all $s \leqslant t$ in S, $v^*(s)=U^*(s,t)v^*(t)$.

DEFINITION 3.1

Let $U(t,s)$, $t \geqslant s$, be a time-inhomogeneous multiplicative process on (V,C) for positive time. $U(t,s)$, $t \geqslant s$, is called weakly ergodic if

$$\lim_{t \to \infty} d(U(t,s)\psi, U(t,s)\phi)=0 \qquad \textit{for all } \psi, \phi \in C^+. \tag{3.2}$$

In addition, if there exist numbers $\varepsilon > 0$, $M(\varepsilon) \geqslant 1$ such that for any $s \in S$,

$$d(U(t,s)\psi, U(t,s)\phi) \leqslant M(\varepsilon)\exp\{-\varepsilon(t-s)\}\, d(\psi,\phi) \qquad \textit{for } d(\psi,\phi) < \infty, \tag{3.3}$$

then $U(t,s)$, $t \geqslant s$, is called exponentially weakly ergodic.

PROPOSITION 3.2

Let $U(t,s)$, $t \geqslant s$, be a weakly ergodic time-inhomogeneous multiplicative process on a Banach lattice (V,C) for positive time, and let $f(t)$ and $g(t)$, $t \in S$, be consistent with $U(t,s)$. If $d(f(s), g(s)) < \infty$, then there exists a nonnegative functional $v^*(s) \in V^*$ such that

$$f(t)=\langle v^*(s), f(s) \rangle g(t) + o(\|g(t)\|), \tag{3.4}$$

where $o(\|g(t)\|)/\|g(t)\| \to 0$ as $t \to \infty$ and $v^*(t)$, $t \in S$, is consistent with the dual process $U^*(s,t)$, $s \leqslant t$.

Proof. Observe that

$$\begin{aligned}
\operatorname{osc}\left(\frac{f(t)}{g(t)}\right) &= \sup\left(\frac{f(t)}{g(t)}\right) - \inf\left(\frac{f(t)}{g(t)}\right) \\
&= \{\exp[d(f(t),g(t))]-1\} \inf\left(\frac{f(t)}{g(t)}\right) \\
&\leqslant \{\exp[d(f(t),g(t))]-1\} \sup\left(\frac{f(s)}{g(s)}\right) \to 0,
\end{aligned}$$

as $t \to \infty$. Since $\sup[f(t)/g(t)]$ is monotone decreasing and $\inf[f(t)/g(t)]$ is monotone increasing, there exists a nonnegative functional $v^*(s)$ defined as

$$\langle v^*(s), f(s)\rangle = \lim_{t\to\infty} \inf\left(\frac{f(t)}{g(t)}\right) = \lim_{t\to\infty} \sup\left(\frac{f(t)}{g(t)}\right).$$

It is easy to verify that $v^*(s)$ is linear. From

$$\inf\left(\frac{f(t)}{g(t)}\right) g(t) \leqslant f(t) \leqslant \sup\left(\frac{f(t)}{g(t)}\right) g(t),$$

we obtain

$$|f(t) - \langle v^*(s), f(s)\rangle g(t)| \leqslant |g(t)| \operatorname{osc}\left(\frac{f(t)}{g(t)}\right).$$

By the monotonicity of $\|f\|$ as a function of $|f|$, (3.4) follows immediately. Furthermore, we have

$$\begin{aligned}
\langle v^*(r), f(r)\rangle &= \lim_{t\to\infty} \inf\left(\frac{U(t,r)f(r)}{U(t,r)g(r)}\right) \\
&= \lim_{t\to\infty} \inf\left(\frac{U(t,r)U(r,s)f(s)}{U(t,r)U(r,s)g(s)}\right) = \langle v^*(s), f(s)\rangle,
\end{aligned}$$

for all $s < r$. Therefore, it follows that

$$\begin{aligned}
\langle v^*(s), f(s)\rangle &= \langle v^*(t), f(t)\rangle = \langle v^*(t), U(t,s)f(s)\rangle \\
&= \langle U^*(s,t)v^*(t), f(s)\rangle,
\end{aligned}$$

which shows that $v^*(s) = U^*(s,t)v^*(t)$, that is, $v^*(t)$ is consistent with $U^*(s,t)$, $s \leqslant t$. ∎

The functional $v^*(t)$ defined above is called the *importance functional* of the process. It is easily seen that the importance functional is essentially unique, that is, any other importance functional must be proportional to $v^*(t)$ (Birkhoff [4]).

PROPOSITION 3.3

Let $U(t,s)$, $t \geqslant s$, be a time-inhomogeneous multiplicative process on (V,C) for positive time. If $U(t,s)$, $t \geqslant s$, is uniformly primitive for positive time, then it is weakly ergodic.

Proof. Since $U(t,s)$, $t \geqslant s$, is uniformly primitive for positive time, then for any $s \in S$ there exist a number $\alpha > 0$ and an infinite sequence of positive numbers $s < t_0 < t_1 < \cdots$ in S, tending to ∞, such that $\Delta(U(t_{2n+1}, t_{2n})) \leqslant \alpha$. Then it is easily seen that for $t \geqslant t_{2n+1}$, $\psi, \phi \in C^+$,

$$d(U(t,s)\phi, U(t,s)\psi) \leqslant \left(\tanh\frac{\alpha}{4}\right)^n \alpha \to 0 \qquad \text{as } n \to \infty.$$

This shows that $U(t,s)$, $t \geqslant s$, is weakly ergodic. ■

Let $f(t), g(t)$ be consistent with a weakly ergodic time-inhomogeneous multiplicative process $U(t,s)$, $t \geqslant s$. From (2.4) and the definition of weak ergodicity, it is easy to see that the following holds:

$$\lim_{t \to \infty} \left\| \frac{f(t)}{\|f(t)\|} - \frac{g(t)}{\|g(t)\|} \right\| = 0. \tag{3.5}$$

Furthermore, if $v^* \in V^*$, then

$$\lim_{t \to \infty} \left| \left\langle v^*, \frac{f(t)}{\|f(t)\|} \right\rangle - \left\langle v^*, \frac{g(t)}{\|g(t)\|} \right\rangle \right| = 0. \tag{3.6}$$

In most applications, the consistent function represents the *state vector* of the physical system, which is evolved by the evolution operator $U(t,s)$. Also, the functionals on the state space (V,C) give observable indices of the states of the system. Proposition 3.3 tells us that the state vectors consistent with a weakly ergodic multiplicative process become asymptotically proportional and the values of the functionals at the normalized state vectors converge with each other as time evolves. Furthermore, we can state the following proposition, although we here omit the proof (the reader may refer to Birkhoff [4]).

PROPOSITION 3.4

Let $U(t,s)$ be a multiplicative process on a Banach lattice (V,C) that is uniformly primitive for negative time. Then there is one and essentially only one *function $f(t)$ with values in C^+ that is consistent with the process in the sense that $f(t) = U(t,s)f(s)$ for all $s \leqslant t$ in S.*

Now we introduce the *growth bound for positive time* related to the projective norm, denoted by $\omega_p^+(U(t,s))$, of the time-inhomogeneous process $U(t,s)$, $t \geqslant s$, in order to measure the rate of convergence. Let $U(t,s)$, $t \geqslant s$, be a positive time-inhomogeneous multiplicative process on (V, C) defined in $S = [s_0, \infty)$. Then there exists the growth bound for positive time $\omega_p^+(U(t,s))$ of $U(t,s)$, $t \geqslant s$, defined as

$$\omega_p^+(U(t,s)) = \lim_{\tau \to \infty} \frac{\log \eta(\tau)}{\tau}, \tag{3.7}$$

where

$$\eta(\tau) \overset{\text{def}}{=} \sup_{s \geqslant s_0} \| U(s+\tau, s) \|_p \leqslant 1.$$

In fact, observe that

$$\begin{aligned} \eta(x+y) &= \sup_{s \geqslant s_0} \| U(s+x+y, s) \|_p \\ &\leqslant \sup_{s \geqslant s_0} \| U(s+x+y, s+x) \|_p \| U(s+x, s) \|_p \\ &\leqslant \sup_{s \geqslant s_0} \| U(s+x+y, s+x) \|_p \sup_{s \geqslant s_0} \| U(s+x, s) \|_p \leqslant \eta(y)\eta(x). \end{aligned}$$

Then $\log \eta(\tau)$ is a subadditive function on $[0, \infty)$ and there exists the limit $\lim_{\tau \to \infty} \log \eta(\tau)/\tau = \omega_p^+(U(t,s))$ (Dunford and Schwartz [14], p. 618, Lemma 4). Furthermore, it is easily seen that if $\xi > \omega_p^+(U(t,s))$, then there exists a number $M(\xi) \geqslant 1$ such that

$$\| U(t,s) \|_p \leqslant M(\xi) \exp\{\xi(t-s)\}. \tag{3.8}$$

This fact shows that if $\omega_p^+(U(t,s)) < 0$, the uniformly primitive process $U(t,s)$, $t \geqslant s$, is exponentially weakly ergodic. The following proposition provides a sufficient condition that guarantees that $U(t,s)$, $t \geqslant s$, is exponentially weakly ergodic.

PROPOSITION 3.5

Let $U(t,s)$, $t \geqslant s$, be a time-inhomogeneous multiplicative process on (V, C) defined in $S = [s_0, \infty)$. If there exist numbers $\xi > 0$, $\alpha > 0$ such that $\Delta(U(s+\xi, s)) \leqslant \alpha$ for all $s \geqslant s_0$, then $U(t,s)$, $t \geqslant s$, is exponentially weakly ergodic and

$$\omega_p^+(U(t,s)) \leqslant \frac{1}{\xi} \log\left(\tanh \frac{\alpha}{4}\right) \tag{3.9}$$

Proof. First we show that $U(t,s)$, $t \geqslant s$, is positive. Assume that there exists $\phi \in C^+$ such that $U(t,s)\phi = 0$. Let $[(t-s)/\xi] = n$. Then

$$U(t, s+n\xi) U(s+n\xi, s+(n-1)\xi) \times \cdots \times U(s+\xi, s)\phi = 0.$$

Since $U(s+j\xi, s+(j-1)\xi)$, $1 \leqslant j \leqslant n$, is uniformly positive, then it must follow that

$$U(t, s+n\xi)\psi = 0, \qquad \psi = U(s+n\xi, s+(n-1)\xi) \times \cdots \times U(s+\xi, s)\phi.$$

Then we obtain that

$$U(s+(n+1)\xi, s+n\xi)\psi = U(s+(n+1)\xi, t) U(t, s+n\xi)\psi = 0.$$

This is a contradiction because $U(s+(n+1)\xi, s+n\xi)$ is uniformly positive. Thus we know that $U(t,s)$ is a uniformly primitive time-inhomogeneous multiplicative process on (V,C) and is weakly ergodic. Under the assumptions, it follows that $\eta(\xi) \leqslant \tanh(\alpha/4)$. For $n\xi \leqslant x < (n+1)\xi$, we have

$$\begin{aligned} \| U(s+x, s) \|_p \leqslant \| U(s+x, s+n\xi) \|_p \| U(s+n\xi, s+(n-1)\xi) \|_p \\ \times \cdots \times \| U(s+\xi, s) \|_p. \end{aligned}$$

Thus $\eta(x) \leqslant \eta(\xi)^n$, and $\lim_{x\to\infty} \log\eta(x)/x \leqslant \log\eta(\xi)/\xi$, which shows that $\omega^+(U(t,s)) \leqslant (1/\xi)\log[\tanh(\alpha/4)] < 0$. From (3.8), we know that $U(t,s)$, $t \geqslant s$, is exponentially weakly ergodic. This completes our proof. ∎

The case that $U(t,s)$, $t \geqslant s$, depends only on the time interval $t-s$ is of much interest. In this special case, we have a one-parameter family of nonnegative operators $T(t)$, $t \geqslant 0$, if we let $U(t,0) = T(t)$. It is clear that $T(t)$, $t \geqslant 0$, satisfies the *semigroup property*

$$T(t)T(u) = T(t+u); \qquad t \geqslant 0, \quad u \geqslant 0.$$

In general, a *time-homogeneous multiplicative process* on (V,C) is a one-parameter semigroup $T(t)$, $t \geqslant 0$, of nonnegative operators. If there exists a number $t_1 > 0$ such that $T(t_1)$ is uniformly positive, it is called uniformly primitive.

DEFINITION 3.6

Let $T(t)$, $t \geqslant 0$, be a positive time-homogeneous multiplicative process on (V,C). If there exists $\phi_0 \in C^+$ such that

$$d(T(t)\phi_0, \phi_0) = 0 \qquad \textit{for all } t \geqslant 0,$$
$$\lim_{t\to\infty} d(T(t)\psi, \phi_0) = 0 \qquad \textit{for all } \psi \in C^+,$$

then $T(t)$, $t \geqslant 0$, is called strongly ergodic and ϕ_0 is called the stable distribution. If there exist numbers $\epsilon > 0$, $M(\epsilon) \geqslant 1$ such that

$$d(T(t)\psi, \phi_0) \leqslant M(\epsilon)\exp(-\epsilon t)\, d(\psi, \phi_0) \qquad \text{for } 0 < d(\psi, \phi_0) < \infty,$$

then $T(t)$, $t \geqslant 0$, is called exponentially strongly ergodic

It is clear that the stable distribution ϕ_0 is an eigenvector of $T(t)$, that is, there exists a function $\lambda(t) > 0$ such that

$$T(t)\phi_0 = \lambda(t)\phi_0 \qquad \text{for all } t \geqslant 0.$$

The positive function $\lambda(t)$ is an eigenvalue of $T(t)$ corresponding to ϕ_0, which satisfies the property $\lambda(t+s) = \lambda(t)\lambda(s)$, $t \geqslant 0$, $s \geqslant 0$.

A function $f(s)$ defined in an unbounded subset S of $(-\infty, +\infty)$ with values $f(s) \in C^+$ is consistent with $T(t)$, $t \geqslant 0$, when $f(t+s) = T(t)f(s)$. In particular, the stable distribution $\lambda(t)\phi_0$ of $T(t)$, $t \geqslant 0$ is consistent with $T(t)$, $t \geqslant 0$. For the time-homogeneous case, we can prove the following, although we omit the proofs (the reader may refer to Birkhoff [2,3]).

PROPOSITION 3.7

Let $T(t)$, $t \geqslant 0$, be a strongly ergodic time-homogeneous multiplicative process on a Banach lattice (V, C), let ϕ_0 be the stable distribution, and let $\lambda(t)$ be the eigenvalue of $T(t)$ corresponding to the eigenvector ϕ_0. Then there exists a functional $v_0^* \in C^*$ such that

$$T(t)\phi = \langle v_0^*, \phi\rangle \lambda(t)\phi_0 + o(\lambda(t))$$

for all $d(\phi, \phi_0) < \infty$, where $o(\lambda(t))/\lambda(t) \to 0$ as $t \to \infty$ and v_0^* is the eigenvector of the dual process $T^*(t)$ associated with the eigenvalue $\lambda(t)$.

PROPOSITION 3.8

Let $T(t)$, $t \geqslant 0$ be a positive time-homogeneous multiplicative process on a Banach lattice (V, C). If $T(t)$, $t \geqslant 0$, is uniformly primitive, it is exponentially strongly ergodic.

PROPOSITION 3.9

Let $T(t)$, $t \geqslant 0$, be a uniformly primitive time-homogeneous multiplicative process on a Banach lattice (V, C), and let $f(s)$, $s \in S = (-\infty, t_0]$, be consistent with $T(t)$, $t \geqslant 0$. Then $f(s)$ is the stable distribution of $T(t)$, $t \geqslant 0$, for all $s \in S$.

4. POPULATION EVOLUTION OPERATOR AND ITS UNIFORM PRIMITIVITY

In what follows, we consider a large-scale one-sex closed population divided into n subclasses. The subclasses of the population may correspond to geographical regions, social status, physiological features, or other classifications of the population. Let $p_i(a,t)$ be the density with respect to age a of the ith subpopulation at time t, that is,

$$F_i(r,t) \stackrel{\text{def}}{=} \int_0^r p_i(a,t)\,da$$

denotes the number of individuals of the ith subclass aged less than r at time t. Defining the population density vector as

$$p(a,t) \stackrel{\text{def}}{=} [p_1(a,t),\ldots,p_n(a,t)]^{\tau}, \tag{4.1}$$

where τ denotes the transpose of the vector, we suppose that the population is governed by the Lotka-McKendrick-Von Foerster system

$$Dp(a,t) = Q(a,t)\,p(a,t), \tag{4.2a}$$

$$p(0,t) = \int_0^{\infty} M(a,t)\,p(a,t)\,da, \tag{4.2b}$$

where $0 \leqslant a < \infty$, $-\infty < t < +\infty$, and the operator D is defined as

$$Df(a,t) = \lim_{h\to 0} \frac{f(a+h,t+h) - f(a,t)}{h}. \tag{4.3}$$

$M(a,t)$, called the fertility matrix, is an $n \times n$ matrix, in which the (i,j)th element $m_{ij}(a,t)$ denotes the average number of offspring of the ith subclass per unit time produced by an individual at age a and time t of the jth subclass, and $Q(a,t)$ is an $n \times n$ matrix in which the (i,j)th element $q_{ij}(a,t)$ is the instantaneous transition rate from the jth state to the ith state at time t and age a. In addition, we require that the following hold:

(H-1) $m_{ij}(\cdot,t)$, $1 \leqslant i,j \leqslant n$, are bounded measurable functions for all $t \in (-\infty,+\infty)$, and $m_{ij}(a,t) = 0$ for $a \geqslant \beta$, where $\beta < +\infty$ denotes the upper bound of the reproductive age.

(H-2) $Q(a+h,t+h)$ is locally integrable for $-a \leqslant h$, and the balance law holds as

$$\sum_{i=1}^{n} q_{ij}(a,t) + \mu_j(a,t) = 0, \qquad 1 \leqslant j \leqslant n, \tag{4.4}$$

where $\mu_j(a,t)$ is the force of mortality of the jth state subpopulation (Inaba [17]).

The state of the multistate population at time t is described as a point of the *state space*, that is, the set of possible density vectors. In the following, we concentrate the dynamics of the population on the reproductive age interval, because the population of postreproductive age cannot affect the size of the population of reproductive age (Assumption H-1). Thus we consider system (4.2) on the domain $[0,\beta]\times(-\infty,+\infty)$. Since the physical interpretation of the density function requires that it should be integrable and the L^1-norm of the density is a natural measure of the size of the population, we choose $L^1_+ = L^1_+(0,\beta; R^n)$ as the state space of the population, where $L^1_+(0,\beta; R^n)$ is the positive cone of the Banach space of Lebesgue integrable functions from $[0,\beta]$ to R^n. The solution of system (4.2) will trace an orbit in the state space as time evolves. Now we can define a family of two-parameter operators $U(t,s)$, $s \leqslant t$, on L^1_+ such as

$$p(\cdot,t) = U(t,s)p(\cdot,s), \qquad t \geqslant s, \tag{4.5}$$

where $p(\cdot,t)$ is the density vector of the population at time t. Then, for every $\phi \in L^1_+$, $(U(t,s)\phi)(a)$ gives the solution of the initial-boundary value problem of the system

$$Dp(a,t) = Q(a,t)p(a,t), \tag{4.6a}$$

$$p(0,t) = \int_0^\beta M(a,t)p(a,t)\,da, \tag{4.6b}$$

$$p(a,s) = \phi(a). \tag{4.6c}$$

In order to determine the solution of system (4.6), we here introduce the survival rate matrix $L(a+h,t+h;a,t)$, which is defined as a solution of a matrix differential equation

$$\frac{d}{dh}L(a+h,t+h;a,t) = Q(a+h,t+h)L(a+h,t+h;a,t),$$

$$L(a,t;a,t) = I.$$

Then it is easy to see that the following hold:

$$p(a+h,t+h) = L(a+h,t+h;a,t)p(a,t), \qquad h \geqslant 0 \tag{4.7}$$

$$\begin{aligned} L(a+h+k,t+h+k;a,t) &= L(a+h+k,t+h+k;a+h,t+h) \\ &\quad \times L(a+h,t+h;a,t); \\ &\qquad h \geqslant 0, \quad k \geqslant 0. \end{aligned} \tag{4.8}$$

If we let $l_{ij}(a+h,t+h;a,t)$ be the (i,j)th element of the survival rate matrix $L(a+h, t+h; a, t)$, then $l_{ij}(a+h, t+h; a, t)$, $h > 0$ denotes the proportion of individuals of age a and in subclass j at time t who are alive at age $a+h$ and in subclass i at time $t+h$. Using the survival rate matrix, we can represent the solution $p(a, t)$ of problem (4.6) as follows:

$$p(a,t) = \begin{cases} L(a,t;0,t-a)\,p(0,t-a), & \text{for } t-s > a, \\ L(a,t;a-t+s,s)\,\phi(a-t+s), & \text{for } t-s \leqslant a. \end{cases} \tag{4.9}$$

Substituting (4.9) into (4.6b), we obtain

$$p(0,t) = \int_0^{t-s} M(a,t)\,L(a,t;0,t-a)\,p(0,t-a)\,da + \int_{t-s}^{\beta} M(a,t)\,L(a,t;a-t+s,s)\,\phi(a-t+s)\,da \tag{4.10}$$

for $t-s \leqslant \beta$ and

$$p(0,t) = \int_0^{\beta} M(a,t)\,L(a,t;0,t-a)\,p(0,t-a)\,da \tag{4.11}$$

for $t-s > \beta$. Let $\xi = t-s$. Then (4.10) and (4.11) yield

$$p(0,s+\xi) = \int_0^{\xi} M(a,s+\xi)\,L(a,s+\xi;0,s+\xi-a)\,p(0,s+\xi-a)\,da + \int_{\xi}^{\beta} M(a,s+\xi)\,L(a,s+\xi;a-\xi,s)\,\phi(a-\xi)\,da, \quad 0 \leqslant \xi \leqslant \beta, \tag{4.12}$$

$$p(0,s+\xi) = \int_0^{\beta} M(a,s+\xi)\,L(a,s+\xi;0,s+\xi-a)\,p(0,s+\xi-a)\,da, \quad \xi > \beta. \tag{4.13}$$

Here we define

$$B(\xi;\phi,s) \overset{\text{def}}{=} p(0,s+\xi),$$

$$\Psi(a,s+\xi) \overset{\text{def}}{=} \begin{cases} M(a,s+\xi)\,L(a,s+\xi;0,s+\xi-a) & \text{for } 0 \leqslant a \leqslant \beta, \\ 0 & \text{for } a > \beta, \end{cases}$$

$$G(\xi;\phi,s) \overset{\text{def}}{=} \begin{cases} \int_{\xi}^{\beta} M(a,s+\xi)\,L(a,s+\xi;a-\xi,s)\,\phi(a-\xi)\,da & \text{for } 0 \leqslant \xi \leqslant \beta, \\ 0 & \text{for } \xi > \beta. \end{cases}$$

Thus $B(\xi; \phi, s)$ is the solution of the Volterra integral equation

$$B(\xi; \phi, s) = G(\xi; \phi, s) + \int_0^{\xi} \Psi(a, s+\xi) B(\xi - a; \phi, s)\, da, \qquad \xi \geqslant 0. \tag{4.14}$$

Let $R(\xi, \rho; s)$ be the resolvent kernel corresponding to the integral kernel $\Psi(\xi - \rho, \xi + s)$; that is, $R(\xi, \rho; s)$ is the solution of the resolvent equation

$$R(\xi, \rho; s) = \Psi(\xi - \rho, \xi + s) + \int_{\rho}^{\xi} \Psi(\xi - \eta, \xi + s) R(\eta, \rho; s)\, d\eta. \tag{4.15}$$

Then we have the solution representation

$$B(\xi; \phi, s) = G(\xi; \phi, s) + \int_0^{\xi} R(\xi, \rho; s) G(\rho; \phi, s)\, d\rho. \tag{4.16}$$

It is clear that $B(\xi; \phi, s)$ is linearly dependent on the initial data ϕ:

$$B(\xi; \phi_1 + \phi_2, s) = B(\xi; \phi_1, s) + B(\xi; \phi_2, s). \tag{4.17}$$

Furthermore, we can rewrite the solution of problem (4.6) as

$$p(a, t) = \begin{cases} L(a, t; 0, t-a) B(t - s - a; \phi, s), & \text{for } t - s > a, \\ L(a, t; a - t + s, s) \phi(a - t + s), & \text{for } t - s \leqslant a. \end{cases} \tag{4.18}$$

Hence we arrive at the direct representation of the nonnegative linear operator $U(t, s)$ in L^1_+ as

$$(U(t, s)\phi)(a) = \begin{cases} L(a, t; 0, t-a) B(t - s - a; \phi, s), & \text{for } t - s > a, \\ L(a, t; a - t + s, s) \phi(a - t + s), & \text{for } t - s \leqslant a. \end{cases} \tag{4.19}$$

$$= \begin{cases} L(a, t; 0, t-a) \Big\{ G(t - s - a; \phi, s) \\ \qquad + \displaystyle\int_0^{t-s-a} R(t - s - a, \rho; s) G(\rho; \phi, s)\, d\rho \Big\}, & t - s > a, \\ L(a, t; a - t + s, s) \phi(a - t + s), & t - s \leqslant a, \end{cases} \tag{4.20}$$

where $\phi \in L^1_+(0, \beta; R^n)$. In the following, we refer to the operator $U(t, s)$ given by (4.19) as the *population evolution operator*, because $U(t, s)$ evolves a population $p(\cdot, s)$ at time s to a population $p(\cdot, t)$ at any later time t. From the above definition, $U(s,s) = I$, where I denotes the identity operator in L^1. The most important feature of the population evolution operator is its *multiplicativity*:

$$U(u,t)U(t,s) = U(u,s) \qquad \text{for all } u \geqslant t \geqslant s, \tag{4.21}$$

which is a consequence of the uniqueness property of the solution of problem (4.6). From (4.21), we know that $U(t,s)$, $t \geqslant s$, is a time-inhomogeneous multiplicative process on a Banach lattice (L^1, L^1_+). The reader may find another way to construct the evolutionary system related to population dynamics in Clément et al. [10].

Next we will show sufficient conditions that guarantee that $U(t,s)$, $t \geqslant s$, is a uniformly primitive multiplicative process on $L^1_+(0,\beta;\, R^n)$. If $t - s > 2\beta$, then for $a \in [0,\beta]$, $\phi \in L^1_+(0,\beta;\, R^n)$, we have the representation

$$(U(t,s)\phi)(a) = L(a,t;0,t-a)\int_0^\beta R(t-s-a,\rho;s)G(\rho;\phi,s)\,d\rho. \tag{4.22}$$

The following lemma was first proposed by Norton [27] for a scalar-type Volterra integral equation.

LEMMA 4.1

Assume that there exist numbers γ_1, γ_2 and a primitive matrix $N \geqslant 0$ such that

$$\Psi(a,t) \geqslant N \qquad \textit{for all } (a,t) \in [\gamma_1,\gamma_2] \times R. \tag{4.23}$$

Then for any δ such that $0 < \delta < (\gamma_2 - \gamma_1)/2$ and for any nonnegative integer $n = 0,1,2,\ldots,$ the following holds:

$$R(\xi,\rho;s) \geqslant N^{n+1}\delta^n \qquad \textit{for } \xi - \rho \in [\gamma_1 + n(\gamma_1+\delta), \gamma_2 + n(\gamma_2 - \delta)]. \tag{4.24}$$

Proof. From the resolvent equation (4.15), it follows that

$$R(\xi,\rho;s) \geqslant \Psi(\xi-\rho,\xi+s) \geqslant N \qquad \text{for } \xi - \rho \in [\gamma_1,\gamma_2], \tag{4.25}$$

which shows that (4.24) holds for $n = 0$. Assume that (4.24) holds for some integer n_0, that is,

$$R(\xi,\rho;s) \geqslant N^{n_0+1}\delta^{n_0} \qquad \text{for } \xi - \rho \in [\gamma_1 + n_0(\gamma_1+\delta), \gamma_2 + n_0(\gamma_2 - \delta)]. \tag{4.26}$$

If $\rho \leqslant \xi - \gamma_2$, then

$$R(\xi, \rho; s) \geqslant \int_{\rho}^{\xi} \Psi(\xi - \eta, \xi + s) R(\eta, \rho; s) \, d\eta$$

$$\geqslant \int_{\xi - \gamma_2}^{\xi - \gamma_1} \Psi(\xi - \eta, \xi + s) R(\eta, \rho; s) \, d\eta. \tag{4.27}$$

Let $J = [\xi - \gamma_2, \xi - \gamma_1] \cap [\rho + \gamma_1 + n_0(\gamma_1 + \delta), \rho + \gamma_2 + n_0(\gamma_2 - \delta)]$. If

$$\xi - \rho \in [\gamma_1 + (n_0 + 1)(\gamma_1 + \delta), \gamma_2 + (n_0 + 1)(\gamma_2 - \delta)],$$

then we obtain

$$\xi - \gamma_1 - (\rho + \gamma_1 + n_0(\gamma_1 + \delta)) \geqslant \delta, \qquad (\rho + \gamma_2 + n_0(\gamma_2 - \delta)) - (\xi - \gamma_2) \geqslant \delta.$$

Then the length of the interval J is greater than δ. Therefore it follows that

$$R(\xi, \rho; s) \geqslant \int_{\rho}^{\xi} \Psi(\xi - \eta, \xi + s) R(\eta, \rho; s) \, d\eta$$

$$\geqslant \int_{J} \Psi(\xi - \eta, \xi + s) R(\eta, \rho; s) \, d\eta$$

$$\geqslant N N^{n_0 + 1} \delta^{n_0} \delta = N^{n_0 + 2} \delta^{n_0 + 1}.$$

Similarly, if $\xi - \gamma_2 \leqslant \rho$, then the length of the intersection of $[\rho, \xi - \gamma_1]$ and $[\rho + \gamma_1 + n_0(\gamma_1 + \delta), \rho + \gamma_2 + n_0(\gamma_2 - \delta)]$ is also greater than δ and

$$R(\xi, \rho; s) \geqslant \int_{L} \Psi(\xi - \eta, \xi + s) R(\eta, \rho; s) \, d\eta \geqslant N^{n_0 + 2} \delta^{n_0 + 1},$$

where $L = [\rho, \xi - \gamma_1] \cap [\rho + \gamma_1 + n_0(\gamma_1 + \delta), \rho + \gamma_2 + n_0(\gamma_2 - \delta)]$. This shows that (4.24) holds for $n_0 + 1$ if it holds for n_0. By mathematical induction, we know that (4.24) holds for every nonnegative integer. ■

For the fertility rate matrix, we suppose that there exists a positive matrix $\overline{M}$ such that

$$\overline{M} \geqslant \Psi(a, t) \qquad \text{for all } (a, t) \in R^+ \times R, \tag{4.28}$$

which is a natural assumption for biological applications. Then the following inequality follows immediately from the resolvent equation (4.15):

$$R(\xi, \rho; s) \leqslant \overline{M} \exp\{\overline{M}(\xi - \rho)\}. \tag{4.29}$$

Furthermore, we define a nonnegative matrix $S(a,t)$ as

$$S(a,t) \stackrel{\text{def}}{=} \int_0^\beta M(a+\rho, t+\rho) L(a+\rho, t+\rho; a, t)\, d\rho \tag{4.30}$$

and let $s_{ij}(a,t)$ denote the (i,j)th element of $S(a,t)$. It is easily seen that

$$\int_0^\beta G(\rho; \phi, s)\, d\rho = \int_0^\beta S(a,s)\phi(a)\, da. \tag{4.31}$$

Then we can prove the following.

PROPOSITION 4.2

Suppose that the assumption of Lemma 4.1 *holds and there exists a positive number* ω_1 *such that*

$$\max_i s_{ij}(a,t) \leqslant \omega_1 \min_i s_{ij}(a,t) \tag{4.32}$$

for all $(a,t) \in [0,\beta] \times R$, $1 \leqslant j \leqslant n$, *where* $\min_{1 \leqslant i \leqslant n} s_{ij}(a,t)$, $1 \leqslant j \leqslant n$, *is almost everywhere positive. Then there exists a number* $\xi > 0$ *such that* $U(s+\xi, s)$ *is a uniformly positive operator for all* s.

Proof. From (4.22), if $t - s > 2\beta$, then we have

$$\begin{aligned}(U(t,s)\phi)(a) &= L(a,t;0,t-a)\int_0^\beta R(t-s-a,\rho;s)G(\rho;\phi,s)\, d\rho \\ &\leqslant L(a,t;0,t-a)\overline{M}\exp\{\overline{M}(t-s)\}\int_0^\beta G(\rho;\phi,s)\, d\rho. \end{aligned}\tag{4.33}$$

If we take an integer n such that $n > (2\beta - \gamma_2 + \gamma_1)/(\gamma_2 - \gamma_1 - 2\delta)$, then there exists a number $\xi > 0$ such that $2\beta + \gamma_1 + n(\gamma_1 + \delta) < \xi < \gamma_2 + n(\gamma_2 - \delta)$. Let $t - s = \xi$. Then $\xi - a - \rho \in [\gamma_1 + n(\gamma_1 + \delta), \gamma_2 + n(\gamma_2 - \delta)]$ when $a, \rho \in [0,\beta]$. Therefore, from Lemma 4.1, we obtain

$$R(\xi - a, \rho; s) \geqslant N^{n+1}\delta^n \qquad \text{for } a, \rho \in [0,\beta]. \tag{4.34}$$

Hence,

$$(U(s+\xi,s)\phi)(a) \geqslant L(a,t;0,t-a)N^{n+1}\delta^n \int_0^\beta G(\rho;\phi,s)\, d\rho. \tag{4.35}$$

From (4.31) and (4.32), we obtain

$$\lambda_s(\phi)e \leqslant \int_0^\beta G(\rho;\phi,s)\, d\rho \leqslant \omega_1 \lambda_s(\phi)e, \tag{4.36}$$

where e denotes a vector all of whose components are 1 and

$$\lambda_s(\phi) \stackrel{\text{def}}{=} \sum_j \int_0^\beta \Big[\min_i s_{ij}(a,s) \Big] \phi_j(a)\, da.$$

Under our assumption, $\lambda_s(\phi)$ is a positive functional on $L^1_+(0,\beta; R^n)$. If we take an integer n sufficiently large, N^{n+1} is a positive matrix. Let $(N^{n+1}\delta^n)_{ij}$ and $(\overline{M}\exp(\overline{M}\xi))_{ij}$ be the (i,j)th element of $N^{n+1}\delta^n$ and $\overline{M}\exp(\overline{M}\xi)$, respectively. Then there exists a number

$$\omega_2 = \max_{i,j} \left[\frac{(\overline{M}\exp(\overline{M}\xi))_{ij}}{(N^{n+1}\delta^n)_{ij}} \right].$$

Thus we arrive at the following inequality:

$$\lambda_s(\phi) f_{\xi,s} \leqslant U(s+\xi, s)\phi \leqslant (\omega_1\omega_2)\lambda_s(\phi) f_{\xi,s}, \tag{4.37}$$

where $f_{\xi,s} = L(a, s+\xi; 0, s+\xi-a) N^{n+1}\delta^n e \in L^1_+(0,\beta; R^n)$. Therefore, we have

$$d\big(U(s+\xi,s)\phi, f_{\xi,s}\big) \leqslant \log(\omega_1\omega_2), \tag{4.38}$$

where the right-hand side is independent of s. Hence, we obtain that $\Delta(U(s+\xi,s)) \leqslant 2\log(\omega_1\omega_2)$. Thus we know that $U(s+\xi,s)$ is a uniformly positive operator on $L^1_+(0,\beta; R^n)$. This completes our proof. ■

From Propositions 4.2 and 3.5, we have the following immediately:

COROLLARY

Under the assumptions of Proposition 4.2, $U(t,s)$, $t \geqslant s$, *is exponentially weakly ergodic and the growth bound for positive time of* $U(t,s)$, $t \geqslant s$, *satisfies the inequality*

$$\omega_p^+(U(t,s)) \leqslant \frac{1}{\xi} \log\left(\frac{\omega_1\omega_2 - 1}{\omega_1\omega_2 + 1} \right). \tag{4.39}$$

Assumption (4.32) is not so unnatural as it looks; it can be satisfied under natural conditions for a fertility and mortality schedule of the population. Suppose that there exist nonnegative functions $\underline{\mu}(a,t)$, $\bar{\mu}(a,t)$, $\underline{m}(a,t)$, and $\overline{m}(a,t)$ such that

$$\underline{\mu}(a,t) \leqslant \mu_j(a,t) \leqslant \bar{\mu}(a,t), \underline{m}(a,t) \leqslant m_{ij}(a,t) \leqslant \overline{m}(a,t)$$

for all $1 \leqslant i, j \leqslant n$. Then it follows that

$$\underline{s}(a,t) \leqslant s_{ij}(a,t) \leqslant \bar{s}(a,t),$$

where

$$\underline{s}(a,t) \stackrel{\text{def}}{=} \int_0^\beta \underline{m}(a+\rho, t+\rho)\underline{l}(a+\rho, t+\rho; a, t)\, d\rho,$$

$$\bar{s}(a,t) \stackrel{\text{def}}{=} \int_0^\beta \overline{m}(a+\rho, t+\rho)\bar{l}(a+\rho, t+\rho; a, t)\, d\rho,$$

$$\underline{l}(a+\rho, t+\rho; a, t) \stackrel{\text{def}}{=} \exp\left(-\int_0^\rho \bar{\mu}(a+s, t+s)\, ds \right),$$

$$\bar{l}(a+\rho, t+\rho; a, t) \stackrel{\text{def}}{=} \exp\left(-\int_0^\rho \underline{\mu}(a+s, t+s)\, ds \right).$$

Furthermore, if we assume that there exists a number $K > 0$ such that $\overline{m}(a,t) \leqslant K\underline{m}(a,t)$, then the following holds:

$$\underline{s}(a,t) \leqslant s_{ij}(a,t) \leqslant K\underline{s}(a,t).$$

If $\underline{s}(a,t)$ is positive almost everywhere, it is clear that Assumption (4.32) is satisfied.

If the vital rates $M(a,t)$ and $Q(a,t)$ are time-independent in system (4.2), then the operator $U(t,s)$ depends only on the time interval $t-s$. Define the one-parameter semigroup $T(t)$, $t \geqslant 0$, as $T(t) = U(t,0)$, called the *population semigroup*. Then we have

$$p(\cdot, t+u) = T(u)p(\cdot, t); \qquad u \geqslant 0, \quad t \geqslant 0.$$

Let Σ be the set of characteristic roots

$$\Sigma = \left\{ \lambda \in C;\ \det\left(I - \int_0^\beta \exp(-\lambda a) M(a) L(a)\, da \right) = 0 \right\}$$

where $L(a)$ is the survival rate matrix, which is given as a solution of a matrix differential equation

$$\frac{d}{da} L(a) = Q(a) L(a), \qquad L(0) = I.$$

Under the assumption of Proposition 4.2, it can be shown that there exists a real $r_0 \in \Sigma$ such that $r_0 > \sup\{\operatorname{Re}\lambda;\ \lambda \in \Sigma - \{r_0\}\}$ and $T(t)$, $t \geqslant 0$, has an eigenvector $\phi_0(a) = \exp(-r_0 a) L(a) \phi_0(0)$ corresponding to an eigenvalue

$\exp(r_0 t)$, where $\phi_0(0)$ is the Frobenius vector of the positive matrix $\int_0^\beta \exp(-r_0 a) M(a) L(a)\, da$ corresponding to the Frobenius root 1 (Inaba [19]). Therefore, it follows that $T(t)$, $t \geqslant 0$, is exponentially strongly ergodic under the assumptions of Proposition 4.2.

5. RESULTS AND DISCUSSION

THE WEAK ERGODIC THEOREM

Let $p^{(i)}(a,t)$, $i=1,2$, be a population governed by system (4.2) with an initial condition

$$p^{(i)}(a,0) = \phi^{(i)}(a) \in L^1_+(0,\infty; R^n).$$

Then we can write

$$(U(t,0)\phi^{(i)})(a) = p^{(i)}(a,t) \qquad \text{for } a \in [0,\beta],\ t \geqslant 0,$$

where $\phi^{(i)}(a)$, $a \in [0,\beta]$ are elements of $L^1_+(0,\beta; R^n)$. Suppose that $U(t,s)$, $t \geqslant s$, is weakly ergodic. From (3.5), we obtain

$$\left\| \frac{p^{(1)}(a,t)}{\| p^{(1)}(a,t) \|} - \frac{p^{(2)}(a,t)}{\| p^{(2)}(a,t) \|} \right\| \to 0 \qquad \text{as } t \to \infty, \tag{5.1}$$

where $\|\cdot\|$ denotes the L^1-norm of $L^1(0,\beta; R^n)$. This shows that the age distribution on the age interval $[0,\beta]$ is asymptotically independent of the initial data. And the rate of convergence is estimated by (4.39). Furthermore, it can be shown that the weak ergodicity holds for any finite age interval of the form $[0,\omega]$, $\omega \geqslant \beta$, as follows: Let $B^{(i)}(t)$ be the birth rate at time t of the population $p^{(i)}(a,t)$. Then

$$B^{(i)}(t) = \int_0^\beta M(a,t)\, p^{(i)}(a,t)\, da. \tag{5.2}$$

Let

$$C(t) = \inf\left[\frac{p^{(2)}(\cdot,t)}{p^{(1)}(\cdot,t)}\right], \qquad D(t) = \sup\left[\frac{p^{(2)}(\cdot,t)}{p^{(1)}(\cdot,t)}\right].$$

From the definition, we have

$$C(t)\, p^{(1)}(a,t) \leqslant p^{(2)}(a,t) \leqslant D(t)\, p^{(1)}(a,t) \qquad \text{for } 0 \leqslant a \leqslant \beta.$$

From (5.2), we obtain

$$C(t)\, B^{(1)}(t) \leqslant B^{(2)}(t) \leqslant D(t)\, B^{(1)}(t).$$

Let $K=\lim_{t\to\infty}C(t)=\lim_{t\to\infty}D(t)$. Then it is easily seen that

$$0\leqslant B^{(2)}(t)-KB^{(1)}(t)\leqslant E(t)B^{(1)}(t),$$

where

$$E(t)\stackrel{\text{def}}{=}D(t)-C(t)=\operatorname{osc}\left(\frac{p^{(2)}(\cdot,t)}{p^{(1)}(\cdot,t)}\right).$$

For any age $\omega\geqslant\beta$, we can write

$$p^{(i)}(a,t)=L(a,t;0,t-a)B^{(i)}(t-a);\qquad t\geqslant\omega,\quad a\in[0,\omega].$$

Then we obtain that for $t\geqslant\omega$,

$$Kp^{(1)}(a,t)\leqslant p^{(2)}(a,t)\leqslant[K+E(t-\omega)]p^{(1)}(a,t)\qquad\text{for all } a\in[0,\omega]$$

and

$$K\|p^{(1)}(\cdot,t)\|_{\omega}\leqslant\|p^{(2)}(\cdot,t)\|_{\omega}\leqslant[K+E(t-\omega)]\|p^{(1)}(\cdot,t)\|_{\omega},$$

where $\|\cdot\|_{\omega}$ denotes the L^1-norm of $L^1(0,\omega;R^n)$. Hence

$$\begin{aligned}\frac{K}{K+E(t-\omega)}\frac{p^{(1)}(a,t)}{\|p^{(1)}(\cdot,t)\|_{\omega}}&\leqslant\frac{p^{(2)}(a,t)}{\|p^{(2)}(\cdot,t)\|_{\omega}}\\&\leqslant\left(1+\frac{E(t-\omega)}{K}\right)\frac{p^{(1)}(a,t)}{\|p^{(1)}(\cdot,t)\|_{\omega}}.\end{aligned}$$

Therefore, it is easy to see that the following holds:

$$\left\|\frac{p^{(2)}(\cdot,t)}{\|p^{(2)}(\cdot,t)\|_{\omega}}-\frac{p^{(1)}(\cdot,t)}{\|p^{(1)}(\cdot,t)\|_{\omega}}\right\|_{\omega}\leqslant\frac{E(t-\omega)}{K}\to0\qquad\text{as } t\to\infty.$$

This shows that the population $p(a,t)$ is weakly ergodic on the age interval $[0,\omega]$.

Next we consider the time-homogeneous case. Let

$$p^{(i)}(a,t)=[T(t)\phi^{(i)}](a),\qquad p^{(i)}(a,0)=\phi^{(i)}(a)\in L^1_+(0,\beta;R^n)$$

and let $\phi^{(1)}(a)=\exp(-r_0a)L(a)\psi$, where ψ is a positive eigenvector of the matrix

$$\int_0^{\beta}\exp(-r_0a)M(a)L(a)\,da$$

associated with the Frobenius root 1. Then we have

$$p^{(1)}(a,t) = \exp[r_0(t-a)]\,L(a)\psi, \qquad B^{(1)}(t) = \exp(r_0 t)\psi.$$

By the same argument as for the time-inhomogeneous case, we conclude that for any age interval of the form $[0,\omega]$,

$$\left\| \frac{p^{(2)}(\cdot,t)}{\|p^{(2)}(\cdot,t)\|_\omega} - \frac{\exp(-r_0 a)\,L(a)\psi}{\|\exp(-r_0 a)\,L(a)\psi\|_\omega} \right\|_\omega \to 0 \qquad \text{as } t \to \infty,$$

which shows that $\exp(-r_0 a)L(a)\psi/\|\exp(-r_0 a)L(a)\psi\|_\omega$ is the stable age distribution of the population on $[0,\omega]$.

THE CONSISTENCY PROBLEM

Finally, we consider the consistency problem. Lotka [24, 25] stated the following theorem regarding the stable age distribution of a closed population.

THEOREM (LOTKA [25])

A closed population that is increasing at a constant rate r per head under the regime of a constant age schedule of mortality and fertility can have no other than a stable age distribution.

This theorem was first stated loosely in [24], and Lotka attempted to give a proof in [25]. However, these papers were ignored until the Samuelson's paper [29] appeared. Although Lotka's proof was incomplete, Samuelson reformulated Lotka's theorem carefully and gave a proof.

Using the theory of multiplicative processes, we can go far beyond the Lotka-Samuelson theorem. Suppose that the population $p(a,t)$ is evolved by the time-homogeneous multiplicative process $T(h)$, $h \geqslant 0$, for infinite negative time:

$$p(a,t) = (T(h)p(\cdot,t-h))(a); \qquad h \geqslant 0, \quad -\infty < t \leqslant t_0,$$

which means that the population is under the regime of a constant schedule of mortality and fertility. Then Proposition 3.9 tells us that if $T(h)$ is uniformly primitive, then the population $p(a,t)$ has a stable age distribution for all time $t \in (-\infty, t_0]$. In contrast to the Lotka-Samuelson theorem, we need not assume that the growth rate is constant.

Next, we consider the time-inhomogeneous case. Suppose that time-series data of the vital rates are given for infinite negative time and the population is governed by a time-inhomogeneous multiplicative process $U(t,s)$, $t \geqslant s$, defined by the vital rates. From Proposition 3.4, we know that if $U(t,s)$ is uniformly primitive, then there exists one and essentially only one time series of the age distributions consistent with the multiplicative process. Thus we can say that an infinite history of the vital rates uniquely determines the history of age structures of the population governed by the uniformly primitive population evolution operator. From a practical standpoint, it is important that if we have information about data of vital rates for a sufficiently long period of past times, the current age structure of the population can be almost completely explained.

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