

On the existence and characterization of the maxent distribution under general moment inequality constraints

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Abstract—A broad set of sufficient conditions that guarantees the existence of the maximum entropy (maxent) distribution consistent with specified bounds on certain generalized moments is derived. Most results in the literature are either focused on the minimum cross-entropy distribution or apply only to distributions with a bounded-volume support or address only equality constraints. The results of this work hold for general moment inequality constraints for probability distributions with possibly unbounded support, and the technical conditions are explicitly on the underlying generalized moment functions. An analytical characterization of the maxent distribution is also derived using results from the theory of constrained optimization in infinite-dimensional normed linear spaces. Several auxiliary results of independent interest pertaining to certain properties of convex coercive functions are also presented.

Keywords: Coercive functions, Constrained optimization, Convex analysis, Cross-entropy, Differential entropy, Maximum entropy methods.

I. INTRODUCTION

Consider the problem of estimating a signal from “noisy” observations when we have complete information about the statistics of the observation process but only partial prior (statistical) information about the signal of interest. Partial prior information about the signal probability distribution might be available in the form of bounds on a restricted set of certain general moment measurements. Incompleteness in the prior information is with regard to the underlying signal probability distribution that is consistent with the measurements. There arises the question of selecting a distribution from the feasible ones that is noncommittal with respect to missing information. The maxent principle provides a selection mechanism that enjoys several appealing optimality properties [1]–[7].

Questions of existence and characterization of the maxent distribution in a collection of probability distributions over a finite-dimensional Euclidean space are, in general, problems in infinite dimensional constrained optimization involving several subtleties, and many derivations in the literature contain

errors¹. Although the form of the maxent distribution subject to general moment *equality* constraints has been known for long, there has been little systematic investigation into its validity and the existence of the maxent distribution. Most results in the literature are either focused on the minimum cross-entropy distribution or apply only to distributions with a bounded-volume support. A key difficulty in extending such existence and characterization results from cross-entropy to differential entropy is that unlike cross-entropy which is always well-defined, nonnegative, and satisfies a joint lower semi-continuity property, differential entropy is not always well-defined and lacks a crucial upper-semicontinuity property that is needed for establishing existence results along the lines of those for cross-entropy.

Building upon results due to Csiszár and Topsøe [1], [9], we provide broad sufficient conditions on *general convex families* of distributions that guarantee the *existence* of the maxent distribution in the family. We also specialize these existence results to specific convex families of probability distributions defined through general moment inequality constraints. We also provide an analytical *characterization* of the maxent distribution for such general moment-constrained families. Our existence and characterization results hold for probability densities over a finite-dimensional Euclidean space, that is, finite-dimensional probability distributions that are absolutely continuous with respect to the Lebesgue measure, although they can be extended to general finite-dimensional sigma-finite measures also. For results pertaining to specific convex families of distributions defined through general moment inequality constraints, a finite number of constraints is assumed although the results can be extended when there are a countable number of constraints. Our results apply for both differential entropy and I-divergence although we state and prove results only for differential entropy.

Existence and characterization results for a family of compactly supported probability densities on the real line with a prescribed mean and variance (moment equality constraints) are presented in [10]. The analysis in [9] is exclusively devoted to I-divergence (which requires a reference measure) and not differential entropy and the existence results were stated only in terms of the convexity and variational completeness of the feasible set of distributions. Unlike the results in [9] which are in terms of general conditions on the convex collections

¹See Borwein and Limber [8] for references to these nonrigorous derivations.

of distributions satisfying general moment constraints with *equality*, which might be difficult to check in practice, our results are for general moment *inequality* constraints, and the technical conditions are *explicitly* on the underlying moment functions². The results presented in [1] hold for probability distributions over a *countable space* and the existence results therein pertain to the *center of attraction* of a convex collection of distributions. The relationship between the center of attraction of a family of densities defined via moment *equality* constraints³ and the maximum–likelihood estimate in an associated exponential family of densities is derived in [11]. Borwein and Limber in [8] also provide a set of sufficient conditions for the existence of the maxent distributions and characterize its form but these results differ from ours in several aspects. Their results are for equality constraints, ours are for inequality constraints. The underlying space in their analysis is the real line, our analysis is on \mathbb{R}^d . Their analysis considered distributions with bounded support. Our analysis allows distributions with unbounded support.

For a collection of distributions satisfying moment–equality constraints, the maxent distribution, when it exists, has an exponential form where the exponent belongs to the closed subspace spanned by the measurement functions [8]. The additional flexibility allowed by inequality constraints leads to a stronger characterization of the maxent distribution. We show, not surprisingly, that under moment inequality constraints and mild regularity assumptions, the maxent distribution has an exponential form where the exponent belongs to the negative cone generated by the measurement functions. In many applications, inequality constraints are perhaps more commonly encountered than equality constraints. With equality constraints, it is often difficult to verify the existence of a maxent solution because of possible errors in the estimated moments. The conditions of our existence and characterization theorem are application–oriented in the sense that if the measurement functions meet certain general requirements, the maxent solution exists and has a special exponential form. We have learnt (thanks to an anonymous reviewer) about another work by Csiszár which addresses inequality constraints [12]. However, those results are for the minimum cross–entropy problem and it is not clear how they could be extended to the maxent problem especially when the support set has unbounded volume – an important consideration in our work. Other general references where inequality constraints have been considered include [13, Section 13.1.4] and [14].

We provide two sets of sufficient conditions on the underlying constraint functions that guarantee the existence of the maxent distribution. In one set of sufficient conditions, the proof hinges on the assumption that the distributions of interest have supports that are contained in a finite volume subset of \mathbb{R}^d that need not be bounded. The second set of sufficient conditions removes this restriction by assuming the presence of a general “stabilizing” moment constraint in the definition of the feasible collection of distributions. We also

²We use the terms moment function and measurement function interchangeably.

³The moments were with respect to a σ –finite reference measure over a general measurable space.

present a rich class of “well–behaved” functions that provide the general “stabilizing” moment constraints guaranteeing the existence of the maxent distribution. Frequently encountered constraints such as mean quadratic energy and mean absolute energy are well–behaved. These well–behaved constraints have several interesting and intuitively appealing properties that are of independent interest.

In Section II we provide some background, define all important terms, and state the maxent problem. In Section III we state the main results of this work – fundamental theorems on the existence and characterization of the maxent distribution consistent with specified moment inequality constraints. Proofs of these theorems and related results of independent interest are presented in the appendices.

II. BACKGROUND AND PROBLEM STATEMENT

Notation: \mathbb{R} denotes the set of real numbers,

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$$

the set of extended real numbers, and \mathbb{R}^d the d –dimensional real Euclidean space. Vectors are denoted by boldface letters, for example, $\mathbf{x} \in \mathbb{R}^d$, and finite dimensional vectors are treated as column vectors. All sets in this work are Lebesgue–measurable. If A and B are Lebesgue–measurable subsets of \mathbb{R}^d , then the statement $A = B$ means that the set of points not simultaneously in both A and B has Lebesgue measure zero and A is said to be equal to B almost everywhere (a.e.). All functions in this work take values in $\overline{\mathbb{R}}$ and are measurable with respect to the Lebesgue measure over \mathbb{R}^d . Inequalities involving measurable functions are to be understood in the a.e. sense. All integrals are in the sense of Lebesgue. A probability density function (pdf) is a measurable function $\pi(\mathbf{x})$ on \mathbb{R}^d that is non–negative almost everywhere (a.e.) and integrates to unity over \mathbb{R}^d . All results in this work are stated for probability densities over finite–dimensional Euclidean spaces, that is, probability distributions that are absolutely continuous with respect to the Lebesgue measure, although they can be extended to general sigma–finite measures on \mathbb{R}^d also. $\mathcal{L}^1(\mathbb{R}^d)$ and $\mathcal{L}^\infty(\mathbb{R}^d)$ respectively denote the set of absolutely–integrable functions over \mathbb{R}^d and the set of essentially bounded functions [15, p. 119] over \mathbb{R}^d . For convenience, we shall often omit the ‘ \mathbf{x} ’ and the ‘ $d\mathbf{x}$ ’ that appear inside an integral. Thus,

$$\int_A f(\mathbf{x})d\mathbf{x}$$

will often be abbreviated to $\int_A f(\mathbf{x})$ or simply $\int_A f$. The symbol π and its variants will denote pdfs and

$$\mathbb{E}_\pi[\phi] := \int_{\mathbb{R}^d} \phi \cdot \pi$$

denotes the mathematical expectation of the function $\phi(\mathbf{x})$ under the pdf $\pi(\mathbf{x})$. The support of a function $f(\mathbf{x})$ is the set of points where it is nonzero⁴ and is denoted by $\text{supp}(f)$. The indicator or characteristic function of a subset A of \mathbb{R}^d denoted by $\mathbf{1}_A(\mathbf{x})$ is the function that is equal to one over A and zero elsewhere. The volume of a Lebesgue–measurable subset

⁴Note that we are working with probability density functions.

S of \mathbb{R}^d is its Lebesgue measure and is denoted by $|S|$. In addition to the arithmetic of the extended reals, the following conventions regarding infinity are adopted in keeping with measure–theoretically consistent operations:

$$\ln 0 = -\infty, \quad \ln \frac{a}{0} = +\infty, \quad \forall a > 0, \quad 0 \cdot (\pm\infty) = 0.$$

Thus $0 \ln 0 = 0$ which also agrees with the limiting value of the quantity $t \ln t$ as the variable t decreases to zero.

In Bayesian inference, signals of interest are modeled as high–dimensional real random vectors with associated pdfs referred to as prior distributions on the signals. Let $\mathbf{X} \in \mathbb{R}^d$ have an underlying d –dimensional pdf denoted by $\pi(\mathbf{x})$. In many applications, only limited information about $\pi(\mathbf{x})$ can be gathered. Moments of probability distributions are often used to describe the underlying statistical structure of a stochastic process. For example, the set of all finite–order moments of a scalar random variable provides, under suitable regularity assumptions, a complete statistical description of the random variable [16, Theorem 30.1, p. 388]. In practice, only a finite set of moments is a priori known or can be estimated (measured) from samples. In many cases even these are not available but bounds on the moments are available. The bounds may be regarded as arising from the impreciseness of moment measurements. For example, for $p > 0$, the empirical mean ℓ^p energies of wavelet coefficients in different subbands are often used to construct statistical models for images [17]–[19]. In general, the limited information will be unable to single out a desirable distribution that is consistent with the moment constraints. The limited information would rather specify a whole class of distributions that satisfy the moment constraints.

Let prior information about a random vector \mathbf{X} be available in terms of upper bounds on the expected values of certain real–valued Lebesgue–measurable (measurement) functions

$$\phi_\gamma : \mathbb{R}^d \rightarrow \mathbb{R}, \quad \gamma \in \Gamma,$$

where Γ is a finite index set⁵. A useful notion is that we can sometimes design these functions $\phi_\gamma(\mathbf{x})$ (that is, the measurements). Each candidate distribution $\pi(\mathbf{x})$ that is consistent with these measurements then belongs to the set

$$\Omega(\mathbf{u}) := \{\text{pdf } \pi : \text{supp}(\pi) \subseteq S, \text{ and for all } \gamma \text{ in } \Gamma, \mathbb{E}_\pi[\phi_\gamma] \leq u_\gamma < +\infty\}, \quad (2.1)$$

where S is a closed Lebesgue–measurable subset of \mathbb{R}^d having nonzero but possibly infinite volume and

$$\mathbf{u} := \{u_\gamma \in \mathbb{R}\}_{\gamma \in \Gamma}$$

is a finite–dimensional, real–valued, vector of moment upper–bounds. We assume that the only prior information available is expressed by the moment constraints of Ω . Since Ω is defined through inequality constraints that are linear in π , *it is a convex set of probability distributions*. It is possible to implicitly incorporate support constraints into Ω through

⁵The focus of this work is on the case when the number of measurement functions is finite but the results can also be extended to the case when there are a countable number of measurement functions.

appropriate moment inequalities without explicitly requiring that $\text{supp}(\pi) \subseteq S$ in the definition. For example, if

$$u_0 = u_1 = 0,$$

and

$$\phi_0(\mathbf{x}) := -\phi_1(\mathbf{x}) := 1 - \mathbf{1}_S(\mathbf{x})$$

then for each π belonging to Ω , we have $|\text{supp}(\pi) \setminus S| = 0$. For clarity of exposition we shall primarily work with the convex collection (2.1). However, it is quite straightforward to extend our results to convex collections having individual lowerbounds $\{l_\gamma \in \mathbb{R}\}_{\gamma \in \Gamma}$ on the moment measurements.

In general, many distributions will satisfy the moment constraints of Ω . The choice of a distribution from this moment consistent class depends upon the goals to be achieved by the selection. For the application of lossless compression, a clear answer can be given. The unique pdf that maximizes the differential entropy functional

$$h(\pi) := -\mathbb{E}_\pi[\ln \pi]$$

over a convex set F , *whenever it exists*, also minimizes the worst–case rate for encoding repeated independent observations of \mathbf{X} “losslessly” [20, pp. 105–106], [7, pp. 61–63], [1, Theorem 3, p. 16] (The results in [1], [7] are for discrete entropy). A similar result holds for high–rate lossy compression [6].

Definition 2.1: (Maximum entropy distribution) Let F be a convex collection of distributions for which

$$F \cap \{\text{pdf } \pi : h(\pi) > -\infty\}$$

is nonempty. The maxent distribution in F whenever it exists is the unique pdf π_{ME} belonging to F satisfying⁶

$$h(\pi_{ME}) = \max_{\pi \in F} h(\pi).$$

It may be noted that since $h(\pi)$ is a concave functional [21], the set

$$\{\text{pdf } \pi : h(\pi) > -\infty\}$$

is convex. The uniqueness of π_{ME} follows from the strict concavity of the differential–entropy functional [21] and the convexity of F .

In addition to being minimax optimal for the application of lossless compression with uncertain source statistics discussed above, the maxent distribution is also “maximally noncommittal” with respect to missing information while satisfying prior constraints [4]. Shore and Johnson in [2] show that if a distribution has to be picked from a class of probability distributions by maximizing a functional satisfying some natural postulates, it must necessarily be the maxent functional. Again, in a study of logically consistent methods of inference, Csiszár demonstrates that the maxent distribution is the only one that satisfies two different intuitively appealing axiom systems [5]. These properties of the maxent distribution make it a desirable choice for signal estimation.

In some applications, based on previous measurements, a reliable reference distribution $r(\mathbf{x})$ for the signal of interest is available. New moment measurements might reveal that the

⁶The subscript ME stands for maximum entropy.

reference distribution has inconsistencies with new information in the form of bounds on moments (2.1). The situation suggests a revision of the reference model while not ignoring earlier measurements. An attractive model selection criterion in this situation is to select the distribution in Ω that is closest to the reference distribution in the sense that it has minimum cross-entropy (MCE) relative to the reference prior:

Definition 2.2: (Cross-entropy [9, p. 146]) The *cross-entropy* of pdf $\pi_1(\mathbf{x})$ with respect to pdf $\pi_2(\mathbf{x})$ (also known as the I-divergence, Kullback–Leibler distance, relative entropy, and information discrimination) denoted by $D(\pi_1||\pi_2)$ is defined as:

$$D(\pi_1||\pi_2) := \begin{cases} \mathbb{E}_{\pi_1}[\ln(\frac{\pi_1}{\pi_2})] & \text{if } \pi_1 \ll \pi_2 \text{ (see Definition A.3)} \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 2.3: (I-projection [9, p. 147]) Let r be a pdf and F a convex collection of priors such that

$$F \cap \{\text{pdf } \pi : D(\pi||r) < +\infty\}$$

is nonempty. The I-projection of r onto F , whenever it exists, is the unique pdf π_{MCE} belonging to F satisfying

$$D(\pi_{MCE}||r) = \min_{\pi \in F} D(\pi||r).$$

The updated distribution π_{MCE} is referred to as the *I-projection* of r onto F . Since $D(\pi||r)$ is strictly convex in π [21], and F is a convex set, π_{MCE} is unique whenever it exists.

Generally speaking, the maxent distribution in Ω (2.1) need not exist. Our goal is to provide a set of sufficient conditions on the measurement functions that guarantee the existence of the maxent prior. We provide such a set of conditions in the following section. We also characterize the form of the maxent prior. Similar existence and characterization results for I-projection under moment *inequality* constraints can be derived along similar lines but are omitted from the present work (see [9], [12], [22], [23]).

III. EXISTENCE AND CHARACTERIZATION OF THE MAXENT DISTRIBUTION

The following theorem proved in Appendix B.1 provides a characterization of the unique maxent distribution in Ω subject to suitable technical conditions.

Theorem 3.1: (Characterization of the maxent distribution) Let $\Omega(\mathbf{u})$ be as in (2.1). Let there exist a pdf π_0 in $\Omega(\mathbf{u})$ such that for all γ in Γ , $\mathbb{E}_{\pi_0}[\phi_\gamma] < u_\gamma$. If the unique maxent pdf π_{MCE} belonging to $\Omega(\mathbf{u})$ exists and $h(\pi_{MCE})$ is finite, then the maxent pdf has the form

$$\pi_{MCE}(\mathbf{x}, \mathbf{u}) = \mathbf{1}_{S_{ME}}(\mathbf{x}) \cdot \exp \left\{ -\alpha(\mathbf{u}) - \sum_{\gamma \in \Gamma} \lambda_\gamma(\mathbf{u}) \phi_\gamma(\mathbf{x}) \right\}, \quad (3.1)$$

where $S_{ME} := \text{supp}(\pi_{MCE}) \subseteq S$ satisfies $\mathbb{E}_\pi[\mathbf{1}_{S \setminus S_{ME}}] = 0$ for every $\pi \in \Omega(\mathbf{u})$ for which $-\infty < h(\pi)$ and

$$\alpha(\mathbf{u}) = \ln \left(\int_{S_{ME}} \exp \left\{ -\sum_{\gamma \in \Gamma} \lambda_\gamma(\mathbf{u}) \phi_\gamma(\mathbf{x}) \right\} d\mathbf{x} \right)$$

is a finite normalization constant. The parameters $\{\lambda_\gamma(\mathbf{u})\}_{\gamma \in \Gamma}$ are all nonnegative, and satisfy

$$\sum_{\gamma \in \Gamma} \lambda_\gamma(\mathbf{u}) (\mathbb{E}_{\pi_{MCE}}[\phi_\gamma] - u_\gamma) = 0. \quad (3.2)$$

Moreover,

$$\begin{aligned} h(\pi_{MCE}) &= \alpha(\mathbf{u}) + \sum_{\gamma \in \Gamma} \lambda_\gamma(\mathbf{u}) \mathbb{E}_{\pi_{MCE}}[\phi_\gamma] \\ &= \alpha(\mathbf{u}) + \sum_{\gamma \in \Gamma} u_\gamma \lambda_\gamma(\mathbf{u}). \end{aligned}$$

Remark 3.1: Note that if π belongs to $\Omega(\mathbf{u})$ and $-\infty < h(\pi)$, then $\pi \ll \pi_{MCE}$. If there exists a pdf π in $\Omega(\mathbf{u})$ with $-\infty < h(\pi)$ and $\text{supp}(\pi) = S$, then the set $S \setminus S_{ME}$ has zero volume; that is, S_{ME} almost everywhere coincides with S and we may take $S_{ME} = S$ in the above theorem.

Remark 3.2: The numbers $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ in Theorem 3.1 above are Lagrange multipliers associated with the moment constraints of $\Omega(\mathbf{u})$ in (2.1). The constraint qualification (3.2) implies that $\lambda_\gamma = 0$ if constraint γ is inactive, that is, $\mathbb{E}_{\pi_{MCE}}[\phi_\gamma] < u_\gamma$.

Remark 3.3: Since S has nonzero volume and π_{MCE} is unique, if the measurement functions $\{\phi_\gamma\}_{\gamma \in \Gamma}$ are linearly independent then there is a unique choice for the parameters $\boldsymbol{\lambda} := \{\lambda_\gamma\}_{\gamma \in \Gamma}$ that satisfies the moment constraints of $\Omega(\mathbf{u})$. In this case, the mapping from the vector of moment bounds \mathbf{u} to the vector of Lagrange multipliers $\boldsymbol{\lambda}$ is a function, that is, it is not a one-to-many map. If the measurement functions are not linearly independent, the characterization theorem still holds, but the Lagrange multipliers need not be unique.

Remark 3.4: The Lagrange multipliers $\boldsymbol{\lambda}(\mathbf{u})$ are usually implicit functions of the moment bounds \mathbf{u} . If for some value of \mathbf{u} a Lagrange multiplier turns out to be zero — that is, $\lambda_\gamma(\mathbf{u}) = 0$ for some $\gamma \in \Gamma$ (a situation that will arise if the associated moment constraint is inactive, that is, $\mathbb{E}_{\pi_{MCE}}[\phi_\gamma] < u_\gamma$) — then the maxent solution corresponding to any larger value of u_γ will remain the same (see Appendix B.2 for a proof). Thus, the map $\boldsymbol{\lambda}(\mathbf{u})$ from moment bounds to Lagrange multipliers is in general not injective. However, see the following remark.

Remark 3.5: The mapping from the moment upper-bounds \mathbf{u} to the Lagrange multipliers $\boldsymbol{\lambda}(\mathbf{u})$ is one-to-one when the domain is restricted to the set of those values of \mathbf{u} for which $\lambda_\gamma(\mathbf{u}) > 0$ for every γ in Γ , that is, *all* the constraints are active. This fact can be seen by the following argument. Suppose that $\{u_\gamma^{(1)}\}_{\gamma \in \Gamma}$ and $\{u_\gamma^{(2)}\}_{\gamma \in \Gamma}$ both map to the same set of strictly positive Lagrange multipliers $\{\lambda_\gamma > 0\}_{\gamma \in \Gamma}$. Then because all constraints are active, due to (3.2), necessarily

$$u_\gamma^{(1)} = \mathbb{E}_{\pi_{MCE}}[\phi_\gamma(\mathbf{x})] = u_\gamma^{(2)}$$

for every γ in Γ .

Theorem 3.1 asserts that whenever the maxent distribution in a moment-consistent class exists then, subject to some mild technical conditions, it has a natural exponential form given by (3.1). The next result proved in Appendix B.3 essentially asserts that if a pdf having the exponential form given by (3.1) is moment consistent then it must be the maxent distribution

for the moment-consistent class. In this sense, the next result is a converse to Theorem 3.1.

Theorem 3.2: (Converse to the characterization theorem) Let $\Omega(\mathbf{u})$ be as in (2.1). Consider a pdf

$$\pi_{\text{exp}}(\mathbf{x}, \boldsymbol{\lambda}) := \mathbf{1}_{S_{\text{exp}}}(\mathbf{x}) \cdot \exp \left\{ -\alpha - \sum_{\gamma \in \Gamma} \lambda_{\gamma}(\mathbf{u}) \phi_{\gamma}(\mathbf{x}) \right\},$$

where S_{exp} is a measurable subset of S , and the vector of nonnegative but finite-valued parameters $\{\lambda_{\gamma}(\mathbf{u})\}_{\gamma \in \Gamma}$ is denoted by $\boldsymbol{\lambda}$. If

- (i) π_{exp} belongs to $\Omega(\mathbf{u})$,
- (ii) $\mathbb{E}_{\pi} [\mathbf{1}_{S \setminus S_{\text{exp}}}] = 0$ for every $\pi \in \Omega(\mathbf{u})$ for which $-\infty < h(\pi)$, and
- (iii)

$$\sum_{\gamma \in \Gamma} \lambda_{\gamma} (\mathbb{E}_{\pi_{\text{exp}}} [\phi_{\gamma}] - u_{\gamma}) = 0,$$

then π_{exp} is the unique maxent pdf in $\Omega(\mathbf{u})$ and

$$h(\pi_{\text{exp}}) = \alpha + \sum_{\gamma \in \Gamma} \lambda_{\gamma}(\mathbf{u}) \mathbb{E}_{\pi_{\text{exp}}} [\phi_{\gamma}]$$

is finite.

Before entering into sufficient conditions for the existence of the maxent distribution, we would like to briefly comment on some practical aspects of computing the Lagrange multipliers from given moment constraints. The infinite-dimensional constrained entropy maximization problem can be converted to a finite-dimensional convex minimization problem by invoking Lagrange duality theory [20, pp. 21–24]. This forms the basis for developing numerical techniques for computing the Lagrange multipliers that characterize the maxent distribution. Several algorithms based on iterative gradient-projection or moment-matching procedures having different convergence properties have been proposed in the literature, for example, Bregman’s balancing method, multiplicative algebraic reconstruction technique, generalized iterative scaling method, Newton’s method, interior-point methods, etc. [24]. However, these algorithms have been largely applied to problems where the underlying space is a finite set and require evaluating moments at each step. This task can be nontrivial if the underlying space is \mathbb{R}^d and d is large, as in the case of images, because moment computation will involve evaluating very high dimensional integrals. One would typically need to take recourse to computationally intensive algorithms like importance sampling or Markov-chain Monte-Carlo for numerically evaluating the high-dimensional integrals at each step. However, in certain situations it might be possible to take advantage of the structure of the specific moment functions to develop fast heuristic approximations for the Lagrange multipliers [20, Chapter 4], [17]–[19].

Theorem 3.3: (Existence of the maxent distribution – finite volume support constraint) Let S be a closed, Lebesgue-measurable subset of \mathbb{R}^d having nonzero but finite volume. If F is a nonempty, convex, \mathcal{L}^1 -complete collection of pdfs over S and $-\infty < h(\pi_0)$ for at least one pdf π_0 belonging to F , then

$$h(F) := \sup_{\pi \in F} h(\pi) \in \mathbb{R},$$

that is, $h(F)$ is finite, and there exists a unique maxent pdf in F .

Corollary 3.4: Let $\Omega(\mathbf{u})$ be as in (2.1). Let $\{\phi_{\gamma}\}_{\gamma \in \Gamma}$ be uniformly bounded from below by $L \in \mathbb{R}$ and S have nonzero but finite volume. If Ω is nonempty and

$$C := \bigcap_{\gamma \in \Gamma} \{\mathbf{x} \in S : \phi_{\gamma}(\mathbf{x}) \leq u_{\gamma}\}$$

has nonzero volume then there exists a unique maxent pdf π_{ME} in $\Omega(\mathbf{u})$ having the exponential form given by Theorem 3.1 with $h(\pi_{ME}) \in \mathbb{R}$.

The proof of Theorem 3.3 appears in Appendix C.1. The proof of Corollary 3.4 appears in Appendix C.2. While the finite measure condition is crucial to the proof of Theorem 3.3 and Corollary 3.4, the next theorem and corollary show that the existence of the maxent distribution is guaranteed by the presence of a “stabilizing” constraint function in the definition of Ω even if the support set’s volume is not finite. The proofs of these results appear in Appendix C.3 and Appendix C.4 respectively. We would like to point out that the sufficient conditions for existence mentioned in [9] and the corollary following Theorem 5.2 in [25] for the cross-entropy problem is not available for differential entropy unless attention is restricted to distributions supported on a set of finite Lebesgue measure due to the lack of a general upper-semicontinuity property for differential entropy. It is not immediately clear how those results can be extended to distributions having an infinite-volume support.

Definition 3.1: (Stable function) A real-valued measurable function $f(\mathbf{x})$ is stable if $\exp\{-\lambda f(\mathbf{x})\}$ belongs to $\mathcal{L}^1(\mathbb{R}^d)$ for all $\lambda > 0$.

Remark 3.6: If $f(\mathbf{x})$ is stable so is $\lambda f(\mathbf{x})$ for all $\lambda \in (0, +\infty)$.

Theorem 3.5: (Existence of the maxent distribution – stabilizing constraint) Let S be a closed, Lebesgue-measurable subset of \mathbb{R}^d having nonzero but possibly infinite volume and F be a nonempty, convex, \mathcal{L}^1 -complete collection of pdfs over S . If

- (i) there exists a π_0 in F such that $-\infty < h(\pi_0)$ and
- (ii) there exist finite reals L, u , with $L \leq u$, and a stable function ψ such that for all π in F , $L \leq \mathbb{E}_{\pi}[\psi] \leq u$,

then $h(F) := (\sup_{\pi \in F} h(\pi)) \in \mathbb{R}$, that is, $h(F)$ is finite, and there exists a unique maxent pdf in F .

Corollary 3.6: Let $\Omega(\mathbf{u})$ be as in (2.1). Let $\{\phi_{\gamma}\}_{\gamma \in \Gamma}$ be uniformly bounded from below by $L \in \mathbb{R}$ and S have nonzero (but possibly infinite) volume. If Ω is nonempty and

- 1) $C := \bigcap_{\gamma \in \Gamma} \{\mathbf{x} \in S : \phi_{\gamma}(\mathbf{x}) \leq u_{\gamma}\}$ has nonzero volume,
- 2) there exists $\gamma_0 \in \Gamma$ for which $u_{\gamma_0} \in \mathbb{R}$ and ϕ_{γ_0} is stable,

then there exists a unique maxent pdf $\pi_{ME} \in \Omega(\mathbf{u})$ having the exponential form given by Theorem 3.1 with $h(\pi_{ME}) \in \mathbb{R}$.

Remark 3.7: In Corollaries 3.4 and 3.6, the condition that the measurement functions $\{\phi_{\gamma}\}_{\gamma \in \Gamma}$ be uniformly bounded from below by $L \in \mathbb{R}$ is sufficient to ensure that $\Omega(\mathbf{u})$ is complete under the $\mathcal{L}^1(\mathbb{R}^d)$ norm (see Proposition C.1 in Appendix C). The condition that

$$C := \bigcap_{\gamma \in \Gamma} \{\mathbf{x} \in S : \phi_{\gamma}(\mathbf{x}) \leq u_{\gamma}\}$$

has nonzero volume is a sufficient condition to ensure that there is at least one pdf π_0 with $-\infty < h(\pi_0)$.

In conclusion, we demonstrate a rich class of “well-behaved” constraint functions for which condition (2) in Corollary 3.6 is satisfied. The main result here is Theorem 3.7 whose proof appears in Appendix D.

Definition 3.2: (Omni-directional unboundedness) A real-valued function on a vector space is *asymptotically positive and unbounded in all directions* if $f(\mathbf{z}) \rightarrow +\infty$ whenever $\|\mathbf{z}\| \rightarrow \infty$. For simplicity we shall refer to this as the *omni-directional unboundedness* property (which is also sometimes referred to as the coercive property [26, Definition A.4(c), p. 653]).

Remark 3.8: In a finite-dimensional Banach space such as \mathbb{R}^d , all norms are equivalent [27, Theorem 23.6, p. 177]. In other words, if $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms, there are positive constants $L > 0$ and $U > 0$ such that

$$L\|\mathbf{x}\|_a \leq \|\mathbf{x}\|_b \leq U\|\mathbf{x}\|_a$$

for all \mathbf{x} in the finite-dimensional Banach space. Thus in \mathbb{R}^d ,

$$\|\mathbf{x}\|_a \rightarrow \infty \iff \|\mathbf{x}\|_b \rightarrow \infty.$$

The definition of omnidirectional unboundedness therefore does not depend upon the specific norm used when the underlying space is finite dimensional.

Definition 3.3: (Well-behaved function) Let $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex and omni-directionally unbounded function. A real-valued function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is well-behaved if there exists a nonnegative real number M such that

$$\begin{aligned} \phi(\mathbf{x}) &\leq \psi(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| > M \text{ and} \\ \sup_{\|\mathbf{x}\| \leq M} |\psi(\mathbf{x})| &< +\infty. \end{aligned}$$

Remark 3.9: A convex and omni-directionally unbounded function is well-behaved. If $f(\mathbf{x})$ is well-behaved so is $\lambda f(\mathbf{x})$ for all λ belonging to the open interval $(0, +\infty)$.

Theorem 3.7: A well-behaved function is stable. If ϕ_{γ_0} is well-behaved and

$$\mathbb{E}_{\pi}[\phi_{\gamma_0}] \leq u_{\gamma_0} < +\infty$$

then $h(\pi)$ exists and

$$h(\pi) \leq u_{\gamma_0} + \ln \|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1} < \infty.$$

Hence, if ϕ_{γ_0} belongs to $\{\phi_{\gamma}\}_{\gamma \in \Gamma}$ in Corollary 3.6 then

$$\sup_{\pi \in \Omega(\mathbf{u})} h(\pi) \leq u_{\gamma_0} + \ln \|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1} < +\infty.$$

Remark 3.10: Suppose that in Corollary 3.6, none of the measurement functions $\{\phi_{\gamma}\}_{\gamma \in \Gamma}$ is well-behaved, but some nonnegative linear combination of the measurement functions

$$\phi_{\boldsymbol{\mu}} := \sum_{\gamma \in \Gamma} \mu_{\gamma} \phi_{\gamma}, \text{ where } 0 \leq \mu_{\gamma} < +\infty \text{ for all } \gamma \in \Gamma,$$

is well-behaved. Let $u_{\boldsymbol{\mu}} := \sum_{\gamma \in \Gamma} \mu_{\gamma} u_{\gamma}$ and

$$\Omega_{\boldsymbol{\mu}} := \{\text{pdf } \pi : \mathbb{E}_{\pi}[\phi_{\boldsymbol{\mu}}] \leq u_{\boldsymbol{\mu}}\}.$$

It is clear that $\Omega(\mathbf{u}) \subseteq \Omega_{\boldsymbol{\mu}}$. Hence, the well-behaved function $\phi_{\boldsymbol{\mu}}$ and the associated moment constraint

$$-\infty < L \leq \mathbb{E}_{\pi}[\phi_{\boldsymbol{\mu}}] \leq u_{\boldsymbol{\mu}}$$

can be included in the set of available moment measurements without affecting the maxent solution. Although this new constraint is redundant, it tells us that Theorem 3.7 can be applied and the maxent distribution in Ω exists under the mild requirements of Corollary 3.6.

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APPENDIX A PRELIMINARIES

Definition A.1: (Convex set) A subset C of a vector space is said to be *convex* if whenever \mathbf{z}_1 and \mathbf{z}_2 are in C , so is $\alpha \mathbf{z}_1 + (1 - \alpha) \mathbf{z}_2$ for every α in the closed interval $[0, 1]$.

Definition A.2: (Convex function) Let V be a vector space. A functional $f : V \rightarrow \overline{\mathbb{R}}$ is said to be convex if for every $\alpha \in [0, 1]$, and for any \mathbf{z}_1 and \mathbf{z}_2 belonging to V ,

$$f(\alpha \mathbf{z}_1 + (1 - \alpha) \mathbf{z}_2) \leq \alpha f(\mathbf{z}_1) + (1 - \alpha) f(\mathbf{z}_2).$$

If equality holds only when $\mathbf{z}_1 = \mathbf{z}_2$ then f is said to be *strictly convex*. If $-f$ is (strictly) convex then f is said to be (strictly) *concave*.

Definition A.3: (Absolute continuity) A pdf π_1 is said to be absolutely continuous relative to a pdf π_2 , in symbols $\pi_1 \ll \pi_2$ or $\pi_2 \gg \pi_1$, if for every Lebesgue measurable subset A of \mathbb{R}^d , $\int_A \pi_2 = 0$ implies $\int_A \pi_1 = 0$ and hence $\text{supp}(\pi_1) \subseteq \text{supp}(\pi_2)$.

Fact A.1: [28, p. 5] The cross-entropy of pdf π_1 relative to pdf π_2 is always well defined and non-negative (it could be $+\infty$). The cross-entropy is zero if and only if $\pi_1 = \pi_2$ almost everywhere.

Fact A.2: [21] Differential entropy $h(\pi)$ is strictly concave in π . Cross-entropy $D(\pi_1 || \pi_2)$ is convex in the pair (π_1, π_2) and strictly convex in π_1 .

Fact A.3: (Joint lower semi-continuity of cross-entropy [29, Section 2.4, Assertion 5]) If the pdfs p_n and q_n converge in $\mathcal{L}^1(\mathbb{R}^d)$ norm to pdfs p and q respectively as $n \rightarrow \infty$, then

$$D(p||q) \leq \liminf_{n \rightarrow \infty} D(p_n||q_n). \quad (\text{A.1})$$

Fact A.4: [20, p.88], [11], [1, Theorem 1, p. 14]: If $F \subseteq \mathcal{L}^1(\mathbb{R}^d)$ is a complete, convex collection of pdfs and $h(F) := \sup_{\pi \in F} h(\pi)$ is finite, then there exists a unique distribution π^* belonging to F such that for every sequence $\{\pi_n\} \subseteq F$ for which $h(\pi_n) \rightarrow h(F)$, we have $\pi_n \rightarrow \pi^*$ in $\mathcal{L}^1(\mathbb{R}^d)$ norm.

Fact A.5: (A fundamental theorem of convex optimization [30, adapted from Theorem 1, p. 217]) Let V be a vector space and F a convex subset of V . Let $f : F \rightarrow \overline{\mathbb{R}}$ be a convex functional on F and $\{g_{\gamma}\}_{\gamma \in \Gamma}$ a finite collection of convex

mappings from F into $\overline{\mathbb{R}}$. Suppose that there exists a point \mathbf{v}_0 in F such that for all $\gamma \in \Gamma$, $g_\gamma(\mathbf{v}_0) < 0$ and

$$m_0 := \inf_{\mathbf{v} \in G} f(\mathbf{v}) \quad (\text{A.2})$$

is finite where

$$G := \{\mathbf{v} \in F : g_\gamma(\mathbf{v}) \leq 0, \forall \gamma \in \Gamma\}.$$

Then there exist nonnegative Lagrange multipliers $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ such that

$$m_0 = \inf_{\mathbf{v} \in F} \{f(\mathbf{v}) + \sum_{\gamma \in \Gamma} \lambda_\gamma g_\gamma(\mathbf{v})\}. \quad (\text{A.3})$$

Furthermore, if the infimum is achieved in (A.2) by \mathbf{v}^* belonging to G , it is also achieved by \mathbf{v}^* in (A.3) and

$$\sum_{\gamma \in \Gamma} \lambda_\gamma g_\gamma(\mathbf{v}^*) = 0. \quad (\text{A.4})$$

APPENDIX B

CHARACTERIZATION OF THE MAXENT DISTRIBUTION

1) *Proof of Theorem 3.1:* We shall apply Fact A.5 with $V = \mathcal{L}^1(\mathbb{R}^d)$,

$$F = \{\text{pdf } \pi : \text{supp}(\pi) \subseteq S\},$$

$f(\pi) := -h(\pi)$, $\mathbf{v}_0 := \pi_0$, and

$$g_\gamma(\pi) := \mathbb{E}_\pi[\phi_\gamma] - u_\gamma$$

for each γ in Γ . Clearly, V is a vector space and F is a convex subset of V . Since $h(\pi)$ is a concave functional, $f(\pi)$ is a convex functional on F . Also, $\{g_\gamma(\pi)\}_{\gamma \in \Gamma}$ is a finite collection of linear (hence convex) functionals on F . By assumption, π_0 belongs to F and for each γ in Γ , $g_\gamma(\pi_0) < 0$. Therefore F is nonempty. The infimum m_0 in Fact A.5 is attained at $\mathbf{v}^* = \pi_{ME}$ and is equal to $h(\pi_{ME})$ which is finite, that is, $m_0 = h(\pi_{ME}) \in \mathbb{R}$. Hence $\pi_{ME} \cdot \ln \pi_{ME}$ is absolutely integrable on \mathbb{R}^d . We have now verified that the conditions of Fact A.5 are fulfilled and as a consequence, we are guaranteed the existence of nonnegative reals $\{\lambda_\gamma\}_{\gamma \in \Gamma}$ so that (A.3) and (A.4) hold, that is,

$$-h(\pi_{ME}) = \min_{\pi \in F} \left[-h(\pi) + \sum_{\gamma \in \Gamma} \lambda_\gamma [\mathbb{E}_\pi[\phi_\gamma] - u_\gamma] \right], \quad (\text{B.1})$$

$$\sum_{\gamma \in \Gamma} \lambda_\gamma [\mathbb{E}_{\pi_{ME}}[\phi_\gamma] - u_\gamma] = 0. \quad (\text{B.2})$$

The last condition is equation (3.2) in Theorem 3.1. Consider perturbations around the minimizer π_{ME} of the form

$$\pi_\theta := \pi_{ME} \cdot (1 + \theta \cdot q)$$

where $\theta \in [0, 1]$,

$$q \in \mathcal{L}^\infty(\mathbb{R}^d), \|q\|_\infty \leq 1, \text{ and } \mathbb{E}_{\pi_{ME}}[q] = 0. \quad (\text{B.3})$$

It can be verified that $\pi_\theta \geq 0$, $\|\pi_\theta\|_{\mathcal{L}^1} = 1$, and $\text{supp}(\pi_\theta) \subseteq S$, that is, π_θ is a pdf with support contained in S for every $\theta \in [0, 1]$. This ensures that the θ -perturbations of π_{ME} along

q lie inside F . In view of (B.2), for all $\gamma \in \Gamma$ for which $\lambda_\gamma > 0$, we must have

$$\mathbb{E}_{\pi_{ME}}[\phi_\gamma] = u_\gamma \in \mathbb{R}$$

which implies that

$$\mathbb{E}_{\pi_{ME}}|\phi_\gamma| < \infty.$$

Hence,

$$\mathbb{E}_{\pi_\theta}|\phi_\gamma| \leq (1 + \theta) \cdot \mathbb{E}_{\pi_{ME}}|\phi_\gamma| < \infty \quad (\text{B.4})$$

since $|1 + \theta q| \leq 1 + \theta$. Furthermore,

$$0 \leq (1 - \theta) \leq 1 + \theta \cdot q \leq 1 + \theta$$

implies that for all $\mathbf{x} \in \mathbb{R}^d$ and for all $\theta \in [0, 1]$,

$$|\ln(1 + \theta \cdot q(\mathbf{x}))| \leq A_\theta := \ln \left(\max \left((1 + \theta), \frac{1}{(1 - \theta)} \right) \right) < \infty.$$

It follows that

$$\begin{aligned} |h(\pi_\theta)| &\leq (1 + \theta) \cdot \mathbb{E}_{\pi_{ME}}|\ln \pi_{ME}| + \\ &+ A_\theta < \infty, \forall \theta \in [0, 1]. \end{aligned} \quad (\text{B.5})$$

This shows that $\pi_\theta \cdot \ln \pi_\theta$ is also absolutely integrable for all θ in $[0, 1]$. In view of (B.1), (B.4), (B.5), and the fact that Γ is a finite index set,

$$\begin{aligned} -\infty &< -h(\pi_{ME}) + \sum_{\gamma \in \Gamma} \lambda_\gamma [\mathbb{E}_{\pi_{ME}}[\phi_\gamma] - u_\gamma] \leq \\ &\leq -h(\pi_\theta) + \sum_{\gamma \in \Gamma} \lambda_\gamma [\mathbb{E}_{\pi_\theta}[\phi_\gamma] - u_\gamma] < +\infty. \end{aligned}$$

Collecting terms together and using (B.2) we arrive at:

$$\begin{aligned} 0 &\leq \theta \cdot \mathbb{E}_{\pi_{ME}}[q \cdot \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma] + \\ &+ \int_S (\pi_\theta \cdot \ln \pi_\theta - \pi_{ME} \cdot \ln \pi_{ME}) < +\infty. \end{aligned}$$

Thus for all $\theta \in (0, 1)$ we have

$$\begin{aligned} 0 &\leq \mathbb{E}_{\pi_{ME}}[q \cdot \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma] + \\ &+ \int_S \frac{(\pi_\theta \ln \pi_\theta - \pi_{ME} \ln \pi_{ME})}{\theta} < +\infty. \end{aligned} \quad (\text{B.6})$$

The function

$$\tau_\theta := \frac{(\pi_\theta \cdot \ln \pi_\theta - \pi_{ME} \cdot \ln \pi_{ME})}{\theta}$$

is integrable for each θ in $(0, 1)$ and is nondecreasing in θ for each \mathbf{x} in S . Furthermore,

$$\tau_{0+} := \lim_{\theta \downarrow 0} \tau_\theta = q \cdot \pi_{ME} \cdot (1 + \ln \pi_{ME})$$

is also integrable. The monotone convergence theorem [15, p. 87] applied to $(\tau_\theta - \tau_{0+})$ shows that

$$\int_S \tau_{0+} = \lim_{\theta \downarrow 0} \int_S \tau_\theta.$$

From (B.6) one therefore obtains:

$$\begin{aligned} 0 &\leq \int_S q \cdot \pi_{ME} \cdot (1 + \ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma) < +\infty \\ &= \int_S q \cdot \pi_{ME} \cdot (\ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma) < +\infty, \quad (\text{B.7}) \end{aligned}$$

since $\mathbb{E}_{\pi_{ME}}[q] = 0$ from (B.3). But if (B.7) holds for q satisfying (B.3), it also holds for $-q$. We are led to the conclusion that for every q belonging to $\mathcal{L}^\infty(\mathbb{R}^d)$ satisfying $\|q\|_\infty \leq 1$, whenever $\int_S q \cdot \pi_{ME} = 0$ we must also have

$$\int_S q \cdot \pi_{ME} \cdot (\ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma) = 0.$$

Thus, for all q belonging to $\mathcal{L}^\infty(\mathbb{R}^d)$, whenever $\int_S q \cdot \pi_{ME} = 0$ we must also have

$$\int_S q \cdot \pi_{ME} \cdot (\ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma) = 0.$$

Let $S_{ME} := \text{supp}(\pi_{ME})$. Now, $\mathbf{1}_{S_{ME}} \cdot \pi_{ME}$, $\mathbf{1}_{S_{ME}} \cdot \pi_{ME} \cdot \ln \pi_{ME}$, and $\{\mathbf{1}_{S_{ME}} \cdot \pi_{ME} \cdot \phi_\gamma\}_{\gamma \in \Gamma}$ all belong to $\mathcal{L}^1(\mathbb{R}^d)$ whose norm-dual [30, p. 106] is $\mathcal{L}^\infty(\mathbb{R}^d)$. If

$$\mathbf{1}_{S_{ME}} \cdot \pi_{ME} \cdot (\ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma)$$

does not belong to the one-dimensional closed subspace spanned by $\mathbf{1}_{S_{ME}} \cdot \pi_{ME}$, then by the Hahn–Banach theorem [30, p. 133], there exists a bounded linear functional q on $\mathcal{L}^1(\mathbb{R}^d)$ which vanishes at $\mathbf{1}_{S_{ME}} \cdot \pi_{ME}$ but not at

$$\mathbf{1}_{S_{ME}} \cdot \pi_{ME} \cdot (\ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma),$$

that is, there exists a q in $\mathcal{L}^\infty(\mathbb{R}^d)$ such that $\int_S q \cdot \pi_{ME} = 0$ but

$$\int_S q \cdot \pi_{ME} \cdot (\ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma) \neq 0$$

contradicting the conclusion of the last paragraph. Hence there exists a real scalar α such that

$$\pi_{ME} \cdot (\ln \pi_{ME} + \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma) = -\alpha \pi_{ME}$$

for all \mathbf{x} in S_{ME} , that is,

$$\pi_{ME}(\mathbf{x}) = \mathbf{1}_{S_{ME}}(\mathbf{x}) \cdot \exp\{-\alpha - \sum_{\gamma \in \Gamma} \lambda_\gamma \phi_\gamma(\mathbf{x})\}.$$

We shall presently show that for each π belongs to Ω with $-\infty < h(\pi)$, we have $D(\pi||\pi_{ME}) < +\infty$, that is, $\pi, \pi \ll \pi_{ME}$. In particular, this would mean that

$$\mathbb{E}_\pi[\mathbf{1}_{S \setminus S_{ME}}] = 0$$

for all $\pi \in \Omega$ with $-\infty < h(\pi)$. To show this, define

$$\pi_k := \left(1 - \frac{1}{k}\right) \pi_{ME} + \frac{1}{k} \pi, \quad k = 1, 2, \dots$$

and note that for each k , (i) π_k belongs to Ω , (ii) $\pi \ll \pi_k$ and $\pi_{ME} \ll \pi_k$, and (iii) $\pi_k \rightarrow \pi_{ME}$, where the convergence

is in the almost everywhere sense and also under the $\mathcal{L}^1(\mathbb{R}^d)$ norm. We have

$$\begin{aligned} +\infty &> h(\pi_{ME}) \geq h(\pi_k) = \\ &= \left(1 - \frac{1}{k}\right) h(\pi_{ME}) + \frac{1}{k} h(\pi) + \\ &\quad + \left(1 - \frac{1}{k}\right) D(\pi_{ME}||\pi_k) + \frac{1}{k} D(\pi||\pi_k) \\ &\geq \left(1 - \frac{1}{k}\right) h(\pi_{ME}) + \frac{1}{k} h(\pi) + \frac{1}{k} D(\pi||\pi_k), \end{aligned}$$

where the first inequality follows from the existence of π_{ME} and because π belongs to Ω , the second equality is an identity, and the third inequality follows from the nonnegativity of cross-entropy (Fact A.1). Hence,

$$h(\pi) + D(\pi||\pi_k) \leq h(\pi_{ME})$$

for all k . Taking limits, noting that π_k converges to π_{ME} in norm, and using the lower semi-continuity property of cross-entropy (Fact A.3) one obtains

$$h(\pi) + D(\pi||\pi_{ME}) \leq h(\pi_{ME}) < \infty.$$

Since $h(\pi) > -\infty$, $D(\pi||\pi_{ME}) < \infty$. The characterization is now complete. \blacksquare

2) *Proof of Remark 3.4:* Let \mathbf{u} map to $\lambda(\mathbf{u})$ and π_{ME} be the maxent pdf in $\Omega(\mathbf{u})$. Define

$$\Gamma_0 := \{\gamma \in \Gamma : \lambda_\gamma = 0\}.$$

Suppose that for all γ in Γ_0 , $u'_\gamma \geq u_\gamma$ and for all γ in $\Gamma \setminus \Gamma_0$, $u'_\gamma = u_\gamma$. Let π'_{ME} be the maxent pdf in $\Omega(\mathbf{u}')$. We shall show that $\pi_{ME} = \pi'_{ME}$. Clearly, $\Omega(\mathbf{u}) \subseteq \Omega(\mathbf{u}')$ implies that $h(\pi_{ME}) \leq h(\pi'_{ME})$. On the other hand, using (B.1) with $\pi = \pi'_{ME}$ it follows that

$$\begin{aligned} h(\pi_{ME}) &\geq h(\pi'_{ME}) - \sum_{\gamma \in \Gamma \setminus \Gamma_0} \lambda_\gamma [\mathbb{E}_{\pi'_{ME}}[\phi_\gamma] - u_\gamma] \\ &\geq h(\pi'_{ME}) \end{aligned}$$

since $\lambda_\gamma > 0$ and

$$\mathbb{E}_{\pi'_{ME}}[\phi_\gamma] \leq u_\gamma$$

for all γ in $\Gamma \setminus \Gamma_0$. Thus $h(\pi_{ME}) = h(\pi'_{ME})$. Since π'_{ME} is unique, the result follows.

3) *Proof of Theorem 3.2:* Let α be the normalization constant for which π_{exp} is a valid pdf. The condition

$$\mathbb{E}_\pi[\mathbf{1}_{S \setminus S_{\text{exp}}}(\mathbf{x})] = 0$$

for all π belonging to $\Omega(\mathbf{u})$ for which $-\infty < h(\pi)$ implies that $\pi \ll \pi_{\text{exp}}$, in particular, $\text{supp}(\pi) \subseteq \text{supp}(\pi_{\text{exp}})$. Hence,

$$0 \leq D(\pi||\pi_{\text{exp}}) = \alpha + \sum_{\gamma \in \Gamma} \lambda_\gamma(\mathbf{u}) \mathbb{E}_\pi[\phi_\gamma(\mathbf{x})] - h(\pi) < \infty.$$

This implies that

$$\begin{aligned} -\infty < h(\pi) &\leq \alpha + \sum_{\gamma \in \Gamma} \lambda_\gamma(\mathbf{u}) \mathbb{E}_\pi[\phi_\gamma] \\ &\leq \alpha + \sum_{\gamma \in \Gamma} \lambda_\gamma u_\gamma < \infty. \end{aligned}$$

Since π_{exp} belongs to $\Omega(\mathbf{u})$ and

$$\sum_{\gamma \in \Gamma} \lambda_{\gamma} (\mathbb{E}_{\pi_{\text{exp}}}[\phi_{\gamma}] - u_{\gamma}) = 0,$$

hence $\mathbb{E}_{\pi_{\text{exp}}}[\phi_{\gamma}] = u_{\gamma}$ for all $\gamma : \lambda_{\gamma} > 0$. Thus, for all π belonging to $\Omega(\mathbf{u})$ for which $-\infty < h(\pi)$,

$$-\infty < h(\pi) \leq \alpha + \sum_{\gamma \in \Gamma} \lambda_{\gamma}(\mathbf{u}) \mathbb{E}_{\pi_{\text{exp}}}[\phi_{\gamma}] = h(\pi_{\text{exp}}),$$

that is,

$$-\infty < h(\pi) \leq h(\pi_{\text{exp}}) < \infty.$$

Hence, for all π in $\Omega(\mathbf{u})$, $h(\pi) \leq h(\pi_{\text{exp}}) < \infty$ and π_{exp} belongs to $\Omega(\mathbf{u})$. \blacksquare

APPENDIX C

PROOF OF EXISTENCE THEOREMS

1) *Proof of Theorem 3.3:* Let $\pi_S(\mathbf{x}) := \frac{1}{|S|} \mathbf{1}_S(\mathbf{x})$ (note that $|S| < \infty$). For all π in F we have

$$0 \leq D(\pi || \pi_S) = \ln |S| - h(\pi),$$

that is,

$$h(\pi) \leq \ln |S| < \infty.$$

Since there exists a pdf π_0 in F for which $-\infty < h(\pi_0)$, it follows that

$$h(F) := \sup_{\pi \in F} h(\pi) \in \mathbb{R},$$

that is, $h(F)$ is finite. Let $\{\pi_k\}_{k=1}^{\infty}$ be any sequence of pdfs in F such that for each k , $h(\pi_k) \in \mathbb{R}$ and $h(\pi_k) \rightarrow h(F)$ as k goes to ∞ . Since F is \mathcal{L}^1 -complete, from Fact A.4 it follows that there exists a unique pdf π^* in F to which π_k converges in norm. Convergence in norm implies convergence in measure which in turn implies the existence of a subsequence which converges almost everywhere [15, Proposition 18, p. 95]. By passing to the subsequence we can assume that, without loss of generality, π_k converges to π^* almost everywhere (and in norm). The lower semi-continuity property of cross-entropy (see (A.1) in Fact A.3) implies that

$$\begin{aligned} \ln |S| - h(\pi^*) = D(\pi^* || \pi_S) &\leq \liminf_{k \rightarrow \infty} D(\pi_k || \pi_S) \\ &= \ln |S| - \lim_{k \rightarrow \infty} h(\pi_k) \\ &= \ln |S| - h(F). \end{aligned} \quad (\text{C.1})$$

This shows that $h(F) \leq h(\pi^*)$. However, $h(\pi^*) \leq h(F)$ because π^* belongs to F . It follows that $h(\pi^*) = h(F)$ and hence $\pi_{ME} = \pi^*$ is the unique maxent pdf in F . \blacksquare

Proposition C.1: (Variational completeness of Ω) Let $\Omega(\mathbf{u})$ be as in (2.1). Let $\{\phi_{\gamma}\}_{\gamma \in \Gamma}$ be uniformly bounded from below by $L \in \mathbb{R}$. Then Ω is a convex collection of pdfs which is complete under the $\mathcal{L}^1(\mathbb{R}^d)$ norm.

Proof: Ω is convex because $\mathbb{E}_{\pi}[\phi_{\gamma}]$ is linear in π . Let $\{\pi_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\Omega \subseteq \mathcal{L}^1(\mathbb{R}^d)$. Since $\mathcal{L}^1(\mathbb{R}^d)$ is complete with respect to the $\|\cdot\|_{\mathcal{L}^1(\mathbb{R}^d)}$ norm [15, Theorem 6, p. 125] and $\{\pi_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there exists $\pi \in \mathcal{L}^1(\mathbb{R}^d)$ such that π_n converges to π under the $\mathcal{L}^1(\mathbb{R}^d)$ norm. We need to show that: (i) $\pi \geq 0$ (ii) $\int_{\mathbb{R}^d} \pi = 1$, and (iii) for all γ in Γ , $\mathbb{E}_{\pi}[\phi_{\gamma}] \leq u_{\gamma}$. Recall that convergence

in $\mathcal{L}^1(\mathbb{R}^d)$ -norm implies convergence in (Lebesgue) measure which in turn implies the existence of a subsequence π_{n_k} converging to π almost everywhere in \mathbb{R}^d [15, Proposition 18, p. 95]. Since each element of the subsequence satisfies (i), so does the limit π . Furthermore,

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (\pi - 1) \right| &= \left| \int_{\mathbb{R}^d} \pi - \int_{\mathbb{R}^d} \pi_n \right| \\ &\leq \int_{\mathbb{R}^d} |\pi - \pi_n| \\ &= \|\pi - \pi_n\|_{\mathcal{L}^1(\mathbb{R}^d)} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ so (ii) holds. Applying Fatou's lemma [15, Theorem 9, p. 86] to the sequence of non-negative functions

$$\pi_n(\mathbf{x}) [\phi_{\gamma}(\mathbf{x}) - L]$$

which converges to

$$\pi(\mathbf{x}) [\phi_{\gamma}(\mathbf{x}) - L],$$

gives

$$\int_{\mathbb{R}^d} \pi \phi_{\gamma} \leq \liminf_n \int_{\mathbb{R}^d} \pi_n \phi_{\gamma} \leq u_{\gamma}.$$

Hence (iii) also holds, and π belongs to Ω . \blacksquare

2) *Proof of Corollary 3.4:* Ω is nonempty by assumption, convex by definition, and \mathcal{L}^1 -complete by Proposition C.1. S has finite volume by assumption. Since C has nonzero volume and $C \subseteq S$ which has finite volume, $|C| < \infty$. If

$$\pi_C(\mathbf{x}) := \frac{1}{|C|} \mathbf{1}_C(\mathbf{x})$$

denotes the distribution that is uniform over the set C , it is clear that π_C belongs to $\Omega(\mathbf{u})$ and

$$h(\pi_C) = \ln |C| > -\infty.$$

Hence by Theorem 3.3,

$$h(\Omega) := \sup_{\pi \in \Omega} h(\pi) \in \mathbb{R},$$

that is, $h(\Omega)$ is finite, in fact

$$-\infty < \ln |C| \leq h(\Omega) \leq \ln |S| < \infty,$$

and there exists a unique maxent pdf π_{ME} belonging to $\Omega(\mathbf{u})$ having the exponential form given by Theorem 3.1. \blacksquare

3) *Proof of Theorem 3.5:* For each $\lambda > 0$, let

$$Z_{\lambda} := \|\exp\{-\lambda\psi\}\|_{\mathcal{L}^1(\mathbb{R}^d)} < +\infty$$

and

$$\pi_{\lambda} := (Z_{\lambda})^{-1} \exp\{-\lambda\psi\}.$$

For all π in F we have

$$0 \leq D(\pi || \pi_{\lambda}) = \lambda \mathbb{E}_{\pi}[\psi] + \ln Z_{\lambda} - h(\pi),$$

that is,

$$h(\pi) \leq \lambda \mathbb{E}_{\pi}[\psi] + \ln Z_{\lambda} \leq \lambda u + \ln Z_{\lambda} < \infty.$$

Since there exists a pdf π_0 in F for which $-\infty < h(\pi_0)$, it follows that

$$h(F) := \sup_{\pi \in F} h(\pi) \in \mathbb{R},$$

that is, $h(F)$ is finite. Since F is \mathcal{L}^1 -complete, following the proof of Theorem 3.3, there exists a unique pdf π^* in F and a sequence π_k in F such that for each k , $h(\pi_k) \in \mathbb{R}$, $h(\pi_k) \rightarrow h(F)$ as k goes to ∞ and $\pi_k \rightarrow \pi^*$ both in norm and in the almost everywhere sense. The lower semi-continuity property of cross-entropy (see (A.1) in Fact A.3) and the moment constraints

$$\{\forall \pi \in F, -\infty < L \leq \mathbb{E}_\pi[\psi] \leq u < \infty\}$$

imply that

$$\begin{aligned} \lambda \mathbb{E}_{\pi^*}[\psi] + \ln Z_\lambda - h(\pi^*) &= D(\pi^* || \pi_\lambda) \\ &\leq \liminf_{k \rightarrow \infty} D(\pi_k || \pi_\lambda) \\ &= \liminf_{k \rightarrow \infty} [\lambda \mathbb{E}_{\pi_k}[\psi] + \ln Z_\lambda \\ &\quad - h(\pi_k)] \\ &\leq \lambda u + \ln Z_\lambda - \lim_{k \rightarrow \infty} h(\pi_k) \\ &= \lambda u + \ln Z_\lambda - h(F). \end{aligned}$$

This shows that

$$h(F) \leq h(\pi^*) + \lambda [u - \mathbb{E}_{\pi^*}[\psi]] \leq h(\pi^*) + \lambda(u - L)$$

for all $\lambda > 0$. Hence for every $\epsilon > 0$ by choosing λ such that $\lambda(u - L) \leq \epsilon$, we obtain $h(F) \leq h(\pi^*) + \epsilon$. Thus $h(F) \leq h(\pi^*)$. However, $h(\pi^*) \leq h(F)$ because π^* belongs to F . It follows that $h(\pi^*) = h(F)$ and hence $\pi_{ME} = \pi^*$ is the unique maxent pdf in F . ■

4) *Proof of Corollary 3.6:* Ω is nonempty by assumption, convex by definition, and \mathcal{L}^1 -complete by Proposition C.1. Let C' be a subset of C having nonzero but finite volume and

$$\pi_{C'}(\mathbf{x}) := \frac{1}{|C'|} \mathbf{1}_{C'}(\mathbf{x}).$$

It is clear that $\pi_{C'}$ belongs to $\Omega(\mathbf{u})$ and

$$h(\pi_{C'}) = \ln |C'| > -\infty.$$

Since ϕ_{γ_0} is uniformly bounded from below by $L \in \mathbb{R}$, for all π in Ω we have

$$-\infty < L \leq \mathbb{E}_\pi[\phi_{\gamma_0}].$$

Again, for all π in Ω ,

$$\mathbb{E}_\pi[\phi_{\gamma_0}] \leq u_{\gamma_0} < \infty.$$

Hence by Theorem 3.3,

$$h(\Omega) := \sup_{\pi \in \Omega} h(\pi) \in \mathbb{R},$$

that is, $h(\Omega)$ is finite, in fact

$$-\infty < \ln |C'| \leq h(\Omega) \leq \inf_{\lambda > 0} [\lambda u + \ln Z_\lambda] < \infty$$

where Z_λ is as in the proof of Theorem 3.5, and there exists a unique maxent pdf π_{ME} belonging to $\Omega(\mathbf{u})$ having the exponential form given by Theorem 3.1. ■

APPENDIX D

PROOF OF THEOREM 3.7

Proposition D.1: If pdf π belongs to $\mathcal{L}^\infty(\mathbb{R}^d)$ then $h(\pi)$ exists and

$$-\infty < -\ln \|\pi\|_{\mathcal{L}^\infty} \leq h(\pi).$$

If pdf π belongs to $\mathcal{L}^2(\mathbb{R}^d)$ then $h(\pi)$ exists and

$$-\infty < 1 - \|\pi\|_{\mathcal{L}^2}^2 \leq h(\pi).$$

Proof: Since $\|\pi\|_{\mathcal{L}^1} = 1$ and π belongs to $\mathcal{L}^\infty(\mathbb{R}^d)$, it follows that $0 < \|\pi\|_{\mathcal{L}^\infty}$. Also,

$$0 \leq \pi(\mathbf{x}) \leq \|\pi\|_{\mathcal{L}^\infty}$$

almost everywhere. Thus,

$$-\infty < -\pi \ln \|\pi\|_{\mathcal{L}^\infty} \leq -\pi \ln \pi.$$

Since for all nonnegative t , $\ln t \leq t - 1$, we have

$$\pi(\mathbf{x}) - (\pi(\mathbf{x}))^2 \leq -\pi(\mathbf{x}) \ln \pi(\mathbf{x})$$

almost everywhere. Since π belongs to $\mathcal{L}^2(\mathbb{R}^d)$ and π is a pdf, the result follows. ■

Remark D.1: The conditions in the above proposition are not necessary for $h(\pi)$ to exist and be strictly greater than $-\infty$. For example, if

$$\pi(t) := 1_{(0,1]}(t) \frac{1}{2\sqrt{t}},$$

then $h(\pi) = \ln(\frac{2}{e})$, where $1_{(0,1]}(t)$ is the characteristic function of the interval $(0, 1]$. The conditions in proposition D.1 do not guarantee that $h(\pi)$ will be finite. For example,

$$\pi(t) := 1_{[e,\infty)}(t) t^{-1} (\ln t)^{-2}$$

is both bounded and square integrable which implies that $h(\pi)$ exists but, $h(\pi) = +\infty$ [31, p. 237]. In the sequel, we shall derive a general moment condition that ensures that $h(\pi)$ when it exists, is less than $+\infty$ (Corollary D.6).

Proposition D.2: (Sufficient condition for integrability.) If $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex and omnidirectionally unbounded, then for all strictly positive a ,

$$0 < Z_\phi(a) := \int_{\mathbb{R}^d} \exp\{-a\phi(\mathbf{x})\} d\mathbf{x} < \infty.$$

In other words, a convex and omni-directionally unbounded function is stable.

Proof: It is clear that for all real-valued a , $0 < Z_\phi(a)$. Since $\phi(\mathbf{x})$ is unbounded in all directions, there exists a strictly positive r such that for all \mathbf{x} satisfying $\|\mathbf{x}\|_{\ell_1} > r$ we have $\phi(\mathbf{x}) > \phi(\mathbf{0})$. Thus,

$$\inf_{\mathbf{x} \in \mathbb{R}^d} \phi(\mathbf{x}) = \inf_{\|\mathbf{x}\|_{\ell_1} \leq r} \phi(\mathbf{x}) = \min_{\|\mathbf{x}\|_{\ell_1} \leq r} \phi(\mathbf{x}) = \phi(\mathbf{x}_0)$$

for some \mathbf{x}_0 in \mathbb{R}^d satisfying $\|\mathbf{x}_0\|_{\ell_1} \leq r$. The second equality follows because ϕ being convex on \mathbb{R}^d is continuous, and the closed ball

$$\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_{\ell_1} \leq r\}$$

is a compact subset of \mathbb{R}^d . Next, define the function

$$\psi(\mathbf{x}) := \phi(\mathbf{x} + \mathbf{x}_0) - \phi(\mathbf{x}_0).$$

Since \mathbf{x}_0 is a global minimizer of $\phi(\mathbf{x})$, it follows that $\psi(\mathbf{x})$ is non-negative, and attains its global minimum value of 0 at the origin. The function $\psi(\mathbf{x})$ also inherits the convexity and omnidirectional unboundedness properties of $\phi(\mathbf{x})$. Hence it suffices to demonstrate that for all strictly positive a , $\exp\{-a\psi(\mathbf{x})\}$ is integrable. Since $\psi(\mathbf{x})$ is non-negative and unbounded in all directions, there exists a $\rho > 0$ such that for all \mathbf{x} satisfying $\|\mathbf{x}\|_{\ell_1} > \rho$ we have $\psi(\mathbf{x}) > 1$. Now

$$\inf_{\|\mathbf{x}\|_{\ell_1}=\rho} \psi(\mathbf{x}) = \min_{\|\mathbf{x}\|_{\ell_1}=\rho} \psi(\mathbf{x}) = \psi(\mathbf{x}^*) \geq 1 \text{ with } \|\mathbf{x}^*\|_{\ell_1} = \rho.$$

The first equality follows from the continuity of $\psi(\mathbf{x})$ and the compactness of the closed sphere of radius ρ in \mathbb{R}^d . The last inequality above follows from the way ρ has been defined. For all \mathbf{x} in \mathbb{R}^d having a norm $\|\mathbf{x}\|_{\ell_1}$ which is strictly larger than ρ , the convexity of $\psi(\mathbf{x})$ and the definition of \mathbf{x}^* imply that

$$1 \leq \psi(\mathbf{x}^*) \leq \psi\left(\frac{\rho\mathbf{x}}{\|\mathbf{x}\|_{\ell_1}}\right) \leq \frac{\rho}{\|\mathbf{x}\|_{\ell_1}}\psi(\mathbf{x}) + \left(1 - \frac{\rho}{\|\mathbf{x}\|_{\ell_1}}\right)\psi(\mathbf{0}).$$

For all $a > 0$ and for all \mathbf{x} in \mathbb{R}^d such that $\|\mathbf{x}\|_{\ell_1} > \rho$ we have

$$0 < a\frac{\|\mathbf{x}\|_{\ell_1}}{\rho} \leq a\psi(\mathbf{x}),$$

since $\psi(\mathbf{0}) = 0$. Thus, for all \mathbf{x} in \mathbb{R}^d such that

$$\begin{aligned} \|\mathbf{x}\|_{\ell_1} > \rho > 0, \\ \exp\{-a\psi(\mathbf{x})\} \leq \exp\left\{-\frac{a}{\rho}\|\mathbf{x}\|_{\ell_1}\right\}. \end{aligned}$$

Finally, since

$$\|\mathbf{x}\|_{\ell_1} := \sum_{i=1}^d |\mathbf{x}(i)|,$$

and the exponential function $\exp\{-|t|\}$, $t \in \mathbb{R}$ is integrable over \mathbb{R} , the result follows. \blacksquare

The conditions on $\psi(\mathbf{x})$ in the previous proposition can be somewhat relaxed as the following corollary demonstrates.

Corollary D.3: A well-behaved function is stable, that is, if $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is well-behaved, then for all $a > 0$,

$$0 < Z_\psi(a) := \int_{\mathbb{R}^d} e^{-a\psi(\mathbf{x})} d\mathbf{x} < +\infty.$$

Proof: Since ψ is well-behaved, there exists a convex, omnidirectionally unbounded function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ and a nonnegative real number M such that for all \mathbf{x} in \mathbb{R}^d whose norm is strictly larger than M we have $\phi(\mathbf{x}) \leq \psi(\mathbf{x})$ and for all \mathbf{x} in \mathbb{R}^d whose norm is no larger than M we have $\psi(\mathbf{x}) < +\infty$. Now,

$$Z_\psi(a) = \int_{\{\mathbf{x}:\|\mathbf{x}\|\leq M\}} e^{-a\psi(\mathbf{x})} + \int_{\{\mathbf{x}:\|\mathbf{x}\|>M\}} e^{-a\psi(\mathbf{x})}.$$

The first term on the right side is bounded since $\psi(\mathbf{x})$ is bounded over the set

$$\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq M\}$$

which has finite measure. Proposition D.2 provides an upper bound for the second term:

$$\int_{\{\mathbf{x}:M<\|\mathbf{x}\|\}} e^{-a\psi(\mathbf{x})} d\mathbf{x} \leq \int_{\{\mathbf{x}:M<\|\mathbf{x}\|\}} e^{-a\phi(\mathbf{x})} d\mathbf{x} < +\infty$$

and the result follows. \blacksquare

Proposition D.4: Let π be a pdf for which there exists a convex and omnidirectionally unbounded function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\phi(\mathbf{x}) \leq -\ln \pi(\mathbf{x})$ for all $\|\mathbf{x}\|$ sufficiently large. Then $h(\pi)$ exists and $h(\pi) < +\infty$. If further, π belongs to $\mathcal{L}^\infty(\mathbb{R}^d)$ or $\mathcal{L}^2(\mathbb{R}^d)$ then $h(\pi)$ exists, and $|h(\pi)| < +\infty$, that is, $-\pi \ln \pi$ belongs to $\mathcal{L}^1(\mathbb{R}^d)$.

Proof: Let

$$P := \{\mathbf{x} \in \mathbb{R}^d : 0 \leq \pi(\mathbf{x}) \leq 1\}$$

be the set over which $-\pi \ln \pi$ is nonnegative. From the assumptions on π there exists a strictly positive real number R such that for all $\|\mathbf{x}\| > R$,

$$0 < \phi(\mathbf{x}) \leq -\ln \pi(\mathbf{x}).$$

Define the set

$$B := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| > R\}.$$

Its complement: B^c is a closed and bounded subset of \mathbb{R}^d and has finite volume. Write

$$\int_P -\pi \ln \pi = \int_{P \cap B^c} -\pi \ln \pi + \int_{P \cap B} -\pi \ln \pi. \quad (\text{D.1})$$

We shall show that each integral on the right side of the above equality is upper bounded by a positive real number. From this it will follow that $h(\pi)$ exists and $h(\pi) < +\infty$. Since for all nonnegative t , $\ln t \leq t$, for all \mathbf{x} in P we have

$$\begin{aligned} 0 \leq \ln \frac{1}{\pi(\mathbf{x})} \leq \frac{1}{\pi(\mathbf{x})} \\ \Rightarrow 0 \leq -\pi(\mathbf{x}) \ln \pi(\mathbf{x}) \leq 1. \end{aligned}$$

Thus the first integral on the right side of (D.1) is upper bounded by the volume of $P \cap B^c$ which is less than the volume of the bounded set B^c . Again, since for all nonnegative t , $\ln t \leq \sqrt{t}$, for all \mathbf{x} in P we have

$$\begin{aligned} 0 \leq -\ln \pi(\mathbf{x}) \leq \frac{1}{\sqrt{\pi(\mathbf{x})}} \\ \Rightarrow 0 \leq -\pi(\mathbf{x}) \ln \pi(\mathbf{x}) \leq \sqrt{\pi(\mathbf{x})}. \end{aligned}$$

Now for all \mathbf{x} in $P \cap B$ we have,

$$0 \leq -\pi(\mathbf{x}) \ln \pi(\mathbf{x}) \leq \sqrt{\pi(\mathbf{x})} \leq e^{-\frac{\phi(\mathbf{x})}{2}}$$

where the last inequality follows from the fact that

$$\phi(\mathbf{x}) \leq -\ln \pi(\mathbf{x})$$

for all \mathbf{x} in B . We are lead to the following inequalities

$$0 \leq \int_{P \cap B} -\pi \ln \pi \leq \int_{P \cap B} e^{-\frac{\phi(\mathbf{x})}{2}} \leq \|e^{-\frac{\phi(\mathbf{x})}{2}}\|_{\mathcal{L}^1} < +\infty,$$

where the last inequality is a consequence of Corollary D.3. From Proposition D.1 it follows that if further π belongs to $\mathcal{L}^\infty(\mathbb{R}^d)$ or to $\mathcal{L}^2(\mathbb{R}^d)$, then $h(\pi) > -\infty$ and hence $|h(\pi)| < +\infty$, that is, $-\pi \ln \pi$ is absolutely integrable. The proof is complete. \blacksquare

Corollary D.5: Let $\psi(\mathbf{x})$ be well-behaved. If we define

$$\pi(\mathbf{x}) := \frac{e^{-\psi(\mathbf{x})}}{Z_\psi(1)},$$

then, π is a pdf, π belongs to both $\mathcal{L}^\infty(\mathbb{R}^d)$ and $\mathcal{L}^2(\mathbb{R}^d)$. Thus π satisfies the conditions and hence the results of Proposition D.4, that is, $h(\pi)$ exists and $|h(\pi)| < +\infty$.

Corollary D.6: Let ϕ_{γ_0} be a well-behaved function. If π is any pdf that satisfies

$$\mathbb{E}_\pi[\phi_{\gamma_0}] \leq u_{\gamma_0} < +\infty$$

then $h(\pi)$ exists and

$$h(\pi) \leq u_{\gamma_0} + \ln \|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1} < +\infty.$$

If further $-\infty < h(\pi)$ then $0 \leq D(\pi||r) < +\infty$ where

$$r(\mathbf{x}) := \frac{e^{-\phi_{\gamma_0}(\mathbf{x})}}{\|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1}}$$

is a pdf.

Proof: Corollary D.3 shows that $r(\mathbf{x})$ is integrable and is hence a valid pdf. It is also clear that $\text{supp}(r) = \mathbb{R}^d$ and hence $\pi \ll r$ for each pdf π . The inequality $\ln t \leq t - 1$ which holds for all nonnegative t when applied to $t = r(\mathbf{x})/\pi(\mathbf{x})$ reveals that for all \mathbf{x} belonging to the support-set of the pdf π ,

$$-\pi(\mathbf{x}) \ln \pi(\mathbf{x}) \leq r(\mathbf{x}) - \pi(\mathbf{x}) + \pi(\mathbf{x})\phi_{\gamma_0}(\mathbf{x}) + \pi(\mathbf{x}) \ln \|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1}.$$

Now since $\mathbb{E}_\pi[\phi_{\gamma_0}] \leq u_{\gamma_0}$ for π belonging to Ω , integrating the above inequality over $\text{supp}(\pi)$ we can conclude that $h(\pi)$ exists and

$$h(\pi) \leq u_{\gamma_0} + \ln \|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1} < +\infty.$$

It is also clear that if $-\infty < h(\pi)$ then

$$\begin{aligned} 0 &\leq D(\pi||r) \\ &\leq \ln \|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1} - h(\pi) + \int_{\mathbb{R}^d} \pi \phi_{\gamma_0} \\ &\leq u_{\gamma_0} + \ln \|e^{-\phi_{\gamma_0}}\|_{\mathcal{L}^1} - h(\pi) \\ &< +\infty. \end{aligned}$$

Theorem 3.7 follows from Corollary D.3 and Corollary D.6. ■

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