ON SOME EXTREMAL PROBLEMS IN GRAPH THEORY

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ABSTRACT. In this paper we are concerned with various graph invariants (girth, diameter, expansion constants, eigenvalues of the Laplacian, tree number) and their analogs for weighted graphs – weighing the graph changes a combinatorial problem to one in analysis. We study both weighted and unweighted graphs which are extremal for these invariants. In the unweighted case we concentrate on finding extrema among all (usually) regular graphs with the same number of vertices; we also study the relationships between such graphs.

1. INTRODUCTION

In this paper we are concerned with various invariants of graphs related to distance (girth, diameter), expansion properties of the graph (Cheeger constant) and spectrum of the Laplacian (the first eigenvalue, the tree number).

The behavior of many of these invariants for random graphs has been extensively studied ([Bol]). For regular graphs, the results about weak convergence of graph spectra to the spectrum of the infinite regular tree ([Kest, Mck81]) allow to get many results on the asymptotic behavior of spectral invariants (as the number of vertices increases).

Often when certain bounds are established for the rate of growth of some invariant, one is interested in studying families graphs with (asymptotically) the *best possible* behavior with respect to a given invariant (two well-known examples are Ramanujan graphs and cages).

A related problem is to study the graphs which maximize (or minimize) a given invariant among all simple connected graphs with n vertices and m edges (we denote the set of such graphs by $\mathcal{G}_{n,m}$), or among k-regular connected simple graphs with n vertices (we denote the set of such graphs by $\mathcal{X}_{n,k}$). The sets $\mathcal{G}_{n,m}$ and $\mathcal{X}_{n,k}$ are finite, so maximizing a given invariant is a discrete optimization problem.

On the other hand, $\mathcal{G}_{n,m}$ (respectively, $\mathcal{X}_{n,k}$) is naturally embedded in the complex $\mathcal{G}_{n,m}^w$ of all *weighted* graphs with n vertices and m edges (respectively, the complex $\mathcal{X}_{n,k}^w$ of weighted k-regular graphs with n vertices); the edge weights are normalized so that their sum is constant. One can

study the extremal problems for weighted analogs of graph invariants (see e.g. [Fie]). A natural question then is which *unweighted* simple graphs in $\mathcal{G}_{n,m}$ or $\mathcal{X}_{n,k}$ (all weights are equal to 1) are local extrema in $\mathcal{G}_{n,m}^w$ or $\mathcal{X}_{n,k}^w$ for a given invariant. In section 2 we study this question for various graph invariants.

In section 3 we discuss some heuristics about the structure of graphs which have many spanning trees. In section 4.1 we give an upper bound on the rate of growth of the number of spanning trees for 6-regular simple graphs on a fixed surface. In section 4.2 we prove a result about the asymptotic properties of certain lattices, associated to graphs by Bacher, de la Harpe and Nagnibeda (cf. [BHN]), for sequences of regular graphs. Finally, in section 5 we explain how the first 8 cubic cages appear naturally in a sequence of graphs produced from K_4 by "greedy" edge insertion with respect to the number of spanning trees.

In the remainder of this section we review the definitions and some results about the asymptotic behavior of various graph invariants.

1.1. **Definitions and notation.** The *adjacency* matrix A of a graph $G \in \mathcal{G}_{m,n}$ is a square matrix of size n where $A_{i,j}$ is the number of edges joining the vertices v_i and v_j. We always consider loopless graphs, so $A_{ii} = 0$; for simple graphs A_{ij} is equal to 0 or 1.

The nearest neighbor *Laplacian* $\mathbf{L}$ acts on functions on the set $V(G)$ of the vertices of G: given $f : V \to \mathbf{R}$, $\mathbf{L}(f)(v) = \sum_{w \sim v}(f(v) - f(w))$. Let $\mathbf{L} = \{L_{ij}\}$ be the matrix of $\mathbf{L}(G)$ of a (not necessarily simple) graph G; then L_{ii} is the degree of the vertex v_i, and $-L_{ij}$ is the number of the edges joining v_i and $v_j \neq v_i$ (equal to 0 or 1 for simple graphs). For k-regular graphs $\mathbf{L} = k \cdot \mathrm{Id} - A$.

Let A be the adjacency matrix of a graph $G \in \mathcal{X}_{n,k}$, and let its spectrum (in the decreasing order) be given by

$$k = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n \geq (-k) \tag{1}$$

(the constant function is the eigenfunction with the eigenvalue k). The last eigenvalue $\lambda_n = -k$ if and only if G is bipartite. The spectrum of $\mathbf{L}(G)$ is $\mu_j = k - \lambda_{j+1}, j = 0, 1, \ldots, n-1$; it satisfies $0 = \mu_0 < \mu_1 \leq \ldots \leq \mu_{n-1} \leq \min\{n, 2k\}$.

1.2. **Spectral properties of random graphs.** Alon and Boppana ([Al86, LPS]) proved that

$$\lim_{n\to\infty} \left(\inf_{G \in \mathcal{X}_{n,k}} \lambda_2(G) \right) \geq 2\sqrt{k-1} \tag{2}$$

(cf. [Nil, Sole] for more precise results); k-regular graphs all of whose $\lambda_j \neq \pm k$ lie inside $[-2\sqrt{k-1}, 2\sqrt{k-1}]$ are called *Ramanujan* graphs (because of the relation to Ramanujan conjectures, cf. [Sar90, Lub]). The first such graphs were constructed in [LPS, Mar88]; many other constructions followed.

1.3. **Complexity of a graph.** An important invariant of G is the number $\tau(G)$ of spanning trees of G; it is sometimes called the *complexity* of G. By Kirckhoff's theorem ([Kir]),

$$n \, \tau(G) \;=\; \mu_1 \mu_2 \cdot \ldots \cdot \mu_{n-1}$$

and $\tau(G)$ is equal to the determinant of any cofactor of the matrix $\mathbf{L}$ and (it is some times called the *determinant of Laplacian*). McKay proved in [Mck83] that

$$\lim_{n\to\infty} \left(\sup_{G \in \mathcal{X}_{n,k}} (\tau(G))^{1/n} \right) \;=\; \sigma_k \;=\; \frac{(k-1)^{k-1}}{(k(k-2))^{k/2-1}} \tag{3}$$

and that a typical graph G in $\mathcal{X}_{n,k}$ satisfies

$$(\tau(G))^{1/n} \;\sim\; \sigma_k$$

The above formula also holds for some Ramanujan graphs ([Sar]).

Alon in [Al90] proved some results concerning

$$\delta_k \;=\; \lim_{n\to\infty} \left(\inf_{G \in \mathcal{X}_{n,k}} (\tau(G))^{1/n} \right) \tag{4}$$

Namely, he showed that for $k \geq 3$, $\delta_k \geq \sqrt{2}$ and that $\delta(k)/k = 1 + o(1)$ as $k \to \infty$. Valdes in [Val] determines the 2-connected cubic graphs with the *minimum* number of spanning trees. It follows from her results that if the limit in (4) is taken over all 2-connected cubic graphs, the answer is $2^{3/4}$ (this was also proved by A. Kostochka in [Kost]).

1.4. **Extremals for girth.** We begin with girth. It is easy to show that a k-regular graph $(k \geq 3)$ with girth $g \geq 3$ has at least $f_0(k,g)$ vertices where

$$f_0(k,g) \;=\; \begin{cases} 1 + k[(k-1)^{(g-1)/2} - 1]/(k-2), & g \ odd \\ 2[(k-1)^{g/2} - 1]/(k-2), & g \ even \end{cases} \tag{5}$$

The k-regular graphs with girth g and $f_0(k,g)$ vertices are called *Moore graphs* (sometimes they are also called *generalized polygons* for even $g \geq 4$); these graphs are unique if they exist. For $g = 3$ those are complete graphs K_{k+1}, for $g = 4$ those are complete bipartite graphs $K_{k,k}$. It is proved in [Big93, Ch. 23] that for $g > 4$ Moore graphs only exist if $g = 5$ and $k = 3$ (Petersen graph), $k = 7$ (Hoffmann-Singleton graph), and (possibly) $k = 57$; or if $g = 6, 8$ or 12. k-regular Moore graphs of girth 6 exist if and only if a projective plane with k points on the line does [Sin].

The k-regular graphs of girth g which have the smallest possible number of vertices are called (k,g) *cages* (such graphs always exist, cf. [HS, Ch. 6]), and the number of their vertices is denoted by $f(k,g)$. For $k \geq 3$ and g such that the corresponding Moore graphs exist, those graphs are (k,g) cages $f(k,g) = f_0(k,g)$; if Moore graphs don't exist, $f(k,g) > f_0(k,g)$. Very few cages other than Moore graphs are known (see [Won], [HS, Ch. 6] and [Big93, Ch. 23] for surveys of known results). For cubic graphs,

$f(3, g)$ is known for $3 \leq g \leq 12$ ([BMS]). For $k \geq 4$, the $f(k, 7)$ is already unknown. The best upper bounds for $f(k, g)$ come from various infinite series of symmetric graphs (cf. [LPS, Mar88, Mor, Chi, BH, LUW]). It is known that

$$1/2 \ \leq \ \liminf_{g \to \infty} \frac{\log_{k-1} f(k, g)}{g} \ \leq \ \limsup_{g \to \infty} \frac{\log_{k-1} f(k, g)}{g} \ \leq \ 3/4 \quad (6)$$

For all known infinite families of regular graphs,

$$\limsup_{g \to \infty} (\log_{k-1} f(k, g))/g = 3/4$$

it is sometimes conjectured that the upper bound in (6) is actually an asymptotic result.

1.5. **Extremals for complexity.** We next turn to the number of spanning trees. It is easy to see that the complete graph on n vertices has the maximum number of trees among all multigraphs with the same number of vertices and edges. In a series of papers (see e.g. [CK, Kel74, Kel76, Kel80, Kel96, Kel97]) Kelmans characterized the graphs, obtained by removing a small number of edges form the complete graph, which have the maximum number of spanning trees among all the graphs with the same number of vertices and edges. Cheng ([Che]) proved that the complete multipartite regular graphs also have this property; see also [Con].

We notice that K_{k+1} and $K_{k,k}$ are, respectively, the unique $(k, 3)$ and $(k, 4)$ cages. One can also check (see [Val]) that the Petersen graph (which is the $(3, 5)$ cage) has the most trees (2000) among all cubic simple graphs with 10 vertices. We conjecture

Conjecture 1.1. *Moore graphs have the maximum number of spanning trees among all simple graphs with the same number of vertices and edges.*

Some evidence supporting this conjecture is discussed in section 5.

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2. Local extrema for weighing the edges

In this chapter we shall discuss several extremal problems for weighted graphs. We will be concerned with finding extrema for various functionals on the *weighted edge polytope P* (which is actually a simplex) defined below. Let G be a simple connected graph, and let $E(G) = \{e_1, \dots, e_m\}$ be the set

of its edges. Then

$$P(G) \; := \; \left\{ (x_1, \dots, x_m) : \; x_j \geq 0, \; \sum_{j=1}^{m} x_j \; = \; m \right\} \tag{7}$$

where the (nonnegative) real number x_j is called the *weight* of e_j; the corresponding graph is denoted by $G(x_1, \dots, x_m)$. The graph $G(1, 1, \dots, 1)$ is called an *unweighted graph*. We introduce the following notation:

$$\underline{\omega} \; = \; (1, \dots, 1).$$

The adjacency matrix of $G(x_1, \dots, x_m)$ is defined by

$$A_{ij} \; = \; \begin{cases} 0, & i = j \text{ or } v_i \not\sim v_j \\ x_k, & e_k = (v_i v_j) \end{cases}$$

The *degree* $\deg(v_i)$ of a vertex v_i is defined by

$$\deg(v_i) \; = \; \sum_{j : (v_i v_j) = e_k \in E(G)} x_k$$

The Laplacian is defined by

$$L_{ij} \; = \; \begin{cases} deg(v_i), & i = j \\ 0, & v_i \not\sim v_j \\ -x_k, & e_k = (v_i v_j) \end{cases} \tag{8}$$

The notation for the eigenvalues of A and $\mathbf{L}$ will be the same as for the unweighted graphs. The proof of Kirchhoff's theorem extends to weighted graphs ([Chu]); we shall continue to call the determinant $\tau(G)$ of any cofactor of L_{ij} the (weighted) tree number of G.

One can regard $\deg(v_i)$ as a combinatorial analog of a curvature at v_i. Motivated by an analogy with metrics of constant curvature, we define another polytope $\tilde{P}(G) \subset P(G)$ by

$$\tilde{P}(G) \; := \; \{(x_1, \dots, x_m) : \; x_j \geq 0, \; \deg(v_i) = d_i \; \forall i\} \tag{9}$$

where $d_i = \#\{j \neq i : v_i \sim v_j\}$ is the degree of v_i in the unweighted graph G.

We will consider the problem of maximizing or minimizing various functionals over P and $\tilde{P}$. Some of the functionals (girth and diameter) will be linear in x_j-s, and so we will be solving a problem in linear programming ([Sch, Ch. 7]). Other functionals (expansion constants) will not be linear in x_j-s on P, but will become linear on $\tilde{P}$.

We shall be often concerned with the following question:

Question: When is $\underline{\omega} = (1, 1, \dots, 1)$ an extremum (in P or $\tilde{P}$)?

In other words, when is the unweighted graph an extremum for a given functional? In some cases we are able answer the above question completely.

2.1. **Girth and diameter.** We turn to ths study of the girth and the diameter. We make several definitions: The *length* of a path in a weighted graph G is the sum of the weights of the edges in the path. The girth $\gamma(G)$ is the length of the shortest cycle in G. The *distance* between two vertices of G is the length of the shortest path connecting them; the *diameter* $\mathrm{diam}(G)$ is equal to the largest such distance.

We are interested in the points $x \in P(G)$ where $\gamma(x)$ takes its maximal value and where $\mathrm{diam}(x)$ takes its minimal value on P. Denote by $C_1, \dots, C_s$ all the cycles of G. Then finding the maximum of the girth leads to solving the following problem in linear programming:

$$\max_{x \in P} \left(\min_{1 \le i \le s} \sum_{e_j \in C_i} x_j \right).$$

We assume that any edge on a cycle is visited at most once, so there are finitely many cycles; we make a similar assumption about the paths in the next definition.

Similarly, for two vertices u, v of G denote by $\{\pi_1(u, v), \dots, \pi_\sigma(u, v)\}$ the set of all paths in G connecting u and v (where $\sigma = \sigma(u, v)$). Finding the minimum of the diameter leads to the following problem:

$$\min_{x \in P} \left(\max_{u, v \in G} \min_{1 \le j \le \sigma(u,v)} \sum_{e_k \in \pi_j} x_k \right).$$

We now fix some $x \in P(G)$ and consider the corresponding graph $G = G(x)$. Two vertices u, v of G are called *diametrically opposite* if $\mathrm{dist}(u, v) = \mathrm{diam}(G)$. Any path connecting two diametrically opposite points which has length $\mathrm{diam}(G)$ is called a *meridian*. Also, any shortest cycle is called a *systole*.

To study how girth changes, we remark that for small weight changes the new systoles will come from among the weighted old systoles. Denote the systoles of G by $S_1, S_2, \dots, S_r$, and form the (m-by-r) edge–cycle incidence matrix S_{ij}. Turning to the diameter, we remark that for small weight changes the new meridians will come from among the weighted old meridians. Denote the meridians of G by $M_1, M_2, \dots, M_t$ and form the (m-by-t) edge-meridian incidence matrix M_{ij}. Denote by $\mathcal{S}$ (respectively, $\mathcal{M}$) the cone formed by all linear combinations of column vectors of S_{ij} (respectively, M_{ij}) with nonnegative coefficients. Clearly, both $\mathcal{C}$ and $\mathcal{M}$ lie inside the first quadrant.

Theorem 2.1. *The graph G is a local maximum for girth (respectively, a local minimum for diameter) if and only if $\underline{\omega}$ lies in $\mathcal{S}$ (respectively, $\mathcal{M}$).*

Proof. Let $G(x^0)$ be the graph in question, and let x^1 be another point in $P(G)$. Denote by y the vector $x^1 - x^0$ connecting the points in P. The graph $G(x^0)$ is a local maximum for the girth if and only if the inequality

$$S\, y \ > \ 0$$

has no solutions for small enough y. By Farkas' lemma ([Sch, Ch. 7]), this condition is equivalent to the condition $\underline{\omega} \in \mathcal{S}$.

Similarly, the graph $G(x^0)$ is a local maximum for the girth if and only if the inequality

$$M \ y \ < \ 0$$

has no solutions for small enough y. Again by Farkas' lemma, this is equivalent to the condition $\underline{\omega} \in \mathcal{M}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

An obvious sufficient condition for the assumptions of the Theorem 2.1 to hold is for every edge to be contained in the same number of systoles (diameters), so that edge-transitive graphs are always satisfy this assumption. Another large class of graphs where every edge is contained in the same number of systoles consists of graphs arising from 4-vertex-connected proper triangulations of surfaces (every edge is contained in exactly two triangles).

On the other hand, the assumptions of Theorem 2.1 are clearly *not* satisfied when there is an edge in G which is not contained in any systole (diameter).

2.2. **Expansion constants.** We now turn to Cheeger and expansion constants. Let G be an unweighted graph. Given a partition of its vertex set $G = A \cup B$ into *disjoint* sets A and B, define $E(A, B)$ to be the set of all edges with one endpoint in A and another in B. Let $|E(A, B)|$ be the sum of the weights of the edges in $E(A, B)$,[1] and let $|A|$ denote the sum of degrees of all vertices in A (also called the *volume* of A).[2] The expansion constant $c(G)$ is defined by

$$c(G) \ = \ \min_{G = A \cup B} \ \frac{|E(A, B)|}{|A| \cdot |B|} \tag{10}$$

The *Cheeger constant* $h(G)$ is defined by

$$h(G) \ = \ \min_{G = A \cup B} \ \frac{|E(A, B)|}{\min(|A|, |B|)} \tag{11}$$

To prove an analog of Theorem 2.1 for the above constants, we find it convenient to maximize Cheeger and expansion constants over $\tilde{P}(G)$ instead of $P(G)$. The reason is that for $A \subset V(G)$ the volume $|A|$ does not depend on $x \in \tilde{P}(G)$ (since the degrees are preserved), and so $c(G)$ and $h(G)$ become linear functionals on $\tilde{P}(G)$.

Let N_{ij} denote the m-by-n edge–vertex incidence matrix. The dimension of $\tilde{P}$ is equal to $m - \text{rank}(N_{ij})$. We denote by $\mathcal{N}$ the cone spanned by linear combination of the column vectors of (N_{ij}) with nonnegative coefficients.[3]

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[1]It is equal to the number of the edges in $E(A, B)$ for unweighted graphs.

[2]Often $|A|$ denotes the number of vertices in A, which for regular graphs is proportional to our definition; our definition allows a straightforward extension to weighted graphs.

[3]Here $\mathcal{N}$ stands for *normalization* conditions; in general, the more conditions we impose, the larger the cone $\mathcal{N}$ is; vector $\underline{\omega}$ should always be in $\mathcal{N}$.

To state the next theorem, we need to define several matrices. Given $x \in \tilde{P}$, let $G(x)$ be the corresponding graph. An edge cut H of $G(x)$ is called an *expansion edge cut* (respectively, a *Cheeger edge cut*) if it is one of the cuts for which the expansion constant (respectively, the Cheeger constant) of G is attained. Let $H_1, H_2, \ldots, H_r$ (respectively, $H^1, H^2, \ldots, H^t$) be the expansion (respectively, Cheeger) edge cuts.

Denote by $A(H)$ and $B(H)$ the disjoint subsets into which H separates $V(G)$. We now define two matrices: an m-by-r *expansion* matrix (E_{ij}) and an m-by-t *Cheeger matrix* (C_{ij}) by

$$E_{ij} \;=\; \begin{cases} 1/(|A(H_j)| \cdot |B(H_j)|), & e_i \in H_j \\ 0, & otherwise. \end{cases} \tag{12}$$

and by

$$C_{ij} \;=\; \begin{cases} 1/\min(|A(H_j)|, |B(H_j)|), & e_i \in H^j \\ 0, & otherwise. \end{cases} \tag{13}$$

Let $\mathcal{E}$ (respectively, $\mathcal{C}$) denote the cone formed by linear combinations of column vectors of E_{ij} (respectively, C_{ij}).

We are now ready to formulate and prove

Theorem 2.2. *A graph G is a local maximum for the expansion constant (respectively, the Cheeger constant) for degree-preserving edge valuations if and only if $\mathcal{E} \cap \mathcal{N}$ (respectively, $\mathcal{C} \cap \mathcal{N}$) contains a nonzero vector.*

Proof : The proof is similar to that of Theorem 2.1. $\qquad\qquad\square$

We remark that edge-transitive graphs satisfy both assumptions of Theorem 2.2 (for such graphs $\underline{\omega} \in (\mathcal{E} \cap \mathcal{N})$ and $\underline{\omega} \in (\mathcal{C} \cap \mathcal{N})$). Also, as in the case of girth and diameter the assumptions are not satisfied if some edge is not contained in any Cheeger (respectively, expansion) edge cut.

2.3. **The tree number.** By the weighted version of Kirckhoff's theorem ([Chu])

$$\tau(G) \;=\; \sum_{T \in \mathcal{T}(G)} \prod_{e_j \in T} x_j \tag{14}$$

where the sum is taken over the set $\mathcal{T}(G)$ of the spanning trees of G.

Finding the maximum of $\tau(G)$ on P becomes a Lagrange multiplier problem with the constraints given by Eq. (7). The condition for $x \in P(G)$ to be a critical point for $\tau(G)$ is

$$\frac{\partial \tau(G)}{\partial x_1} \;=\; \frac{\partial \tau(G)}{\partial x_2} \;=\; \ldots \;=\; \frac{\partial \tau(G)}{\partial x_m} \tag{15}$$

The partial derivatives above are given by

$$\tau_j \;=\; \frac{\partial \tau(G)}{\partial x_j} \;=\; \sum_{T \in \mathcal{T}(G)} \prod_{k \neq j; e_k \in T} x_k.$$

The ratio $\tau_j/\tau(G)$ is called the *effective resistance* of e_j.

We have thus proved:

Proposition 2.3. *The graph $G(x)$ is a critical point for $\tau(G)$ if and only if the effective resistances of all edges are the same.*

If an unweighted graph satisfies the assumptions of Proposition 2.3 then every edge of this graph is contained in *the same* number of spanning trees. Such graphs were studied by Godsil in [God]; he calls these graphs *equiarboreal*. Obviously, all edge-transitive graphs (the automorphism group acts transitively on the edges) are equiarboreal.[4] Godsil gives several more sufficient conditions for a graph to be equiarboreal; in particular, any distance-regular graph and any color class in an association scheme is equiarboreal (the least restrictive condition Godsil gives is for a graph to be *1-homogeneous*). By an easy counting argument one can show that for an unweighted equiarboreal graph

$$T_1 \;=\; T_2 \;=\; \ldots \;=\; \tau(G) \cdot (n-1)/m \tag{16}$$

(this is actually the result of Foster, cf. [Fos]) so the necessary condition for a graph to be equiarboreal is

$$m \mid (n-1)\tau(G).$$

We remark that the graphs which have the most spanning trees among the regular graphs with the same number of vertices are not necessarily equiarboreal, and vice versa. For example, the 8-vertex Möbius wheel (cf. section 5) which has the most spanning trees among the 8-vertex cubic graphs is not equiarboreal (cf. also [Val]), while the *cube* (which is certainly edge-transitive, hence equiarboreal) has the *second biggest* number of spanning trees among the 8-vertex cubic graphs.

We next study the function $\tau(G)$ on $P(G)$. Surprisingly, it turns out to be concave, so there is at most one critical point, and if there is one it has to be a global maximum. The eigenspace E_0 corresponding to the eigenvalue $\mu_0 = 0$ of $\mathbf{L}$ on $G(x)$ is the same for all x in the interior of P (and consists of constant functions). Let Π_0 be the operator of projection onto E_0; its matrix is given by $(1/n)\,J$, where J is the matrix of 1-s. The eigenvalues of

$$M \;=\; \mathbf{L} \;+\; \Pi_0$$

are $\{1, \mu_1, \ldots, \mu_n\}$, hence

$$n^2\,\tau(G) \;=\; \det(J + \mathbf{L}). \tag{17}$$

This result is due to Temperley ([Tem]). Thus it suffices to study the concavity of the positive-definite operator M.

Theorem 2.4. *The function $\tau(G)$ is concave on (the interior of) P.*

The theorem immediately implies

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[4]See [Bou] for examples of edge-transitive graphs which are not vertex-transitive.

Corollary 2.5. *If z is a critical point of $\tau(G)$ in the interior of P, then it is the global maximum for $\tau(G)$. In particular, the unweighted graph is the global maximum for equiarboreal graphs.*

Proof of Theorem 2.4. It suffices to prove that the determinant is a concave function on the set of positive definite symmetric matrices. Let Q be such a matrix, and let

$$Q(t) \;=\; Q + tB, \;\; t \in \mathbf{R}$$

be a line of symmetric matrices through Q. Then

$$\frac{d \log \det(Q(t))}{dt} \;=\; \operatorname{tr}(BQ^{-1}),$$

and

$$\frac{d^2 \log \det(Q(t))}{dt^2} \;=\; - \operatorname{tr}(BQ^{-1}BQ^{-1}).$$

It suffices to show that the last trace is strictly positive. The matrix $R = Q^{-1}$ is positive definite. Conjugate R to a diagonal matrix with eigenvalues $\underline{\mathrm{w}} = (w_1, \ldots, w_n)$. Then

$$\operatorname{tr}(BRBR) \;=\; \underline{\mathrm{w}} S \, \underline{\mathrm{w}}^t$$

where $S_{ij} = R_{ij}R_{ij}$ is the *Hadamard product* of R with itself. By Schur product theorem ([HJ, Theorem 7.5.3]), S is positive definite, which finishes the proof. □

We remark that Theorem 2.4 can be considered as an analogue of the convexity of the determinant of Laplacian in a conformal class of metrics on surfaces of genus greater than or eqaul to one, proved by Osgood, Phillips and Sarnak in [OPS]. It would be interesting to characterize graphs G where $\tau(G)$ achieves its maximum in the interior of $P(G)$.

2.4. **Eigenvalues of the Laplacian.** We introduce some notation which will be used in studying the Taylor expansion of the eigenvalues.

Given two points $x^0, x^1 \in P(G)$, let $y = x^1 - x^0$ and let

$$x(t) \;=\; x^0 \;+\; ty, \;\; 0 \le t \le 1 \tag{18}$$

be the segment (call it $\pi(t)$) connecting the two points in P. The normalization condition reads

$$y \cdot \underline{\omega}^t \;=\; 0 \tag{19}$$

The basis of all y-s satisfying (19) is given by vectors

$$\underline{\mathrm{b}}_{ij} \;=\; \underline{\mathrm{e}}_i - \underline{\mathrm{e}}_j \tag{20}$$

for $i \neq j$ where $\underline{\mathrm{e}}_i$-s are the standard basis vectors.

We next turn to studying how the eigenvalues of the Laplacian depend on t. We look at one parameter family $\mathbf{L}(t)$ of Laplacians on G corresponding to the points on $\pi(t)$. The Laplacian is given by the matrix

$$\tilde{\mathbf{L}}(t) \;=\; \mathbf{L} \;+\; tB$$

By standard results in perturbation theory ([Kat]), there exists an orthonormal basis of the eigenvectors of Q such that the the eigenvalues and the eigenvectors admit Taylor expansions in t.

Let $\mu = \mu(0) > 0$ be an eigenvalue of the multiplicity $p \geq 1$. Let E_μ be the corresponding (p-dimensional) eigenspace, and let P_μ denote the corresponding projection. The condition for the graph G to be a critical point for μ is that the matrix $P_\mu B P_\mu$ be indefinite for all B. We now compute the eigenvalues of $P_\mu B P_\mu$ for the edge weightings given by (18) with $\underline{b}$ as in (20). We distinguish between two cases:

a) edges e_1 and e_2 are disjoint.
b) edges e_1 and e_2 share an endpoint

We first consider the case a). Let $f_1, f_2, \ldots, f_p : V \to \mathbf{R}$ be a basis of eigenvectors of E_μ. For convenience, we denote $f_j(v_1) - f_j(v_2)$ by z_j and $f_j(v_3) - f_j(v_4)$ by w_j. The eigenvalues of $P_\mu B P_\mu$ are given by the eigenvalues of the linear form $F(\alpha_1, \ldots, \alpha_p)$ defined by

$$F(\alpha_1, \ldots, \alpha_p) = \langle B(\alpha_1 f_1 + \ldots + \alpha_p f_p), f_j \rangle \tag{21}$$

It is easy to see that the matrix of $F = F_{ij}$ is given by

$$F_{ij} = z_i z_j - w_i w_j$$

The eigenvalue μ is critical if F is indefinite for any choice of B.

We remark that F is a difference of two rank one matrices, $F = z \otimes z - w \otimes w$ where $z = (z_1, \ldots, z_p)$ and $w = (w_1, \ldots, w_p)$ and so has rank at most two. Computing the coefficient C_{p-2} of t^{p-2} in the characteristic polynomial of F, we find that

$$C_{p-2} = - \sum_{1 \leq i < j \leq p} (z_i w_j - z_i w_j)^2.$$

Therefore, $C_{p-2} \leq 0$ and is strictly negative unless x and y are proportional or one of them is zero. Now, if $C_{p-2} < 0$ then F is indefinite (since it has at most two nonzero eigenvalues). If $C_{p-2} = 0$ (and F has rank one) and if μ is critical then $F \equiv 0$.

Consider first the case of a simple eigenvalue ($p = 1$). Then μ is critical if $F = F_{11} = 0$ for every choice of B and therefore $z_1^2 - w_1^2 = 0$ or $z_1 = \pm w_1$. Recalling the definition of z_1 and w_1, we see that for the eigenvector $f_1 = f$

$$|f(v_1) - f(v_2)| = |f(v_3) - f(v_4)| \tag{22}$$

The following proposition is proved by repeating the argument for all pairs of nonadjacent edges; if some edge e of G is adjacent to every other edge, we prove an analogue of (22) for the case b) of adjacent edges (it was proved before for the case a) of non-adjacent edges).

Proposition 2.6. *A simple eigenvalue μ of the Laplacian on G is critical if and only if the corresponding eigenvector $f : V \to \mathbf{R}$ satisfies*

$$|f(u) - f(v)| \equiv const \tag{23}$$

We now study the graphs which admit an eigenvector $f : V \to \mathbf{R}$ satisfying (23) for some $c \geq 0$. If $c = 0$ then f is a multiple of a constant vector and so has eigenvalue zero which is a contradiction. If $c \neq 0$ then it is easy to see that the graph G cannot have odd cycles and hence is bipartite. Namely, let $u_1 u_2 \ldots u_l$ be a cycle. Then (putting $u_l = u_0$) $\sum_{i=1}^{l}(f(u_i) - f(u_{i-1})) = 0$. But each term in the sum is equal to $\pm c$, and since the number of terms in the sum is odd, they cannot add up to 0.

We now want to study the unweighted k-regular graphs which have an eigenvector (corresponding to an eigenvalue $\mu > 0$) satisfying (23) (without necessarily assuming that μ is simple). We shall rescale the eigenvector so that $c = 1$ in (23). From (8) and (23) it follows that for each vertex u the expression $\mu \cdot f(u)$ can only take one of the values $k, k-2, k-4, \ldots, -k+2, -k$. Consider first the vertex u_0 where $f(u)$ takes its maximal value a (by changing the sign if necessary we can assume that $a > 0$). It follows that f takes value $a - 1$ on all the neighbors of u_0, hence

$$a\mu = k$$

Next, consider any neighbor u_1 of u. The value of f at any neighbor of u_1 can be either a (let there be $r_1 \geq 1$ such neighbors; u_0 is one of them); or $a - 2$ (it follows that there are $k - r_1$ such neighbors). From (8) it follows that

$$\mu(a - 1) = k - 2r_1$$

It follows from the last two formulas that

$$\mu = 2r_1 \tag{24}$$

where $r_1 \geq 1$ is a positive integer. If $r_1 = k$, then $\mu = 2k$ is the largest eigenvalue of $\mathbf{L}$.

We next define the *level* of a vertex u to be equal to j if $f(u) = a - j$; we denote the set of all vertices of G at level j by G_j. It is easy to see that if $u \in G_j$ has r_j neighbors where f takes value $a - j + 1$ then

$$\mu(a - j) = k - 2r_j$$

It follows that r_j is the same for all $u \in G_j$. Using (24) we see that

$$r_1 \cdot j = r_j$$

Consider now a "local minimum" $u \in G_N$. Then $r_N = k$, and we see that

$$r_1 \mid k \tag{25}$$

Let n_j denote the number of vertices in G_j. Counting the vertices connecting G_j and G_{j+1} in two different ways, we see that for all $0 \leq j \leq N-1$,

$$n_j(k - r_j) = n_{j+1}r_{j+1}$$

Consider the case $r_1 = 1, \mu = 2$. It follows from the previous calculations that $r_j = j$ and that $N = k$. Accordingly, $n_j = n_0 \binom{k}{j}$ and

$$|G| = 2^k n_0 \tag{26}$$

We next describe a class of graphs admitting an eigenvector of $\mathbf{L}$ with $\mu = 2$ satisfying (23).

An obvious example of such a k-regular graph is the k-cube, and any such graph has the same number of vertices as a disjoint union of n_0 cubes by (26). Start now with such a union, choose the partition of the vertices of each cube into "levels" and take two edges $u_1 u_2$ and $u_3 u_4$ in two different cubes such that u_1, u_3 are both in level j while u_2, u_4 are both in level $j+1$. If we perform an edge switch

$$(u_1 u_2), (u_3 u_4) \ \rightarrow \ (u_1 u_4), (u_3 u_2)$$

then the number of the connected components of our graph will decrease while the eigenvector f will remain an eigenvector with the same eigenvalue.

Performing sequences of edge switches as described above, we obtain examples of connected graphs satisfying (23) and (26) for any n_0. Conversely, it is easy to show that starting from a graph satisfying (23) and (26) and having chosen a partition of its vertices into levels one can obtain n_0 disjoint k-cubes by performing a sequence of edge switches as above.

We now want to consider the case when $\mu = \mu_1$ is the lowest eigenvalue of the Laplacian. The first remark is that then necessarily $\mu \leq k$, and $\mu = k$ only if $G = K_{k,k}$. Next, we want to consider "small" k for which $k - 2\sqrt{k-1}$ (the "Ramanujan bound") is less than 2 (this happens for $3 \leq k \leq 6$). It then follows from the results of Alon ([Nil]) that the diameter of G (and hence the number of vertices in G) is bounded above.

Proposition 2.7. *For $3 \leq k \leq 6$ there are finitely many k-regular graphs for which the condition (23) is satisfied for an eigenvector of μ_1.*

We next discuss graphs which have an eigenvector satisfying (23) with the eigenvalue $\mu = 2r_1 > 2$. Recall that by (25) $r_1 | k$. By counting the edges connecting the vertices in two consecutive levels one can show (as for $\mu = 2$) that the number of vertices satisfies

$$|G| \ = \ 2^{(k/r_1)} n_0 \tag{27}$$

Also, since any vertex $u_1 \in G_1$ has r_1 distinct neighbors in G_0,

$$n_0 \geq r_1.$$

It is easy to construct examples of regular graphs which have eigenvectors with the eigenvalue $\mu > 2$ satisfying (23); the construction is similar to that for $\mu = 2$.

We summarize the previous results:

Theorem 2.8. *Let G be a k-regular graph which has an eigenvector of $\mathbf{L}$ with an eigenvalue μ satisfying (23). Then G is bipartite, $\mu = 2l$ is an even integer dividing $2k$, the number of vertices of G is divisible by $2^{(k/l)}$, and for $n_0 \geq l$ there exist such graphs with $n = 2^{(k/l)} n_0$ vertices.*

3. Some heuristics for the number of spanning trees

Here we shall give some heuristic arguments concerning some properties that graphs with many spanning trees are expected to have. One such property is having large girth; the sequences of cubic graphs described in section 5 seem to support that expectation. We shall give some heuristic arguments in its support.

We start with the following formula from [Mck83, p. 153] for the number $\tau(G)$ of spanning trees of a k-regular graph G with n vertices:

$$\tau(G) \;=\; kI_k(t) \lim_{t \to 0_+} \left(\frac{n}{n-1} + \frac{1}{(n-1)I_k(t)} \sum_{r=3}^{\infty} \binom{t}{r} (-1)^r u_r k^{-r} \right)^{1/t}, \tag{28}$$

where $I_k(t)$ is a function depending on the valency k only (and hence *the same* for all k-regular graphs), while u_r is the number of walks which are not *totally reducible* (i.e. which don't "come" from the infinite tree T_k). In particular, $u_r = 0$ for $r < \gamma(G)$.

The coefficients $(-1)^r \binom{t}{r}$ in the sum in (28) are always negative for small t, so each nonzero term in the sum *decreases* the value of the expression in brackets in (28). Heuristically then, the more u_r-s are equal to zero, the larger the expression in brackets is in (28) and so the larger is the limit which is $\tau(G)$. Now, if we want to compare $\tau(G)$ for different graphs in $\mathcal{X}_{n,k}$ those with larger girth $\gamma(G)$ have more u_r-s equal to zero and therefore are the natural candidates for large values of $\tau(G)$, and even more so Moore graphs, since they are the unique graphs in $\mathcal{X}_{n,k}$ having the largest girth.

Of course, we have only looked at the first nonzero term in the infinite series, but we remark that the subsequent u_r-s get multiplied by $(-1)^r \binom{t}{r} k^{-r}$ which is exponentially decreasing in r. This seems to give at least a partial heuristic explanation of the cages appearing in section 5.

It also seems reasonable to conjecture that graphs with higher connectivity have more spanning trees. Indeed, one can show that if an edge e is a bridge in a one-connected graph $G = G_1 \cup \{e\} \cup G_2$ and if G_1 (say) is two-connected, then one can make an edge switch in G which will increase the number of spanning trees (cf. [Kel97] for the discussion of operations on graphs increasing their number of spanning trees). However, it seems to be more difficult to prove that for a two-connected graph there exists an edge switch which simultaneously increases the number of spanning trees. We believe this to be true in essentially all cases, however.

4. Graphs on a fixed surface

4.1. **The tree number for regular graphs on a surface.** In this section we shall give an upper bound for the number of spanning trees for 6-regular graphs on a fixed surface S which is strictly smaller than the McKay's bound (3) for general 6-regular graphs.

Let G be a k-regular simple graph on n vertices embedded in an orientable surface S_g of genus g. Let V, E, F denote the number of vertices, edges and

faces of G. By Euler's formula,

$$F \ = \ E - V + 2 - 2g \ \geq \ n(k/2 - 1) + 2 - 2g.$$

(the inequality becomes an equality if all faces are simply-connected). It follows easily that if infinitely many simple k-regular graphs can be embedded in a fixed surface S, then $k \leq 6$. We denote by $\mathcal{X}_{n,k,g}$ the set of all simple k-regular graphs embeddable on the orientable surface S_g.

McKay proves in [Mck83] that for a random k-regular graph on n vertices, $(\tau(G))^{1/n} \to \sigma_k = (k-1)^{k-1}/(k(k-2))^{k/2-1}$ as $n \to \infty$. We now state the main result of this section:

Theorem 4.1. *Let* $\mathcal{X}_{n,k,g}$ *be as above. Then*

$$\lim_{n \to \infty} \left(\sup_{G \in \mathcal{X}_{n,6,g}} \tau(G)^{1/n} \right) \ \leq \ \sigma_3^2 \ < \ \sigma_6. \tag{29}$$

To prove the theorem we first establish a proposition estimating the ratio of the tree number of G to the tree number of its dual graph G^*. We next remark that the dual of a 6-regular graph on n vertices is "roughly" a 3-regular graph on $2n$ vertices; an application of McKay's theorem finishes the proof.

Proposition 4.2. *Let* $\{G_j\}$ *be a sequence of simple graphs on an orientable surface S of genus g such that their maximal degrees are bounded. Let $\tau(G)$ and $\tau(G^*)$ be the corresponding numbers of spanning trees, and let $n_j \to \infty$ be the number of vertices of G_j. Then*

$$n_j^{-2g} \ \ll \ \frac{\tau(G_j)}{\tau(G_j^*)} \ \ll \ n_j^{2g}. \tag{30}$$

Proof. We first define a mapping from the spanning trees of G_j into (subsets of) the spanning trees of G_j^* as follows: take a spanning tree T of G_j, and consider the complement $S = G_j - T$ of T in G_j. It is easy to see ([Big71]) that S is connected, spans G_j^* and that one can get a spanning tree of G_j^* by removing at most $2g$ edges of S. We denote by $F(T)$ the set of all spanning trees of G_j^* which can be obtained from T by the above procedure. By reversing the construction, it is easy to see that the mapping $T \to F(T)$ is onto.

Let $|V(G_j)| = n_j$, and let the maximal degree be bounded above by δ. Let T be a spanning tree of G_j, and let $S = S(T) = G_j - T$. Then $|S| = |E(G_j)| - n_j + 1 \leq n_j(\delta/2 - 1) + 1$ and so

$$|F(T)| \ \leq \ \binom{|S(T)|}{2g} \ \ll \ n_j^{2g},$$

Adding the inequalities for all T proves one of the bounds in (30). The other inequality is proved similarly. We remark that the constants in (30) depend on δ, k and g only.

$\square$

Consider now a simple 6-regular graph G (embedded on a surface S_g) with n vertices and $m = 3n$ edges, and let G^* be the dual graph (of genus $h \leq g$). G^* has $m' = 3n$ edges and at least $2n + 2 - 2h \geq 2n + 2 - 2g$ vertices; moreover, since G is simple, every face of G has at least three sides, so the minimal degree of G^* is at least three. It is easy to see that G^* can be obtained from a (possibly disconnected) graph G_1 of maximal degree at most 3 with at most $2n + 2$ vertices by adding $2h$ "new" edges between the vertices of G_1.

Proof of Theorem 4.1. Let G be a graph in $\mathcal{X}_{n,6,g}$. By Proposition 4.2, it suffices to estimate $\tau(G^*)$. Consider a spanning tree T of G^* containing l of the "new" edges $G^* - G_1$, where $0 \leq l \leq 2h \leq 2g$. Removing the l edges from T produces a spanning forest of G_1 with $l \leq 2h$ components. It suffices to establish the estimate (29) for l-component spanning forests of G_1 for each l separately. For $l = 0$ the estimate follows easily from McKay's estimate (3) for cubic graphs on at most $2n + 2$ vertices.

To establish the estimate for $l \geq 1$, we first remark that the number of the connected components of G_1 is at most $2h \leq 2g$; let $H_1, \ldots, H_p$ be denote these components, and let H_k have q_k vertices, where $q_1 + \ldots + q_p \leq 2n + 2$. Each component $H = H_k$ is a graph of maximal degree at most 3.

Let F be an l-component spanning forest of G_1. Let $F \cap H_k = F_k$ be a spanning forest of H_k with l_k components, where $l_1 + \ldots + l_p = l$. F_j can be obtained from a spanning tree of H by removing l_k edges from the tree. Each tree has $q_k - 1$ edges, so one can get at most $\binom{q_k-1}{l_k}$ spanning forests by removing l_k edges from it, so the total number of such forests is at most

$$\tau(H_k) \binom{q_k - 1}{l_k}$$

Multiplying the inequalities for all H_k-s we estimate the number of l-component spanning trees of G_1 by

$$\prod_{k=1}^{p} \tau(H_k) \prod_{k=1}^{p} \binom{q_k - 1}{l_k} \tag{31}$$

Now, let a sequence $G_j \in \mathcal{X}_{6,n_j,g}$ be a sequence of 6-regular graphs on S_g, and let $n_j \to \infty$. Taking the n_j-th root in (31) and letting $n_j \to \infty$ shows that it suffices to prove that

$$\limsup_{n_j \to \infty} \left(\prod_{k=1}^{p_j} \tau(H_{k,j}) \right)^{1/n_j} \leq \sigma_3^2 \tag{32}$$

where the subscript j refers to the graph G_j.

In order to arrive at a contradiction, we shall assume that there exists a sequence of graphs G_j on S_g such that the limit in (32) is strictly greater than σ_3^2. Passing to a subsequence if necessary we may assume that the number $1 \leq p_j \leq 2g$ of components of $G_{1,j}$ is actually constant and equal to p. We shall order the components $H_{k,j}$ so that $q_{k,j}$-s are non-decreasing.

Passing once again to a subsequence if necessary, we may assume that there exists a nonnegative number $r < p$ and a sequence $1 \leq q_1 \leq q_2 \leq \ldots \leq q_r$ such that $q_{k,j} = q_k$ for $k \leq r$, and $q_{k,j} \to \infty$ for $k > r$. Now applying McKay's bound (3) for each $k > r$ (with $n = q_{k,j}$) and estimating each of the first r terms in (32) by an absolute constant leads easily to a contradiction, finishing the proof of Theorem 4.1.

□

The bound proved in Theorem 4.1 is not optimal. We define the optimal bound for $3 \leq k \leq 6$:

$$\lim_{n \to \infty} \left(\sup_{G \in \mathcal{X}_{n,k,g}} (\tau(G))^{1/n} \right) := \sigma_{k,g} \tag{33}$$

It follows from the proof of Theorem 4.1 that for $g_1 < g_2$,

$$\sup_{G \in \mathcal{X}_{n,k,g_2}} \tau(G) \ll n^{2g_2} \left(\sup_{G \in \mathcal{X}_{n,k,g_1}} \tau(G) \right).$$

It follows that the limit in (33) does not depend on g (as long as $\mathcal{X}_{n,k,g}$ is not empty for large n, which can happen for small g). Accordingly, we define $\sigma_s(k)$ to be[5] the limit $\sigma_{k,g}$ achieved in (33) for some fixed $g \gg 0$ and $3 \leq k \leq 6$. Noga Alon has informed the authors that he can show that

$$\sigma_s(k) < \sigma_k$$

(where σ_k is defined in (3)).

It follows from the proof of Theorem 4.1 that $\sigma_s(6) \leq \sigma_s(3)^2$. One can actually show by a similar argument that

$$\sigma_s(6) = \sigma_s(3)^2$$

We make a conjecture about the graphs that achieve the limit in (33) for $k = 3, 4$:

Conjecture 4.3. *The limit $\sigma_s(3)$ (respectively, $\sigma_s(4)$) is achieved by quotients of the hexagonal lattice (respectively, the square lattice) in $\mathbf{R}^2$.*

It would follow from Conjecture 4.3 (cf. [CDS, pp. 246–247]) that

$$\sigma_s(4) = e^{4G/\pi} \approx 3.21 < 3.375 = \sigma_4 \tag{34}$$

Here G is Catalan's constant. The exponent $4G/\pi$ in (34) is equal to the toplogical entropy

$$\int_0^1 \int_0^1 \log(4 - 2\cos(2\pi x) - 2\cos(2\pi y)) \, dx dy$$

of the *essential spanning forest process* on the nearest neighbor graph on $\mathbf{Z}^2$ ([Solo]). The integral above is equal to the *Mahler measure* of the polynomial $4 - x - 1/x - y - 1/y$; it is equal to

$$2L'(\chi, -1)$$

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[5]Here the subscript s stands for *surface*.

where $L(\chi, s)$ is the Dirichlet L-function associated to the quadratic extension $\mathbf{Q}(\sqrt{-3})/\mathbf{Q}$ (cf. [RV]). It would be interesting to find candidates for achieving $\sigma_s(5)$.

4.2. **Jacobians of regular graphs.** In this section we want to establish an analog for graphs of a result of Buser and Sarnak ([BS]) about the rate of growth of the minimal norm of a period matrix of a Riemann surface. They have shown that as the genus of a surface increases, the minimal norm grows at most logarithmically in the genus.

Bacher, de la Harpe and Nagnibeda in their work ([BHN, Nag]) defined two lattices associated to any unweighted graph, *the lattice of integral flows* Λ^1 and *the lattice of integral coboundaries* N^1. The quotient $(\Lambda^1)^{\#}/\Lambda^1$ is a finite abelian group called a *Jacobian* of a graph (in an analogy to Jacobians of Riemann surfaces).

We shall now consider k-regular graphs for some fixed $k \geq 3$. Let $G \in \mathcal{X}_{n,k}$; the dimension of $\Lambda^1(G)$ is equal to $kn/2 - n + 1$. The determinant of $\Lambda^1(G)$ is equal to the tree number $\tau(G)$, and the (unnormalized) minimal norm $\tilde{\nu}(G)$ of $\Lambda^1(G)$ is equal to the girth of G.

We rescale Λ^1 so that the volume of its fundamental domain is equal to 1; the (normalized) minimal norm $\nu(G)$ is given by

$$\nu(G) \;=\; \gamma(G)\,\tau(G)^{2/(kn/2-n+1)}.$$

From (3) and (4) we conclude that for every $\varepsilon > 0$,

$$\delta_k^{2/(k/2-1)} - \varepsilon \;<\; \tau(G)^{2/(kn/2-n+1)} \;<\; \sigma_k^{2/(k/2-1)} + \varepsilon$$

for n large enough.

We want to study the behavior of $\nu(G)$ when k is fixed and $n \to \infty$. Using the known results about the behavior of the girth, we conclude:

Proposition 4.4. *The minimal norm $\nu(G)$ of the Jacobian of a k-regular graph $G \in \mathcal{X}_{n,k}$ satisfies*

$$\nu(G) \;=\; O(\log_{k-1} n)$$

This establishes an analogue of the results of Buser and Sarnak.

5. Some explicit constructions of cubic graphs

It is known ([BG, Kotz, Tut]) that every 3-connected cubic graph may be constructed from a tetrahedron by a sequence of *edge insertions*: given a cubic graph on $2n$ vertices, choose a pair of edges, subdivide them by putting a single vertex in the middle, and connect the two subdivision vertices to get a cubic graph on $2n + 2$ vertices. So, every 3-connected cubic graph G may be constructed from the tetrahedron provided we know the "history," i.e. the sequence in which insertions were applied to successive pairs of edges. Clearly, that "history" is not unique. However, one may hope that if G is an extremal graph, among all its histories there exist ones with some nice properties. Below we shall give some examples when that is indeed the case.

The first example is the following sequence S: starting from a tetrahedron, insert edges in such a way that the number of spanning trees is maximized after each application. When the number of vertices is small, one may find the graphs in S from the table of graphs in [Val]. The 6-vertex graph is $K_{3,3}$, the 8-vertex graph is a Möbius ladder (an 8-cycle with the pairs of "opposite" vertices connected, cf. [Big93, §3e]); it has 392 trees. The 10-vertex graph is the Petersen graph P (2000 trees), and the 12-vertex graph is obtained from P by applying A to a pair of edges at distance 2 from each other (this is graph # 84 in [Val]; it has 9800 spanning trees). The last graph also appears in [Hol]. From the table of graphs in [Val] one can check that the graphs in the sequence have the most trees among the cubic graphs with the same number of vertices.

The rest of the graphs in the sequence S are shown in the Appendix; they were obtained by a computer program. It is remarkable that the sequence contains all $(3, g)$-cages for $3 \leq g \leq 8$. Namely, the 14-vertex graph in S is the Heawood graph, the 24-vertex one is the McGee graph, and the 30-vertex one is the Tutte-Coxeter graph (cf. [Won] or [HS, Ch. 6]). Unfortunately, the graphs in S with 32 or more vertices seem to be less interesting. The first few graphs in the sequence appear in [AS] (but without the reference to the tree number); however, the inclusion of the 12-vertex graph in the sequence and the graphs with 16 or more vertices appear to be new. The authors hope that similar constructions may lead to discovery of new extremal graphs.[6]

Edge insertion has its disadvantages: for example, certain cages of even girth have the property that every two edges are contained in a common shortest cycle ([KT]); inserting an edge will then certainly decrease the girth. So, one might try adding several new vertices and edges simultaneously. For cubic graphs, one can try subdividing several edges of an "old" graph (with one, two or more points on an edge) to get an *even* number of "new" vertices of valence 2, then choosing a perfect matching for the new vertices and connecting pairs of these vertices according to the perfect matching to get a new cubic graph.

Several cubic cages can be obtained by taking uniform subdivisions of smaller graphs and inserting edges:[7]

- The Petersen graph can be constructed from a tetrahedron by subdividing its 6 edges;
- The Heawood graph can be obtained from the unique loopless two-vertex cubic graph (G_2) by putting 4 new vertices on each of its 3 edges;

---

[6]Other sequences of cubic graphs might be constructed using other graph invariants (e.g. mean distance – cf. [CCMS] – instead of the tree number) in the algorithm; one can also start from a "large" cubic graph and delete an edge together with its endpoints so that a given invariant is maximized. For regular graphs of degree $k \geq 4$ one can use operations similar to the edge insertion to increase the number of vertices and construct sequences of k-regular graphs by maximizing a certain invariant.

[7]Similar examples appear in [AS].

- The Tutte graph can be obtained from $2K_{3,3}$ by subdividing each of its 18 edges.

This section describes a sequence of cubic graphs generated (by a computer) from the tetrahedron by edge insertion where at each stage the insertion was chosen so that the number of spanning trees of he new graph was the maximal possible. The first six graphs are shown in Figure 1; the endpoints of the new edge inserted into a cubic graph on $2n$ vertices were numbered $2n + 1$ and $2n + 2$.
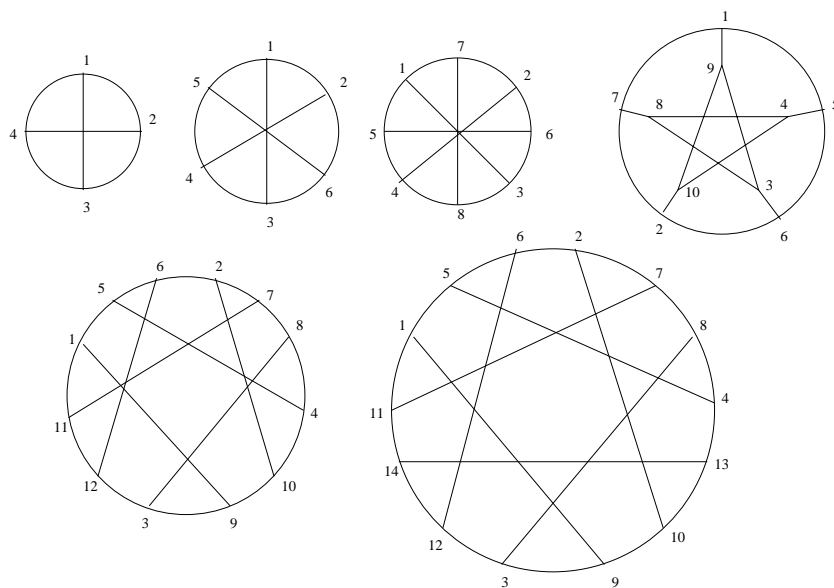

Figure 1. The first six graphs in a sequence S.

The graphs with 18 and 24 vertices (as they appear in S) are shown in Figure 2.
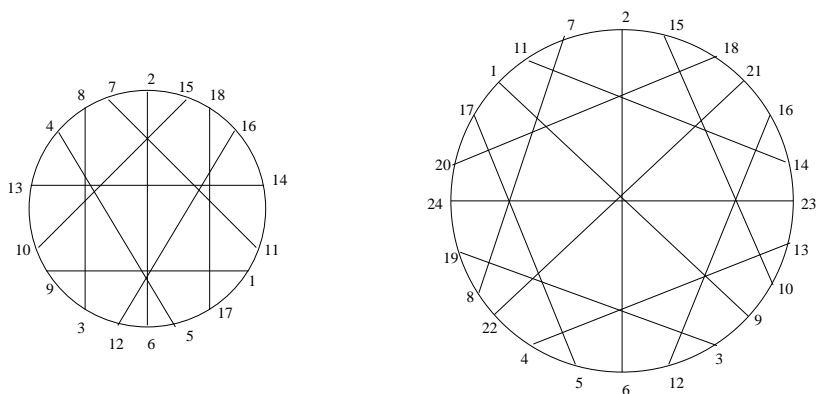
Figure 2. The 18-vertex graph and the McGee graph.

The Tutte-Coxeter graph (as it appears in S) is shown on Figure 3.
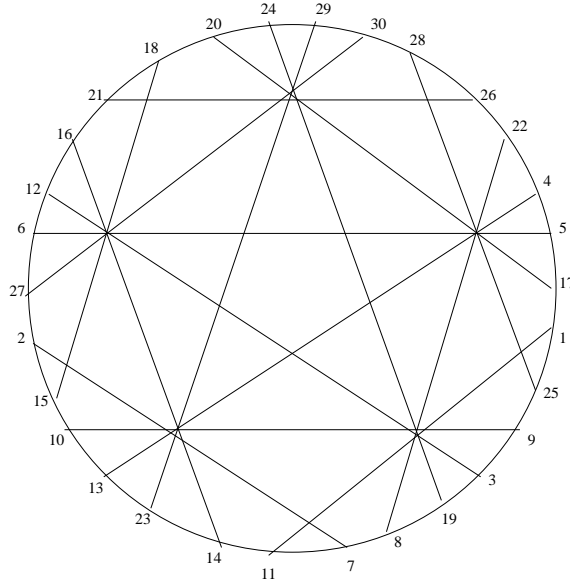

Figure 3. The $(3, 8)$-cage.

In Figure 4 we show how to construct Petersen and Heawood graphs by edge insertions into the tetrahedron and the graph G_2 respectively. The vertices of K_4 and G_2 are shown in black.
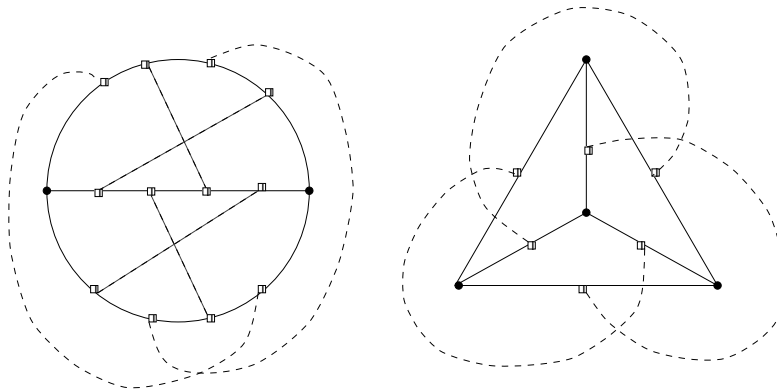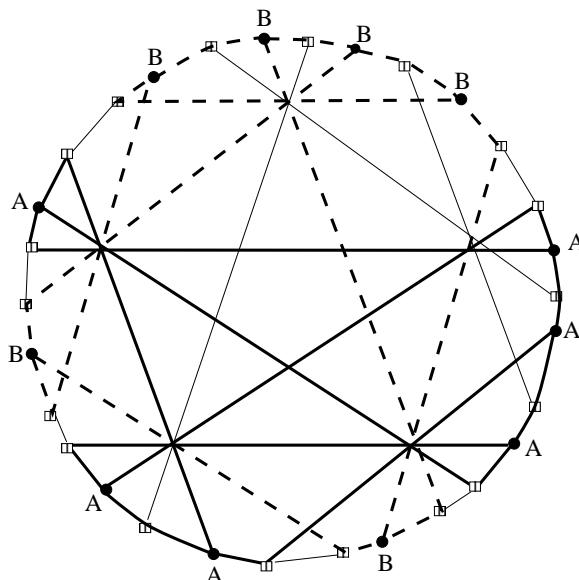

Figure 4. Petersen and Heawood graphs.

In Figure 5 we show how to construct the Tutte graph by edge insertions into $2K_{3,3}$. The vertices and the edges of $2K_{3,3}$ are shown in black; the vertices of the two copies of $K_{3,3}$ are marked A and B respectively.

Figure 5. The Tutte graph obtained from $2K_{3,3}$.

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