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ERGODIC PROPERTIES OF LINEAR DYNAMICAL SYSTEMS*

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Abstract. The Multiplicative Ergodic theorem, which gives information about the dynamical structure of a cocycle Φ, or a linear skew product flow π, over a suitable base space $\mathbf{M}$, asserts that for every invariant probability measure μ on $\mathbf{M}$ there is a measurable decomposition of the vector bundle over $\mathbf{M}$ into invariant measurable subbundles, and that every solution with initial conditions in any of these subbundles has strong Lyapunov exponents. These exponents depend on the measure μ, and when μ is ergodic, they are constant (almost everywhere) on $\mathbf{M}$ and form a finite set meas $\Sigma(\mu)$. The dynamical spectrum dyn Σ consists of those values $\lambda \in \mathbf{R}$ for which the shifted flow π_λ fails to have an exponential dichotomy over $\mathbf{M}$. The Spectral theorem for linear skew product flows states that when $\mathbf{M}$ is compact and dynamically connected then dyn Σ is the finite union of k disjoint compact intervals and the vector bundle over $\mathbf{M}$ is the sum of k continuous invariant subbundles. We show that

$$\text{Boundary dyn } \Sigma \subseteq \bigcup_{\mu} \text{meas } \Sigma(\mu) \subseteq \text{dyn } \Sigma$$

where the union above is over all ergodic measures μ on $\mathbf{M}$. Also we show that the measurable invariant subbundles which arise in the Multiplicative Ergodic theorem form a refinement of the continuous invariant subbundles described in the Spectral theorem. A new proof of the Multiplicative Ergodic theorem is presented here. This proof is a substantial simplification over other arguments. Applications of the theory of Lyapunov exponents to "spiral" systems, products of "random" matrices, stochastic differential equations, and the almost periodic Schrödinger operator are included.

Key words. ergodic properties, linear dynamical system, Lyapunov exponents

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1. Introduction. Nearly two decades ago Oseledec (1968) published his proof of the Multiplicative Ergodic theorem. This theorem, which is one of the milestones in the study of ergodic properties of dynamical systems, has had far-reaching applications, including its role in the work of Margulis (1975) on arithmeticity in Lie groups, in the theory of Pesin (1977) on Bernoullian substructures for diffeomorphisms, in the theory of Katok (1980) on entropy and periodic points, in the study of Kotani (1982) on spectral measures for Schrödinger operators, in the work of Constantin and Foias (1983) on attractors in the Navier–Stokes equations, and in the study of Novikov (1975) and Millionscikov (1978) on almost reducible systems with almost periodic coefficients. As a testimony to the importance of this theorem one finds several alternative proofs including the contemporaneous paper of Millionscikov (1968), and those of Raghunathan (1979), Ruelle (1979) and Crauel (1981), as well as the anticipatory paper of Liao (1966).

The Multiplicative Ergodic theorem gives information about the dynamical structure of a cocycle Φ, or a linear skew product flow π, over a suitable base space $\mathbf{M}$. In typical applications the base space $\mathbf{M}$ is either an attractor, a compact invariant set, or the space of coefficients for a diffeomorphism, a differential equation, or a vector field. This theorem asserts that for every invariant probability measure μ on $\mathbf{M}$ there is a measurable decomposition of the vector bundle over $\mathbf{M}$ into invariant measurable

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subbundles, and that every solution with initial conditions in any of these subbundles has strong Lyapunov exponents. These exponents, or growth rates, depend on the measure μ, and when μ is ergodic, they are constant (almost everywhere) on $\mathbf{M}$ and form a finite set meas $\Sigma(\mu)$, the measurable (Millionscikov–Oseledec) spectrum.

The main objective in this paper is to study the connection between the measurable spectrum meas $\Sigma(\mu)$ and the dynamical spectrum dyn Σ introduced by Sacker and Sell (1975), (1978), (1980). (Also see Daletskii and Krein (1974), as well as Selgrade (1975).) The dynamical spectrum dyn Σ consists of those values $\lambda \in \mathbf{R}$ for which the shifted flow π_λ fails to have an exponential dichotomy over $\mathbf{M}$. It follows from the Spectral theorem for linear skew-product flows, Sacker and Sell (1978), that the dynamical spectrum is the finite union of disjoint compact intervals when $\mathbf{M}$ is compact and dynamically connected.

The dynamical spectrum and the theory of exponential dichotomies are central concepts in wide-ranging branches of analysis including the perturbation theories for invariant manifolds (see Sacker (1969), Fenichel (1971) and Hirsch, Pugh and Shub (1977), the bifurcation theories of Chenciner and Iooss (1979) and Sell (1979), the characterization of the spectrum of Schrödinger operators in Johnson (1982), linearization theories near invariant manifolds in Sell (1984), the study of transversal homoclinic orbits in Palmer (1984), as well as the study of inertial manifolds for dissipative systems in Foias, Sell and Temam (1985).

It is important therefore to understand the connection between these two spectral concepts. We will show, in § 8, that

$$\text{boundary dyn}\,\Sigma \subseteq \bigcup_{\mu} \text{meas}\,\Sigma(\mu) \subseteq \text{dyn}\,\Sigma, \tag{1.1}$$

where the union above is over all ergodic measures μ on $\mathbf{M}$. We actually derive much more than (1.1). We show that the measurable invariant subbundles which arise in the Multiplicative Ergodic theorem form a refinement of the continuous invariant subbundles described in the Spectral theorem. The relationship (1.1) also leads to good methods for computing the Lyapunov exponents and the continuous spectral bundles (see Perry (1986)).

Another objective is to show that the cocycle Φ, itself, has a strong Lyapunov exponent (almost everywhere) and that this agrees with max meas $\Sigma(\mu)$. Although simple, this fact is very important because it forms the foundation for deriving an approximation theory which leads to the numerical evaluation of the measurable and dynamical spectra. The approximation theory and the related numerical coding is described in the University of Minnesota Ph.D. thesis of David Perry (1986).

While doing this investigation we discovered a new proof of the Multiplicative Ergodic theorem. Since our proof is a substantial simplification over other arguments, we present it here. In addition to this simplification, our proof has some interesting geometrical features which may be useful elsewhere. While our proof of the Multiplicative Ergodic theorem is restricted to cocycles over a compact base space $\mathbf{M}$, we will see that this includes practically every application. Among other things, our theory applies to linear stochastic differential equations with bounded measurable coefficients, as well as to the linearized flow near an attractor in a nonlinear dynamical system.

A final objective of this paper is the presentation of several applications of these spectral theories. One of these theories, the theory of Lyapunov exponents for "spiral" systems, is central to any numerical investigation of Lyapunov exponents. Other applications include products of "random" matrices, stochastic differential equations, and the almost periodic Schrödinger operator.

This paper is organized as follows: In § 2 we present the statements of the main theorems in this paper. Section 3 is concerned with a number of technical details which shall be used in the proofs of our theorems. One may wish to skip this on the first reading. In § 4 we present the basic triangularization method as it applies to linear skew product flows. Section 5 is concerned with a brief review of some basic facts about invariant measures, and in § 6 we present our proof of the Multiplicative Ergodic theorem. The ergodic properties of the induced flow on the projective bundle are presented in § 7. In § 8 we derive (1.1) which describes the connection between the measurable and the dynamical spectra. In § 9 we study the theory of wedge-product flows and show how this can be used to compute the measurable spectrum, and in § 10 we present the applications discussed above. The paper concludes with an Appendix which contains some comments on related geometric properties of linear skew product flows.

2. Statement of main theorems. Let $\mathbf{M}$ be a compact Hausdorff space and let $\mathbf{T}$ denote either the integers $\mathbf{Z}$ or the reals $\mathbf{R}$. Assume that $\theta \cdot t$ is a flow on $\mathbf{M}$, i.e. the mapping $(\theta, t) \to \theta \cdot t$ of $\mathbf{M} \times \mathbf{T}$ into $\mathbf{M}$ is continuous and satisfies $\theta \cdot 0 = \theta$, and $\theta \cdot (t+s) = (\theta \cdot t) \cdot s$. The Krylov-Bogoliubov theorem (see Nemytskii and Stepanov (1960)) assures us that there is an invariant probability measure μ on $\mathbf{M}$. This means that $\mu(A \cdot t) = \mu(A)$ for all Borel sets $A \subseteq \mathbf{M}$ and all $t \in \mathbf{T}$, where $A \cdot t = \{\theta \cdot t \colon \theta \in A\}$. The invariant measure μ is *ergodic* if $\mu(A \triangle A \cdot t) = 0$ for all $t \in \mathbf{T}$ implies that $\mu(A) = 0$ or $\mu(A) = 1$. Recall that $A \triangle B = (A \backslash B) \cup (B \backslash A)$ is the symmetric difference. For an integer $m \geqq 1$ let $\mathscr{GL}(m)$ denote the group of all isomorphisms on $\mathbf{R}^m$, i.e., the group of nonsingular $(m \times m)$ matrices with entries in $\mathbf{R}$. A *cocycle* on $\mathbf{M}$ is a continuous mapping $\Phi \cdot \mathbf{M} \times \mathbf{T} \to \mathscr{GL}(m)$ that satisfies

$$\Phi(\theta, t+s) = \Phi(\theta \cdot t, s)\Phi(\theta, t) \tag{2.1}$$

for all $\theta \in \mathbf{M}$ and $s, t \in \mathbf{T}$. We note that Φ is a cocycle on $\mathbf{M}$ if and only if

$$\pi(x, \theta, t) := (\Phi(\theta, t)x, \theta \cdot t) \tag{2.2}$$

is a linear skew product flow on $R^m \times \mathbf{M}$.

If $\mathbf{T} = \mathbf{R}$ we shall say that the flow π is *smooth* provided the mapping

$$A : \theta \to \frac{d}{dt}\Phi(\theta, t)\big|_{t=0}$$

exists and is continuous. In this case the cocycle $\Phi(\theta, t)$ is simply the fundamental matrix solution of

$$x' = A(\theta \cdot t)x \qquad (x \in \mathbf{R}^m) \tag{2.3}$$

that satisfies $\Phi(\theta, 0) = I$. This is the prototypical example of a cocycle.

Let Φ and Ψ be two cocycles on $\mathbf{M}$ with range in $\mathscr{GL}(m)$. We shall say that Φ and Ψ are *cohomologous* if there is a continuous mapping $F : \mathbf{M} \to \mathscr{GL}(m)$ that satisfies

$$\Phi(\theta, t) = F(\theta \cdot t)\Psi(\theta, t)F(\theta)^{-1} \tag{2.4}$$

for all $\theta \in \mathbf{M}$, $t \in \mathbf{T}$, where (-1) denotes the matrix inverse.

Let Φ be a cocycle on $\mathbf{M}$. Let $x \in \mathbf{R}^m$ $(x \neq 0)$ and $\theta \in \mathbf{M}$ and define the four *Lyapunov exponents* $\lambda_i^{\pm}(x, \theta)$, $\lambda_s^{\pm}(x, \theta)$ by

$$\lambda_s^{\pm}(x, \theta) = \limsup_{t \to \pm\infty} \frac{1}{t} \log |\Phi(\theta, t)x|, \qquad \lambda_i^{\pm}(x, \theta) := \liminf_{t \to \pm\infty} \frac{1}{t} \log |\Phi(\theta, t)x|.$$

If it happens that the following two limits exist *and* are equal

$$\lim_{t \to +\infty} \frac{1}{t} \log |\Phi(\theta, t)x| = \lim_{t \to -\infty} \frac{1}{t} \log |\Phi(\theta, t)x|, \tag{2.5}$$

then we shall denote the common value as $\lambda(x, \theta)$. *In the future when we write the symbol $\lambda(x, \theta)$ this should be interpreted as an assertion that both limits in* (2.5) *exist and $\lambda(x, \theta)$ is the common value.* In this case one says that (x, θ) has a *strong* Lyapunov exponent.

Let Φ and Ψ be two cohomologous cocycles on $\mathbf{M}$ that satisfy (2.4), and let $x = F(\theta)y$. Then one has

$$\limsup_{t \to \pm\infty} \frac{1}{t} \log |\Phi(\theta, t)x| = \limsup_{t \to \pm\infty} \frac{1}{t} \log |\Psi(\theta, t)y|$$

and

$$\lim_{|t| \to \infty} \frac{1}{t} \log |\Phi(\theta, t)x| = \lim_{|t| \to \infty} \frac{1}{t} \log |\Psi(\theta, t)y|.$$

In other words, cohomologous cocycles have the same Lyapunov exponents.

For $0 \leqq k \leqq m$ let $\mathcal{G}(m, k)$ denote the Grassman manifold of k-planes in $\mathbf{R}^m$, and let $\mathcal{G}(m) = \bigcup_{k=0}^{m} \mathcal{G}(m, k)$ denote the disjoint union of these compact manifolds. For $k \in \{1, \cdots, m\}$ we shall let $N(k)$ denote those vectors $\vec{m} = (m_1, \cdots, m_k)$ with $1 \leqq m_i$ and $m_1 + \cdots + m_k = m$.

The first two theorems are statements of the Multiplicative Ergodic theorem.

THEOREM 2.1. *Let $\mathbf{M}$ be a compact Hausdorff space with a flow $\theta \cdot t$ and let μ be an invariant probability measure on $\mathbf{M}$. Let ϕ denote a cocycle on $\mathbf{M}$. Then there exist*:

(i) *an invariant set $\mathbf{M}_\mu \subseteq \mathbf{M}$ with $\mu(\mathbf{M}_\mu) = 1$;*

(ii) *a measurable decomposition $\mathbf{M}_\mu = \bigcup \mathbf{M}_\mu(p)$ where each $\mathbf{M}_\mu(p)$ is invariant and the union is taken over all pairs $p = (k, \vec{m})$ where $1 \leqq k \leqq m$, and $\vec{m} \in N(k)$;*

(iii) *measurable mappings $\lambda_1, \cdots, \lambda_k : \mathbf{M}_\mu(p) \to \mathbf{R}$, where*

$$\lambda_1(\theta) < \lambda_2(\theta) < \cdots < \lambda_k(\theta)$$

for $\theta \in \mathbf{M}_\mu(p)$; and

(iv) *measurable mappings $\mathcal{W}_i : \mathbf{M}_\mu(p) \to \mathcal{G}(m, m_i)$, $1 \leqq i \leqq k$, where*

$$\vec{m} = (m_1, \cdots, m_k),$$

such that for $\theta \in \mathbf{M}_\mu(p)$ one has:

(v) *$\{\mathcal{W}_1(\theta), \cdots, \mathcal{W}_k(\theta)\}$ is linearly independent;*

(vi) *$R^m = \mathcal{W}_1(\theta) + \cdots + \mathcal{W}_k(\theta)$;*

(vii) *if $x \in \mathcal{W}_i(\theta)$, $x \neq 0$, then $\lambda(x, \theta) = \lambda_i(\theta)$ for $1 \leqq i \leqq k$.*

If, in addition, μ is an ergodic measure, then precisely one $\mathbf{M}_\mu(p)$ has positive measure, and the mappings $\lambda_i : \mathbf{M}_\mu \to \mathbf{R}$ are constant, $1 \leqq i \leqq k$.

The results in the last theorem extend readily to linear skew product flows on arbitrary vector bundles, Sacker and Sell (1978).

THEOREM 2.2. *Let $\mathscr{E}$ be a vector bundle over a compact Hausdorff space* $\mathbf{M}$ *and let* π *be a linear skew product flow on* $\mathscr{E}$. *Let* μ *be an invariant probability measure on* $\mathbf{M}$. *Then the conclusions of Theorem* 2.1 *remain valid where* $\mathscr{W}_i$, $1 \leqq i \leqq k$, *now assume values in the appropriate Grassman bundles over* $\mathbf{M}$.

The *measurable spectrum* meas $\Sigma(\mu)$ is defined to be the collection $\{\lambda_1, \cdots, \lambda_k\}$ when μ is ergodic. The numbers $m_1, \cdots, m_k$ are the *multiplicities* of the spectral values $\lambda_1, \cdots, \lambda_k$. When μ is not ergodic, then the spectrum is meas $\Sigma(\mu, \theta) = \{\lambda_1(\theta), \cdots, \lambda_k(\theta)\}$ and the multiplicities $\{m_1, \cdots, m_k\}$ depend on $\theta \in \mathbf{M}_\mu$ and $\theta \in \mathbf{M}_\mu(p)$. For an ergodic measure μ, the *measurable bundle* associated with a spectral value λ_i, $1 \leqq i \leqq k$ is

$$\mathscr{W}_i = \{(x, \theta): x \in \mathscr{W}_i(\theta), \theta \in \mathbf{M}_\mu\}.$$

If μ is not ergodic, then the measurable bundles are defined similarly on each of the invariant sets $\mathbf{M}_\mu(p)$.

The next theorem compares the measurable spectrum and the measurable bundles with the dynamical (or continuous) spectrum and associated continuous spectral subbundles arising in the theory of exponential dichotomies in linear skew product flows; see Sacker and Sell (1978), (1980).

Let $\pi(x, \theta, t)$ be a linear skew product flow on $\mathbf{R}^m \times \mathbf{M}$, where $\mathbf{M}$ is a compact, connected space, and for $\lambda \in \mathbf{R}$ let

$$\pi_\lambda(x, \theta, t) := (\Phi_\lambda(\theta, t)x, \theta \cdot t) \tag{2.6}$$

be the shifted flow, where $\Phi_\lambda(\theta, t) := e^{-\lambda t}\Phi(\theta, t)$. Recall that π_λ has an *exponential dichotomy over* $\mathbf{M}$ if there is a (continuous) projector $P(x, \theta) = (P(\theta)x, \theta)$ on $\mathbf{R}^m$ and constants $K \geqq 1$, $\alpha > 0$ such that

$$|\Phi_\lambda(\theta, t)P(\theta)\Phi_\lambda^{-1}(\theta, s)| \leqq K e^{-\alpha(t-s)}, \qquad s \leqq t,$$

$$|\Phi_\lambda(\theta, t)[I - P(\theta)]\Phi_\lambda^{-1}(\theta, s)| \leqq K e^{-\alpha(s-t)}, \qquad t \leqq s$$

for all $\theta \in \mathbf{M}$ and $s, t \in \mathbf{T}$. The set $\lambda \in \mathbf{R}$ for which π_λ fails to have an exponential dichotomy over $\mathbf{M}$ is defined to be dyn Σ, the dynamical spectrum. The Spectral theorem (Sacker and Sell (1978)) assures us that dyn $\Sigma = \bigcup_{i=1}^k [a_i, b_i]$ is the union of k-nonoverlapping compact intervals, where $1 \leqq k \leqq m$. Also corresponding to each spectral interval $[a_i, b_i]$ there is an invariant spectral subbundle $\mathscr{V}_i$ of $\mathbf{R}^m \times \mathbf{M}$ with dim $\mathscr{V}_i(\theta) \geqq 1$, where $\mathscr{V}_i(\theta) = \{x \in \mathbf{R}^m: (x, \theta) \in V_i\}$, $1 \leqq i \leqq k$, the spaces $\{\mathscr{V}_1(\theta), \cdots, \mathscr{V}_k(\theta)\}$ are linearly independent and $\mathbf{R}^m \times \mathbf{M} = \mathscr{V}_1 + \cdots + \mathscr{V}_k$ (as a Whitney sum). The boundary of dyn Σ is the finite collection of end points $\{a_1, \cdots, a_k, b_1, \cdots, b_k\}$. The next theorem, which describes the connection between the measurable and dynamical spectra, is proved in § 7.

THEOREM 2.3. *Let* π *be a linear skew product flow on* $\mathbf{R}^m \times \mathbf{M}$ *where* $\mathbf{M}$ *is compact and connected, and let* dyn Σ *denote the dynamical spectrum of* π. *Then one has*

$$\text{boundary dyn } \Sigma \subseteq \bigcup_\mu \text{meas } \Sigma(\mu) \subseteq \text{dyn } \Sigma \tag{2.7}$$

where the union is either over all invariant probability measures μ *on* $\mathbf{M}$ *or over all ergodic measures on* $\mathbf{M}$. *Let* μ *be a given invariant probability measure on* $\mathbf{M}$ *and let* meas $\Sigma(\mu, \theta) = \{\lambda_1(\theta), \cdots, \lambda_k(\theta)\}$ *be the measurable spectrum for* $\theta \in \mathbf{M}_\mu(p)$. *Then for each* λ_j *there is precisely one spectral interval* $[a_i, b_i]$ *with* $\lambda_j(\theta) \in [a_i, b_i]$ *for all* $\theta \in \mathbf{M}_\mu(p)$. *Also the associated measurable bundle* $\mathscr{W}_j(\theta)$ *satisfies* $\mathscr{W}_j(\theta) \subseteq \mathscr{V}_i(\theta)$ *for all* $\theta \in \mathbf{M}_\mu(p)$.

Finally one has $\mathcal{V}_i(\theta)=\sum \mathcal{W}_j(\theta)$ *for all* $\theta \in \mathbf{M}_\mu(p)$ *where the summation is over all j with* $\lambda_j(\theta)\in[a_i, b_i]$.

If $\mathbf{T}=\mathbf{Z}$, then the last theorem is valid when $\mathbf{M}$ is compact and "dynamically connected," where the latter means that $\mathbf{M}$ cannot be written as the union of two disjoint nonempty closed invariant sets. Also, as in the spirit of Theorem 2.2, we note that Theorem 2.3 extends to linear skew product flows on general vector bundles.

Our next theorem is concerned directly with the problem of computing the measurable spectrum meas $\Sigma(\mu, \theta)$. The point is that one is able to do this without computing the basis elements $e_1, \cdots, e_m$. The key idea here is the notion of a wedge product, cf. Matshushima (1972). For $1 \leqq k \leqq m$ let $\Lambda^k\mathbf{R}^m$ denote the vector space generated by all k-fold wedge products $x_1 \wedge \cdots \wedge x_k$ where $x_i \in \mathbf{R}^m$, $1 \leqq i \leqq k$. Recall that the wedge product $x_1 \wedge \cdots \wedge x_k$ is linear in each factor and antisymmetric, i.e. $x \wedge y = -y \wedge x$. If $L: \mathbf{R}^m \to \mathbf{R}^m$ is linear, then this induces a linear mapping $\Lambda^k L$ on $\Lambda^k\mathbf{R}^m$ by the formula

$$\Lambda^k L(x_1 \wedge \cdots \wedge x_k) := Lx_1 \wedge \cdots \wedge Lx_k.$$

Since one has $\Lambda^k(LM)=(\Lambda^k L)(\Lambda^k M)$, we see that if $\Phi(\theta, t)$ is a cocycle on $\mathbf{M}$ then $\Lambda^k\Phi(\theta, t)$ is also cocycle, for $1 \leqq k \leqq m$.

In the statement of the next theorem reference will be made to the notation of Theorem 2.1. In particular for $\theta \in \mathbf{M}_\mu(p)$ the growth rates

$$\lambda_1(\theta)<\lambda_2(\theta)<\cdots<\lambda_k(\theta)$$

with multiplicities $m_1, \cdots, m_k$ will be rewritten in the form

$$\gamma_1(\theta)\leqq\gamma_2(\theta)\leqq\cdots\leqq\gamma_m(\theta) \tag{2.8}$$

where $\lambda_i(\theta)$ is repeated m_i-times in (2.8), $1 \leqq i \leqq k$.

THEOREM 2.4. *Let* $\mathbf{M}$ *be a compact Hausdorff space with a flow* $\theta \cdot t$ *and let* μ *be an invariant probability measure on* $\mathbf{M}$. *Let* Φ *denote a cocycle on* $\mathbf{M}$ *and adopt the conclusions and notation of Theorem* 2.1. *Let* $\gamma_1, \cdots, \gamma_m$ *satisfy* (2.8) *for* $\theta \in \mathbf{M}_\mu(p)$. *Then for all* $\theta \in \mathbf{M}_\mu(p)$ *one has*:

(i) $\lim_{t\to+\infty}(1/t)\log|\Phi(\theta, t)|=\gamma_m(\theta)$,

(ii) $\lim_{t\to+\infty}(1/t)\log|\Lambda^k\Phi(\theta, t)|=\gamma_{m+1-k}(\theta)+\cdots+\gamma_m(\theta)$, *for* $2 \leqq k \leqq m$,

(iii) $\lim_{t\to-\infty}(1/t)\log|\Phi(\theta, t)|=\gamma_1(\theta)$,

(iv) $\lim_{t\to-\infty}(1/t)\log|\Lambda^k\Phi(\theta, t)|=\gamma_1(\theta)+\cdots+\gamma_k(\theta)$, *for* $2 \leqq k \leqq m$.

The last theorem extends to linear skew product flows on a vector bundle $\mathscr{E}$ over a compact Hausdorff space $\mathbf{M}$. In this case the wedge product of vectors $x_1, \cdots, x_k \in \mathscr{E}(\theta)$ forms a new bundle $\Lambda^k\mathscr{E}$, $1 \leqq k \leqq m$ over $\mathbf{M}$. Also the flow π on $\mathscr{E}$ induces a flow $\Lambda^k\pi$ on $\Lambda^k\mathscr{E}$. This extension is a direct consequence of the proof of the last theorem together with Lemma 3.4 below. We will omit the details.

Remark 2.1. For simplicity of exposition we have formulated these theorems for cocycles with values in $\mathscr{GL}(m, \mathbf{R})$. The theorems are valid for cocycles with values in $\mathscr{GL}(m, \mathbf{C})$, and the proofs we give below extend with only trivial modifications.

3. Some technicalities. Before we turn our attention to the proofs of the main theorems, we need to dispense with some technical details which will enable us to simplify our arguments. We begin with a proof of the following facts:

1. One can assume, without loss of generality, that the base space $\mathbf{M}$ is a compact metric space instead of a compact Hausdorff space. (This fact simplifies substantially some of the measure theoretic considerations.)

2. If $\mathbf{T}=\mathbf{R}$ one can assume that the cocycle Φ is the fundamental solution matrix of an ordinary differential equation on $\mathbf{M}$ with continuous coefficients. We call such a cocycle *smooth.*

3. A linear skew product flow on an arbitrary vector bundle $\mathscr{E}$ over a compact Hausdorff space $\mathbf{M}$ can be imbedded into a linear skew product flow on $\mathbf{R}^m \times \mathbf{M}$ for some $m \geqq 1$.

The argument in each of these three cases is based on the same principle, viz. one can show that the given flow is cohomologous to the desired flow. The resulting cohomology preserves all the desired properties of our main theorems. In particular if Φ_1 and Φ_2 are two cohomologous cocycles on a compact Hausdorff space $\mathbf{M}$ that satisfy

$$F(\theta \cdot t)\Phi_1(\theta, t) = \Phi_2(\theta, t)F(\theta)$$

where $F:\mathbf{M} \to \mathscr{GL}(m)$ is continuous, then as noted above Φ_1 and Φ_2 have the same collection of Lyapunov exponents. Furthermore (x, θ) has a strong Lyapunov exponent for Φ_1 if and only if $(F(\theta)x, \theta)$ has a strong Lyapunov exponent for Φ_2. Thus F preserves the measurable spectrum meas $\Sigma(\mu, \theta)$ and it maps the measurable bundles of Φ_1 onto those of Φ_2. In addition F preserves the dynamical spectrum, and it sets up a one-to-one correspondence between the continuous spectral subbundles.

The situation is, in fact, more general. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two compact Hausdorff spaces and let $f:\mathbf{M}_1 \to \mathbf{M}_2$ be a flow epimorphism. Next let Φ_i be a cocycle on $\mathbf{M}_i$ $(i=1,2)$ and let $F:\mathbf{M}_1 \to \mathscr{GL}(m)$ satisfy

$$F(\theta_1 \cdot t)\Phi_1(\theta_1, t) = \Phi_2(f(\theta_1), t)F(\theta_1).$$

Then Φ_1 and Φ_2 have the same measurable and dynamical spectra and F sets up a one-to-one correspondence between the associated spectral bundles.

Our first step is to show that we can replace a compact Hausdorff base space $\mathbf{M}_1$ with a compact metric space $\mathbf{M}_2$. We use an argument of Ellis (1969).

LEMMA 3.1. *Let Φ_1 be a cocycle over a compact Hausdorff space $\mathbf{M}_1$ with a flow $\theta_1 \cdot t$. Then there is* (i) *a compact metric space $\mathbf{M}_2$ with a flow $\theta_2 \cdot t$,* (ii) *a flow epimorphism $f:\mathbf{M}_1 \to \mathbf{M}_2$ and* (iii) *a cocycle Φ_2 over $\mathbf{M}_2$ such that $\Phi_1(\theta_1, t) = \Phi_2(f(\theta_1), t)$.*

Proof. Let $\{t_n\}$ be a countable dense subset of $\mathbf{T}$. Let $\mathscr{A}$ be the closed subalgebra of $\mathscr{C}(\mathbf{M}_1, \mathbf{R})$ generated by all functions of the form $\{\theta_1 \to \phi_{ij}(\theta_1, t_n)\}$ where ϕ_{ij} are the components of Φ. Then $\mathscr{A}$ is a separable subalgebra of $\mathscr{C}(\mathbf{M}_1, \mathbf{R})$. Since $\mathscr{A}$ is closed it contains all mappings $\{\theta_1 \to \phi_{ij}(\theta_1, \tau)\}$ where $\tau \in T$; in fact $\mathscr{A}$ is also the closed subalgebra generated by all such mappings for $\tau \in T$. Because of the cocycle identity (2.1) we see that $\mathscr{A}$ is invariant, in the sense that if $g \in \mathscr{A}$ then $g_\tau \in \mathscr{A}$, where $g_\tau(\theta_1) = g(\theta_1 \cdot \tau)$. The Stone theorem, cf. Hewitt and Ross (1963, pp. 483–484), says that $\mathscr{A} = \mathscr{C}(\mathbf{M}_2, \mathbf{R})$, where $\mathbf{M}_2$ is the maximal ideal space of $\mathscr{A}$. Since $\mathscr{A}$ is separable, $\mathbf{M}_2$ is a compact metric space. Recall that $\mathbf{M}_2$ can be realized as the space of equivalence classes $[\theta_1]$ where $\theta_1 \sim \tilde{\theta}_1$ provided $\phi_{ij}(\theta_1, t_n) = \phi_{ij}(\tilde{\theta}_1, t_n)$ for all i, j and all t_n. Note that if $\theta_1 \sim \tilde{\theta}_1$ then $\theta_1 \cdot \tau \sim \tilde{\theta}_1 \cdot \tau$ for all $\tau \in T$. Consequently a flow on $\mathbf{M}_2$ is given by $[\theta_1] \cdot \tau = [\theta_1 \cdot \tau]$. Also the mapping $f(\theta_1) := [\theta_1]$ from $\mathbf{M}_1$ to $\mathbf{M}_2$ is an epimorphism because $\mathscr{A}$ is invariant. Finally we see that for each $t \in \mathbf{T}$ the cocycle $\Phi_1(\theta_1, t)$ depends only on the equivalence class $[\theta_1]$. So we conclude the proof by defining Φ_2 by $\Phi_2([\theta_1], t) := \Phi_1(\theta_1, t)$. Q.E.D.

The following lemma appears in Ellis and Johnson (1982), but we include a proof for completeness of exposition.

LEMMA 3.2. *Let Φ be a cocycle over a compact Hausdorff space $\mathbf{M}$ with $\mathbf{T}=\mathbf{R}$. Then Φ is cohomologous to a smooth cocycle Ψ over $\mathbf{M}$, i.e. $\Psi(\theta, t)$ is a fundamental matrix solution to $x' = A(\theta \cdot \tau)x$ where A is given by* (3.1) *below.*

Proof. Let $V \subseteq \mathscr{GL}(m)$ be a compact convex neighborhood of the identity I and choose $r > 0$ so that $\Phi(\theta, t) \in V$ for all $\theta \in \mathbf{M}$ and $0 \leqq t \leqq r$. Define $F(\theta) := (1/r) \int_0^r \Phi(\theta, s)\, ds$. Then $F(\theta)$ is invertible, and it is easily verified that the cocycle

$$\Psi(\theta, t) := F(\theta \cdot t)\Phi(\theta, t)F(\theta)^{-1} = \frac{1}{r} \int_t^{t+r} \Phi(\theta, s)\, ds\, F(\theta)^{-1},$$

which is cohomologous to Φ, is the fundamental matrix solution to $x' = A(\theta \cdot t)x$ where

$$A(\theta) := \frac{1}{r}[\Phi(\theta, r) - I]F(\theta)^{-1}. \tag{3.1}$$

Q.E.D.

The next lemma will allow us to conclude that the Multiplicative Ergodic theorem 2.2 is valid for linear skew product flows on a vector bundle $\mathscr{E}$ over a compact Hausdorff space $\mathbf{M}$. The same lemma shows that Theorems 2.3 and 2.4 extend to vector bundles as well. Before stating this we need to derive the following general fact concerning smooth approximations to continuous mappings on a compact invariant set.

LEMMA 3.3. *Let* $\mathbf{M}$ *be a compact Hausdorff space with a flow* $\theta \cdot t$ *and let* $f : \mathbf{M} \to \mathbf{N}$ *be a continuous mapping where* $\mathbf{N}$ *is a smooth compact Riemannian manifold. Then for every* $\delta > 0$ *there is a continuous function* $g : \mathbf{M} \to \mathbf{N}$ *with the following properties*:

(i) $\sup \{\operatorname{dist}(f(\theta), g(\theta)) : \theta \in \mathbf{M}\} \leqq \delta$,

(ii) *for every* $\theta \in \mathbf{M}$, *the mapping* $\theta \to (d/dt)g(\theta \cdot t)|_{t=0}$ *of* $\mathbf{M}$ *into the tangent bundle* $\mathbf{TN}$ *is a continuous mapping in* θ.

Proof. Let $\delta > 0$ be given. The Tubular Neighborhood theorem, see Guillemin and Pollack (1974), assures us that for a sufficiently large $m \geqq 1$ there is a smooth imbedding $h : \mathbf{N} \to \mathbf{R}^m$, an open set $W \supseteq h(\mathbf{N})$ and a smooth retract $R : W \to h(\mathbf{N})$. Now choose $\eta > 0$ so that if $\phi_1, \phi_2 \in h(\mathbf{N})$ and $|\phi_1 - \phi_2| \leqq \eta$ then $\operatorname{dist}(h^{-1}(\phi_1), h^{-1}(\phi_2)) \leqq \delta$. Next choose $\tau > 0$ so that

$$V(\theta) := \operatorname{Co}\{h(f(\theta \cdot t)) : 0 \leqq t \leqq \tau\} \subseteq W$$

for every $\theta \in \mathbf{M}$, where Co refers to the closed convex hull, and $|h(f(\theta)) - R(y)| \leqq \eta$ for every $\theta \in \mathbf{M}$ and $y \in V(\theta)$. We now define $g : \mathbf{M} \to \mathbf{N}$ by

$$g := h^{-1} \circ R\left(\frac{1}{\tau} \int_0^\tau h \circ f(\theta \cdot s)\, ds\right).$$

Since $1/\tau \int_0^\tau hf(\theta \cdot s)\, ds \in V(\theta)$ we see that $|f(\theta) - g(\theta)| \leqq \delta$ for all $\theta \in \mathbf{M}$. Furthermore it is easy to conclude that g is C^1 along trajectories and the mapping $\theta \to (d/dt)g(\theta \cdot t)|_{t=0}$ is continuous. Q.E.D.

LEMMA 3.4. *Let* $\mathscr{E}$ *be a finite dimensional vector bundle over a compact Hausdorff base space* $\mathbf{M}$ *and let* $\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta \cdot t)$ *be a linear skew product flow on* $\mathscr{E}$. *Then for any* $\lambda \in R$ *there exists an integer* $m \geqq 1$, *a monomorphism* $H : \mathscr{E} \to \mathbf{R}^m \times \mathbf{M}$, *a smooth cocycle* $\Psi : \mathbf{M} \to \mathscr{GL}(m)$ *and an orthogonal invariant resolution of the identity* $Q = (Q_1, Q_2)$ *such that* $H(\mathscr{E}) = \text{Range } Q_1$ *and*

$$Q_1(\theta \cdot t)\Psi(\theta, t) = \Psi(\theta, t)Q_1(\theta), \qquad H(\theta \cdot t)\Phi(\theta, t) = \Psi(\theta, t)H(\theta),$$
$$Q_2(\theta \cdot t)\Psi(\theta, t) = \Psi(\theta, t)Q_2(\theta) = e^{\lambda t}Q_2(\theta).$$

Proof. Since $\dim \mathscr{E}(\theta)$ is constant on the components of $\mathbf{M}$, there is no loss in generality in assuming that $\dim \mathscr{E}(\theta) = k$ for all $\theta \in \mathbf{M}$. The first step is to apply a standard result in the theory of vector bundles, Atiyah (1967, p. 25), which states that there is an integer $m > 0$ and a projector $P_1 : \mathbf{R}^m \times \mathbf{M} \to \mathbf{R}^m \times \mathbf{M}$ such that the vector bundle Range P_1 is isomorphic to $\mathscr{E}$. Let $\hat{H} : \mathscr{E} \to \text{Range } P_1 \subseteq \mathbf{R}^m \times \mathbf{M}$ be the isomorphism. Without any loss of generality we can assume that $P_1(\theta)$ is an orthogonal projection on $\mathbf{R}^m$ for all $\theta \in \mathbf{M}$. Let $P_2 := I - P_1$.

The mapping $\hat{W}_1 : \theta \to \text{Range } P_1(\theta)$ defines a continuous mapping of $\mathbf{M}$ into the smooth manifold $\mathscr{G}(m, k)$ of k-planes in $\mathbf{R}^m$. By Lemma 3.3, there is a smooth mapping $W_1 : \mathbf{M} \to \mathscr{G}(m, k)$ that is close to $\hat{W}_1$. Define $Q_1(\theta)$ to be the orthogonal projection with $W_1(\theta) = \text{Range } Q_1(\theta)$. Since W_1 is smooth this means that $\theta \to (d/dt)Q_1(\theta \cdot t)|_{t=0}$ is continuous. Also $Q_2(\theta) = I - Q_1(\theta)$ is smooth. Since Q_1 is close to P_1 it follows that $H = Q_1 \hat{H}$ is an isomorphism of $\mathscr{E}$ onto Range Q_1.

Next we define a flow on $\mathbf{R}^m \times \mathbf{M}$ under which Q_1 and Q_2 are invariant. Let $S(\theta) = Q_1'(\theta)Q_1(\theta) + Q_2'(\theta)Q_2(\theta)$ and let $\Psi_1(\theta, t)$ be the fundamental matrix solution of $x' = S(\theta \cdot t)x$ satisfying $\Psi_1(\theta, 0) = I$. Then as shown by Daletskii and Krein (1974) one has

$$Q_i(\theta \cdot t)\Psi_1(\theta, t) = \Psi_1(\theta, t)Q_i(\theta)$$

for all $\theta \in \mathbf{M}$, $t \in \mathbf{R}$, and $i = 1, 2$. Define a cocycle Ψ on $\mathbf{M}$ by

$$\Psi(\theta, t)Q_1(\theta) = H(\theta \cdot t)\Phi(\theta, t)H^{-1}(\theta),$$

$$\Psi(\theta, t)Q_2(\theta) = e^{\lambda t}\Psi_1(\theta, t)Q_2(\theta).$$

It is now straightforward to check the remaining details. Lemma 3.2 assures us that Ψ can be chosen to be smooth. Q.E.D.

4. Triangularization of cocycles. We turn next to the theory of the Gram-Schmidt factorization of isomorphisms on $\mathbf{R}^m$, where $\mathbf{R}^m$ has the Euclidean inner product $\langle \ , \ \rangle$. Let $\mathscr{GL}(m)$ denote the group of all isomorphisms of $\mathbf{R}^m$. Each element $L \in \mathscr{GL}(m)$ is identified with the $(m \times m)$ matrix whose column vectors satisfy $\text{col}_i\, L = Le_i$, $1 \leqq i \leqq m$, where $\{e_1, \cdots, e_m\}$ is a fixed orthonormal basis in $\mathbf{R}^m$. Let $\mathscr{O} = \mathscr{O}(m)$ denote the subgroup of $\mathscr{GL}(m)$ consisting of all orthogonal linear transformations, and let $\mathscr{T}^+(m)$ denote the subcollection of all upper triangular matrices $L \in \mathscr{GL}(m)$ with positive entries on the main diagonal. Then $\mathscr{T}^+(m)$ is also a subgroup of $\mathscr{GL}(m)$ and one has

$$\mathscr{O}(m) \cap \mathscr{T}^+(m) = \{I\}. \tag{4.1}$$

The Gram-Schmidt orthogonalization process assures us that for every $A \in \mathscr{GL}(m)$ there are unique matrices $G(A) \in \mathscr{O}(m)$ and $T(A) \in \mathscr{T}^+(m)$ such that

$$G(A) = AT(A). \tag{4.2}$$

Since the entries in $T(A)$ are algebraic functions of $\langle \text{col}_i\, A, \text{col}_j\, A \rangle$ we see that both $T(A)$ and $G(A)$ are smooth functions of A.

Next we note that one has

$$G(AB) = G(AG(B)), \qquad T(AB) = T(B)T(ABT(B)). \tag{4.3}$$

In order to prove (4.3), we define $U, V \in \mathscr{O}(m)$ by

$$U := G(AB) = ABT(AB), \qquad V := G(AG(B)) = ABT(B)T(ABT(B)),$$

where (4.2) is used above. One then has

$$U^{-1}V = T(AB)^{-1}T(B)T(ABT(B)) \in \mathscr{O}(m) \cap \mathscr{T}^+(m). \tag{4.4}$$

Hence by (4.1) $U^{-1}V = I$, which proves (4.3).

Let $\Phi : \mathbf{M} \to \mathscr{GL}(m)$ be a cocycle on $\mathbf{M}$. Then (4.2) admits the factorization

$$G(\Phi(\theta, t)U) = \Phi(\theta, t)UT(\Phi(\theta, t)U) \tag{4.5}$$

for every $U \in \mathcal{O}(m)$. This permits us to define a new flow on $\mathbf{H} := \mathbf{M} \times \mathcal{O}(m)$ as follows: Let $\phi := (\theta, U) \in \mathbf{M} \times \mathcal{O}(m)$ and define

$$\phi \cdot t := (\theta \cdot t, G(\Phi(\theta, t)U)). \tag{4.6}$$

LEMMA 4.1. *Equation* (4.6) *defines a flow on* $\mathbf{H} = \mathbf{M} \times \mathcal{O}(m)$.

Proof. It suffices to verify the group property

$$G[\Phi(\theta \cdot t, s)G(\Phi(\theta, t)U)] = G(\Phi(\theta, t+s)U).$$

However, this is an immediate consequence of (2.1) and (4.3). Q.E.D.

We noted in § 2 that the cocycle $\Phi(\theta, t)$ on $\mathbf{M}$ defines a linear skew product flow on $\mathbf{R}^m \times \mathbf{M}$ by $\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta \cdot t)$. By using (4.6) we see that π can be lifted to a new flow $\hat{\pi}$ on $\mathbf{R}^m \times \mathbf{H}$ by

$$\hat{\pi}(x, \phi, t) := (\Phi(\theta, t)x, \phi \cdot t)$$

where $\phi = (\theta, U)$. Let $q: \mathbf{M} \times \mathcal{GL}(m) \to \mathcal{GL}(m)$ and $r: \mathbf{M} \times \mathcal{GL}(m) \to \mathbf{M}$, (or $r: \mathbf{H} \to \mathbf{M}$) be the natural projections. Define $\Psi(\phi, t)$ by

$$\Psi(\phi, t) := q(\phi \cdot t)^{-1}\Phi(\theta, t)q(\phi) = G(\Phi(\theta, t)U)^{-1}\Phi(\theta, t)U \tag{4.7}$$

where the (-1) denotes the matrix inverse and $\phi = (\theta, U)$. Since $\phi \cdot t$ is a flow on $\mathbf{H}$, it follows that Ψ is a cocycle on $\mathbf{H}$, and

$$\tilde{\pi}(x, \phi, t) := (\Psi(\phi, t)x, \phi \cdot t) \tag{4.8}$$

is a linear skew product flow on $\mathbf{R}^m \times \mathbf{H}$ which is cohomologous to $\hat{\pi}$. The following lemma is now an immediate consequence of (4.2) and (4.7).

LEMMA 4.2. *Let* Φ *be a cocycle on* $\mathbf{M}$ *and define* $\phi \cdot t$ *and* $\Psi(\phi, t)$ *by* (4.6) *and* (4.7). *Then one has*

$$\Psi(\phi, t) = T(\Phi(\theta, t)U)^{-1} \in \mathcal{T}^+(m) \tag{4.9}$$

for all $\phi = (\theta, U) \in \mathbf{H}$ *and* $t \in \mathbf{T}$.

Remark 4.1. The triangularization method described above is directly related to the familiar technique developed by Lyapunov (1892), Perron (1930) and Diliberto (1950). Let $\mathbf{T} = \mathbf{R}$ and let $\Phi(\theta, t)$ be a smooth cocycle and (therefore) the fundamental solution matrix of a differential equation

$$x' = A(\theta \cdot t)x, \qquad x \in \mathbf{R}^m, \quad \theta \in \mathbf{M}, \tag{4.10}$$

where A is a continuous $(m \times m)$ matrix valued function defined on $\mathbf{M}$. Then $\Psi(\phi, t)$ is the fundamental solution matrix of

$$y' = B(\phi \cdot t)y, \qquad y \in \mathbf{R}^m, \quad \phi \in \mathbf{H}, \tag{4.11}$$

where $\phi = (\theta, U)$, $B = G^{-1}(AG - G')$, $G = G(\Phi(\theta, t)U)$ and $G' = (d/dt)G$. The change of variables which maps solutions of (4.11) onto those of (4.10) is

$$x = P(t)y = G(\Phi(\theta, t)U)y.$$

Also since the fundamental matrix solution of (4.11) is Ψ, an upper triangular matrix, we see that B is also upper triangular.

Remark 4.2. For $\mathbf{T} = \mathbf{Z}$ this is basically the triangularization method described in Oseledec (1968).

5. Invariant measures. In this section we record for reference a number of known results concerning invariant measures associated with the flows on $\mathbf{M}$ and $\mathbf{H}$. Let r be the natural projection of $\mathbf{H}$ onto $\mathbf{M}$. By (4.6) we see that r is a flow epimorphism, i.e. $r(\phi) \cdot t = r(\phi \cdot t)$.

Because of Lemma 3.1 we see that there is no loss in generality in assuming **M** (and therefore **H**) to be compact metric spaces. The Riesz Representation theorem says that for any compact metric space **M** there is an isomorphism between bounded positive linear functionals l on $\mathscr{C}(\mathbf{M}, \mathbf{R})$ satisfying $l(1)=1$ with the (regular, positive, Borel, probability) measures μ on **M**, and this isomorphism is given by the formula

$$l(f) = \int_M f(\theta)\mu(d\theta).$$

Hereafter we will interchange freely such functionals and the associated measures and write $\mu(f)$ in place of $l(f)$.

The measure μ is invariant for the flow $\theta \cdot t$ if and only if $\mu(f_\tau)=\mu(f)$ for all $f \in \mathscr{C}(\mathbf{M}, \mathbf{R})$ and all $\tau \in \mathbf{T}$ where $f_\tau(\theta)=f(\theta \cdot \tau)$. Also μ is ergodic if and only if for $f \in \mathscr{L}^1(\mathbf{M}, \mathbf{R})$ one has

$$\mu(f_\tau)=\mu(f) \quad \text{for all } \tau \in \mathbf{T} \Leftrightarrow f \equiv \text{constant}.$$

The Krylov-Bogoliubov method, cf. Nemytskii and Stepanov (1960), is a method for constructing invariant measures. Let us review this for the case $\mathbf{T}=\mathbf{R}$. Let μ be a given measure on **M** and define

$$\mu_T(f) := \frac{1}{T}\int_0^T \mu(f_\tau)\, d\tau \tag{5.1}$$

for $T>0$. Let $T_n \to +\infty$, and suppose (by choosing a subsequence if necessary) that μ_{T_n} converges weakly to a measure $\hat{\mu}$. Then $\hat{\mu}$ is easily seen to be invariant.

If the original measure μ is a δ-measure, i.e. $\mu(f)=\delta_\theta(f)=f(\theta)$, then (5.1) becomes

$$\mu_T(f) := \frac{1}{T}\int_0^T f(\theta \cdot \tau)\, d\tau. \tag{5.2}$$

Notice that if the original measure μ has support in a closed invariant set $\mathbf{M}_0$, then the induced invariant measure $\hat{\mu}$ has support in $\mathbf{M}_0$ as well.

Let μ be a given invariant measure on **M**. Let $I(\mu)$ denote the collection of all invariant measures ν on **H** that cover μ, i.e. $\nu \in I(\mu)$ if it is invariant and $r(\nu)=\mu$. If μ is an ergodic measure on **M** we let $E(\mu)$ denote the ergodic measures $\nu \in I(\mu)$. By using the Krylov-Bogoliubov method we see that $I(\mu)$ is nonempty. Indeed if l is any measure on $\mathcal{O}(m)$, then $\mu \times l$ is a measure on **H**. Now form

$$(\mu \times l)_T(g) = \frac{1}{T}\int_0^T (\mu \times l)(g_\tau)\, d\tau,$$

and let ν be a resulting invariant measure. In order to show that ν covers μ we need to show that $\nu(f)=\mu(f)$ whenever $f=f(\theta)$ depends only on the coordinate $\theta \in \mathbf{M}$. However in this case one has

$$(\mu \times l)(f_\tau)=\mu(f_\tau)=\mu(f)=(\mu \times l)_T(f)$$

since μ is invariant. Hence the limit ν satisfies $\nu(f)=\mu(f)$. Since $I(\mu)$ is nonempty, compact and convex it has extreme points. The extreme points in $I(\mu)$ are ergodic measures ν when μ is ergodic.

6. Proof of the Multiplicative Ergodic theorem. Throughout this section we will adopt without any loss of generality the following Standing Hypotheses which will lead to a proof of Theorems 2.1 and 2.2: Let $\pi(x, \theta, t)=(\Phi(\theta, t)x, \theta \cdot t)$ be a given linear skew-product flow on the trivial (Lemma 3.4) vector bundle $\mathbf{R}^m \times \mathbf{M}$, where $\mathbf{M}$ is a compact metric space (Lemma 3.1). If $\mathbf{T} = \mathbf{R}$ we assume that $\Phi(\theta, t)$ is smooth and is the fundamental solution matrix (Lemma 3.2) of

$$x' = A(\theta \cdot t)x, \qquad x \in \mathbf{R}^m, \quad \theta \in \mathbf{M}. \tag{6.1}$$

Let $\hat{\pi}(x, \phi, t) = (\Psi(\phi, t)x, \phi \cdot t)$ be the cohomologous triangular flow induced on $\mathbf{R}^m \times \mathbf{H}$ (§ 4). If $\mathbf{T} = \mathbf{R}$ then $\Psi(\phi, t)$ is also smooth and is the fundamental solution matrix of

$$y' = B(\phi \cdot t)y, \qquad y \in \mathbf{R}^m, \quad \phi \in \mathbf{H} \tag{6.2}$$

where B is a continuous upper triangular matrix. Let μ be a given invariant measure on $\mathbf{M}$ and let $\nu \in I(\mu)$ be any invariant measure on $\mathbf{H}$ that covers μ. If μ is ergodic we assume that ν is ergodic (§ 5). Also $r: \mathbf{H} \to \mathbf{M}$ is the natural projection.

We shall say that a point $\theta \in \mathbf{M}$ (or $\phi \in \mathbf{H}$) is a *Lyapunov point for* π (or $\hat{\pi}$) if there are real numbers $\gamma_1, \cdots, \gamma_m$ and a basis $e_1, \cdots, e_m$ of $\mathbf{R}^m$ such that

$$\lambda(e_i, \theta) := \lim_{|t| \to \infty} \frac{1}{t} \log |\Phi(\theta, t)e_i| = \gamma_i, \tag{6.3}$$

$$(\text{or } \lambda(e_i, \phi) := \lim_{|t| \to \infty} \frac{1}{t} \log |\Psi(\phi, t)e_i| = \gamma_i) \tag{6.4}$$

for $1 \leqq i \leqq m$.

Roughly speaking, the Multiplicative Ergodic theorem asserts that there are *many* Lyapunov points (i.e. $\mu(\mathbf{M}_\mu) = 1$) and that they fit together in a measurable manner. As we now show this follows from the triangularization technique described in § 4.

LEMMA 6.1. *Let $\phi = (\theta, U) \in \mathbf{H}$ be a Lyapunov point for $\hat{\pi}$. Then $\theta \in \mathbf{M}$ is a Lyapunov point for π.*

Proof. Choose $\gamma_1, \cdots, \gamma_m$ in $\mathbf{R}$ and a basis $e_1, \cdots, e_m$ in $\mathbf{R}^m$ so that (6.4) is satisfied. Define $f_1, \cdots, f_m$ by $f_i = Ue_i$, $1 \leqq i \leqq m$. Equation (4.7) yields

$$G(\Phi(\theta, t)U)\Psi(\phi, t) = \Phi(\theta, t)U. \tag{6.5}$$

Since $G(\Phi(\theta, t)U)$ is an orthogonal matrix one has

$$|\Phi(\theta, t)f_i| = |\Psi(\phi, t)e_i|, \qquad 1 \leqq i \leqq m.$$

It follows that (6.3) is satisfied with the same γ_i when e_i is replaced by f_i, $1 \leqq i \leqq m$. Q.E.D.

The next lemma is the key step in our proof.

LEMMA 6.2. *Let $\phi = (\theta, U) \in \mathbf{H}$ be fixed. Assume that the diagonal entries $\Psi(\phi, t)$ satisfy*

$$\lim_{|t| \to \infty} \frac{1}{t} \log |\psi_{ii}(\phi, t)| = \gamma_i \tag{6.6}$$

for some constants γ_i, $1 \leqq i \leqq m$. Then ϕ is a Lyapunov point for $\hat{\pi}$ where the growth rates $\gamma_1, \cdots, \gamma_m$ are given by (6.6), *and the associated matrix V of basis vectors $\{e_i, \cdots, e_m\}$ is an upper triangular matrix given by* (6.7) *with $v_{ii} = 1$, $1 \leqq i \leqq m$.*

If $\mathbf{T} = \mathbf{R}$, then $\psi_{ii}(\phi, t) = \exp\left(\int_0^t b_{ii}(\phi \cdot s)\, ds\right)$ where b_{ii}, $1 \leqq i \leqq m$, are the diagonal entries of the triangular matrix B in (6.2). In this case (6.6) becomes

$$\lim_{|t| \to \infty} \frac{1}{t} \log |\psi_{ii}(\phi, t)| = \lim_{|t| \to \infty} \frac{1}{t} \int_0^t b_{ii}(\phi \cdot s)\, ds = \gamma_i, \qquad 1 \leqq i \leqq m.$$

Also if $\mathbf{T}=\mathbf{Z}$, then the diagonal elements of ψ satisfy

$$\psi_{ii}(\phi, t)=\prod_{s=0}^{t-1} \psi_{ii}(\phi \cdot s, 1), \qquad t>0$$

with a similar expression valid for $t<0$. For $t>0$ one has

$$\frac{1}{t} \log |\psi_{ii}(\phi, t)|=\frac{1}{t} \sum_{s=0}^{t-1} \log |\psi_{ii}(\phi \cdot s, 1)|.$$

We see then that for both $\mathbf{T}=\mathbf{R}$ and $\mathbf{T}=\mathbf{Z}$, the limits in (6.6) are time-averages of continuous real-valued functions defined on $\mathbf{H}$. This fact will be used later when we apply the Birkhoff Ergodic theorem.

Proof. The argument we now give applies to any triangular cocycle Ψ over any compact metric space $\mathbf{H}$. We will not use the special form of the flow on $\mathbf{H}$.

Let i satisfy $1 \leqq i \leqq m$. For any upper triangular $(m \times m)$ matrix T we let T_i denote the lower-right $(k \times k)$-dimensional block where $k=(m-i+1)$. Thus $T_1=T$ and $T_m=(t_{mm})$. For the matrix B given by (6.2) we let β_i denote the $(m-i)$-dimensional row vector that satisfies

$$B_i=\begin{pmatrix} b_{ii} & \beta_i \\ 0 & B_{i+1} \end{pmatrix}$$

for $1 \leqq i \leqq m-1$.

The upper triangular matrix V of basis vectors is obtained by constructing the V_i inductively starting with $V_m=(1)$. Suppose $1 \leqq i \leqq m-1$ and that V_{i+1} has been constructed with the properties that its diagonal elements are 1 and

$$\lambda(\operatorname{col}_j (V_{i+1}), \phi)=\gamma_j$$

for $i+1 \leqq j \leqq m$. To construct V_i with the corresponding properties we first define $v_{ii}=1$. For $1 \leqq i \leqq m-1$ and $i+1 \leqq j \leqq m$ we define[1]

$$\text{(6.7)} \qquad v_{ij}=v_{ij}(\phi):=\int_{\tau}^{0} \psi_{ii}^{-1}(\phi, s) \beta_i(\phi \cdot s) \Psi_{i+1}(\phi, s) \operatorname{col}_j (V_{i+1}(\phi))\, ds$$

$$\operatorname{col}_j (V_i):=\begin{pmatrix} v_{ij} \\ \operatorname{col}_j (V_{i+1}) \end{pmatrix}$$

where

$$\tau=\tau_{ij}:=\begin{cases} \infty & \text{if } \gamma_i>\gamma_j, \\ 0 & \text{if } \gamma_i=\gamma_j, \\ -\infty & \text{if } \gamma_i<\gamma_j. \end{cases}$$

Equations (6.4) and (6.6) and the induction hypothesis imply that for every $\varepsilon>0$ there are constants K_1 and K_2 such that for $t \geqq 0$ one has

$$|\psi_{ii}(\theta, t)| \leqq K_2 \exp [(\gamma_i+\varepsilon) t], \qquad |\psi_{ii}^{-1}(\theta, t)| \leqq K_2 \exp [(-\gamma_i+\varepsilon) t],$$
$$K_1 \exp [(\gamma_j-\varepsilon) t] \leqq |\Psi_{i+1}(\phi, t) \operatorname{col}_j (V_{i+1})| \leqq K_2 \exp [(\gamma_j+\varepsilon) t]$$

for $i+1 \leqq j \leqq m$. Since $|\beta_i|$ is uniformly bounded on $\mathbf{H}$, it follows that the infinite integral in (6.7) is well defined.

[1] We assume for the moment that $\mathbf{T}=\mathbf{R}$. The modification of our argument needed for the case $\mathbf{T}=\mathbf{Z}$ is described in the last paragraph of the proof.

The variation of constants formula for the block-triangular system $u' = B_i(\phi \cdot t)u$ yields

$$\Psi_i(\phi, t)\,\mathrm{col}_j\,(V_i) = \begin{pmatrix} \psi_{ii}(\phi, t)\int_\tau^t \psi_{ii}^{-1}(\phi, s)\beta_i(\phi \cdot s)\Psi_{i+1}(\phi, s)\,\mathrm{col}_j\,(V_{i+1})\,ds \\ \Psi_{i+1}(\phi, t)\,\mathrm{col}_j\,(V_{i+1}) \end{pmatrix} \tag{6.8}$$

for $i+1 \leqq j \leqq m$ and $\Psi_i(\phi, t)\,\mathrm{col}_i\,(V_i) = \mathrm{col}\,(\psi_{ii}(\theta, t), 0, \cdots, 0)$. Let $v_{ij}(t)$ denote the first entry in (6.8). One then has $\lambda(\mathrm{col}_i\,(V_i), \phi) = \gamma_i$. While it is known that

$$\lambda(\mathrm{col}_j\,(V_i), \phi) = \lambda(\mathrm{col}_j\,(V_{i+1}), \phi) = \gamma_j \tag{6.9}$$

for $i+1 \leqq j \leqq m$, cf. Millionscikov (1968), we shall include a proof for completeness. Indeed it follows from (6.8) and the inequalities after (6.7) that there is a constant K such that

$$|v_{ij}(t)| \leqq K \exp\left[(\gamma_j + 3\varepsilon)t\right]$$

for $i+1 \leqq j \leqq m$ and $t \geqq 0$. Since ε is arbitrary one has

$$\limsup_{t\to+\infty} \frac{1}{|t|} \log |v_{ij}(t)| \leqq \gamma_j, \qquad i+1 \leqq j \leqq m, \tag{6.10}$$

and therefore by (6.8) we have

$$\gamma_j = \lim_{t\to+\infty} \frac{1}{t} \log |\Psi_{i+1}(\phi, t)\,\mathrm{col}_j\,(V_{i+1})| \leqq \liminf_{t\to+\infty} \frac{1}{t} \log |\Psi_i(\phi, t)\,\mathrm{col}_j\,(V_i)|$$

$$\leqq \limsup_{t\to+\infty} \frac{1}{t} \log |\Psi_i(\phi, t)\,\mathrm{col}_j\,(V_i)| \leqq \limsup_{t\to+\infty} \frac{1}{t} \log |v_{ij}(t)| \leqq \gamma_j.$$

A similar argument applies as $t \to -\infty$. Also (6.10) is valid as $t \to -\infty$.

This completes the argument for $\mathbf{T} = \mathbf{R}$. If $\mathbf{T} = \mathbf{Z}$ the integrals in (6.7)-(6.8) are replaced by sums. For example by the variation of constants formula in Sacker and Sell (1976b), $v_{ij}(t)$ takes the form

$$v_{ij}(t) = \psi_{ii}(\phi, t) \sum_{s=\tau}^{t-1} \psi_{ii}^{-1}(\phi, s+1)\beta_i(\phi \cdot s)\Psi_{i+1}(\phi, s)\,\mathrm{col}_j\,(V_{i+1})$$

where $\beta_i = \beta_i(\phi)$ is the $(m-i)$-dimensional row vector that satisfies

$$\Psi_i(\phi, 1) = \begin{pmatrix} \psi_{ii}(\phi, 1) & \beta_i(\phi) \\ 0 & \Psi_{i+1}(\phi, 1) \end{pmatrix}$$

for all $\phi \in \mathbf{H}$. We will omit the details, which are easily verified. Q.E.D.

LEMMA 6.3. *Let $\phi = (\theta, U)$ satisfy the hypotheses of Lemma 6.2 and let V be the matrix of basis vectors constructed above. Then there are upper triangular $(m \times m)$ matrices $S(\phi, t)$ and $D(\phi, t)$ that satisfy*

(i) $\Psi(\phi, t)V = S(\phi, t)D(\phi, t)$,

(ii) $D(\phi, t) = \mathrm{diag}\,(\psi_{11}(\phi, t), \cdots, \psi_{mm}(\phi, t))$,

(iii) $\limsup_{|t|\to\infty} (1/|t|) \log |S(\phi, t)| \leqq 0$,

(iv) $\limsup_{|t|\to\infty} (1/|t|) \log |S^{-1}(\phi, t)| \leqq 0$.

Proof. S and D are uniquely determined by (i) and (ii) and

$$S(\phi, t) = \begin{pmatrix} 1 & \psi_{22}^{-1}(\phi, t)v_{12}(t) & \cdots & \psi_{mm}^{-1}(\phi, t)v_{1m}(t) \\ 0 & 1 & \cdots & \psi_{mm}^{-1}(\phi, t)v_{2m}(t) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

It follows from (6.10) that

$$\limsup_{|t|\to+\infty} \frac{1}{|t|} \log |\psi_{jj}^{-1}(\phi, t) v_{ij}(t)| \leqq 0$$

for $1 \leqq i \leqq m-1$ and $i+1 \leqq j \leqq m$. Parts (iii) and (iv) then follow from the last inequality and the fact that the entries in S^{-1} are polynomials in the entries of S. Q.E.D.

Next we apply the Birkhoff Ergodic theorem which assures us that there are a good supply of Lyapunov points in $\mathbf{H}$.

LEMMA 6.4. *There is a Borel measurable invariant set* $\mathbf{H}_\nu \subseteq \mathbf{H}$ *with* $\nu(\mathbf{H}_\nu) = 1$ *and such that every point* $\phi \in \mathbf{H}_\nu$ *is a Lyapunov point.*

Proof. As noted above the limits

$$\lim_{|t|\to\infty} \frac{1}{t} \log |\psi_{ii}(\phi, t)|, \qquad 1 \leqq i \leqq m,$$

are time-averages of continuous functions defined on $\mathbf{H}$. The Birkhoff Ergodic theorem, see Nemytskii and Stepanov (1960), asserts that there is a Borel set $\mathbf{H}_\nu \subseteq \mathbf{H}$ with $\nu(\mathbf{H}_\nu) = 1$, and there exist bounded Borel measurable invariant functions $\rho_1, \cdots, \rho_m : \mathbf{H}_\nu \to \mathbf{R}$ with the property that

$$\lim_{|t|\to\infty} \frac{1}{t} \log |\psi_{ii}(\phi, t)| = \rho_i(\phi), \qquad 1 \leqq i \leqq m, \tag{6.11}$$

for all $\phi \in \mathbf{H}_\nu$. It then follows from Lemma 6.2 that each $\phi \in \mathbf{H}_\nu$ is a Lyapunov point. Q.E.D.

Let $\rho_1, \cdots, \rho_m$ satisfy (6.11). For each integer k, $1 \leqq k \leqq m$, let $N(k)$ denote the collection of vectors $\vec{m} = (m_1, \cdots, m_k)$ with integer entries that satisfy $1 \leqq m_j$, $1 \leqq j \leqq k$, and $m_1 + \cdots + m_k = m$. We will now construct a measurable decomposition of $\mathbf{H}_\nu$. Fix $\phi \in \mathbf{H}_\nu$. We then note that there is an integer k, $1 \leqq k \leqq m$, and an $\vec{m} \in N(k)$ such that the following two properties hold:

(i) There are exactly k distinct values in the collection $\{\rho_1(\phi), \cdots, \rho_m(\phi)\}$, which we rewrite as $\{\lambda_1(\phi), \cdots, \lambda_k(\phi)\}$ where

$$\lambda_1(\phi) < \lambda_2(\phi) < \cdots < \lambda_k(\phi). \tag{6.12}$$

(ii) The cardinality of the set $\{i : 1 \leqq i \leqq m \text{ and } \rho_i(\phi) = \lambda_j(\phi)\}$ is m_j for each j, $1 \leqq j \leqq k$.

We denote the ordered pair $(k, \vec{m})$ briefly by p and define $\mathbf{H}_\nu(p)$ to be the set of all $\phi \in \mathbf{H}_\nu$ to which k and $\vec{m}$ correspond as above. The set $\mathbf{H}_\nu(p)$ is Borel measurable since it is the pullback of the closed set

$$\{(x_1, \cdots, x_m) : x_1 = x_2 = \cdots = x_{m_1} < x_{m_1+1} = \cdots = x_{m_2} < \cdots < x_{m_{k-1}+1} = \cdots = x_{m_k}\}$$

by the Borel measurable function that is the composition of the Borel measurable function $\phi \to (\rho_1(\phi), \cdots, \rho_m(\phi))$ with the continuous function σ that maps $(x_1, x_2, \cdots, x_m)$ onto its permutation $(x_1', x_2', \cdots, x_m')$ where $x_1' \leqq x_2' \leqq \cdots \leqq x_m'$.

It is easy to see that $\mathbf{H}_\nu = \bigcup \mathbf{H}_\nu(p)$, where the union is taken over all such points $p = (k, \vec{m})$, is a measurable decomposition of $\mathbf{H}_\nu$. Since each ρ_i is invariant we see that the sets $\mathbf{H}_\nu(p)$ are invariant. If ν is an ergodic measure, then all but one $\mathbf{H}_\nu(p)$ has ν-measure 0. The following result is a consequence of the measurability and invariance of $\rho_1, \cdots, \rho_m$.

LEMMA 6.5. *The functions* $\lambda_1, \cdots, \lambda_k : \mathbf{H}_\nu(p) \to \mathbf{R}$ *are Borel measurable and invariant.*

Let $\phi \in \mathbf{H}_\nu(p)$ and let $\rho_1, \cdots, \rho_m$ and $\lambda_1, \cdots, \lambda_k$ be given as above. Let $e_1, \cdots, e_m$ be the basis in $\mathbf{R}^m$ constructed in Lemma 6.2. Thus one has $\lambda(e_i, \phi) = \rho_i(\phi)$, $1 \leqq i \leqq m$. For $1 \leqq j \leqq k$ define

$$\mathscr{W}_j(\phi) := \operatorname{Span}\{e_i \colon \lambda(e_i, \phi) = \lambda_j(\phi)\}.$$

One then has $\dim \mathscr{W}_j(\phi) = m_j$ and $R^m = \mathscr{W}_1(\phi) + \cdots + \mathscr{W}_k(\phi)$.

The next lemma shows that every $y \in \mathscr{W}_j(\phi)$, $y \neq 0$, has $\lambda_j(\phi)$ as a strong Lyapunov exponent.

LEMMA 6.6. *Let $\phi \in \mathbf{H}$ satisfy the hypothesis of Lemma 6.2. Then for all $y \in W_j(\phi)$, $y \neq 0$, one has*

$$\lim_{|t| \to +\infty} \frac{1}{t} \log |\Psi(\phi, t) y| = \lambda_j(\phi), \qquad 1 \leqq j \leqq k. \tag{6.13}$$

Moreover the limit in (6.13) *is uniform for* $|y| = 1$.

Proof. We will use Lemma 6.3. Fix j and let $y \in \mathscr{W}_j(\phi)$, $x \neq 0$. Then

$$y \in \operatorname{Span}\{\operatorname{col}_i(V) \colon \lambda_i(\phi) = \lambda_j(\phi)\}.$$

If $\{e_1, \cdots, e_m\}$ is the natural basis in $\mathbf{R}^m$ then $z = V^{-1} y$ satisfies

$$z \in \operatorname{Span}\left\{e_i \colon \lim_{|t| \to \infty} \frac{1}{t} \log |\psi_{ii}(\phi, t)| = \lambda_j(\phi)\right\}.$$

Consequently one has

$$\lim_{|t| \to +\infty} \frac{1}{t} \log |D(\phi, t) z| = \lambda_j(\phi),$$

and the last limit is uniform for $|y| = 1$. Since one has

$$|S^{-1}(\phi, t)|^{-1} |D(\phi, t) z| \leqq |\Psi(\phi, t) V z| = |\Psi(\phi, t) y| \leqq |S(\phi, t)| \, |D(\phi, t) z|$$

and

$$\liminf_{t \to +\infty} \frac{1}{t} \log |S^{-1}(\phi, t)|^{-1} = -\limsup_{t \to +\infty} \frac{1}{t} \log |S^{-1}(\phi, t)| \geqq 0$$

(by Lemma 6.3), we get

$$\begin{aligned}
\lambda_j(\phi) &\leqq \liminf_{t \to +\infty} \frac{1}{t} \log |S^{-1}(\phi, t)|^{-1} + \liminf_{t \to +\infty} \frac{1}{t} \log |D(\phi, t) z| \\
&\leqq \liminf_{t \to +\infty} \frac{1}{t} \log |\Psi(\phi, t) y| \leqq \limsup_{t \to +\infty} \frac{1}{t} \log |\Psi(\phi, t) y| \\
&\leqq \limsup_{t \to +\infty} \frac{1}{t} \log |S(\phi, t)| + \limsup_{t \to +\infty} \frac{1}{t} \log |D(\phi, t) y| \\
&\leqq \lambda_j(\phi).
\end{aligned}$$

A similar argument applies as $t \to -\infty$. Q.E.D.

LEMMA 6.7. *Let $\phi \in \mathbf{H}_\nu(p)$ and let $y \in \mathbf{R}^m$, $y \neq 0$. Assume that $\lambda(y, \phi) = \gamma$. Then there is a j, $1 \leqq j \leqq k$ such that $\lambda(y, \phi) = \lambda_j(\phi)$ and $y \in \mathscr{W}_j(\phi)$.*

Proof. Since $e_1, \cdots, e_m$ is a basis one has

$$y = \alpha_1 e_1 + \cdots + \alpha_m e_m \tag{6.14}$$

where $\alpha_1, \cdots, \alpha_m$ are scalars. It is an easy exercise to see that one has

$$\lim_{t\to\infty} \frac{1}{t} \log |\Psi(\phi, t)y| = \max \{\rho_i(\phi): 1 \leqq i \leqq m \text{ and } \alpha_i \neq 0\},$$

$$\lim_{t\to-\infty} \frac{1}{t} \log |\Psi(\phi, t)y| = \min \{\rho_i(\phi): 1 \leqq i \leqq m \text{ and } \alpha_i \neq 0\}.$$

Therefore if the two-sided limit $\lambda(y, \phi) = \gamma$ exists, the only nonzero α's in (6.14) are coefficients of basis vectors used to define a single $\mathscr{W}_j(\phi)$. Hence $\lambda(y, \phi) = \lambda_j(\phi)$ and $y \in \mathscr{W}_j(\phi)$. Q.E.D.

Let $\phi_1 = (\theta, U_1)$, $\phi_2 = (\theta, U_2)$ be two points in $\mathbf{H}$ with the same θ-coordinate. From (6.5) one has

$$G(\Phi(\theta, t)U_1)\Psi(\phi_1, t)U_1^{-1} = \Phi(\theta, t) = G(\Phi(\theta, t)U_2)\Psi(\phi_2, t)U_2^{-1}. \tag{6.15}$$

Therefore if $y \in \mathbf{R}^m$ then $|\Psi(\phi_1, t)y| = |\Psi(\phi_2, t)Vy|$ where $V = U_2^{-1}U_1$. We see that

$$\lambda(y, \phi_1) = \gamma \Leftrightarrow \lambda(Vy, \phi_2) = \gamma. \tag{6.16}$$

Now assume further that $\phi_1 \in \mathbf{H}_\nu$. Then ϕ_1 is a Lyapunov point by Lemma 6.2 and (6.4). Let $\gamma_1, \cdots, \gamma_m$ be the strong Lyapunov exponents and let $e_1, \cdots, e_m$ be a basis with $\lambda(e_i, \phi_1) = \gamma_i$, $1 \leqq i \leqq m$. It follows from (6.16) that $Ve_1, \cdots, Ve_m$ is a basis for which $\lambda(Ve_i, \phi_2) = \gamma_i$, $1 \leqq i \leqq m$. We have just proved the following result:

LEMMA 6.8. *Let $\phi = (\theta, U) \in \mathbf{H}_\nu$, and let $\gamma_1, \cdots, \gamma_m$ be the set of strong Lyapunov exponents given by Lemma 6.2 and (6.4). Then every point $\hat{\phi} = (\theta, \hat{U})$ in the fiber over θ is a Lyapunov point with precisely the same set of strong Lyapunov exponents.*

By combining Lemmas 6.7 and 6.8 and (6.15) we immediately have the following:

LEMMA 6.9. *Let $\phi_1 = (\theta, U_1)$ and $\phi_2 = (\theta, U_2)$ be two points in $\mathbf{H}_\nu$ with the same θ-coordinate. Then ϕ_1 and ϕ_2 lie in the same set $\mathbf{H}_\nu(p)$, and for $1 \leqq j \leqq k$ one has*

$$\lambda_j(\phi_1) = \lambda_j(\phi_2), \qquad U_1\mathscr{W}_j(\phi_1) = U_2\mathscr{W}_j(\phi_2). \tag{6.17}$$

Hence $\lambda_j(\phi_1)$ and $U_1\mathscr{W}_j(\phi_1)$ depend only on the θ-coordinate.

By using (6.16) together with Lemma 6.6 we see that if $\phi_1 = (\theta, U_1) \in \mathbf{H}_\nu$ then for any $\phi_2 = (\theta, U_2)$, with the same θ-coordinate, we have

$$\lambda(y, \phi_1) = \lambda(Vy, \phi_2) = \lambda_j(\phi_1)$$

for all $y \in \mathscr{W}_j(\phi_1)$, $y \neq 0$, where $V = U_2^{-1}U_1$.

We now use $r: \mathbf{H} \to \mathbf{M}$ to project $\mathbf{H}_\nu$ and $\mathbf{H}_\nu(p)$ to $\mathbf{M}$. Define $\mathbf{M}_\mu$ and $\mathbf{M}_\mu(p)$ by

$$r(\mathbf{H}_\nu) := \mathbf{M}_\mu, \qquad r(\mathbf{H}_\nu(p)) := \mathbf{M}_\mu(p).$$

Note that since $\mathbf{M}$ and $\mathbf{H}$ are compact metric spaces, and $\mathbf{H}_\nu$ and $\mathbf{H}_\nu(p)$ are Borel measurable sets in $\mathbf{H}$, the images $\mathbf{M}_\mu$ and $\mathbf{M}_\mu(p)$ are μ-measurable sets in $\mathbf{M}$, see Federer (1969, Chap. 2). Furthermore one has $\mu(\mathbf{M}_\mu) = 1$. (Strictly speaking, $\mathbf{M}_\mu$ and $\mathbf{M}_\mu(p)$ depend on the choice of $\nu \in I(\mu)$. Since $\mu(r(\mathbf{H}_\nu)) = \nu(\mathbf{H}_\nu) = 1$ we see that any two such sets $\mathbf{M}_\mu$ agree except on a set of μ-measure 0.)

Let $\phi = (\theta, U) \in \mathbf{H}_\nu(p)$. Then $\theta \in \mathbf{M}_\nu(p)$. Next define

$$\lambda_j(\theta) := \lambda_j(\phi), \quad \mathscr{W}_j(\theta) := U\mathscr{W}_j(\phi), \quad 1 \leqq j \leqq k.$$

From Lemma 6.9 we see that $\lambda_j(\theta)$ and $U(W_j(\phi))$ depend only on the θ-coordinate. Also from Lemmas 6.5 and 6.9 we see that $\lambda_1, \cdots, \lambda_k: \mathbf{M}_\mu(p) \to \mathbf{R}$ are μ-measurable and invariant. For $\theta \in \mathbf{M}_\mu(p)$ we see that the spaces $\mathscr{W}_1(\theta), \cdots, \mathscr{W}_k(\theta)$ satisfy conclusions (v)–(vii) of Theorem 2.1.

The only point that remains to be proven is that the mappings $\mathcal{W}_i: \mathbf{M}_\mu(p) \to \mathcal{G}(m, m_i)$, $1 \leqq i \leqq k$, are μ-measurable. Because of Lemma 6.9 it suffices to show that each $\mathcal{W}_i$ is Borel measurable on $\mathbf{H}_\nu(p)$, $1 \leqq i \leqq k$. We will do this by noting that the basis matrix $V = \{e_1, \cdots, e_m\}$ constructed in (6.7) is Borel measurable in ϕ since the coefficients in the integral depend continuously in ϕ, and therefore the integral is measurable[2] in ϕ, cf. Federer (1969). This completes the proof of Theorem 2.1.

Theorem 2.2, the Multiplicative Ergodic theorem on a vector bundle $\mathscr{E}$, follows directly from Lemma 3.4 and Theorem 2.1. In Lemma 3.4 one can choose the $\lambda \in \mathbf{R}$ arbitrarily. A good choice for λ is $\lambda \notin \operatorname{dyn} \Sigma(\mathscr{E})$, where $\operatorname{dyn} \Sigma(\mathscr{E})$ is the dynamical spectrum of the linear skew product flow on $\mathscr{E}$. With this choice one knows that the measurable subbundle associated with λ is Range (Q_2) and is disjoint from Range (Q_1). (See Theorem 8.1 below.)

Remark 6.1. The uniformity described in Lemma 6.6 can be strengthened. Let $\phi = (\theta, U) \in \mathbf{H}_\nu(p)$, let $\lambda_1, \cdots, \lambda_k: \mathbf{H}_\nu(p) \to \mathbf{R}$ be the growth rates with $\lambda_1(\phi) < \cdots < \lambda_k(\phi)$, and let

$$\mathbf{R}^m = \mathcal{W}_1(\phi) + \cdots + \mathcal{W}_k(\phi)$$

be the decomposition of $\mathbf{R}^m$ into the measurable bundles. Then every $y \in \mathbf{R}^m$ can be written uniquely as $y = y_1 + \cdots + y_k$ where $y_i \in \mathcal{W}_i(\phi)$, $1 \leqq i \leqq k$. Furthermore for $y \neq 0$ one has

$$\lim_{t \to +\infty} \frac{1}{t} \log |\Psi(\phi, t) y| = \lambda_b(\phi), \tag{6.18}$$

$$\lim_{t \to -\infty} \frac{1}{t} \log |\Psi(\phi, t) y| = \lambda_a(\phi), \tag{6.19}$$

where $a = \min\{i: y_i \neq 0\}$ and $b = \max\{i: y_i \neq 0\}$. (See Lemma 6.7.) By using the argument of Lemma 6.6, it is easily seen that the limits in (6.18) and (6.19) are uniform on compact sets of the form

$$\{y \in \mathbf{R}^m: 0 < \alpha \leqq |y_b|, |y| \leqq \beta\}, \qquad \{y \in \mathbf{R}^m: 0 < \alpha \leqq |y_a|, |y| \leqq \beta\}.$$

These considerations extend immediately to the cocycle $\Phi(\theta, t)$ over $\mathbf{M}$, where $\mathcal{W}_i(\phi)$ is replaced by $U\mathcal{W}_i(\phi)$, $1 \leqq i \leqq k$. (See Lemma 6.9.)

Remark 6.2. As noted by Oseledec (1968) the uniformity condition in Lemma 6.6 implies that the limits

$$\lim_{|t| \to +\infty} \frac{1}{|t|} \log \beta_i(\theta, t), \qquad 1 \leqq i \leqq m, \tag{6.20}$$

exist almost everywhere, where $\beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_m$ are the eigenvalues of the positive self-adjoint matrix $\Phi^*(\theta, t)\Phi(\theta, t)$.

Remark 6.3. The basis $e_1, \cdots, e_m$, which we construct in Lemma 6.2, is very closely related to Lyapunov's concept of "regularity" or "biregularity", see Lyapunov (1892) and Bylov et al. (1966). Note that if $\theta \in \mathbf{M}_\mu$ then there are real numbers $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ and a splitting

$$\mathbf{R}^m = W_1 + \cdots + W_k$$

such that if $x \in W_i$, $x \neq 0$, then $\lambda(x, \theta) = \gamma_i$, $1 \leqq i \leqq k$, and

$$\sum_{i=1}^{k} m_i \gamma_i = \lim_{|t| \to +\infty} \frac{1}{t} \log |\det (\Phi(\theta, t))|, \tag{6.21}$$

[2] Discontinuities in ϕ can arise from the definition of τ in Lemma 6.2.

where $m_i = \dim W_i$, $1 \leqq i \leqq k$. If Φ is a smooth cocycle, i.e. if Φ is a fundamental solution matrix of (6.1), then (6.21) becomes

$$\sum_{i=1}^{k} m_i \gamma_i = \lim_{|t| \to +\infty} \frac{1}{t} \int_0^t \operatorname{tr} A(\theta \cdot s) \, ds.$$

Also see Vinograd (1956).

Remark 6.4. *Other proofs of the Multiplicative Ergodic theorem.* The proof of Oseledec (1968) uses many features of our argument, including the triangularization method described in § 4 and the theory of regularity described above. Some complication in Oseledec's argument seems to be due to the fact that he used neither (6.7) nor the factorization technique described in Lemmas 6.3 and 6.6. Also, Oseledec did not assume the base space $\mathbf{M}$ to be a compact metric space, and consequently his proof of the measurability (with respect to θ) of the bundles $\mathcal{W}_i(\theta)$ leaves some unanswered questions.

A portion of the Multiplicative Ergodic theorem was derived by Millionscikov (1968) for the case where μ is an ergodic measure. He constructed the measurable spectrum meas $\Sigma(\mu)$ and showed that it was constant almost everywhere. Equation (6.7) was used by Millionscikov; however, he did not derive Lemma 6.6, nor did he address the question of the measurability of the bundles $\mathcal{W}_i(\theta)$.

Raghunathan (1979), Ruelle (1979), Crauel (1981), and Kifer (1985) give alternative proofs of the Multiplicative Ergodic theorem. Their approach is based on either a theorem of Furstenberg and Kesten (1960) (see § 10) or the Subadditive Ergodic theorem, which was proved by Kingman (1968). Ruelle, for example, first shows that the limits in (6.20) exist almost everywhere. By using the eigenspaces of the associated self-adjoint operator $\Phi^*(\theta, t)\Phi(\theta, t)$, he constructs the measurable subbundles $\mathcal{W}_i(\theta)$.

The proof by Ruelle is more general than ours in that it applies to certain linearized semiflows generated by evolutionary equations on an infinite dimensional Hilbert space. Ruelle does not assume the base space $\mathbf{M}$ to be compact; instead he uses a logarithmic-boundedness condition on the cocycle Φ. This boundedness condition is automatically satisfied when the base space is compact. As we shall see in § 10, the assumption that $\mathbf{M}$ be compact is not a serious restriction, since this can be satisfied in practically every application.

7. Flow on the projective bundle. In this section we shall study the ergodic properties of the induced flow on the projective bundle, see Johnson (1978), (1980b) and Crauel (1981).

As in § 6, we let $\phi = (\theta, U) \in \mathbf{H}_\nu(p)$ and let $\lambda_1(\phi) < \cdots < \lambda_k(\phi)$ be the growth rates with multiplicity $\vec{m} = (m_1, \cdots, m_k)$, where $m_1 + \cdots + m_k = m$. By Lemma 6.9 we recall that $\lambda_i(\phi)$ depends only on θ, $1 \leqq i \leqq k$. Next define

$$U_i^{\pm}(\phi) = \operatorname{Span} \left\{ y \in \mathbf{R}^m : y \neq 0 \text{ and } \limsup_{t \to \pm\infty} \frac{1}{|t|} \log |\Psi(\phi, t) y| \leqq \lambda_i(\phi) \right\}.$$

Then $\dim U_i^+(\phi) = m_1 + \cdots + m_i$ and $\dim U_i^-(\phi) = m_i + \cdots + m_k$. Also one has $\mathcal{W}_i(\phi) = U_i^+(\phi) \cap U_i^-(\phi)$, $\dim \mathcal{W}_i(\phi) = m_i$ and $\mathbf{R}^m = \mathcal{W}_1(\phi) + \cdots + \mathcal{W}_k(\phi)$. By Lemma 6.9 we see that $\mathcal{W}_i(\theta) := U\mathcal{W}_i(\phi)$ depends only on θ.

Let $\mathbf{P}^{m-1}(\mathbf{R})$ denote the projective space of lines in $\mathbf{R}^m$ containing the origin, with the usual topology. We define the induced flow $\hat{\pi}$ on the projective bundle $\mathbf{N} = \mathbf{P}^{m-1}(\mathbf{R}) \times \mathbf{M}$ by $(l, \theta) \cdot t = (\Phi(\theta, t) l, \theta \cdot t)$. (Since Φ is linear it maps lines onto lines.)

If $\mathbf{T} = \mathbf{R}$, we define $f : \mathbf{N} \to \mathbf{R}$ by $f(l, \theta) = \langle A(\theta) x, x \rangle$, where A is the matrix function (6.1), $\langle\ ,\ \rangle$ is the Euclidean inner product on $\mathbf{R}^m$, and $x \in l$ satisfies $|x| = 1$. Then for

$|x|=1$ one has

$$\log|\Phi(\theta, t)x| = \int_0^t f((l, \theta)\cdot s)\,ds. \tag{7.1}$$

If $\mathbf{T}=\mathbf{Z}$, we define $f:\mathbf{N}\to\mathbf{R}$ by $f(\theta, l)=\frac{1}{2}\log\langle A^*(\theta)A(\theta)x, x\rangle$, where $A(\theta)=\Phi(\theta, 1)$ and $|x|=1$. Then one has

$$(7.2)\qquad \begin{aligned} \log|\Phi(\theta, t)x| &= \sum_{s=0}^{t-1} f((l,\theta)\cdot s), && t\geqq 1,\\ \log|\Phi(\theta, t)x| &= \sum_{s=1}^{|t|} f((l,\theta)\cdot -s), && t<0. \end{aligned}$$

Next define the time-averages

$$f^+(l,\theta)=\limsup_{t\to+\infty}\frac{1}{t}\int_0^t f((l,\theta)\cdot s)\,ds,$$

and

$$f^-(l,\theta)=\liminf_{t\to-\infty}\frac{1}{|t|}\int_0^t f((l,\theta)\cdot s)\,ds$$

when $\mathbf{T}=\mathbf{R}$. (For $\mathbf{T}=\mathbf{Z}$, $f^\pm$ are defined similarly by using (7.2).) Then by (7.1) and (7.2) $f^+(l,\theta)$ and $f^-(l,\theta)$ are the Lyapunov exponents $\lambda_s^+(x,\theta)$ and $\lambda_i^-(x,\theta)$, respectively, where $x\in l$, $x\neq 0$. Also the functions $f^\pm:\mathbf{N}\to\mathbf{R}$ are Borel measurable.

For $\theta\in\mathbf{M}_\mu(p)=r(\mathbf{H}_\nu(p))$ and $1\leqq i\leqq k$ we define

$$u_i^+(\theta)=\{l\in\mathbf{P}^{m-1}(\mathbf{R})|f^+(l,\theta)\leqq\lambda_i(\theta)\},$$
$$u_i^-(\theta)=\{l\in\mathbf{P}^{m-1}(\mathbf{R})|f^-(l,\theta)\leqq-\lambda_i(\theta)\}.$$

For θ fixed, $u_i^\pm(\theta)$ are closed subsets of $\mathbf{P}^{m-1}(\mathbf{R})$, and in fact are the "traces" in $\mathbf{P}^{m-1}(\mathbf{R})$ of the vector subspaces $U_i^\pm(\theta)$ of $\mathbf{R}^m$ defined above. This leads us to introduce the space $\mathcal{K}$ of closed subsets of $\mathbf{P}^{m-1}(\mathbf{R})$, with the Hausdorff topology. Thus $F_n\to F$ in $\mathcal{K}\Leftrightarrow$ to each $x\in F$, there corresponds a sequence $x_n\in F_n$ so that $x_n\to x$ in $\mathbf{P}^{m-1}(\mathbf{R})$. Observe that the "trace" of $\mathcal{W}_i(\theta)$ in $\mathbf{P}^{m-1}(\mathbf{R})$ is $u_i^+(\theta)\cap u_i^-(\theta)$.

Fix the pair $p=(k,\vec{m})$ and restrict attention to $\mathbf{M}_\mu(p)$. The following proposition is a direct consequence of the measurability of the exponents $\lambda_1,\cdots,\lambda_k$.

LEMMA 7.1. *For every $r>0$, there is a compact set $Z\subseteq\mathbf{M}_\mu(p)$ such that $\mu(\mathbf{M}_\mu(p)\backslash Z)<r$ and the restriction $\lambda_i|_Z$ is continuous, $1\leqq i\leqq k$.*

We will now show that the functions $u_i^\pm$ are μ-measurable. (The measurability of $\mathcal{W}_i$ is also a consequence of this fact.) Consider u_i^+ and $\mathbf{T}=\mathbf{R}$. (The arguments for u_i^- and $\mathbf{T}=Z$ are similar and we will omit them.) Define $g_t(l,\theta):=(1/t)\int_0^t f((l,\theta)\cdot s)\,ds$. Then g_t is continuous on $\mathbf{N}$, and $\limsup_{t\to+\infty} g_t(l,\theta_0)=f^+(l,\theta)$ for all $(l,\theta)\in\mathbf{P}^{m-1}(\mathbf{R})\times\mathbf{M}_\mu(p)$. Let $r>0$ be given, and let $Z\subseteq\mathbf{M}_\mu(p)$ be a compact set with $\mu(\mathbf{M}_\mu(p)\backslash Z)<r$, where λ_i is continuous on Z, $1\leqq i\leqq k$. Choose δ so that

$$0<3\delta<|\lambda_i(\theta)-\lambda_j(\theta)| \tag{7.3}$$

for all $i\neq j$ and $\theta\in Z$. Finally define

$$v_t^+(\theta):=\{l\in\mathbf{P}^{m-1}(\mathbf{R})\,|\,g_t(l,\theta)\in(-\infty,\lambda_i(\theta)+\delta]\}.$$

Then $v_t^+(\theta)\in\mathcal{K}$ for $\theta\in Z$. It is not difficult to verify that $v_t^+:Z\to\mathcal{K}$ is a Borel measurable function. (In general it is not continuous.)

We claim that $v_t^+(\theta) \to u_i^+(\theta)$ in $\mathscr{K}$ as $t \to +\infty$, for each $\theta \in Z$. Assume on the contrary that there is a monotone subsequence $t_k \to +\infty$ and an element $Q \in \mathscr{K}$ such that $Q \neq u_i^+(\theta)$ and $v_{t_k}^+(\theta) \to Q$. Then $u_i^+(\theta) \subseteq Q$, since $l \in u_i^+(\theta) \Rightarrow l \in v_t^+(\theta)$ for large t. On the other hand, let $l \in Q \backslash u_i^+(\theta)$. Since $v_{t_k}^+(\theta) \to Q$ in the Hausdorff topology, there is a sequence $l_k \in v_{t_k}^+(\theta)$ with $l_k \to l$. Therefore l_k is eventually in every neighborhood of l in $\mathbf{P}^{m-1}(\mathbf{R})$. By Lemma 7.2 (below) we conclude that $g_{t_k}(l_k, \theta) \geqq \lambda_i(\theta) + 2\delta$, which contradicts the definition of $v_{t_k}^+(\theta)$.

LEMMA 7.2. *Let $l \in \mathbf{P}^{m-1}(\mathbf{R})$ with $l \notin u_i^+(\theta)$ where $\theta \in Z$. Then given any $\delta > 0$ there is a neighborhood $N(l)$ of l and a $\tau \in \mathbf{T}$ such that for all $t \geqq \tau$ one has $g_t(\tilde{l}, \theta) \geqq \lambda_i(\theta) + 2\delta$ for all $\tilde{l} \in N(\tilde{l})$.*

Proof. We will use the notation of Remark 6.1. Let $x \in l$, $x \neq 0$. Since $l \notin u_i^+(\theta)$ one has $i < b$ where

$$\lim_{t \to +\infty} \frac{1}{t} \log |\Phi(\theta, t) x_b| = \lambda_b(\theta)$$

and $|x_b| = 2\alpha > 0$. Let $N(l)$ be those lines $\tilde{l}$ in $\mathbf{P}^{m-1}(\mathbf{R})$ with the property that $\tilde{x} \in \tilde{l}$, $|\tilde{x}| = 1$, satisfies $|\tilde{x}_b| > \alpha$. The uniformity assertion in Remark 6.1 implies that for every $\beta > 0$ there is a $\tau \in \mathbf{T}$ such that

$$g_t(\tilde{l}, \theta) = \frac{1}{t} \log |\Phi(\theta, t)\tilde{x}| \geqq \lambda_b(\theta) - \beta$$

for all $t \geqq \tau$ and all $\tilde{x} \in \tilde{l} \in N(l)$ with $|\tilde{x}| = 1$. Now set $\beta = \delta$, then the lemma follows from (7.3). Q.E.D.

We see then that u_i^+ is the point-wise limit of a sequence of Borel measurable functions on Z. By the Lusin theorem, it follows that u_i^+ is measurable on $\mathbf{M}_\mu(p)$.

8. Comparison with the continuous spectrum. Let Φ be a cocycle on a compact, connected Hausdorff space $\mathbf{M}$. Let $\operatorname{dyn} \Sigma = \bigcup_{i=1}^k [a_i, b_i]$ be the dynamical spectrum with the corresponding Whitney decomposition of $\mathbf{R}^m \times \mathbf{M}$ into continuous spectral subbundles $\mathbf{R}^m \times \mathbf{M} = \mathscr{V}_1 + \cdots + \mathscr{V}_k$. Let $\mathscr{V}_i(\theta)$ denote the fiber of $\mathscr{V}_i$ in $\mathbf{R}^m$, $1 \leqq i \leqq k$. The following result is proved in Sacker and Sell (1978):

THEOREM 8.1. *The spectral subbundles $\mathscr{V}_i$ are characterized by*

$$V_i(\theta) = \operatorname{Span}\{x \in \mathbf{R}^m \colon x \neq 0 \text{ and } \lambda_s^\pm(x, \theta), \lambda_i^\pm(x, \theta) \in [a_i, b_i]\}$$

for $1 \leqq i \leqq k$, *where*

$$\lambda_s^\pm(x, \theta) = \limsup_{t \to \pm\infty} \frac{1}{t} \log |\Phi(\theta, t)x|, \qquad \lambda_i^\pm(x, \theta) = \liminf_{t \to \pm\infty} \frac{1}{t} \log |\Phi(\theta, t)x|.$$

We will next give a proof of Theorem 2.3. The essence of the argument is to verify (2.7), i.e.

$$\text{boundary } \operatorname{dyn} \Sigma \subseteq \bigcup_\mu \operatorname{meas} \Sigma(\mu) \subseteq \operatorname{dyn} \Sigma,$$

and to show that the measurable subbundle decomposition implied by the Multiplication Ergodic theorem leads to a refinement of the continuous decomposition given by the Spectral theorem.

It follows immediately from Theorem 8.1 that for any invariant probability measure μ on $\mathbf{M}$ one has $\operatorname{meas} \Sigma(\mu, \theta) \subseteq \operatorname{dyn} \Sigma$ for all $\theta \in \mathbf{M}_\mu$. In particular one has $\operatorname{meas} \Sigma(\mu) \subseteq \operatorname{dyn} \Sigma$ for every ergodic measure μ on $\mathbf{M}$. Furthermore the measurable bundles $\mathscr{W}_j(\theta)$ are contained in the associated continuous spectral bundle $\mathscr{V}_i(\theta)$ when $\lambda_j(\theta) \in [a_i, b_i]$, and $\theta \in \mathbf{M}_\mu$. Since the sum of both the $\mathscr{W}_j(\theta)$'s and the $\mathscr{V}_i(\theta)$'s span $\mathbf{R}^m$ for $\theta \in \mathbf{M}_\mu$, it follows that for all $\theta \in \mathbf{M}_\mu$ one has $\mathscr{V}_i(\theta) = \sum \mathscr{W}_j(\theta)$, where the summation is over all j with $\lambda_j(\theta) \in [a_i, b_i]$, $1 \leqq i \leqq k$. (This also follows from applying Theorem 2.2 to the spectral subbundle $\mathscr{V}_i$.)

It remains to show that if $\beta \in$ boundary dyn Σ then $\beta \in$ meas $\Sigma(\mu)$ for some ergodic measure μ. Let β be an endpoint of one of the spectral intervals $[a_i, b_i]$ in dyn Σ. Let $\mathscr{V}_i$ be the continuous spectral subbundle associated with $[a_i, b_i]$. As noted in Lemma 3.1 there is no loss in generality in assuming $\mathbf{M}$ to be a compact metric space. Let X_i be the trace of $\mathscr{V}_i$ in the projective bundle $\mathbf{N} = \mathbf{P}^{m-1}(\mathbf{R}) \times \mathbf{M}$, i.e.

$$X_i = \{(l, \theta): l \text{ is a line in } \mathscr{V}_i(\theta)\}.$$

Since $\mathscr{V}_i$ is invariant under the flow $\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta \cdot t)$, X_i is invariant under the induced flow $\hat{\pi}$ on $\mathbf{N}$. Also X_i is compact. Let $f(l, \theta)$ be given as in § 7. Recall that the time-average of $f(l, \theta)$ along orbits in $\mathbf{N}$ determines the Lyapunov exponents of the solution $\Phi(\theta, t)x$ where x is on the line l, $x \neq 0$.

Let J be the set of all invariant measures η on X_i. We claim that

$$a_i \leqq \int_{X_i} f\, d\eta \leqq b_i \tag{8.1}$$

for all $\eta \in J$. If, on the contrary, (8.1) is false for some $\eta \in J$, then for η-almost all $(l, \theta) \in X_i$ one has

$$\lim_{|t| \to \infty} \frac{1}{t} \log |\Phi(\theta, t)x| = \hat{f}(l, \theta)$$

where f is an invariant function defined on $\mathbf{N}$ with $\int_{X_i} \hat{f}\, d\eta = \int_{X_i} f\, d\eta$, x is on l, $x \neq 0$, and $\int_{X_i} f\, d\eta \notin [a_i, b_i]$. This implies that $\hat{f}(l, \theta) \notin [a_i, b_i]$ on some invariant set of positive η-measure, which contradicts Theorem 8.1.

Next we claim that there is a measure $\eta \in J$ such that $\int_{X_i} f\, d\eta = \beta$. To see this, assume for definiteness that $\beta = b_i$ is the right endpoint of $[a_i, b_i]$. Recall that J is compact, and that the mapping $\eta \to \int_{X_i} f\, d\eta$ is continuous. Therefore if there is no $\eta \in J$ with $\int_{X_i} f\, d\eta = \beta$, then it follows from (8.1) that there is an $\varepsilon > 0$ such that $\int_{X_j} f\, d\eta \leqq \beta - \varepsilon$ for all $\eta \in J$. It follows from the Krylov–Bogoliubov method described in § 5 that for every $(x, \theta) \in \mathscr{V}_i$ with $x \neq 0$ one has

$$\limsup_{t \to \infty} \frac{1}{t} \log |\Phi(\theta, t)x| = \limsup_{t \to \infty} \frac{1}{t} \int_0^t f(\pi(l, \theta, s))\, ds \leqq \beta - \varepsilon.$$

It then follows from Sacker and Sell (1978, Lemma 4) that $\beta \notin$ dyn Σ, a contradiction. A similar argument works for $\beta = a_i$.

We want to show next that η can be chosen to be an ergodic measure on X_i. Now fix $\eta \in J$ with $\int_{X_j} f\, d\eta = \beta$. Since X_j is a metric space, we can use the Choquet Representation theorem, Phelps (1966), to find a probability measure A_η on the set of *ergodic* measures E in J such that

$$\int_{X_j} g\, d\eta = \int_E \left(\int_{X_j} g\, d\sigma \right) dA_\eta(\sigma)$$

for each $g \in \mathscr{C}(X_i, \mathbf{R})$. In particular for $f = g$ one has

$$\beta = \int_{X_j} f\, d\eta = \int_E \left(\int_{X_j} f\, d\sigma \right) dA_\eta(\sigma).$$

From (8.1) we see that one cannot have $\int_{X_j} f\,d\sigma < \beta = b_i$ for all $\sigma \in E$. It follows then that there is an ergodic measure $\sigma \in E$ with $\int_{X_j} f\,d\sigma = \beta$.

Finally let μ be the projection of σ to $\mathbf{M}$. Then μ is an ergodic measure on $\mathbf{M}$, and from § 6 we see that $\beta \in \operatorname{meas} \Sigma(\mu)$. This completes the proof of Theorem 2.3. Q.E.D.

Remark 8.1. In general, one cannot find a single ergodic measure μ such that $a_i, b_i \in \operatorname{meas} \Sigma(\mu)$ for all endpoints a_i, b_i, even if $\mathbf{M}$ is minimal. Here is a simple example. According to Furstenberg (1961), there is a discrete flow on the 2-torus $\mathbf{M} = \mathbf{T}^2$ with more than one (in fact uncountably many) ergodic measures. Moreover, there is a continuous function g on $\mathbf{M}$ so that $\int_{\mathbf{M}} g\,d\mu_1 \neq \int_{\mathbf{M}} g\,d\mu_2$ for distinct ergodic measures μ_1, μ_2. Define $\Phi: \mathbf{M} \times \mathbf{Z} \to \mathbf{R}$ by $\Phi(y, 1) = \exp g(y)$. The dynamical spectrum of the cocycle Φ is $[a, b]$, where $a = \inf \int_{\mathbf{M}} g\,d\mu$, $b = \sup \int_{\mathbf{M}} g\,d\mu$, and the inf and sup are taken over all ergodic measures on $\mathbf{M}$. However, the measurable spectrum contains just the point $\{\int_{\mathbf{M}} g\,d\mu\}$ for each ergodic measure μ.

Remark 8.2. If there is only one ergodic measure μ on $\mathbf{M}$, for example if the flow $\theta \cdot t$ on $\mathbf{M}$ is almost periodic, then $\operatorname{meas} \Sigma(\mu)$ is a subset of the $\operatorname{dyn} \Sigma$ and *all* endpoints a_i, b_i of $\operatorname{dyn} \Sigma$ are in $\operatorname{meas} \Sigma(\mu)$. For $m = 2$ we conclude that $\operatorname{meas} \Sigma(\mu) =$ boundary $\operatorname{dyn} \Sigma$. An example in Johnson (1986) shows that for $m = 3$, even if $\mathbf{M}$ is almost periodic, the measurable spectrum need not consist entirely of endpoints a_i, b_i.

9. Computation of the measurable spectrum. Wedge product flows. Let $\theta \in \mathbf{M}$ be a Lyapunov point and let $\gamma_1(\theta) \leqq \cdots \leqq \gamma_m(\theta)$ denote the growth rates with associated basis $e_1, \cdots, e_m$. Thus one has $\lambda(e_i, \theta) = \gamma_i(\theta)$, $1 \leqq i \leqq m$. By a standard argument, see Naylor and Sell (1982, p. 268) for example, there is a constant K such that for any vector $x \in \mathbf{R}^m$ one has $x = \alpha_1 e_1 + \cdots + \alpha_m e_m$ and

$$|\Phi(\theta, t)x| \leqq K \max \{|\Phi(\theta, t)e_i| : \alpha_i \neq 0\}|x\} \tag{9.1}$$

for all $t \in \mathbf{T}$. It follows from (9.1) that

$$\limsup_{t \to +\infty} \frac{1}{t} \log |\Phi(\theta, t)| \leqq \gamma_m(\theta).$$

On the other hand one has

$$|\Phi(\theta, t)e_m| \leqq |\Phi(\theta, t)|\,|e_m|,$$

which implies that

$$\gamma_m(\theta) = \lim_{t \to +\infty} \frac{1}{t} \log |\Phi(\theta, t)e_m| \leqq \liminf_{t \to \infty} \frac{1}{t} \log |\Phi(\theta, t)|,$$

and hence one has

$$\lim_{t \to \infty} \frac{1}{t} \log |\Phi(\theta, t)| = \gamma_m(\theta). \tag{9.2}$$

A similar argument yields

$$\lim_{t \to -\infty} \frac{1}{t} \log |\Phi(\theta, t)| = \gamma_1(\theta). \tag{9.3}$$

An early version of (9.2) for stationary stochastic process of $(m \times m)$ matrices appears in Furstenberg and Kesten (1960).

The same considerations extend to the induced wedge product cocycles $\Lambda^k\Phi(\theta, t)$ on $\Lambda^k\mathbf{R}^m$, where $2 \leqq k \leqq m$. If $\Psi(\phi, t)$ satisfies (4.7), then one has

$$\Lambda^k\Psi(\phi, t) = (\Lambda^k q(\phi \cdot t))^{-1}(\Lambda^k\Phi(\theta, t))(\Lambda^k q(\phi)).$$

Hence the cocycles $\Lambda^k\Psi$ and $\Lambda^k\Phi$ are cohomologous. Therefore $\Lambda^k\Psi$ and $\Lambda^k\Phi$ have the same Lyapunov exponents.

For $1 \leqq k \leqq m$ let Ord (k, m) denote the collection of all strictly monotone mappings $\sigma:\{1, \cdots, k\} \to \{1, \cdots, m\}$. We will use the lexicographic ordering on Ord (k, m); thus $\sigma < \tau$, where $\sigma, \tau \in$ Ord (k, m), provided there is a j, $1 \leqq j \leqq k$ such that $\sigma(i) = \tau(i)$ for $1 \leqq i \leqq j-1$ and $\sigma(j) < \tau(j)$. If $\{e_1, \cdots, e_m\}$ is any basis for $\mathbf{R}^m$, then $\{e_\sigma\colon \sigma \in \text{Ord}\,(k, m)\}$ is a basis for $\Lambda^k\mathbf{R}^m$ where

$$e_\sigma = e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(k)}. \tag{9.4}$$

Furthermore if T is an upper-triangular $(m \times m)$ matrix (with respect to the basis $\{e_1, \cdots, e_m\}$), then $\Lambda^k T$ is an upper-triangular matrix with respect to the basis e_σ, $\sigma \in$ Ord (k, m). Also the diagonal entry $t_{\sigma\sigma}$ (in the σth position on the diagonal) is given by the product

$$t_{\sigma\sigma} = t_{\sigma(1)\sigma(1)} \cdot \cdots \cdot t_{\sigma(k)\sigma(k)}.$$

If, in addition, one has $t_{ii} > 0$ for $1 \geqq i \geqq m$, then $t_{\sigma\sigma} > 0$ for all $\sigma \in$ Ord (k, m).

Let us return to the triangular cocycle $\Psi(\phi, t)$. It follows from the last paragraph that if $\phi = (\theta, U)$ is fixed and if the diagonal entries of $\Psi(\phi, t)$ satisfy (6.6), then the diagonal entry $\psi_{\sigma\sigma}$ of $\Lambda^k\Psi$ satisfies

$$\lim_{|t|\to\infty} \frac{1}{t} \log |\psi_{\sigma\sigma}(\phi, t)| = \gamma_{\sigma(1)} + \cdots + \gamma_{\sigma(k)}, \tag{9.5}$$

where $\sigma \in$ Ord (k, m) and $2 \leqq k \leqq m$. The collection of numbers given by (9.5), where σ varies over Ord (k, m), represents the Lyapunov exponents of $\Lambda^k\Psi$. This analysis applies for every $\phi \in \mathbf{H}_\nu$, where $\mathbf{H}_\nu$ is given by Lemma 6.4. For $\phi \in \mathbf{H}_\nu(p)$ we shall rewrite the growth rates in the form (2.8) where

$$\gamma_1(\phi) \leqq \gamma_2(\phi) \leqq \cdots \leqq \gamma_m(\phi). \tag{9.6}$$

It then follows from (9.5) that the largest growth rate for $\Lambda^k\Psi$ is

$$\gamma_{m+1-k}(\phi) + \cdots + \gamma_m(\phi)$$

and the smallest is

$$\gamma_1(\phi) + \cdots + \gamma_k(\phi).$$

The argument in the first paragraph in this section now applies to $\Lambda^k\Psi$, which completes the proof of Theorem 2.4.

Remark 9.1. One can give a precise description of the measurable bundles $W_\tau^{(k)}(\phi)$ corresponding to $\Lambda^k\Psi$. Fix $\phi \in \mathbf{H}_\nu(p)$, where the growth rates of Ψ satisfy (9.6), and let $e_1(\phi), \cdots, e_m(\phi)$ be a basis in $\mathbf{R}^m$ that satisfies $\lambda(e_i(\phi), \phi) = \gamma_i(\phi)$, $1 \leqq i \leqq m$. For $\tau \in$ Ord (k, m) we define

$$\gamma_\tau(\phi) = \gamma_{\tau(1)}(\phi) + \cdots + \gamma_{\tau(k)}(\phi).$$

Then one has

$$W_\tau^{(k)}(\phi) = \text{span}\,\{e_\sigma\colon \lambda(e_\sigma, \phi) = \gamma_\tau(\phi)\}, \tag{9.7}$$

where e_σ is defined by (9.4) for $\sigma \in$ Ord (k, m). We will omit the proof of (9.7), which is a simple application of the techniques developed in § 6.

10. Applications and illustrations. In this section we collect several illustrative examples of the theory described above. Included here is a discussion of spiral systems, products of "random" matrices, conservative second-order Schrödinger equations with almost periodic potentials, and linear stochastic differential equations with bounded measurable coefficients.

(A) *Spiral systems.* The theory above applies to every compact invariant set $\mathbf{N}$ in $\mathbf{M}$. In this case the dynamical spectrum $\operatorname{dyn}\Sigma(\mathbf{N})$ depends on $\mathbf{N}$. Next we want to study the case where $\mathbf{N}$ is a single orbit together with its ω-limit set, i.e. a *spiral system.* More precisely let $\mathbf{M}$ be a compact Hausdorff space with a flow $\theta \cdot t$. Let θ_0 be a given point in $\mathbf{M}$ and define

$$\mathbf{N} = H^+(\theta_0) = \text{closure}\,\{\theta_0 \cdot t\colon t \geqq 0\}.$$

Then $\mathbf{N}$ is positively invariant and the ω-limit set $\Omega = \bigcap_{\tau \geqq 0} H^+(\theta_0 \cdot \tau)$ is a compact invariant set. We are interested in the case where $\theta_0 \notin \Omega$. Thus the positive trajectory $\theta_0 \cdot t$ forms a spiral. See Fig. 1.

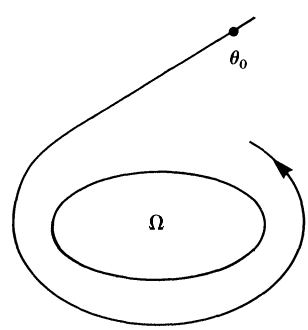

FIG. 1. N: *A spiral system.*

Let $\Phi(\theta, t)$ be a cocycle defined on $\mathbf{M}$. Then the theory described above applied directly to the restriction of Φ to the ω-limit set Ω. The problem we wish to study here is the limiting behavior (as $s, \tau \to +\infty$) of the cocycles $\Lambda^k\Phi(\theta_0 \cdot s, t)$, $1 \leqq k \leqq m$, along the spiral trajectory θ_0. In particular we want to show that this limiting behavior can be used to evaluate the measurable spectrum of the ω-limit set Ω.

Before doing the analysis it should be noted that this study addresses a basic question which arises naturally when one is doing a numerical evaluation of the measurable (or dynamical) spectrum. The initial point θ_0 in $\mathbf{M}$ is determined by the code or program. If by good fortune it happens to lie in the set $\mathbf{M}_\mu$ (Theorem 2.1), then Theorem 2.4 explains how to compute the spectrum. With our present understanding, one does not know whether or not we have had good fortune. However, what is always true is that θ_0 does determine a spiral system.

For $k = 1, \cdots, m$ we define

$$b^k := \limsup_{s, \tau \to +\infty} \frac{1}{\tau} \log |\Lambda^k\Phi(\theta_0 \cdot s, \tau)|.$$

Also let Σ^k denote the dynamical spectrum of the linear skew-product flow $(\Lambda^k\Phi(\theta, t), \theta \cdot t)$ over Ω, and define

$$a^k := \max \Sigma^k.$$

We will now prove the following result:

THEOREM 10.1. *The following statements are valid*:

(A) *For* $k=1,\cdots,m$ *one has* $a^k=b^k$.

(B) *If the flow on* Ω *is uniquely ergodic* (*e.g. almost periodic*) *then* meas $\Sigma=\{\gamma_1,\cdots,\gamma_m\}$ *where* $\gamma_m=b^1$ *and*

$$\gamma_{m-k}=b^{k+1}-(\gamma_m+\cdots+\gamma_{m-k+1}),\qquad 1\leqq k\leqq m-1.$$

Proof. We will give the proof of (A) for $k=1$. The argument for $k\geqq 2$ is the same. Statement (B) is an immediate corollary of part (A) and Theorems 2.1, 2.3 and 2.4.

Let a and b denote a^1 and b^1. For $S\geqq 0$ and $T\geqq 0$ we define

$$\beta(S,T)=\sup_{\substack{S\leqq s\\ T\leqq\tau}}\frac{1}{\tau}\log|\Phi(\theta_0\cdot s,\tau)|.$$

Then $b=\lim_{S,T\to+\infty}\beta(S,T)$. Fix $\varepsilon>0$ and choose $S\geqq 0$, $T\geqq 0$ so that $\beta(S,T)\leqq b+\varepsilon$. For $t=s+\tau$ one then has

$$|\Phi(\theta_0\cdot s,\tau)|=|\Phi(\theta_0,t)\Phi^{-1}(\theta_0,s)|\leqq K\,e^{(b+\varepsilon)(t-s)},\qquad 0\leqq s\leqq t$$

where $K=\max\{|\Phi(\theta_0\cdot s,\tau)|: 0\leqq s\leqq S,\ 0\leqq\tau\leqq T\}$. It then follows directly, see Sacker and Sell (1974, Thms. 2 and 5) and (1976a, Lemma 4) for example, that $a\leqq b+\varepsilon$ for every $\varepsilon>0$. Hence $a\leqq b$.

If one has $a<b$, then we can replace $\Phi(\theta,t)$ by the shifted flow $e^{\lambda t}\Phi(\theta,t)$ where $3\lambda=2a+b$. This has the effect of shifting a and b to $a-\lambda$ and $b-\lambda$, respectively. Without any loss of generality, therefore, we can assume then that $a<0<b$ and set $b=3\alpha$. Since $a<0$, the linear skew-product flow $(\Phi(\theta,t)x,\theta\cdot t)$ has an exponential dichotomy over Ω with $\mathscr{S}=\mathbf{R}^m\times\Omega$. This means that there is a constant K such that

$$|\Phi(\theta,t)\Phi^{-1}(\theta,s)|\leqq K\,e^{-\sigma(t-s)}$$

for all $\theta\in\Omega$, and $s\leqq t$, $s,t\in\mathbf{T}$. In particular it follows from Sacker and Sell (1974, p. 452) that $\mathscr{B}=\{0\}\times\Omega$, $\mathscr{S}=\mathbf{R}^m\times\Omega$ and $\mathscr{U}=\{0\}\times\Omega$, where $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{U}$ are defined to be those $(x,\theta)\in\mathbf{R}^m\times\Omega$ that satisfy

$$\sup_{t\in\mathbf{T}}|\Phi(\theta,t)x|<\infty,\quad \lim_{t\to+\infty}|\Phi(\theta,t)x|=0\quad\text{and}\quad\lim_{t\to-\infty}|\Phi(\theta,t)x|=0,$$

respectively.

Since $b>0$ there are sequences $s_n\to+\infty$, $\tau_n\to+\infty$ such that

$$|\Phi(\theta_0\cdot s_n,\tau_n)|\geqq e^{\alpha\tau_n}.$$

Let e_n be a vector with $|e_n|=1$ and

$$|\Phi(\theta_0\cdot s_n,\tau_n)e_n|\geqq e^{\alpha\tau_n}.$$

Fix σ_n so that $0\leqq\sigma_n\leqq\tau_n$ and

$$|\Phi(\theta_0\cdot s_n,t)e_n|\leqq|\Phi(\theta_0\cdot s_n,\sigma_n)e_n|,\qquad 0\leqq t\leqq\tau_n.$$

Set $\xi_n=\Phi(\theta_0\cdot s_n,\sigma_n)e_n$. Then $|\xi_n|\geqq e^{\alpha\tau_n}$, and

$$|\xi_n|^{-1}|\Phi(\theta_0\cdot s_n,t)e_n|\leqq 1,\qquad 0\leqq t\leqq\sigma_n.$$

Since $|\xi_n|\to+\infty$ one has $\sigma_n\to+\infty$. Also one has $e_n=\Phi(\theta_n,-\sigma_n)\xi_n$ where $\theta_n=\theta_0\cdot(s_n+\sigma_n)$. By choosing subsequences (if necessary) we can assume that $\theta_n\to\theta\in\Omega$ and $|\xi_n|^{-1}\xi_n\to e$ where $|e|=1$. It follows from Sacker and Sell (1976a, Lemma 4) that $(e,\theta)\in\mathscr{U}$, which contradicts the fact that $\mathscr{U}=\{0\}\times\Omega$. We conclude that $a=b$.

Remark 10.1. The conclusions of Theorem 10.1 can be reformulated in another manner. The numbers a^k represent the "largest possible" Lyapunov exponents for the cocycle $\Lambda^k\Phi(\theta, t)$, where $\theta\in\Omega$ and $1\leqq k\leqq n$. By the inequality (2.7) and Theorem 2.4 we see that there are ergodic measures μ_k on Ω with the property that a^k is the largest value in the measurable spectrum generated by $\Lambda^k\Phi$, $1\leqq k\leqq n$. Consequently if μ is any invariant measure on Ω and $\gamma_1(\theta)\leqq\cdots\leqq\gamma_m(\theta)$ are the growth rates satisfying (9.6) for $\theta\in\Omega_\mu$, then one has

$$\gamma_m(\theta)+\cdots+\gamma_{m-k+1}(\theta)\leqq a^k$$

for all $\theta\in\Omega_\mu$.

Remark 10.2. The ergodic measures μ_k referred to in the last paragraph need not be the same. Let Ω be a minimal set in the flow $\theta\cdot t$, and assume that this flow is not uniquely ergodic. Then there are ergodic measures μ_1 and μ_2 on Ω and a continuous real-valued function g that satisfies $\int g\,d\mu_1\neq\int g\,d\mu_2$. Since g can be replaced by $g+c$ and/or αg, where c and α are constants with $\alpha\neq 0$, we can assume that μ_1, μ_2 and g are chosen to satisfy

$$-1=\int g\,d\mu_2\leqq\int g\,d\mu\leqq\int g\,d\mu_1=3$$

for every ergodic measure μ. Now set $h=-2g$ and consider the linear skew-product flow on $\mathbf{R}^2\times\Omega$ generated by

$$x'=\operatorname{diag}(g(\theta\cdot t), h(\theta\cdot t))x.$$

In the notation introduced above one then has $a^1=3$, $a^2=2$, and $a^i=\max\operatorname{meas}\Sigma(\mu_i)$, $i=1,2$.

Remark 10.3. It may happen that all the limits

$$\lim_{t\to+\infty}\frac{1}{t}\log|\Lambda^k\Phi(\theta_0, t)|=c^k,\qquad 1\leqq k\leqq m,$$

exist. If so, then one has $c^k\leqq a^k$, $1\leqq k\leqq m$. By using the associated triangular flow and the Krylov-Bogoliubov method described in § 5, one can show that there is an invariant measure μ on Ω with the property that $\operatorname{meas}\Sigma(\mu)=\{\gamma_1,\cdots,\gamma_m\}$ where $\gamma_1\leqq\cdots\leqq\gamma_m$ and

$$\gamma_{m-k+1}+\cdots+\gamma_m=c^k,\qquad 1\leqq k\leqq m.$$

Remark 10.4. The conclusions of Theorem 10.1 and the above remarks are related to results of Pelikan (1983) who has analyzed the structure of Bowen-Ruelle invariant measures on certain attractors.

(B) *Products of random matrices.* In this example we show that the theory of products of matrices considered in Furstenberg and Kesten (1960) is often imbeddable in our theory. Let $\mathbf{K}$ denote a fixed compact subset of $\mathcal{GL}(m)$ and let $\{X_1, X_2,\cdots\}$ be a given sequence of $(m\times m)$ matrices with values in $\mathbf{K}$. For $n=1,2,\cdots$ form the product

$$Y_n=X_nX_{n-1}\cdots X_1.$$

We wish to study the limiting behavior of

$$\frac{1}{n}\log|Y_n|$$

as $n\to\infty$. This will be done by imbedding this problem into a spiral system, where $Y_n=\Phi(\theta_0, n)$ for an appropriate cocycle Φ, and using Theorem 10.1.

Let $\mathbf{M}:=\mathbf{K}^{\mathbf{Z}}$ denote the collection of all two-sided sequences

$$\theta=(\cdots, A_{-2}, A_{-1}; A_0; A_1, A_2, \cdots) \tag{10.1}$$

with entries $A_i\in\mathbf{K}$, $i\in\mathbf{Z}$. (We will use semi-colons to designate the zeroth position of θ.) Then $\mathbf{M}$ is a compact metric space with the shift flow $\theta\cdot n$ where

$$\theta\cdot n=(\cdots, A_{n-2}, A_{n-1}; A_n; A_{n+1}, A_{n+2}, \cdots)$$

for $n\in\mathbf{Z}$. Define $F:\mathbf{M}\to\mathbf{K}$ by $F(\theta):=A_0$ where θ is given by (10.1). Then one constructs a cocycle Φ over $\mathbf{M}$ by defining $\Phi(\theta, 0)=I$

$$\Phi(\theta, n)=F(\theta\cdot(n-1))\cdots F(\theta), \qquad n\geqq 1,$$

$$\Phi(\theta, n)=[F(\theta\cdot(-1))\cdots F(\theta\cdot n)]^{-1}, \qquad n\leqq -1.$$

It is not difficult to see that the cocycle identity (2.1) is valid for $t, s\in\mathbf{Z}$.

The distinguished sequence $\{X_1, X_2, \cdots\}$ is imbedded into this flow as a spiral, i.e. let A be a fixed element in $\mathbf{K}$ and set

$$\theta_0=(\cdots, A, A; X_1; X_2, X_3, \cdots),$$

where every negative entry in θ_0 is A. Then $\theta_0\in\mathbf{M}$ and $Y_n=\Phi(\theta_0, n)$ for $n\geqq 1$. By Theorem 10.1 we see that

$$\limsup_{m,n\to+\infty}\frac{1}{m}\log|\Phi(\theta\cdot n, m)|=\limsup_{m,n\to+\infty}\frac{1}{m}\log|Y_{m+n}Y_n^{-1}| \tag{10.2}$$

exists and this is the maximum value of the dynamical spectrum over Ω, the ω-limit set of θ_0.

When the distinguished sequence $\{X_1, X_2, \cdots\}$ is a stationary stochastic process, then the expectation satisfies $E(\log^+|X_1|)<\infty$ since X_1 assumes values in the compact set $\mathbf{K}$. If, in addition, this stochastic process is metrically transitive (i.e. ergodic) then Theorem 2.4 is applicable, and one concludes that

$$\lim_{n\to\infty}\frac{1}{n}\log|Y_n| \tag{10.3}$$

exists with probability 1. Also the limits in (10.2) and (10.3) agree. We refer the reader to Furstenberg and Kesten (1960) for more details.

Remark 10.5. Some interesting applications of products of random matrices to problems in demographics can be found in Cohen (1979).

(C) *Schrödinger equation.* In the study of the Schrödinger equation

$$\mathrm{Ly}=\left(-\frac{d^2}{dt^2}+q(t)\right)y=\lambda y,$$

where $q(t)$ is real and Bohr almost periodic, it is of interest to compute the "Lyapunov number" $\beta(\lambda)$ as a function of $\lambda\in R$. $\beta(\lambda)$ is defined as follows: First introduce the *hull* $\mathbf{M}$ of q by

$$\mathbf{M}=\text{closure}\,\{q_\tau\,|\,\tau\in\mathbf{R}\},$$

where $q_\tau(t)=q(t+\tau)$, and the closure is in the uniform topology. Then $\mathbf{M}$ is a compact metric space with translation flow $\theta\cdot\tau=\theta_\tau$. In fact $\mathbf{M}$ is a compact topological group, and the normalized Haar measure μ is the unique invariant measure on $\mathbf{M}$. Define

$Q(\theta) = \theta(0)$, and consider the operators $L_\theta = -(d^2/dt^2) + Q(\theta \cdot t)$ and the associated equations

$$x' = \begin{pmatrix} 0 & 1 \\ -\lambda + Q(\theta \cdot t) & 0 \end{pmatrix} x, \qquad x = \begin{pmatrix} y \\ y' \end{pmatrix}. \tag{10.4}$$

Since the trace of the coefficient matrix is zero, one obtains from Liouville's formula that

$$\text{meas}\, \Sigma(\mu) = \{-\beta(\lambda), \beta(\lambda)\}$$

where $\beta(\lambda) \geqq 0$. This defines $\beta(\lambda)$. Also, as noted in Remark 8.2, one has boundary dyn $\Sigma = \text{meas}\, \Sigma(\mu)$.

Spectral properties of the self-adjoint linear operators L_θ on $L^2(-\infty, \infty)$ are reflected in the dynamics of (10.4). For example, λ is in the resolvent set for L_θ if and only if (10.4) admits an exponential dichotomy, Johnson (1982). Also if $\beta(\lambda) > 0$ for all λ in an interval I, then for μ-almost all θ, the (functional analytic) spectrum of L_θ has no absolutely continuous component in I, Pastur (1980), Ishii (1973). Moreover if $\beta(\lambda) = 0$ for all λ in I, then I is in the purely absolutely continuous spectrum of L_θ for μ-almost all θ, Kotani (1982).

The numerical computation of $\beta(\lambda)$ when $q(t) = \cos t + \cos \pi t$, for example, is a challenging problem. An investigation of this problem is described in Perry (1986). The basic idea is to use Theorem 2.4 to estimate $\beta(\lambda)$. Also special properties of second order linear equations, as described in Johnson (1980a) and Johnson and Moser (1982), can be exploited to help determine whether or not (10.4) admits an exponential dichotomy for $\lambda = 0$. This, in turn, leads to a resolution of the question of whether or not one has dyn $\Sigma = \text{meas}\, \Sigma(\mu)$.

Another method for computing $\beta(\lambda)$, which was suggested by R. Helleman, is to use the theory of Johnson and Moser (1982). In this setting one extends λ to the complex plane so that for $\text{Im}\, \lambda > 0$, $\beta(\lambda)$ is the real part of a holomorphic function $w(\lambda)$, called the Floquet exponent of (10.4). When $\text{Im}\, \lambda > 0$, (10.4) has an exponential dichotomy. One can compute $\beta(\lambda)$ for real λ by a limiting formula:

$$\beta(\lambda) = \lim_{\eta \to 0^+} \beta(\lambda + i\eta).$$

(D) *Linear stochastic differential equations.* An interesting variation of the Schrödinger equation occurs when the potential $q(t)$ is a stochastic variable. More generally consider the m-dimensional case $x' = A(t)x$, $x \in \mathbf{R}^m$, where the entries $a_{ij}(t)$ are stochastic variables in t. We will show how this can be imbedded in a linear skew-product flow on $\mathbf{R}^m \times \mathbf{M}$, where $\mathbf{M}$ is a compact space, under the assumption that the coefficients $a_{ij}(t)$ are bounded and measurable in t, i.e. $a_{ij} \in L^\infty(\mathbf{R})$. (See Kurzweil (1957), Miller and Sell (1970) and Sacker and Sell (1974) for more information.)

The set $\mathbf{M}$ is the hull of A and is defined by

$$\mathbf{M} = \text{closure}\, \{A_\tau : \tau \in \mathbf{R}\},$$

where $A_\tau(t) = A(\tau + t)$ and the closure is taken in the "weak L^1-local" topology. That is, a generalized sequence $\{A_n\}$ converges to a limit B if for every $\tau \in \mathbf{R}$ and every $\phi \in L^1[\tau, \tau+1]$ one has

$$\int_\tau^{\tau+1} A_n(t)\phi(t)\, dt \to \int_\tau^{\tau+1} B(t)\phi(t)\, dt.$$

Since $A(t)$ is bounded and measurable, the hull of A is a compact Hausdorff space. If $B \in \mathbf{M}$ we let $\Phi(B, t)$ be the fundamental solution matrix for $x' = B(t)x$. Then $\Phi: \mathbf{M} \times \mathbf{R} \to \mathscr{GL}(m)$ is continuous and the associated linear skew-product flow is

$$\pi(B, x, \tau) = (\Phi(B, \tau)x, B_\tau).$$

When the coefficients a_{ij} are stationary stochastic variables, it is not difficult to show (by using the ideas of § 5) that a given underlying invariant probability measure μ lifts to an invariant measure ν on $\mathbf{M}$. If the coefficients are metrically transitive, i.e. if μ is ergodic, then the lifted measure ν can be chosen to be ergodic.

Appendix. Further geometric properties of cocycles. A *projector* is a continuous mapping $P(x, \theta) = (P(\theta)x, \theta)$ on $\mathbf{R}^m \times \mathbf{M}$ where $\mathbf{M}$ is a compact Hausdorff space and $P(\theta)$ is a linear projection on $\mathbf{R}^m$. A *resolution of the identity* on $\mathbf{R}^m \times \mathbf{M}$ is a k-tuple $P = (P_1, \cdots, P_k)$, $k \geqq 1$, satisfying: (i) each P_i is a projector, (ii) $P_iP_j = \vec{0}$ when $i \neq j$ and (iii) $I = P_1 + \cdots + P_k$. (Here we define $\vec{0}(x, \theta) := (0, \theta)$.) Let $P = (P_1, \cdots, P_k)$ be a k-tuple of projectors on $\mathbf{R}^m \times \mathbf{M}$ and define

$$\mathscr{R}_i := \text{Range}(P_i) := \{(x, \theta) \in \mathbf{R}^m \times \mathbf{M}: P_i(\theta)x = x\}, \qquad 1 \leqq i \leqq k.$$

Then P is a resolution of the identity if and only if $\mathbf{R}^m \times \mathbf{M} = \mathscr{R}_1 + \cdots + \mathscr{R}_k$ (as a Whitney sum). A resolution of the identity P is said to be *orthogonal* if $\mathscr{R}_i \perp \mathscr{R}_j$ whenever $i \neq j$, i.e. the Euclidean inner product $\langle \cdot, \cdot \rangle$ satisfies $\langle x, y \rangle = 0$ for all $(x, \theta) \in \mathscr{R}_i$, $(y, \theta) \in \mathscr{R}_j$ when $i \neq j$. The latter is equivalent to saying that each $P_i(\theta)$ is an orthogonal projection on $\mathbf{R}^m$.

Now let Φ be a cocycle on $\mathbf{R}^m \times \mathbf{M}$ and assume $\mathbf{R}^m \times \mathbf{M} = \mathscr{V}_1 + \cdots + \mathscr{V}_k$ as a Whitney sum. Let $P = (P_1, \cdots, P_k)$ be the induced resolution of the identity where Range $(P_i) = \mathscr{V}_i$, $1 \leqq i \leqq k$. Then the subbundles $\mathscr{V}_i$ are invariant under the linear skew product flow induced by Φ if and only if one has

$$P_i(\theta \cdot t)\Phi(\theta, t) = \Phi(\theta, t)P_i(\theta), \qquad 1 \leqq i \leqq k \tag{A.1}$$

for all $\theta \in \mathbf{M}$ and $t \in \mathbf{T}$. We shall say that a resolution of the identity $P = (P_1, \cdots, P_k)$ is *invariant* if (A.1) is satisfied. It is not always the case that an invariant resolution of the identity is orthogonal; however, the next lemma shows that one can replace Φ with a cohomologous flow in which the new invariant resolution of the identity is orthogonal.

LEMMA A. *Let Φ be a cocycle over a compact Hausdorff space $\mathbf{M}$ and let $P = (P_1, \cdots, P_k)$ be an invariant resolution of the identity. Then there is a continuous self-adjoint mapping $R: \mathbf{M} \to \mathscr{GL}(m)$ with the property that $Q = (Q_1, \cdots, Q_k)$ is an orthogonal resolution of the identity, where*

$$Q_i(\theta) = R(\theta)P_i(\theta)R^{-1}(\theta), \qquad 1 \leqq i \leqq k. \tag{A.2}$$

Furthermore Q is invariant under the cocycle

$$\Psi(\theta, t) = R(\theta \cdot t)\Phi(\theta, t)R^{-1}(\theta). \tag{A.3}$$

Proof. Define $S(\theta)$ by

$$S(\theta) := \sum_{i=1}^{k} P_i^*(\theta)P_i(\theta)$$

where P^* denotes the adjoint operation. Then $S(\theta)$ is positive definite and self-adjoint, so it has a unique positive definite, self-adjoint square root $R(\theta)$, i.e. $R^2(\theta) = S(\theta)$. If $Q_i(\theta)$ is defined by (A.2) and Ψ is given by (A.3) it is easy to verify that $Q_i^*(\theta) = Q_i(\theta)$ and $\Psi(\theta, t)Q_i(\theta) = Q_i(\theta \cdot t)\Psi(\theta, t)$, $1 \leqq i \leqq k$. Q.E.D.

Lemma 3.4 also gives information about the case where one has a linear skew-product flow on a vector bundle $\mathscr{E}$ over a compact Hausdorff space $\mathbf{M}$ where $\mathscr{E} = \mathscr{V}_1 + \cdots + \mathscr{V}_k$ is a Whitney sum of invariant subbundles. Each of these subbundles $\mathscr{V}_i$ can be separately imbedded in a trivial bundle $\mathbf{R}^{m_i} \times \mathbf{M}$ where $m = m_1 + \cdots + m_k$. The construction of Lemma 3.4 shows that one can construct a cocycle on $\mathbf{R}^m \times \mathbf{M}$ so that the given flow on $\mathscr{E}$ is cohomologous to a flow on a subbundle of $\mathbf{R}^m \times \mathbf{M}$.

If π is a discrete flow on a vector bundle $\mathscr{E}$, i.e. if $\mathbf{T} = \mathbf{Z}$, then Lemma 3.4 can be extended to this case by first suspending the discrete flow to get an equivalent continuous-time flow on a new vector bundle. See Ellis and Johnson (1982) for the suspension construction. One should note that even if the original bundle is trivial, the suspended bundle may be nontrivial.

The triangularization technique can be used to put some cocycles into a block-diagonal, upper-triangular form. Let Φ be a cocycle on $\mathbf{M}$ and let $P = (P_1, \cdots, P_k)$ be an invariant partition of unity of $\mathbf{R}^m \times \mathbf{M}$. Because of Lemma A we can assume P to be orthogonal. For any point $\phi = (\theta, U) \in \mathbf{H}$ we define the P-partition of U to be the partitioning of U into block matrices $U = (U_1, \cdots, U_k)_P$ where the number m_i of column vectors in U_i is $\dim \operatorname{Range} P_i(\theta)$, $1 \leqq i \leqq k$. Let $\mathbf{H}_P$ denote the set of all $\phi = (\theta, U) \in \mathbf{H}$ with the property that the P-partition $U = (U_1, \cdots, U_k)_P$ satisfies

$$P_i(\theta) U_j = \delta_{ij} U_i, \qquad 1 \leqq i, \quad j \leqq k. \tag{A.4}$$

By using Lemma A one can easily verify the following:

LEMMA B. *$\mathbf{H}_P$ is a compact invariant set in $\mathbf{H}$ in the flow $\phi \cdot t$. Furthermore if $\phi = (\theta, U) \in \mathbf{H}_P$ then the column vectors in $\Phi(\theta, t) U_i$ are orthogonal to those in $\Phi(\theta, t) U_j$ when $i \neq j$.*

For $\phi = (\theta, U) \in \mathbf{H}_P$ let $T(\Phi(\theta, t) U)$ be given by (4.5). The P-partition of U prescribes an induced block partition of $T(\Phi(\theta, t) U)$ where the diagonal blocks are square matrices of size $(m_i \times m_i)$, $1 \leqq i \leqq k$. Since the off-diagonal blocks of $T(\Phi(\theta, t) U)$ depend on the inner products of column vectors of $\Phi(\theta, t) U_i$ and $\Phi(\theta, t) U_j$ for $i \neq j$, it follows from Lemma B that these off-diagonal blocks are zero. Hence $\Psi(\phi, t) = T(\Phi(\theta, t) U)^{-1}$ is a block-diagonal, upper-triangular matrix.

The block-diagonalization of Ψ involves an "untwisting" of the spectral subbundles of Φ. A similar untwisting with additional useful structures appears in Ellis and Johnson (1982). Also compare with Coppel (1967), Palmer (1980) and Vinograd et al. (1977).

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