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HOW MANY SHUFFLES TO MIX A DECK?*

JOSEPH B. KELLER†

Abstract. A simple probabilistic model is devised to determine the number of shuffles required for the bottom card of a deck to become uniformly distributed with a specified tolerance. This number is a lower bound on the number of shuffles needed for the entire deck to be randomly mixed.

Key words. card shuffling, probabilistic models, mixing

AMS subject classifications. 60G99, 60K99

We wish to find a lower bound on the number of riffle shuffles required to mix a deck of k cards. In such a shuffle, the deck is cut at random and then the two parts are riffled together. For simplicity we shall assume that the location of the cut is uniformly distributed, so that the probability is $1/(k-1)$ that the cut is just below any one of the first $k-1$ cards. Furthermore, we assume that the bottom cards of the two parts are equally likely to be the bottom card after the riffle. Therefore each has probability $1/2$ of being the new bottom card.

To obtain a lower bound on the number of shuffles, we consider the original bottom card. It will be the bottom card of one of the two parts after the first cut, so it has probability $1/2$ of still being on the bottom after one shuffle. Therefore, the probability that it remains on the bottom throughout n shuffles is $1/2^n$. When a 52-card deck is well mixed, the probability that any given card is on the bottom must lie between $\frac{1}{52}(1-\varepsilon)$ and $\frac{1}{52}(1+\varepsilon)$ for some $\varepsilon > 0$, which indicates how close the deck is to random. Therefore, for the deck to be mixed, it is necessary that $1/2^n < \frac{1}{52}(1+\varepsilon)$, so $n > \log_2[\frac{52}{(1+\varepsilon)}]$. This shows that for $\varepsilon \le \frac{20}{32} = .625$, n must be ≥ 6. For a deck of k cards this argument yields $n \ge \log_2 k - \log_2(1+\varepsilon) \approx \log_2 k - \varepsilon/\ln 2$. The last expression holds for $\varepsilon \ll 1$.

We shall now calculate exactly the probability p_n that the original bottom card is on the bottom after n shuffles of a k card deck. To do so we observe that p_{n+1} is given by

$$p_{n+1} = \frac{1}{2}\left[p_n + (1-p_n)\frac{1}{k-1}\right], \quad n \ge 1; \quad p_0 = 1. \tag{1}$$

The first term in brackets is the probability that the card was on the bottom after n shuffles. The second term is the probability $1-p_n$ that it was not on the bottom, times the probability $\frac{1}{k-1}$ that the deck was cut just below it, which brings it to the bottom of the upper pile before the $(n+1)$st shuffle. The factor $\frac{1}{2}$ is the probability that it stays on the bottom after this shuffle.

The values of p_n given by (1) for $k = 52$ and $n = 0, 1, \ldots, 10, \infty$ are shown in the table. The asymptotic value is $p_\infty = \frac{1}{52} = .01923$, and p_9 exceeds this by $8\frac{1}{2}\%$.

The solution of the recursion equation (1), with the initial condition $p_o = 1$, is

$$p_n = \frac{1}{k} + \left[\frac{k-2}{2(k-1)}\right]^n \left(\frac{k-1}{k}\right). \tag{2}$$

From (2) we see that p_n decreases monotonically from 1 to $\frac{1}{k}$ as $n \to \infty$. The difference $p_n - \frac{1}{k}$ falls below ε/k for the first time when $n = \{n_\varepsilon\}$, where $\{n_\varepsilon\}$ denotes the least integer greater than or equal to n_ε. Here n_ε is defined by

$$n_\varepsilon = \frac{\log_2[(k-1)/\varepsilon]}{\log_2[\frac{2(k-1)}{k-2}]}. \tag{3}$$

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†Departments of Mathematics and Mechanical Engineering, Stanford University, Stanford, California 94305-2125. This research was supported in part by the Air Force Office of Scientific Research, the National Science Foundation, and the Office of Naval Research.

TABLE

$p_{n+1} = \frac{25}{51} p_n + \frac{1}{102}, n \geq 1.$

$p_0 = 1$	$p_6 = .03284$
$p_1 = .5$	$p_7 = .02590$
$p_2 = .2549$	$p_8 = .02250$
$p_3 = .13476$	$p_9 = .02083$
$p_4 = .07486$	$p_{10} = .02001$
$p_5 = .04699$	$p_\infty = .01923$

When $k \gg 1$, (3) simplifies to

$$n_\varepsilon \sim \log_2 k - \log_2 \varepsilon. \tag{4}$$

This result (4) has the same $\log_2 k$ term as we obtained by the simple argument in the second paragraph. However, the term $-\log_2 \varepsilon$ makes n_ε increase to infinity as ε decreases to zero, in contrast to that result. Thus (4) yields a much stronger necessary condition on n than does the simple result.

Bayer and Diaconis [1] have used a more sophisticated and more complicated argument to analyze a somewhat different model of shuffling. They obtained the stronger result that for large k, about $\frac{3}{2}\log_2 k$ + a constant shuffles are necessary and sufficient to mix a deck of k cards. The $\frac{3}{2}\log_2 k$ term in this expression exceeds that in (4) by the factor $3/2$. The additional shuffles are needed to make sure that all positions in the deck are mixed, not just the bottom position. For $k = 52$, their result shows that more than $\frac{3}{2}\log_2 k +$ a constant $= 8.58 +$ a constant shuffles are needed.

REFERENCES

[1] D. BAYER AND P. DIACONIS, *Trailing the dovetail shuffle to its lair*, Ann. Appl. Prob., 2 (1992), pp. 294–313.