# On Connected Diagrams and Cumulants of Erdős-Rényi Matrix Models

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**Abstract**

Regarding the adjacency matrices of $n$-vertex graphs and related graph Laplacian, we introduce two families of discrete matrix models constructed both with the help of the Erdős-Rényi ensemble of random graphs. Corresponding matrix sums represent the characteristic functions of the average number of walks and closed walks over the random graph. These sums can be considered as discrete analogs of the matrix integrals of random matrix theory.

We study the diagram structure of the cumulant expansions of logarithms of these matrix sums and analyze the limiting expressions as $n \to \infty$ in the cases of constant and vanishing edge probabilities.

*Keywords:* Graph Laplacian, Erdős-Rényi random graphs, discrete matrix models, cumulant expansion, connected diagrams, Lagrange equation

## 1 Introduction

It counts about thirty years from the time when the map enumeration problems of theoretical physics were related with the diagram representation of the formal cumulant expansions of the form

$$\frac{1}{N^2} \log \mathbf{E} \left\{ e^{gV_N} \right\} = \sum_{k \geq 1} \frac{g^k}{k!} \frac{1}{N^2} Cum_k(V_N), \tag{1.1}$$

where the average is taken with respect to the gaussian measure over the space of hermitian $N$-dimensional matrices $\{H_N\}$ and $V_N$ is given by $\text{Tr}\, H_N^q$, or more generally, by a linear combination of such traces with certain degrees $q \geq 3$ . This relation between maps, diagrams and random matrices observed first by t'Hooft is still a source of numerous results that reveal deep links between various branches of mathematics and theoretical physics (see e.g. papers [2, 6, 22] for the references and reviews of results). In these studies, the nature of the large-$N$ limit of the right-hand side of (1.1) and existence of

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its asymptotic expansion in degrees of $1/N^2$ are the problems of the primary importance [3, 7, 10].

In present paper we examine the diagram structure of a discrete analog of (1.1), where the role of $H_N$ is played by the $n \times n$ adjacency matrix $A_n$ of the Erdős-Rényi random graphs with the edge probability $p_n$. In this setting, $V_N$ of (1.1) is replaced by random variables

$$X_n^{(q)} = \operatorname{Tr} A_n^q = \sum_{i=1}^{n} (A_n^q)_{ii} \quad \text{and} \quad Y_n^{(q)} = \sum_{i,j=1}^{n} (A_n^q)_{ij}, \quad q \geq 2 \tag{1.2}$$

and this gives two different discrete matrix models related with the powers of adjacency matrix and of the Laplace operator on graphs. Variables $X$ and $Y$ represent the numbers of close walks of $q$ steps and walks of $q$ steps over the graphs, respectively.

We study the limiting behavior of the cumulants of $X_n^{(q)}$ and $Y_n^{(q)}$ in the following three asymptotic regimes, when $p_n = O(1)$, $\frac{1}{n} \ll p_n \ll 1$, or $p_n = O(1/n)$ as $n \to \infty$. Our main result is that in all of these three regimes, the following limits exist (cf. (1.1))

$$\frac{1}{p_n n^2} Cum_k \left( g_n Y_n^{(q)} \right) \to F_k^{(q)}(\omega), \quad n \to \infty, \tag{1.3}$$

with the appropriate choice of the normalizing factors $g_n$. Limiting expressions depend on asymptotic regime indicated by $\omega \in \{1, 2, 3\}$. The same statement as (1.3) holds for random variables $X_n^{(q)}$ with corresponding changes of $g_n$ and the limiting expressions.

To prove convergence (1.3), we develop a diagram technique similar to that commonly adopted in the random matrix theory. We show that the diagram structure of the cumulants of (1.3) is closely related with the trees with $k$ labeled edges. In the simplest case of $q = 2$ and $\omega = 2$, we derive explicit recurrent relations that determine the limits (1.3) and prove that the corresponding exponential generating function verifies the Lagrange (or Pólya) equation. Regarding other values of $q$ and $\omega$, we obtain several generalizations of this equation.

Using (1.3), we show that the Central Limit Theorem is valid for the centered and renormalized variables $X_n^{(q)}$ and $Y_n^{(q)}$. This describes asymptotic properties of the averaged numbers of walks and closed walks over the random graphs. Our results imply CLT for the normalized spectral measure of the adjacency matrices of Erdős-Rényi random graphs. This improves known results about the convergence of the normalized spectral measure [1, 16]. Also, we indicate an asymptotic regime when the CLT does not hold for variables $X_n^{(q)}$ and $Y_n^{(q)}$.

The paper is organized as follows. In Section 2 we determine two families of matrix sums that can be called the Erdős-Rényi matrix models. The key observation here is that the graph Laplacian generates the Erdős-Rényi measure on graphs. This allows us to identify $Y_n^{(2)}$ as a natural analog of the quartic potential $V_N = \operatorname{Tr} H_N^4$ of (1.1). In Section 3, we develop a general diagram technique to study the cumulant expansions of the form (1.1) for discrete Erdős-Rényi matrix models of the first and the second type (1.2). In Section 4, we prove convergence of normalized cumulants in three main asymptotic regimes. The issue of the Central Limit Theorem is discussed at the end of Section 4. In Section 5, we study the classes of connected diagrams and derive recurrent relations for their numbers. In Section 6, we complete the study of limiting expressions $F_k^{(q)}(\omega)$ (1.3) and consider the formal limiting transition for the free energy. This free energy takes different forms in dependence on the edge probability as $n \to \infty$. Section 7 contains a summary of our results.

## 2 Discrete Erdős-Rényi matrix models

In (1.1), one can rewrite mathematical expectation as follows

$$\mathbf{E}\left\{\exp\left(\frac{g}{N} \operatorname{Tr} H_N^q\right)\right\} = \frac{1}{C_N}\int_{\mathcal{H}_N} \exp\left\{-\beta \operatorname{Tr} H_N^2 + \frac{g}{N} \operatorname{Tr} H_N^q\right\} dH_N,\ \beta > 0 \tag{2.1}$$

where $H_N$ is $N \times N$ hermitian matrix, $C_N = C_N(\beta)$ is a normalization constant, and the matrix integral runs over the space $\mathcal{H}_N$ of all hermitian matrices with respect to the Lebesgue measure $dH_N$. The ensemble of random matrices $\{H_N\}$ distributed according to the gaussian measure with the density $\exp\{-\beta \operatorname{Tr} H_N^2\}$ is known as the Gaussian Unitary invariant Ensemble abbreviated as GUE . This ensemble plays a fundamental role in the random matrix theory (see monograph [19] and references therein). The matrix integral of (2.1) is known as the partition function of the matrix model with the potential $V_N = \operatorname{Tr} H_N^q$.

Regarding $\operatorname{Tr} H_N^2$ of (2.1) as a formal kinetic energy term, one can speculate about the trace of the Laplace operator; being restricted to the space of functions on graphs, it leads immediately to a discrete analog of the matrix integral of (2.1). In this case the integral $\int_{\mathcal{H}_N}$ of (2.1) is replaced by the sum over the set of all $n$-dimensional adjacency matrices of graphs. This approach is fairly natural and one benefits from two important counterparts of it.

From one hand, the use of graph Laplacian indicates a natural analog of the matrix models with quartic potentials and clarify relations between the weights generated by $\operatorname{Tr} H^2$ and $\operatorname{Tr} H^4$. This helps to distinguish two matrix models, the traditional one given by $X_n^{(q)}$ and the new one determined by $Y_n^{(q)}$ (1.2).

From another hand, a simple but important reasoning relates the graph Laplacian with the Erdős-Rényi probability measure on graphs. On this way, the gaussian measure of GUE in the average of the left-hand side of (2.1) is replaced by a measure with nice properties we are going to describe.

## 2.1 Graph Laplacian and Erdős-Rényi random graphs

Given a finite graph with the set of $n$ labeled vertices $\mathcal{V}_n = \{v_1, \dots, v_n\}$ and the set of simple non-oriented edges $E_m = \{e^{(1)}, \dots, e^{(m)}\}$, the discrete analog of the Laplace operator $\Delta(\gamma)$ on the graph $\gamma$ is defined by relation

$$\Delta(\gamma) = \partial^* \partial, \tag{2.2}$$

where $\partial$ is the difference operator determined on the space of complex functions on vertices $\mathcal{V}_n \to \mathbf{C}$ and $\partial^*$ is its conjugate determined on the space of complex functions on edges $E_m \to \mathbf{C}$ (see [21] for more details).

It can be easily shown that in the canonical basis, the linear operator $\Delta(\gamma) = \Delta_n$ has $n \times n$ matrix with the elements

$$\Delta_{ij} = \begin{cases} \deg(v_j), & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and } (v_i, v_j) \in E, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

where $\deg(v)$ is the vertex degree. If one considers the $n \times n$ adjacency matrix $A = A(\gamma)$ of the graph $\gamma$,

$$A_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E,\ i \neq j, \\ 0, & \text{otherwise}, \end{cases}$$

then one can rewrite the definition of $\Delta$ (2.2) in the form

$$\Delta_{ij} = B_{ij} - A_{ij} \quad \text{with} \quad B_{ij} = \delta_{i,j} \sum_{l=1}^{n} A_{il}, \tag{2.4}$$

where $\delta_{i,j}$ is the Kronecker $\delta$-symbol

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

It follows from (2.2) that $\Delta(\gamma_n)$ has positive eigenvalues.

Let us consider the set $\Gamma_n$ of all possible simple non-oriented graphs $\gamma_n$ with the set $\mathcal{V} = \mathcal{V}_n$ of $n$ labeled vertices. Obviously, $|\Gamma_n| = 2^{n(n-1)/2}$. Given an element $\gamma \in \Gamma_n$, it is natural to consider the trace $\operatorname{Tr} \Delta(\gamma)$ as the "kinetic energy" of the graph $\gamma$. Then we can assign to each graph $\gamma_n$ the Gibbs weight $\exp\{-\beta \operatorname{Tr} \Delta(\gamma_n)\}$, $\beta > 0$ and introduce the discrete analog of the integral (2.1) by relation

$$Z_n(\beta, Q) = \sum_{\gamma_n \in \Gamma_n} \exp\{-\beta \operatorname{Tr} \Delta_n + Q(\gamma_n)\}, \tag{2.5}$$

where $\Delta_n = \Delta(\gamma_n)$ and $Q$ is an application: $\Gamma_n \to \mathbf{R}$ that we specify later.

Let us note that one should normalize the sum (2.5) by $|\Gamma_n|$, but this does not play any role with respect to further results. In what follows, we omit subscript $n$ in $\Delta_n$. Relation (2.4) implies that

$$\mathrm{Tr}\,\Delta = \sum_{i=1}^{n} \Delta_{ii} = \sum_{i,j=1}^{n} A_{ij} = 2 \sum_{1 \le i < j \le n} A_{ij}. \tag{2.6}$$

Then we can rewrite (2.5) in the form

$$Z_n(\beta, Q) = \sum_{\gamma_n \in \Gamma_n} e^{Q(\gamma_n)} \prod_{1 \le i < j \le n} e^{-2\beta A_{ij}}. \tag{2.7}$$

It is easy to see that

$$Z_n(\beta, 0) = \left(1 + e^{-2\beta}\right)^{n(n-1)/2}. \tag{2.8}$$

Then the normalized partition function can be represented as

$$\hat{Z}_n(\beta, Q) = Z_n(\beta, Q)/Z_n(\beta, 0) = \mathbf{E}_\beta \left\{ e^{Q(\gamma)} \right\}, \tag{2.9}$$

where $\mathbf{E}_\beta\{\cdot\}$ denotes the mathematical expectation with respect to the measure supported on the set $\Gamma_n$. This measure assigns to each element $\gamma \in \Gamma_n$ probability

$$P_n(\gamma) = \frac{e^{-2\beta|E(\gamma)|}}{(1 + e^{-2\beta})^{n(n-1)/2}},$$

where $E(\gamma)$ denotes the set of edges of the graph $\gamma$.

Given a couple $(i, j)$, $i, j \in \{1, \ldots, n\}$, one can determine a random variable $a_{ij}$ on the probability space $(\Gamma_n, P_n)$ that is the indicator function of the edge $(v_i, v_j)$

$$a_{ij}(\gamma) = \begin{cases} 1, & \text{if } (v_i, v_j) \in E(\gamma), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to show that the random variables $\{a_{ij}\}_{1 \le i \le j \le n}$ are jointly independent and are of the same Bernoulli distribution depending on $\beta$ such that

$$a_{ij}^{(\beta)} = \begin{cases} 1 - \delta_{ij}, & \text{with probability } \dfrac{e^{-2\beta}}{1 + e^{-2\beta}} = p, \\ 0, & \text{with probability } 1 - p. \end{cases} \tag{2.10}$$

The term $1 - \delta_{ij}$ reflects the property that graphs $\gamma$ have no loops.

The probability space $(\Gamma_n, P_n)$ is known as the Erdős-Rényi (or Bernoulli) ensemble of random graphs with the edge probability $p$ [11]. Since the series of pioneering papers by Erdős and Rényi, the asymptotic properties of graphs $(\Gamma_n, P_n)$, such as the size and the number of connected components, the maximal and minimal vertex degree and many others, are extensively studied (see [4, 11]). Spectral properties of corresponding random matrices $A$ (2.3) and $\Delta$ (2.4) are considered in a series of papers (in particular, see [1, 13, 14, 15, 16, 20, 24]). In present paper we study the random graph ensemble $(\Gamma_n, P_n)$ from another point of view motivated by the asymptotic behavior of partition functions (2.9).

## 2.2 Quartic potential and Erdős-Rényi matrix models

Let us determine the discrete analog of the integral (2.1) with quartic potential $\mathrm{Tr}\, H_N^4$. Once $\mathrm{Tr}\, H^2$ replaced by $\mathrm{Tr}\,(\partial^*\partial) = \mathrm{Tr}\,\Delta$ (2.2), it is natural to consider

$$\mathrm{Tr}\,(\partial^*\partial\partial^*\partial) = \mathrm{Tr}\,\Delta^2$$

as the analog of $\mathrm{Tr}\, H^4$. Then the partition function (2.5) reads as

$$Z_n(\beta, g) = \sum_{\gamma_n \in \Gamma_n} \exp\{-\beta \;\; \mathrm{Tr}\,\Delta_n + g_n \;\mathrm{Tr}\,\Delta^2\}, \tag{2.11}$$

where $g_n$ is to be specified. It follows from (2.3) and (2.4) that

$$\mathrm{Tr}\,\Delta^2 = \mathrm{Tr}\, B^2 + \mathrm{Tr}\, A^2 = \sum_{i,j=1}^{n} (A^2)_{ij} + \sum_{i,j=1}^{n} A_{ij}.$$

Then, using (2.6) and repeating computations of (2.7) and (2.8), we obtain representation

$$\hat{Z}_n(\beta, g_n) = Z_n(\beta, g_n)/Z_n(\beta, 0) = \left(\frac{1+e^{-2\beta'}}{1+e^{-2\beta}}\right)^{n(n-1)/2} \mathbf{E}_{\beta'}\{e^{g_n Y_n}\}, \tag{2.12}$$

In this relation, we have denoted $\beta' = \beta - g_n$ and introduced variable

$$Y_n = \sum_{i,j,l=1}^{n} a_{il} a_{lj}, \tag{2.13}$$

where $a_{ij}$ are jointly independent random variables of the law (2.10) with $\beta$ replaced by $\beta'$. The average $\mathbf{E}_{\beta'}$ denotes the corresponding mathematical expectation. In what follows, we omit the subscripts $\beta$ and $\beta'$ when they are not necessary.

On can generalize (2.11) and consider mathematical expectation

$$\mathbf{E}\left\{\exp(g_n Y_n^{(q)})\right\}, \quad \text{with} \quad Y_n^{(q)} = \sum_{i,j=1}^{n} (A^q)_{ij}, \tag{2.14}$$

as the normalized partition function of the discrete Erdős-Rényi matrix model that we will refer to as $q$-step walks model. Also one can consider the average

$$\mathbf{E}\left\{\exp(g_n X_n^{(q)})\right\}, \quad \text{with} \quad X_n^{(q)} = \mathrm{Tr}\,(A^q), \tag{2.15}$$

that we relate with the discrete Erdős-Rényi model for $q$-step closed walks. These models are different analogs of the matrix integral (2.1).

Finally, let us point out one more analogy between the discrete model (2.11) and the gaussian matrix integrals (2.1). We mean the invariance property of the probability measure with respect to a group of the space transformations [19].

Let $\mathcal{A}_n$ denotes the set of of all $n$-dimensional symmetric matrices whose elements are equal to 0 or 1 and the diagonal elements are zeros. It is not hard to see that if $n \times n$ orthogonal matrix $\Upsilon$ is such that $\Upsilon A \Upsilon^{-1} \in \mathcal{A}_n$ for all $A \in \mathcal{A}_n$, then $\Upsilon$ verifies the following properties

a) all elements of $\Upsilon$ take values 0 or 1;

b) any given line of $\Upsilon$ contains only one non-zero element and

c) any given column of $\Upsilon$ contains only one non-zero element.

Clearly, the set of all such orthogonal matrices $\mathcal{Y}_n$ is in one-to-one correspondence with the symmetric group of permutations $\mathcal{S}_n$.

It is easy to see that $\Upsilon \in \mathcal{Y}_n$ determines a basis change in $\mathbf{R}^N$ because $\Upsilon$ re-enumerates the vectors of the canonical basis associated with the graph. Equality $\mathrm{Tr}\,(\Upsilon \Delta \Upsilon^T) = \mathrm{Tr}\,\Delta$ shows that the probability measure $P$ on $\mathcal{A}_n$

$$P(A) = C^{-1} \exp\{-\beta \ \mathrm{Tr}\,\Delta\}$$

is invariant with respect the group of transformations $\mathcal{Y}_n$,

$$P(\Upsilon A \Upsilon^T) = P(A). \qquad (2.16)$$

It is not hard to prove the inverse statement: if the probability measure $P$ on the set $\mathcal{A}_n$ verifies (2.16) for all $\Upsilon \in \mathcal{Y}_n$, then the random variables given by the matrix elements of $A$ are jointly independent.

This is in complete analogy with the well-known fact of random matrix theory that the invariant gaussian distribution on the space of hermitian (or real symmetric matrices) generates independent random variables [19]. We do not pursue this topic here and return to asymptotic behavior of the sums (2.11) in the limit of infinite $n$.

## 3 Cumulant expansions and connected diagrams

In this section we develop a diagram technique to study the terms of the expansion

$$\log \mathbf{E}\left\{ e^{gV_n} \right\} = \sum_{k \geq 1} \frac{g^k}{k!}\ Cum_k(V_n), \qquad (3.1)$$

where $V_n$ represents represents $X_n$ (2.15) or $Y_n$ (2.14). Let us stress that the random variables $a_{ij}$ are bounded and therefore the series (3.1) is absolutely convergent for any finite $n$ and sufficiently small $g$.

Given a vector $\alpha = (i_1, \ldots, i_r)$, we introduce random variables $U_\alpha$

$$U_\alpha = \begin{cases} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_q i_1}, & r = q \quad \text{for } X\text{-model} \\ a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_q i_{q+1}}, & r = q+1 \quad \text{for } Y\text{-model} \end{cases} \tag{3.2}$$

such that $V_n = \sum_{\{\alpha\}_1^n} U_\alpha$, where the sum runs over all possible values of $i_s,\ s = 1, \ldots, r$. Then one can write relation

$$Cum_k(V_n) = \sum_{\{\alpha_1\}_1^n} \cdots \sum_{\{\alpha_k\}_1^n} Cum\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}, \tag{3.3}$$

where

$$Cum\{U_{\alpha_1}, \ldots, U_{\alpha_k}\} = \frac{d^k}{dz_1 \cdots dz_k} \log \mathbf{E}\{\exp(z_1 U_{\alpha_1} + \ldots + z_k U_{\alpha_k})\}|_{z_l=0}.$$

Coefficient $Cum\{U_{\alpha_1}, \ldots, U_{\alpha_k}\} = Cum\{\mathcal{U}(\vec{\alpha}_k)\}, \vec{\alpha}_k = (\alpha_1, \ldots, \alpha_k)$ is known as the semi-invariant of the family $\mathcal{U}(\vec{\alpha}_k) = \{U_{\alpha_j}\}_{j=1}^k$ [18].

To simplify (3.3), we separate the set $\{1, \ldots, n\}^{\otimes(rk)}$ into the classes of equivalence according to the properties of the family $\mathcal{U}(\vec{\alpha}_k)$. The rule is that given $\vec{\alpha}_k$, we pay major attention not to the values of variables $i_j$ but rather to the presence of copies of the same random variable $a$ in $\mathcal{U}(\vec{\alpha}_k)$. This approach is fairly common in random matrix theory [9, 27]. It leads to the diagram representation of the classes of equivalence (see e.g. [6] for the review and for mathematical description). This method is used to study random matrices with gaussian or centered random variables. We modify it to the study of cumulants (3.3) of the Erdős-Rényi models.

## 3.1 Connected diagrams

The diagrams we construct for $X$-model and $Y$-model are very similar and we describe them in common. Let us consider a graph $\lambda$ with $r$ labeled vertices $\{\theta_1, \ldots, \theta_r\}$ with $r$ determined by (3.2). The graph $\lambda$ contains $q$ edges: these are $\varepsilon_j = (\theta_j, \theta_{j+1}),\ j = 1, \ldots, q$; for the $X$-model we have $\theta_{q+1} = \theta_1$. Given $\alpha = (i_1, \ldots, i_r)$, one can assign to each $\theta_j$ the value of $i_j$, $j = 1, \ldots, r$ and then each edge $\varepsilon_j$ denotes a random variable $a_{i_j, i_{j+1}}$.

To study $cum\{\mathcal{U}(\vec{\alpha_k})\}$ with given $\vec{\alpha}_k$, we consider the set $\Lambda_k = \{\lambda_1, \ldots, \lambda_k\}$ of $k$ labeled graphs $\lambda$; we will say that $\lambda_l$ is the **element** number $l$ of the diagram we construct and denote by $\{\varepsilon_j^{(l)}\}_{j=1}^r$ the edges of this element. The diagram consists of $k$ elements and a number of **arcs** $\sigma$ that join some of the edges of $\lambda$'s. We draw an arc $\sigma_{j,l}^{(j',l')}$ that joins $\varepsilon_j^{(l)}$ and $\varepsilon_{j'}^{(l')}$ when in $\vec{\alpha}_k$

$$i_j^{(l)} = i_{j'}^{(l')}, i_{j+1}^{(l)} = i_{j'+1}^{(l')} \quad \text{or} \quad i_j^{(l)} = i_{j'+1}^{(l')}, i_{j+1}^{(l)} = i_{j'}^{(l')}. \tag{3.4}$$

Here we assume that $l < l'$. It follows from (3.4) that the arc can have on of the two orientations that we call the direct and the inverse one, respectively.

If $l = l'$, we assume $j < j'$ and again consider the direct and the inverse arcs.

We say that the edges $\varepsilon_j^{(l)}$ and $\varepsilon_{j'}^{(l')}$ represent the *feet* of the arc $\sigma_{j,l}^{(j',l')}$ and denote this by relation $\varepsilon_j^{(l)} \simeq \varepsilon_{j'}^{(l')}$. Clearly, this relation separates the set of edges

$$\mathcal{E}(k,r) = \{\varepsilon_j^{(l)}, j = 1,\ldots,r; l = 1,\ldots,k\}$$

into classes of equivalence that we call the **color groups** of edges. It is possible that a class contains one edge only. In this case we say that there is a **simple color group**. Certainly, we color different classes in different colors.

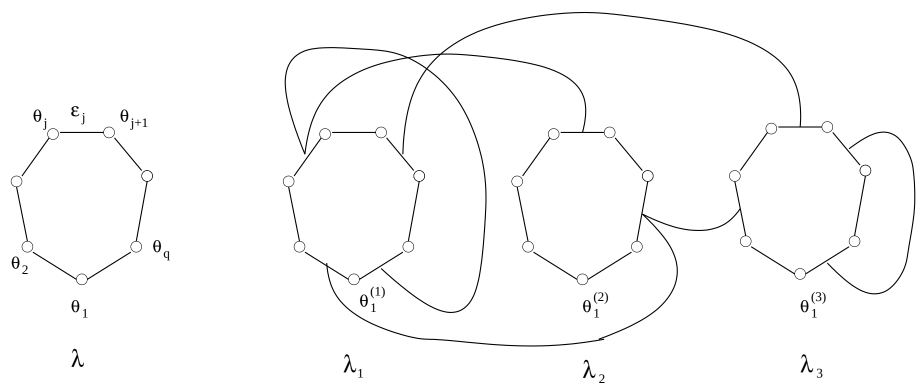


Figure 1: A graph $\lambda$ and a diagram $\delta_3$ for $X$-model with $q = 7$.

An important remark is that it is sufficient to consider the **reduced diagrams**; this means that if there are three edges that belong to the same color group, $\varepsilon_j^{(l)} \simeq \varepsilon_{j'}^{(l')} \simeq \varepsilon_{j''}^{(l'')}$, with $l < l' < l''$, then we draw the arcs $\sigma_{j,l}^{(j',l')}$ and $\sigma_{j',l'}^{(j'',l'')}$ between the nearest neighbors $\varepsilon_j^{(l)}, \varepsilon_{j'}^{(l')}$ and $\varepsilon_{j'}^{(l')}, \varepsilon_{j''}^{(l'')}$ only. The same concerns the arcs $\sigma_{j,l}^{(j',l)}$ whose feet belong both to $\lambda_l$. Regarding a reduced diagram, each color group has its **minimal edge** determined in the obvious manner.

To summarize, we consider the set $\Lambda_k$ and define the application $\vec{\alpha}_k \to \delta(\vec{\alpha}_k)$ of the set of all $\vec{\alpha}_k$ to the set of diagrams by drawing all arcs prescribed by $\vec{\alpha}_k$ and reducing of set of arcs obtained to the set of arcs between nearest neighbors $\Sigma(\vec{\alpha}_k)$. As a result, we get a diagram $\delta_k = \delta(\vec{\alpha}_k) = (\Lambda_k, \Sigma(\vec{\alpha}_k))$ (see Figure 1).

We say that a diagram $\delta_k$ is non-connected when there exist at least two disjoint subsets $\tau_1$ and $\tau_2$ of $\{1,\ldots,n\}$ such that $\tau_1 \cup \tau_2 = \{1,\ldots,n\}$ and there is no arc with one foot in $\Lambda(\tau_1) = \{\lambda_j, j \in \tau_1\}$ and another in $\Lambda(\tau_2) = \{\lambda_j, j \in \tau_2\}$. If there is no such subsets, then we say that the diagram is connected. The following statement is a well-known fact from the probability theory.

**Lemma 3.1** . *If $\vec{\alpha}_k$ is such that the corresponding diagram $\delta_k = \delta_k(\vec{\alpha}_k)$ is non-connected, then $Cum\{\mathcal{U}(\vec{\alpha}_k)\} = 0$.*

*Proof.* It is clear if there exist two subsets $\tau_1$ and $\tau_2$ as described before, then the $\sigma$-algebras generated by random variables $\{U_\mu, \mu \in \Lambda_1\}$ and $\{U_\nu, \nu \in \Lambda_2\}$ are independent. The fundamental property of semi-invariants is that in this case $Cum\{\mathcal{U}(\vec{\alpha}_k)\}$ vanishes [18].

## 3.2 Cumulants and sums over connected diagrams

It is clear that for any given set $\vec{\alpha}_k = \{\alpha_1, \dots, \alpha_k\}$ there exists only one diagram $\delta_k = \delta(\vec{\alpha}_k)$. We agree that two diagrams $\delta_k = (\Lambda_k, \Sigma_k)$ and $\delta'_k = (\Lambda'_k, \Sigma'_k)$ are not equal, $\delta_k \neq \delta'_k$ if in obvious bijection $\mathcal{J}(\Lambda_k) = \Lambda'_k$ preserving orderings, we have $\mathcal{J}(\Sigma_k) \neq \Sigma'_k$.

Let us say that two vectors $\vec{\alpha}_k$ and $\vec{\alpha}'_k$ are equivalent, $\vec{\alpha}_k \sim \vec{\alpha}'_k$ if $\delta(\vec{\alpha}_k) = \delta(\vec{\alpha}'_k)$. Relation $\sim$ separates the set $\{1, \dots, n\}^{\otimes(rk)}$ into the classes of equivalence that we denote by $\mathcal{C}(\delta_k), \delta_k = \delta(\vec{\alpha}_k)$. Let $\mathcal{N}(\delta_k) = |\mathcal{C}(\delta_k)|$ be the cardinality of the equivalence class $\mathcal{C}(\delta_k)$.

Relation (3.4) means that two edges $\varepsilon$ and $\varepsilon'$ belong to one color group if and only if random variables $a$ and $a'$ assigned to these edges by $\vec{\alpha}_k$ are equal. This is true for any given $\vec{\alpha}'_k \in \mathcal{C}(\delta_k)$. Let us denote by $m(\delta_k)$ the number of color groups of $\delta_k$. Clearly, random variables that belong to different groups are jointly independent.

**Lemma 3.2.** *The right-hand side of relation (3.3) can be represented as*

$$\sum_{\{\alpha_1\}_1^n} \cdots \sum_{\{\alpha_k\}_1^n} Cum\{U_{\alpha_1}, \dots, U_{\alpha_k}\} = \sum_{\delta_k \in \mathcal{D}_k} \mathcal{N}(\delta_k) W(\delta_k), \qquad (3.5)$$

*where $\mathcal{D}_k$ is the set of all possible connected reduced diagrams of the form $(\Lambda_k, \Sigma)$ and*

$$W(\delta_k) = \sum_{\pi_s \in \Pi_k} (-1)^{s-1}(s-1)! \; (\mathbf{E}a)^{m(\delta_k)} \, (\mathbf{E}a)^{\chi(\pi_s, \delta_k)} . \qquad (3.6)$$

*In this relation $\pi_s$ denotes a partition $\pi_s = (\tau_1, \dots \tau_s)$ of the set $\{1, 2, \dots, k\}$ into $s$ subsets; $\Pi_k$ is the set of all possible partitions, and $\chi(\pi_s, \delta_k)$ is the number of additional color groups generated by $\pi_s$.*

*Proof.* By definition of the semi-invariant [18], we have that

$$Cum\{U_{\alpha_1}, \dots, U_{\alpha_k}\} = \sum_{\pi_s \in \Pi_k} (-1)^s (s-1)! \, \mathbf{E}\{\mathbf{U}(\tau_1)\} \cdots \mathbf{E}\{\mathbf{U}(\tau_s)\}, \quad (3.7)$$

where we denoted $\mathbf{U}(\tau_j) = \prod_{\mu \in \tau_j} U_\mu$. It is easy to see that that the right-hand side of (3.7) depends on $\delta_k$ only, because it does not change when $\vec{\alpha}_k$

is replaced by another $\vec{\alpha}'_k$ from the same equivalence class $\mathcal{C}(\delta_k)$. Denoting by $W(\delta_k)$ the right-hand side of (3.7), we get (3.5).

To prove (3.6), let us choose an element $\vec{\alpha}_k^{(0)}$ of the equivalence class $\mathcal{C}(\delta_k)$ and consider first the trivial partition $\pi_0 = (\tau_0)$ with $\tau_0 = \{1, \ldots, k\}$. Then $\chi(\pi_s, \delta_k) = 0$ and

$$\mathbf{E}\left\{V_{\alpha_1^{(0)}} \cdots V_{\alpha_k^{(0)}}\right\} = (\mathbf{E}a)^{m(\delta_k)},$$

where we have use the independence of random variables $\{a_{ij}\}$ that belong to different color groups of $\delta_k$ and the fact that $a \in \{0, 1\}$.

Now let us consider a partition $\pi_s \neq \pi_0$. It is clear that there exist at least one color group of two or more edges such that its elements belong to two or more different subsets $\Lambda(\tau_j) = \{\lambda_t,\, t \in \tau_j\}$. In this case we say that this color group is separated into subgroups; each subgroup contains edges of this color that belong to one subset $T_j$. We color the edges of each new subgroup in the same color, the edges of the different subgroups - into different colors. It follows from (3.7) that the color group separated by $\pi_s$ into $v+1$ new subgroups provides the factor $(\mathbf{E}a)^{v+1}$ to the right-hand side of (3.6). We say that $v$ is the number of additional color groups generated by $\pi_s$. Regarding all initial color groups, we get factor $(\mathbf{E}a)^{m(\delta_k)}\,(\mathbf{E}a)^{\Sigma v}$ with $\sum v = \chi(\pi_s, \delta_k)$. Lemma 3.2 is proved.

To study $\mathcal{N}(\delta_k)$, let us consider the set of vertices of $\Lambda_k$

$$\Theta_k = \{\theta_j^{(l)}, l = 1, \ldots, k;\ j = 1, \ldots, r\}.$$

Given $\vec{\alpha}_k$, we draw the arcs according to the rule (3.4) and get $\delta_k = \delta_k(\vec{\alpha}_k)$. Let us say that equality $i_j^{(l)} = i_{j'}^{(l')}$ of (3.4) identifies corresponding vertices $\theta_j^{(l)}$ and $\theta_{j'}^{(l')}$ and denote this by

$$\theta_j^{(l)} \cong \theta_{j'}^{(l')}. \tag{3.8}$$

It is easy to see that relation (3.8) separates $\Theta_k$ into classes of equivalence. We determine the minimal element of the class as the vertex whose numbers $j$ and $l$ take minimal values among those of vertices that belong to this class. If there is no $\theta_{j'}^{(l')}$ such that $\theta_j^{(l)} \cong \theta_{j'}^{(l')}$, then we say that $\theta_j^{(l)}$ belongs to the class of equivalence consisting of one element. Let us denote the total number of the classes of equivalence by $\nu(\delta_k)$.

**Lemma 3.3**. *Given $\delta_k \in \mathcal{D}_k$ the cardinality $\mathcal{N}(\delta_k) = |\mathcal{C}(\delta_k)|$ verifies asymptotic relation*

$$\mathcal{N}(\delta_k) = n^{\nu(\delta_k)}(1 + o(1)), \tag{3.9}$$

*in the limit $n \to \infty$.*

*Proof.* It is clear that two equivalent vectors $\vec{\alpha}_k \sim \vec{\alpha}'_k$ generate the same partition of $\Theta_k$ into classes of equivalence. Inversely, given a diagram $\delta_k$ and regarding corresponding partition of $\Theta_k$, we get a separation of the set of variables

$$I_k = \{i_j^{(l)},\ j = 1, \dots, r;\ l = 1, \dots, k\}$$

into $\nu(\delta_k)$ groups. Variables that belong to the same group are equal between them. To get all possible $\vec{\alpha}_k$ from the same equivalence class, we allow variables $i_j^{(l)}$ to take all possible values from 1 to $n$ with obvious restriction that variables from different groups take different values. Then obviously

$$\mathcal{N}(\delta_k) = n(n-1)\cdots(n-\nu(\delta_k)+1) = n^{\nu(\delta_k)}(1+o(1)), \quad \text{as } n \to \infty. \tag{3.10}$$

Lemma 3.3 is proved.

Let us complete this subsection with the following useful remark. The set of diagrams $\mathcal{D}_k$ we constructed gives a graphical representation of vectors $\vec{\alpha}_k$ and describes the classes of equivalence of such vectors. One can push forward this representation and consider the set of *graphs of diagrams* $G_k = G(\delta_k)$ generated in natural way by diagrams $\delta_k$ by gluing the edges of elements $\lambda$ in the way prescribed by the arcs of $\delta_k$. Actually, this is what is usually done in the standard diagram approach of random matrix theory.

The graph $G(\delta_k) = (\hat{\Theta}, \hat{\mathcal{E}})$ has $\nu(\delta_k)$ labeled vertices $\hat{\theta}$ that correspond to the minimal elements of the classes of equivalence of $\Theta_k$. The vertices $\hat{\theta}$ and $\hat{\theta}'$ are joined by an edge $(\hat{\theta}, \hat{\theta}') \in \hat{\mathcal{E}}$ if there is an edge $\varepsilon \in \mathcal{E}(k, r)$ that joins corresponding classes of equivalence of $\Theta_k$. Certainly, the number of edges of $G(\delta_k)$ is equal to the number of all color groups $m(\delta_k)$.

The graph representation is very useful when $\delta_k$ contains $k-1$ arcs only. We say that $\delta_k$ are tree-like because in this case, the graphs $G(\delta_k)$ of $Y$-model are trees with color edges.

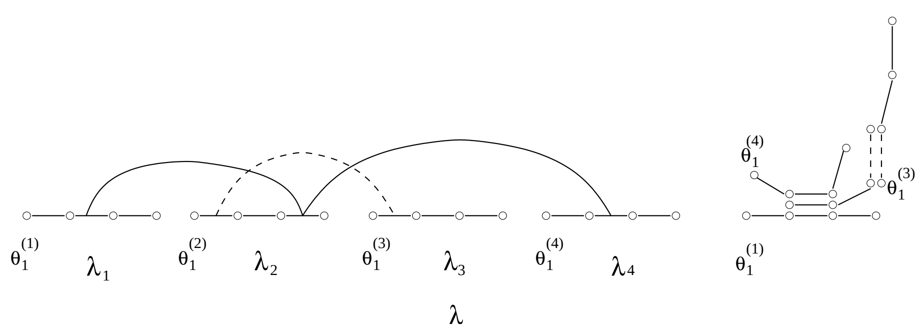


Figure 2: Tree-like diagram and corresponding tree for $Y$-model with $q = 3$

Remembering that the arcs $\sigma \in \Sigma$ have the direct and the inverse orientation (3.4), we agree to consider the graphs $G(\delta_k)$ with the edges of $\delta_k$ glued in the inverse sense only. In what follow, we will use representations of $\vec{\alpha}_k$ by diagrams $\delta_k$ and graphs $G(\delta_k)$ both. Note that $^{G}(\delta_k)$ has no loops.

# 4 Diagrams and limits of the cumulants

In this section we characterize the classes of connected diagrams that provide the leading contribution to the cumulants of $X$ and $Y$ models. The following three asymptotic regimes known from the spectral theory of random matrices are also distinguished in present context. We refer to them as to

- the "full random graphs" regime, when $p = Const$ as $n \to \infty$;
- the "dilute" regime, when $p = c_n/n$ and $1 \ll c_n \ll n$;
- the "sparse" regime, when $p = c/n$ and $c = const$ as $n \to \infty$.

One can consider also the fourth asymptotic regime of the "very sparse" random graphs, when $p = c_n/n$ and $c_n \to 0$ as $n \to \infty$. In this regime, the diagrams of the leading contribution to cumulants are degenerated, so we formulate corresponding results as remarks to the proofs of the main theorems. We omit subscripts in $c_n$ when they are not necessary.

## 4.1 Asymptotic behavior of cumulants

In this subsection we formulate the main results on the asymptotic behavior of the cumulants $Cum_k(U_n)$. We start with the $Y$-model. In what follows, we denote by $u_n \asymp v_n$ asymptotic relation $u_n = v_n(1 + o(1))$ as $n \to \infty$.

**Theorem 4.1** *Asymptotic behavior of $Cum_k(Y_n^{(q)})$ is determined by the following classes of diagrams:*

*A) in the full random graphs regime, the leading contribution to the sum (3.5) is given by the diagrams $\mathcal{D}_k^{(1)}(Y)$ that have exactly $k-1$ arcs; the graph $G(\delta_k), \delta_k \in \mathcal{D}_k^{(1)}(Y)$ is a tree; then*

$$Cum_k(Y_n^{(q)}) \asymp \sum_{\delta_k \in \mathcal{D}_k^{(1)}(Y)} W(\delta_k)\ n^{(q-1)k+2} \tag{4.1}$$

*where $W(\delta_k)$ is given by (3.6) considered with $\delta_k \in \mathcal{D}_k^{(1)}(Y^{(q)})$;*

*B) in the dilute random graphs regime, the leading contribution to (3.5) is given by the same diagrams as in (A), $\delta_k \in \mathcal{D}_k^{(2)}(Y^{(q)}) = \mathcal{D}_k^{(1)}(Y^{(q)})$, and*

$$Cum_k(Y_n^{(q)}) \asymp \sum_{\delta_k \in \mathcal{D}_k^{(1)}(Y)} (\mathbf{E}a)^{(q-1)k+1}\ n^{(q-1)k+2}\,; \tag{4.2}$$

*C) in the sparse random graphs regime, the leading contribution to (3.5) is given by the diagrams $\mathcal{D}^{(3)}(Y^{(q)})$ such that the graph $G(\tilde{\delta}_k), \tilde{\delta}_k \in \mathcal{D}^{(3)}(Y^{(q)})$ is a tree $T_l$ with $l$ edges, $1 \le l \le (q-1)k+1$; then*

$$Cum_k(Y_n^{(q)}) \asymp \sum_{l=1}^{(q-1)k+1} \sum_{\delta_k \in \mathcal{D}^{(3)}(Y), G(\delta_k)=T_l} (\mathbf{E}a)^l n^{l+1}. \tag{4.3}$$

*Corollary of Theorem 4.1.* Remembering that $\mathbf{E}a = p = p_n$, we can reformulate the results of Theorem 4.1 in the following form:

$$\lim_{n\to\infty} \frac{1}{p_n n^2}\, Cum_k \left( \frac{1}{(np_n)^{q-1}} Y_n^{(q)} \right) = F_k^{(q)}(\omega), \quad \omega = 1, 2, 3, \tag{4.4}$$

where the numbers $F_k^{(q)}(i)$ represent the contributions of corresponding families of diagrams (weighted by $W(\delta_k)$ in the full random graphs regime). We discuss relation (4.4) in more details in subsection 4.3. Explicit expressions for some of $F_k^{(q)}(\omega)$ will be obtained in Section 6.

Let us consider the $X$-model. Now the difference between the cases of the even and odd numbers $q$ becomes crucial.

**Theorem 4.2** *Asymptotic behavior of $Cum_k(X_n)$ is determined by the following classes of diagrams:*

*A) in the full random graphs regime, the leading contribution to the sum (3.5) is given by the diagrams $\mathcal{D}_k^{(1)}(X^{(q)})$ that have exactly $k-1$ arcs; in this case*

$$Cum_k(X_n^{(q)}) \asymp \sum_{\delta_k \in \mathcal{D}_k^{(1)}(X)} W(\delta_k)\ n^{(q-2)k+2}\,, \tag{4.5}$$

*where expressions for $W(\delta_k)$ is given by (3.6) with $\delta_k \in \mathcal{D}_k^{(1)}(X_n^{(q)})$;*

*B) in the dilute random graphs regime, the leading contribution to (3.5) in the case of $X$-model with even $q = 2q'$is given by the diagrams $\mathcal{D}_k^{(2)}(X)$ such that the graph $G(\delta'_k), \delta'_k \in \mathcal{D}_k^{(2)}(X)$ is a tree with the maximal possible number of edges; then*

$$Cum_k(X_n^{(2q')}) \asymp \sum_{\delta_k \in \mathcal{D}_k^{(2)}(X^{(2q')})} (\mathbf{E}a)^{(q'-1)k+1}\ n^{(q'-1)k+2}; \tag{4.6}$$

*in the case of $X$-model with odd $q$ the leading contribution to (3.5) is given by the diagrams $\tilde{\mathcal{D}}_k^{(2)}(X)$ such that the graph $G(\tilde{\delta}'_k), \tilde{\delta}'_k \in \tilde{\mathcal{D}}_k^{(2)}(X)$ is a cycle with he maximal possible number of edges; then*

$$Cum_k(X_n^{(2q'+1)}) \asymp \sum_{\delta_k \in \tilde{\mathcal{D}}_k^{(2)}(X^{(2q'+1)})} (\mathbf{E}a)^{2q'+1}\ n^{2q'+1}; \tag{4.7}$$

*C) in the sparse random graphs regime, the leading contribution to (3.5) in the case of even $q = 2q'+1$ is given by the diagrams $\mathcal{D}^{(3)}(X)$ such that the graph $G(\delta''_k), \delta''_k \in \mathcal{D}^{(3)}(X)$ is a tree $T_l$ with $l$ edges, $1 \le l \le (q'-1)k+1$; then*

$$Cum_k(X_n^{(2q')}) \asymp \sum_{l=1}^{(q'-1)k+1} \sum_{\delta_k \in \mathcal{D}^{(3)}(X), G(\delta_k)=T_l} (\mathbf{E}a)^l n^{l+1}; \tag{4.8}$$

*in the case of X-model with odd $q = 2q' + 1$, the leading contribution to (3.5) is given by the diagrams $\tilde{\mathcal{D}}^{(3)}(X)$ such that $G(\delta_k)$ is a graph $T_l^{(1)}$ with $l$ edges and one cycle only; then*

$$Cum_k(X_n^{(2q'+1)}) \asymp \sum_{l=1}^{(q'-1)k+1} \sum_{\delta_k \in \mathcal{D}^{(3)}(X), G(\delta_k)=T_l^{(1)}} (\mathbf{E}a)^l n^l. \qquad (4.9)$$

*Corollary of Theorem 4.2.* In the full random graphs regime,

$$\lim_{n\to\infty} \frac{1}{pn^2}\; Cum_k\left(\frac{1}{p^{q-1}n^{q-2}} X_n^{(q)}\right) = \Phi_k^{(q)}(1), \qquad (4.10)$$

and in the dilute and sparse regimes we have, respectively,

$$\lim_{1\ll c\ll n} \frac{1}{cn}\; Cum_k\left(\frac{1}{c^{q'-1}} X_n^{(2q')}\right) = \Phi_k^{(2q')}(\omega), \quad \omega = 2, 3, \qquad (4.11)$$

and

$$\lim_{1\ll c\ll n} \frac{1}{c^{2q'+1}} Cum_k(X_n^{(2q'+1)}) = \Phi_k^{(2q'+1)}(2) \qquad (4.12)$$

and

$$\lim_{n\to\infty, c=const} Cum_k(X_n^{(2q'+1)}) = \Phi_k^{(2q'+1)}(3) \qquad (4.13)$$

We discuss relations (4.10)-(4.13) in Subsection 4.3. Explicit expressions for some of $\Phi_k^{(q)}(\omega)$ will be obtained in Section 6.

## 4.2 Proof of Theorems 4.1 and 4.2

*Proof of Theorem 4.1.* In the full random graphs regime, $\mathbf{E}a$ is a constant and all the terms of the sum (3.6) are of the same order of magnitude. The leading contribution to (3.5) comes from the diagrams $\delta_k$ such that the graph $G(\delta_k)$ has maximally possible number of vertices. In this case the number of arcs in $\delta_k$ is minimal, i.e. is equal to $k-1$ and $G(\delta_k)$ is a tree. This proves (4.1).

In the case of dilute and sparse graphs, relation $\mathbf{E}a^2 = o(\mathbf{E}a)$ shows that the leading contribution to the sum (3.6) is obtained from those partitions $\pi_s$ that $\chi(\pi_s;\delta_k) = 0$. Since $\delta_k$ is connected, then only trivial partition $\pi_0$ verifies this condition. Let us denote by $l$ the number of edges in $G(\delta_k)$. If $G(\delta_k)$ is given by a tree, then $\nu(\delta_k) = l+1$ and $(\mathbf{E}a)^l|\mathcal{N}(\delta_k)| = nc^l(1+o(1))$. If $G(\delta_k)$ is not a tree, then by the Euler theorem, $\nu(\delta_k) < l$ and such diagrams provide a contribution to (3.5) of the order $o(nc^l)$. In the case of dilute random graphs, $c \to \infty$ and the leading contribution is obtained from the graphs with$l = (q-1)k+1$; other trees do not contribute. This proves (4.2). To show (4.3), it remains to note that in the sparse random graphs regime all trees with $1 \leq l \leq (q-1)k+1$ provide contributions of the same order of magnitude. Theorem 4.1 is proved.

*Proof of Theorem 4.2.* In the full random graph regime, the leading contribution to (3.5) comes from those $\delta_k$ that have $k-1$ arcs. This implies relation (4.5).

It is easy to see that in the dilute and sparse regimes, the only trivial partition $\pi_0$ contributes to (3.6). In the case of $X^{(2q')}$, we repeat arguments of the proof of Theorem 4.1 and get relations (4.6) and (4.8).

Let us pass to the case of odd $q = 2q'+1$ and consider diagrams $\delta_1$ with one element $\lambda_1$. It is clear that graphs $G(\delta_1)$ always contain at least one cycle. Then $G(\delta_k)$ also contain at least one cycle. If the number of edges of $G(\delta_k)$ is equal to $l$, then the contribution of such a diagram is of the order $(\mathbf{E}a)^l n^{l+1-w}$, where $w$ is the number of cycles in $G(\delta_k)$. The graphs with one cycle only provide the leading contribution of the order $(\mathbf{E}a)^l n^l$. In the dilute random graphs regime the leading contribution comes from the graphs with maximal number of edges. Then we conclude that in this case $G(\delta_k)$ represents a cycle with $2q'+1$ edges and $2q'+1$ vertices. This proves (4.7). In the case of sparse regime, we consider $\delta_k$ such that $G(\delta_k)$ has exactly one cycle. This implies (4.9). Theorem 4.2 is proved.

*Remark.* It follows from the proof of theorems 4.1 and 4.2 that if $p_n = c/n$ and $c \to 0$ as $n \to \infty$, then the leading contribution to $Cum_k(Y_n^{(q)})$ and $Cum_k(Y_n^{(q)})$ is given by the connected diagrams $\delta_k$ such that $G(\delta_k)$ have minimal number of edges. It is easy to see that in this case

$$Cum_k(Y_n^{(q)}) = 2^{k-1} p_n n^2 (1+o(1)) = 2^{k-1} cn(1+o(1)), \tag{4.14}$$

and

$$Cum_k(X_n^{(2q')}) = 2^{k-1} p_n n^2 (1+o(1)) = 2^{k-1} cn(1+o(1)). \tag{4.15}$$

Also we have $Cum_k(X_n^{(2q'+1)}) = O(p_n^3 n^3) = O(c^3)$ as $n \to \infty, c \to 0$.

### 4.3 Central Limit Theorem for variables $X$ and $Y$

Regarding the definition of the cumulants, it is easy to see that for any constant $C$,

$$Cum_k(V_n + C) = Cum_k(V_n) + C\delta_{k,1}.$$

Also, if there exists such a sequence $b_n$ that

$$\frac{1}{b_n} Cum_k(V_n) \to \phi_k, \quad k \geq 1, \tag{4.16}$$

then

$$Cum_k\left(\frac{V_n - \mathbf{E}V_n}{\sqrt{b_n}}\right) \to \begin{cases} \phi_k, & \text{if } k = 2; \\ 0, & \text{if } k \neq 2. \end{cases} \tag{4.17}$$

Relation (4.17) means that the probability law of the centered random variable $\tilde{V}_n = (V_n - \mathbf{E}V_n)/\sqrt{b_n}$ converges to the Gaussian distribution $\mathcal{N}(0, \phi_2)$. Regarding centered and normalized variables $X_n^{(q)}$ and $Y_n^{(q)}$, we formulate the Central Limit Theorem.

**Theorem 4.3** *If $n \to \infty$, and $p_n$ determines one of the three asymptotic regimes indicated by $\omega$, the following random variables converge in law to the Gaussian distribution:*

$$\frac{1}{\sqrt{p_n n^2}} \cdot \frac{Y_n^{(q)} - \mathbf{E}Y_n^{(q)}}{(p_n n)^{q-1}} \to \mathcal{N}(0, F_2^{(q)}(\omega)), \quad \omega = 1, 2, 3, \tag{4.18}$$

*and*

$$\frac{1}{\sqrt{p_n n^2}} \cdot \frac{X_n^{(2q')} - \mathbf{E}X_n^{(2q')}}{(p_n n)^{q'-1}} \to \mathcal{N}(0, \Phi_2^{(2q')}(\omega)), \quad \omega = 2, 3. \tag{4.19}$$

*Also convergence to the standard normal distribution holds in the case of $X$ model in the full random graphs asymptotic regime;*

$$\frac{1}{\sqrt{p_n n^2}} \cdot \frac{X_n^{(q)} - \mathbf{E}X_n^{(q)}}{(p_n n)^{q-2}} \to \mathcal{N}(0, \Phi_2^{(q)}), \tag{4.20}$$

*and in the regime of dilute random graphs for $X^{(q)}$ with odd $q = 2q' + 1$:*

$$\frac{1}{c^{q'+1/2}} \left( X_n^{(2q'+1)} - \mathbf{E}X_n^{(2q'+1)} \right) \to \mathcal{N}(0, \Phi_2^{(2q'+1)}(B)). \tag{4.21}$$

*There is no convergence to the normal distribution of $X_n^{(2q'+1)}$ in the regime of sparse random graphs.*

*Proof.* The proof follows immediately from relations (4.4) and (4.11) with properties (4.16) and (4.17) taken into account.

Let us discuss relations of Theorem 4.3 with the spectral theory of random matrices. Remembering that $p_n n = c$ and introducing the normalized adjacency matrices

$$\hat{A}_{(n,c)} = \frac{1}{\sqrt{c}} A_n,$$

we can introduce variables $M_{2q'}^{(n,c)} = \frac{1}{n} \operatorname{Tr} \hat{A}_{(n,c)}^{2q'}$ and rewrite (4.19) in the form

$$\sqrt{nc} \left( M_{2q'}^{(n,c)} - \mathbf{E}M_{2q'}^{(n,c)} \right) \to \mathcal{N}(0, \Phi_2^{(2q')}(\omega)), \quad \omega = 2, 3. \tag{4.22}$$

Also, we deduce from relations (4.11)-(4.13) that

$$\lim_{n\to\infty} \mathbf{E}M_{2q'}^{(n,c)} = \Phi_1^{(2q')}(\omega) = m_{2q'}(\omega) \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}M_{2q'+1}^{(n,c)} = 0. \tag{4.23}$$

Variables $M_q^{(n,c)}$ represent the moments of the normalized eigenvalue counting measure of random matrices $\hat{A}_{(n,c)}$,

$$\sigma_n(\lambda) = \#\{j : \lambda_j^{(n,c)} \le \lambda\} n^{-1}.$$

Convergence (4.23) implies the weak convergence of the measures $\sigma_n$. This convergence is established in the present and more general settings in papers [1, 13] and [16]. Central Limit Theorem (4.22) improves results of these papers.

Let us note that relations (4.14) and (4.15) imply that

$$\frac{1}{cn} Cum_k(Y_n^{(q)}) \to 2^{k-1} \quad \text{as} \quad n \to \infty, c \to 0. \tag{4.24}$$

The same is true for the cumulants of $X_n^{(2q')}$. This means that the Central Limit Theorem does not hold for these variables in the asymptotic regime of very sparse random graphs.

Returning to variables $Y_n^{(q)}$, we rewrite (4.4) with $k = 1$ in the form

$$\lim_{n\to\infty} \frac{1}{nc^q} \, \mathbf{E}\, Y_n^{(q)} = F_1^{(q)}(\omega) = 1, \tag{4.25}$$

since there exists only one connected diagram. Variable $Y_n^{(q)}$ counts the number of $q$-step walks over the graph, and relation (4.25) shows that this number is asymptotically proportional to $nc^q$. It is natural to expect this result because the average degree of a vertex in the Erdős-Rényi random graph converges to $c$, as $n \to \infty$ [11]. It follows from (4.18) that

$$\frac{1}{nc^q} Y_n^{(q)} - \mathbf{E}\left\{ \frac{1}{nc^q} Y_n^{(q)} \right\} \sim \frac{\gamma_n}{\sqrt{nc}},$$

where $\gamma_n$ converges in law to a gaussian random variable. Then convergence with probability 1 holds

$$\frac{1}{nc^q} Y_n^{(q)} \to 1$$

as $n \to \infty$ in the dilute and sparse random graphs regimes. The limiting variance of $\gamma_n$ depends on the asymptotic regime. We return to this question in Section 6.

# 5 Number of tree diagrams

In this section we derive recurrent relations that determine the number of connected diagrams $\delta_k = (\Lambda_k, \Sigma)$ with minimal number of arcs,

$$\mathcal{D}_k^{(1)} = \{(\Lambda_k, \Sigma) : \ |\Sigma| = k - 1\}.$$

In the case of $Y$-model, the graphs $G(\delta_k)$ have the tree structure and we refer to $\delta_k$ as to the tree diagrams. In the case of $X$ model we refer to such diagrams as to the tree-like diagrams. It is clear that

$$|\mathcal{D}_k^{(1)}(X^{(q)})| = |\mathcal{D}_k^{(1)}(Y^{(q)})| = d_k^{(q)},$$

so we consider the case of the $Y$-model only.

### 5.1 Recurrent relations and Lagrange equation

First let us remind that the arcs in $\delta_k$ are interpreted by $G(\delta_k)$ as gluing between edges that make a color group. Let us say that the edges that are not glued are not colored and stay grey. So, we forget the color of the simple color groups consisting of one edge only. The different color groups and grey edges correspond to the edges of $G(\delta_k)$. It is convenient to color the arcs that join the edges of one group in the same color as the edges of this group. We start with the following simple statement.

**Lemma 5.1.** *Let a diagram $\delta_k \in \mathcal{D}_k^{(1)}(Y^{(q)})$ have arcs of $s$ different colors. Then there are $(q-1)k - s + 1$ grey edges in $\delta_k$.*

*Proof.* Let us assume that there are $\mu_j$ arcs of each color. Obviously, $\mu_1 + \ldots + \mu_s = k - 1$ and the total number of colored edges is $k - 1 + s$. Taking into account that the total number of edges in $\delta_k$ is $qk$, we obtain the result. Lemma is proved.

**Lemma 5.2** *Given $q \geq 2$, the numbers $d_k^{(q)} = |\mathcal{D}_k^{(1)}(Y^{(q)})|$ with $k \geq 1$ are given by equalities*

$$d_k^{(q)} = \frac{k!}{(q-1)k+1} h_k^{(q)}, \tag{5.1}$$

*where the sequence $\{h_k^{(q)}\}_{k\geq 0}$ is determined by the recurrent relation*

$$h_k^{(q)} = \frac{(q-1)k+1}{k} \cdot \left( \sum_{\substack{j_1+\ldots+j_q=k-1 \\ j_i \geq 0}} h_{j_1}^{(q)} \cdots h_{j_q}^{(q)} \right), \ k \geq 1 \tag{5.2}$$

*with initial condition $h_0^{(q)} = 1$.*

*Proof.* Let us remind that since our diagrams are reduced, then each color group has a unique maximal edge. Now let us count the number of the diagrams $\bar{\delta}_k$ such that the last element $\lambda_k$ contains one and only one edge that is the maximal edge of a color group. This means that only one arc ends at $\lambda_k$ by its right foot (or leg). Let us denote this arc by $\bar{\sigma}$.

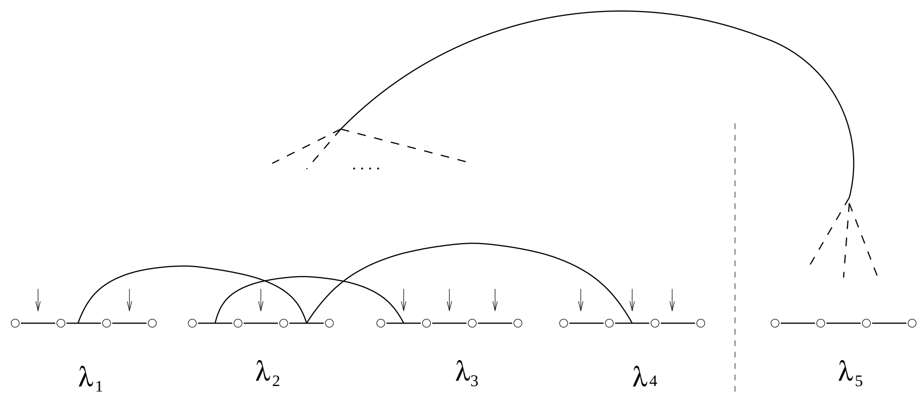


Figure 3: Example of $\bar{\delta}_k$ with possible positions for for the left foot of $\bar{\sigma}$

Obviously, one can choose one of the $q$ edges of $\lambda_k$ to put this right leg. The left leg of $\bar{\sigma}$ joins $\lambda_k$ with connected tree diagram $\delta_{k-1} \in \mathcal{D}^{(1)}_{k-1}(Y^{(q)})$. By Lemma 5.1, there are $(q-1)(k-1)-s+1$ grey edges and $s$ maximal edges of $s$ color groups, where one can put the left foot of $\bar{\sigma}$. Then we obtain relation

$$|\bar{\mathcal{D}}^{(1)}_k| = ((q-1)(k-1)+1)\, d^{(q)}_{k-1}.$$

Now let us consider the case when there are $l$ arcs $\bar{\sigma}_1, \ldots, \bar{\sigma}_l$ that end at $\lambda_k$ with $2 \le l \le \min(q, k-1)$. There is $\binom{q}{l}$ possibilities to choose the emplacements for the right legs of these arcs. To put the left legs of these arcs, one has to choose the $l$ subsets $\Lambda^{(1)}, \ldots, \Lambda^{(l)}$ of $j_i = |\Lambda^{(i)}|$ elements such that $j_1 + \ldots + j_l = k-1$ and $j_i \ge 1$, then to create $l$ connected sub-diagrams $(\Lambda^{(i)}, \Sigma^{(i)}), i = 1, \ldots, l$ and to choose the emplacements for the left legs of $\bar{\sigma}_i$ in $(\Lambda^{(i)}, \mathcal{S}^{(i)})$. This produces

$$\binom{q}{l} \sum_{\substack{j_1+\ldots+j_l=k-1\\ j_i\ge 1}} \frac{(k-1)!}{j_1!\cdots j_l!} \prod_{i=1}^{l} ((q-1)j_i+1) d^{(q)}_{j_i}$$

diagrams. Summing up, we derive for numbers $\{d^{(q)}_k, k \ge 1\}$ with given $q \ge 2$ the following recurrent relation

$$d^{(q)}_k = \sum_{l=1}^{q} I_{[1,k-1]}(l) \times \binom{q}{l} \sum_{\substack{j_1+\ldots+j_l=k-1\\ j_i\ge 1}} \frac{(k-1)!}{j_1!\cdots j_l!} \prod_{i=1}^{l} ((q-1)j_i+1) d^{(q)}_{j_i}, \quad (5.3)$$

with initial condition $d^{(q)}_1 = 1$. Here we denoted by $I_{[1,k-1]}(\cdot)$ the indicator function of the interval $[1, k-1]$.

Introducing the auxiliary numbers $h_j$, such that

$$h_j = \frac{(q-1)j+1}{j!} d^{(q)}_j, \quad j \ge 1,$$

we reduce (5.3) to relation

$$\frac{k}{(q-1)k+1} h_k = \sum_{l=1}^{q} I_{[1,k-1]}(l) \times \binom{q}{l} \sum_{\substack{j_1+\ldots+j_l=k-1 \\ j_i \ge 1}} h_{j_1} h_{j_2} \cdots h_{j_l}, \ k \ge 1. \quad (5.4)$$

If we introduce an auxiliary number $h_0 = 1$, then we can rewrite (5.4) in more compact form. Indeed, we can make an agreement that each labeled edge $\varepsilon_l^{(k)}$ of the element $\lambda_k$ serves as the right foot of the corresponding arc $\bar{\sigma}_l$ but some of these arcs can have the right leg "empty", i.e. with no elements attached to their right feet. Then the variable $l$ of the sum of (5.4) represents the number of non-empty arcs, and $\binom{q}{l}$ stands for the choice of these non-empty arcs from the set $\bar{\sigma}_1, \ldots, \bar{\sigma}_q$. This corresponds to the choice of the variables $j_i$ in the product $h_{j_1} \cdots h_{j_q}$ that take zero value. Finally, we set $h_j^{(q)} = h_j, j \ge 0$ and conclude that relation (5.4) is equivalent to recurrent relation (5.3). Lemma 3.5 is proved.

**Lemma 5.3.**
*The generating function*

$$H_q(z) = \sum_{k=0}^{+\infty} h_k^{(q)} z^k \quad (5.5)$$

*is regular in the domain* $\{z : |z| \le 1/(q^2 e)\}$ *and verifies there the analog of the equation*

$$H_q(z) = \exp\{qz\,(H_q(z))^{q-1}\}. \quad (5.6)$$

*It follows from (5.3) that the numbers* $h_k^{(q)}$ *can be found from recurrent relations*

$$\sum_{\substack{j_1+\ldots+j_{q-1}=k \\ j_i \ge 0}} h_{j_1}^{(q)} \cdots h_{j_{q-1}}^{(q)} = q^k (q-1)^k \frac{(k+1)^{k-1}}{k!}, \quad h_0^{(q)} = 1. \quad (5.7)$$

*Remark.* Relation (5.6) with $q = 2$ is known as the Lagrange (or Pólya) equation [17, 23, 25]. In the next subsection we give another derivation of (5.6) by using the notion of color trees. In Section 6 we obtain one more generalization of the Lagrange equation.

*Proof of Lemma 5.3.* It is easy to deduce from (5.2) that $H_q(z)$ verifies integral equation

$$H_q(z) - 1 = z(q-1)(H_q(z))^q + \int_0^z (H_q(\zeta))^q \, d\zeta.$$

that is equivalent to the differential equation

$$\frac{dH_q(z)}{dz} = \frac{q\,(H_q(z))^q}{1-(q^2-q)z\,(H_q(z))^{q-1}}, \quad H_q(0) = 1. \quad (5.8)$$

Substitution

$$\psi(z) = z\,(H_q(z))^{q-1} \tag{5.9}$$

transforms (5.8) to an elementary equation

$$\psi'(z) = \frac{1}{z} \cdot \frac{\psi(z)}{1-(q^2-q)\psi(z)}.$$

Resolving this equation with obvious initial condition, we conclude that $\psi(z)$ verifies the algebraic equation

$$\psi(z) = ze^{q(q-1)\psi(z)} \tag{5.10}$$

known as the Pólya equation. Then (5.6) easily follows from (5.9) and (5.10).

Using the standard method of the contour integration, we deduce from (5.10) explicit expressions for the coefficients $\psi_k$ of the generating function $\psi(z) = \sum_{k\ge 0} \psi_k z^k$. The first observation is that the inverse function $\psi^*(w) = we^{-q(q-1)w}$ is regular in the vicinity of the origin. By the Cauchy formula, we have

$$\psi_k = \frac{1}{2\pi\ \mathrm{i}} \oint \frac{\psi(z)}{z^{k+1}} dz.$$

By changing variables by $z = \psi^*(w)$, we get $dz = (1-q(q-1)w)e^{-q(q-1)w}dw$ and find that

$$\psi_k = \frac{1}{2\pi\ \mathrm{i}} \oint \frac{1-q(q-1)w}{w^k} e^{q(q-1)wk} dw.$$

Then

$$\psi_k = (q(q-1))^{k-1} \frac{k^{k-2}}{(k-1)!}, \tag{5.11}$$

and (5.7) follows. Lemma 5.3 is proved.

### 5.2 Color trees

In this subsection we give another derivation of the Lagrange equation (5.6) based on the graph representation $G(\delta_k)$ of the diagrams $\delta_k$. For simplicity, we consider the case of $Y$-model with $q = 2$ only.

If $\delta_k$ has $k-1$ arcs, then $G(\delta_k)$ is a tree. We say that the arcs that indicate the groups of edges to be glued are all of the same color. Then the corresponding edges of $G(\delta_k)$ are of this color. We say that the edges that are not glued are grey.

Let us derive expressions for the number of the elements of the set

$$T_k = |\mathcal{G}_k|, \quad \mathcal{G}_k = \{G(\delta_k), \delta_k \in \mathcal{D}^{(1)}(Y^{(2)})\}.$$

We denote by $\hat{T}_m$ the number of rooted trees of the form $G(\delta_m)$.

Let us consider left element of $\lambda_1$ as the root edge $\rho$ for the trees we construct (see Figure 4). We can attach to this root $r$ elements $\nu_1, \dots \nu_r$ that we choose by $\binom{m}{r}$ ways. We glue these elements by their right edges.

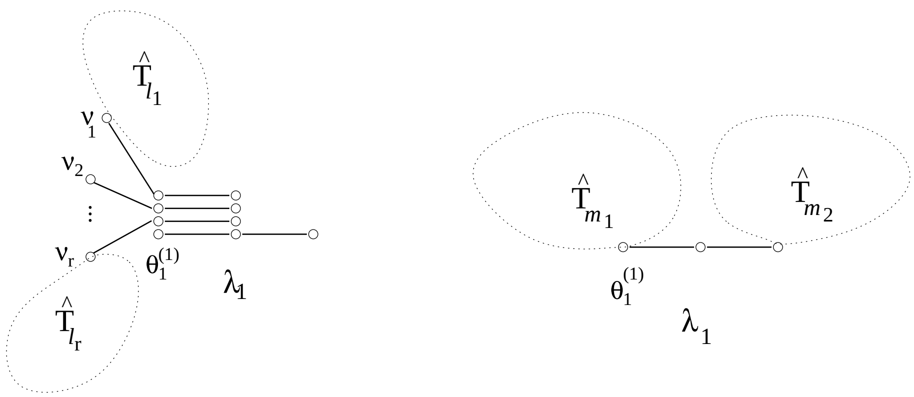


Figure 4: Rooted color tree and color tree

Then we separate $m-r$ elements into the groups of $l_1, l_2, \dots, l_r$ elements and construct rooted trees and attach them by their roots to the free elements of $\nu_1, \dots, \nu_r$. There are $\frac{(m-r)!}{l_1!\cdots l_r!}$ possibilities to separate $m-r$ elements into these groups. Then we get

$$\hat{T}_m = \sum_{r=1}^{m} 2^r \binom{m}{r} \sum_{l_1+\dots+l_r=m-r} \frac{(m-r)!}{l_1!\cdots l_r!} \hat{T}_{l_1}\cdots\hat{T}_{l_r}, \quad m \geq 1, \tag{5.12}$$

where the sum runs over all $l_i \geq 0$ and we accepted that $\hat{T}_0 = 1$.

Simplifying (5.12) and taking into account that $\hat{T}_0 = 1$, we derive from (5.12) the following relation for the generating function $t(z) = \sum_{k\geq 0} t_k z^k$, $t_k = \hat{T}_k/k!$ (cf (5.6)):

$$t(z) = \exp\{2zt(z)\}. \tag{5.13}$$

Then $\hat{T}_m = 2^m(m+1)^{m-1}$ and the numbers $t_m$ are given by recurrent relations

$$t_m = \frac{m+1}{m}\sum_{j=0}^{m-1} t_j t_{m-1-j}, \quad t_0 = 1. \tag{5.14}$$

Now let us return to the construction of the tree $G(\delta_k)$. Now the left edge of $\lambda_1$ can serve as the root for the left rooted tree of $m_1$ elements and the right element serve as the root for the rooted tree of $m_2 = k-1-m_1$ elements. Then

$$T_k = \sum_{\substack{m_1+m_2=k-1 \\ m_i \geq 0}} \frac{(k-1)!}{m_1! m_2!}\, \hat{T}_{m_1}\, \hat{T}_{m_2},$$

where $\hat{T}_0 = 1$. Then

$$\frac{T_k}{(k-1)!} = \sum_{j=0}^{k-1} t_j t_{k-1-j} = \frac{k}{k+1} t_k = \frac{2^k(k+1)^{k-2}}{(k-1)!}.$$

Then equality $T_k = 2^k (k+1)^{k-2}$ follows.

In the general case $q \geq 2$, one can easily repeat this procedure of tree construction and derive (5.6) directly, without use of the numbers $h_k^{(q)}$.

# 6 Limits of the cumulants

In this section we specify the limits of the cumulants of the $Y$-model $F_k^{(q)}(i)$ (4.4) and of the $X$-model $\Phi_k^{(q)}(i)$ (4.10)-(4.13). We start with the case of full random graphs regime, but the most attention is paid to the dilute and sparse random graphs regimes.

## 6.1 Cumulants of full $X$ and $Y$ models

In Section 4 we have shown that the leading contribution to the cumulants of (4.1) and (4.5) is given by those diagrams $\delta_k$ that have exactly $k-1$ arcs and the graph $G(\delta_k)$ with the maximal number of vertices. The weight of the diagram $W(\delta_k)$ is determined by relation (3.6), where all terms $(Ea)^l$ are of the same order of magnitude (see the proof of Lemma 3.2). Regarding $Y$-model and taking into account that in this case $m(\delta_k) = (q-1)k+1$ (see Lemma 5.1), we combine (3.6) with (4.1) and conclude that (4.4) is true with

$$F_k^{(q)}(1) = \sum_{\delta_k \in \mathcal{D}_k^{(1)}(Y)} \sum_{\pi_s \in \Pi_k} (-1)^{s-1}(s-1)!\, p^{\chi(\pi_s;\delta_k)}, \qquad (6.1)$$

where $\chi(\pi_s; \delta_k)$ is determined in Lemma 3.2.

Considering $X$-model, it is easy to see that (3.6) gives the same expression for $W(\delta_k)$ as for the $Y$-model. Using (4.5), we arrive at the conclusion that (4.10) is true with $\Phi_k^{(q)}(A) = F_k^{(q)}(A)$ (6.1).

Restricting ourself to the first two cumulants, it is easy to compute that

$$F_1^{(q)}(1) = \Phi_1^{(q)}(1) = 1$$

and that

$$F_2^{(q)}(1) = \Phi_2^{(q)}(1) = 2q^2(1-p).$$

## 6.2 Cumulants of dilute $Y$-model

Regarding (4.2) with $\mathbf{E}a = c/n$ and using results of Section 5, we conclude that (4.4) takes the form of

$$F_k^{(q)}(2) = \lim_{n,c\to\infty} \frac{1}{nc} Cum_k \left( \frac{1}{c^{q-1}} Y_n^{(q)} \right) = 2^{k-1} |\mathcal{D}_k^{(1)}(Y)| = 2^{k-1} d_k^{(q)}, \quad (6.2)$$

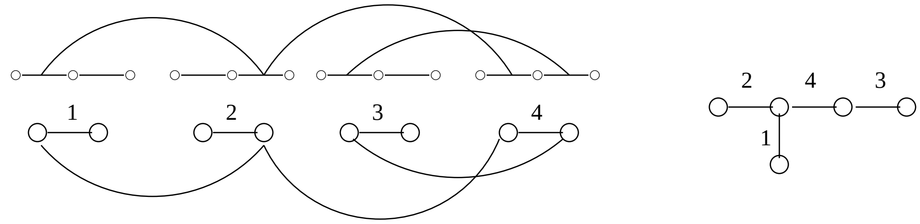


Figure 5: Dual diagrams and dual trees in the case of $q = 2$, $k = 4$

where $d_k^{(q)}$ are determined by relations (5.1) and (5.2). In (6.2), we have taken into account that the $k-1$ arc of the diagrams $\delta_k$ can be drawn in the direct and inverse sense. This produces the factor $2^{k-1}$.

Let us consider in more details the case of $q = 2$ that corresponds to the continuous matrix model (2.1) with the quartic potential. In this case relation (5.2) takes the form

$$h_k^{(2)} = \frac{k+1}{k} \sum_{j_1+j_2=k_1, j_i \geq 0} h_{j_1}^{(2)} h_{j_2}^{(2)}, \quad h_0^{(2)} = 1.$$

It follows from (5.7) that in this case $h_k^{(2)} = 2^k (k+1)^{k-1}/k!$ and

$$d_k^{(2)} = 2^k (k+1)^{k-2}. \tag{6.3}$$

The combinatorial meaning of $d_k^{(2)}$ can be explained with the help of the notion of the dual diagrams that we briefly describe below.

Regarding $\delta_k \in \mathcal{D}^{(1)}(Y^{(2)})$, we construct the dual diagram $\hat{d}_k$ by transforming the edges of elements $\lambda_j$ into vertices joined by an edge; the arcs remain without changes and indicate the vertices of $\hat{\delta}_k$ to be glued (see Figure 5). It is easy to see that the graph $G(\hat{\delta}_k)$ is a tree. Then $d_k^{(2)}$ is equal to the number of all non-rooted trees constructed with the help of $k$ labeled edges. Since the edges of $\delta_k$ are labeled to be the left and the right one, we can think about the edges of $\hat{G}(\delta_k)$ are the oriented one.

Regarding the dual diagrams in the general case of $q \geq 2$, we see that these are trees when $q = 2$ and trees constructed from oriented chains of $q$ edges when $q > 2$. It is not clear, whether it is possible to derive from (5.7) explicit expressions for $d_k^{(q)}$ with $q > 2$.

### 6.3 Cumulants of sparse $Y$-model

The diagrams of $\mathcal{D}^{(3)}(Y)$ are of more complicated structure than those of $\mathcal{D}^{(1)}(Y)$ and we can not obtain recurrent relations to determine $F_k^{(q)}(3)$ in the general case of $q \geq 2$ and $k \geq 1$. We present the results for $F_k^{(2)}(3)$ and $F_1^{(q)}(3)$ only.

**Lemma 6.1** *The coefficients* $F_k^{(2)}(3), k \geq 1$ *are determined by relations*

$$F_k^{(2)}(3) = \frac{2^{k-1}(k-1)!}{c^k} w_k, \tag{6.4}$$

*where* $w_k$ *are determined by relation*

$$w_k = \sum_{s=1}^{k} \frac{1}{(s-1)!} \sum_{j=0}^{k-s} \hat{w}_j \, \hat{w}_{k-s-j}, \quad k \geq 1, \tag{6.5}$$

*and the numbers* $\hat{w}_j, j \geq 0$ *are such that the generating function* $\hat{W}(z) = \sum_{k\geq 0} \hat{w}_k z^k$ *verifies equation*

$$\hat{W}(z) = 1 - ce^{2z} + c \exp\left\{2z[e^{2z}\hat{W}(z) - 1]\right\}. \tag{6.6}$$

*Proof.* Following the lines of subsection 5.2, we consider first the set of the rooted trees of the type we are interested in and denote their contribution by $\hat{T}_m$, where $m$ is the number of elements $\lambda$.

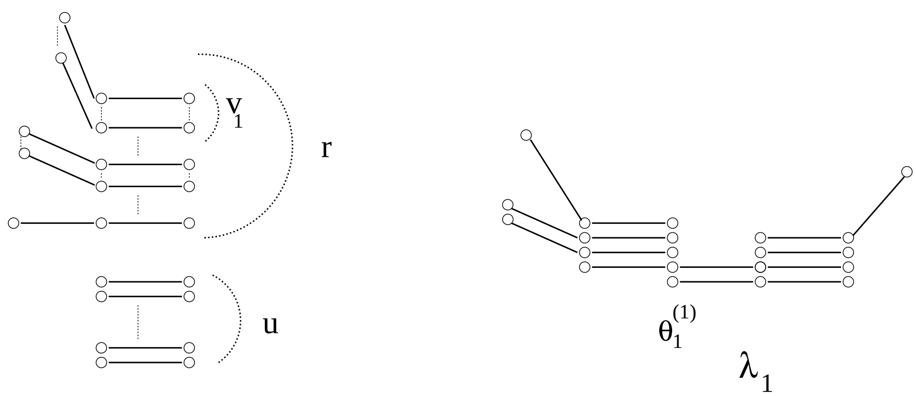


Figure 6: Elements to construct a rooted tree and example of the tree

In the present case the trees are constructed with the help of elements $\lambda$, but there can be double elements that are glued to one of the edge, and there can be simple elements, and there are elements glued to simple elements all the long (see Figure 6). Let us consider the root edge $\rho$. Denoting by $u$ the number of double elements attached to $\rho$ and by $r$ the number of simple elements attached to $\rho$ by one edge. Also we denote by $v_i$, $i = 1, \ldots, r$ the numbers of elements glued all the long to these $r$ simple elements. Taking into account possibilities to choose $u, r$, and $v_i$ elements from the set of $m$ labeled elements, we obtain the recurrent formula

$$\hat{T}_m = c \sum_{u=0}^{m} 2^u \binom{m}{u} \cdot \sum_{r=1}^{m-u} 2^r \binom{m-u}{r} \sum_{v_1+\ldots+v_r=V,\, V=1}^{m-u-r} 2^{v_1} \cdots 2^{v_r} \binom{m-u-r}{v_1, \ldots, v_r}$$

$$\times \sum_{l_1+\ldots+l_r=m-u-r-V} \binom{m-u-r-V}{l_1,\ldots,l_r} \hat{T}_{l_1}\cdots\hat{T}_{l_r}, \tag{6.7}$$

where we denoted the multinomial coefficients

$$\binom{m-u-r}{v_1,\ldots,v_r} = \frac{(m-u-r)!}{v_1!\cdots v_r!\,(m-u-r-V)!}.$$

Factor $c$ in the right-hand side of (6.7) takes into account the contribution (or weight) of the root $\rho$. Simplifying (6.7) and denoting $\hat{w}_m = \hat{T}_m/m!$ for $m \geq 1$, we get relation

$$\hat{w}_m = c\sum_{u=0}^{m} \frac{2^u}{u!}\cdot \sum_{r=1}^{m-u} \frac{2^r}{r!}\cdot \sum_{V=v_1+\ldots+v_r, V=1}^{m-u-r} \frac{2^{v_1}\cdots 2^{v_r}}{v_1!\cdots v_r!} \sum_{l_1+\ldots+l_r=m-u-r-V} \hat{w}_{l_1}\cdots\hat{w}_{l_r}.$$

Passing to the generating function $\hat{W}(z)$, we derive equality

$$\hat{W}(z) - 1 = ce^{2z}\left[\exp\left\{2ze^{2z}\hat{W}(z)\right\} - 1\right], \tag{6.8}$$

that is equivalent to (6.6). Let us note here that (6.8) generalizes the Pólya equation (5.13).

Returning to the set $\mathcal{D}_k^{(3)}(Y)$, we consider $\lambda_1$ as the simple root element. Assuming that there are $s-1$ elements glued all the long this root element, we dispose of $k-s$ elements to construct the rooted trees using the left and the right edge of $\lambda_1$ as the roots $\rho_1$ and $\rho_2$. Denoting the number $T_k = |\mathcal{D}_k^{(3)}(Y)|$, we get relation

$$T_k = \sum_{s=1}^{k} \binom{k-1}{s-1} \sum_{\substack{m_1+m_2=k-s \\ m_i\geq 0}} \binom{k-s}{m_1, m_2} \hat{T}_{m_1}\ \hat{T}_{m_2}.$$

Simplifying this relation and passing to variables $\hat{w}_m = \hat{T}_m/(m-1)!$, we obtain equality (6.5). This completes the proof of Lemma 6.1.

In the general case of $q \geq 2$ we determine the first cumulant only,

$$F_1^{(q)}(3) = \lim_{\substack{n\to\infty, p=c/n \\ c=Const}} \sum_{i_1,\ldots,i_{q+1}=1}^{n} \mathbf{E}\left\{A_{i_1i_2}A_{i_2i_3}\cdots A_{i_qi_{q+1}}\right\}. \tag{6.9}$$

The vertices $\theta$ of $\lambda$ are ordered and we denote $\theta_1 = \rho$ and $\theta_2 = \nu$. It follows from Theorem 4.1 that $F_1^{(q)}(3)$ is determined by the number of diagrams $\delta_1$ consisting of one element $\lambda$ such that corresponding graph $G(\delta_k)$ is a tree of $l$ vertices, $2 \leq l \leq q$. This problem is closely related with the studies done in [1, 13] and formalized in [16]. These studies are related with the $X$

model and we present the results in the next subsection. In what follows, we describe briefly corresponding approach and modify it to suit our situation.

To study the right-hand side of (6.8), it is useful to introduce the notion of a walk $\xi$. A walk $\xi_q$ of $q$ steps is an ordered sequence of $q$ letters, starting with $\rho$ and followed by $\nu$ always. Given a sequence $I_{q+1} = (i_1, i_2, \dots, i_{q+1})$, we construct corresponding $\xi_q = \xi(I_{q+1})$ by the following recurrent rule: regarding the value $i_{s+1}, s \ge 1$, we compare it with the values of $\{i_1, \dots, i_s\}$ and write a letter number $s+1$ in $\xi$. If there is no $i_j, 1 \le j \le s$ such that $i_{s+1} = i_j$, the we write a new letter that is not present in $\xi_s$. If there is such $j$, $1 \le j \le s$ such that $I_{s+1} = i_j$, then we write on the $s+1$-th place the letter number $j$ of $\xi_s$.

We can represent the walks graphically, by drawing the root vertex $\rho$, the next vertex $\nu$ and corresponding edge and then passing along $\xi_q$ and creating the new vertices at the instants, when the new letters occur in $\xi_q$. We also draw the new edges in this case.

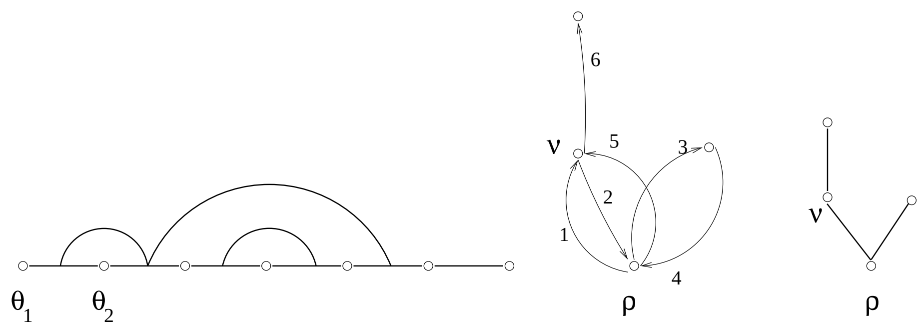


Figure 7: A diagram $\delta_1$, the walk $\xi_q$ and corresponding graph $G(\delta_1) = \mathcal{G}(\xi_q)$

It is clear that there is one-to-one correspondence between the diagrams $\delta_1$ and walks $\xi_q$. Also, the walk $\xi_q$ generates in natural way the graph $\mathcal{G}(\xi_q)$ isomorphic to $G(\delta_k)$.

**Lemma 6.2**. *The number $F_1^{(q)}(3) = F_q\, c^{1-q}$ is given by the sum*

$$F_q = cF_{q-1} + \sum_{r=2}^{q} F_q(r), \tag{6.10}$$

*where the numbers $F_q(r), r \ge 2$ are determined by the following recurrent relations*

$$F_q(r) = c \sum_{v=2}^{r} \sum_{u=0}^{q-v} \sum_{s=0}^{q-v-u} \binom{[\frac{v-1}{2}] + [\frac{s}{2}]}{[\frac{v-1}{2}]} \binom{[\frac{r}{2}] - 1}{[\frac{v}{2}] - 1} F_{q-u-v}(s) F_u(r-v). \tag{6.11}$$

*In (6.11), $[x]$ denotes the largest integer less or equal to $x$ and the following initial conditions are assumed: $F_l(0) = \delta_{l,0}$, $F_{r-j}(r) = 0$ for $j > 0$ and $F_2(2) = 1$.*

*Remark.* Relations (6.10) and (6.11) are similar to those derived in [1, 13] for the number of walks of even number of steps such that their graph is a tree.

*Proof.* Let us consider the set $\Xi_q(r)$ of walks such that there are $r$ steps that start or end in $\rho$. We denote by $f_q(r) = |\Xi_q(r)|$ the cardinality of this set.

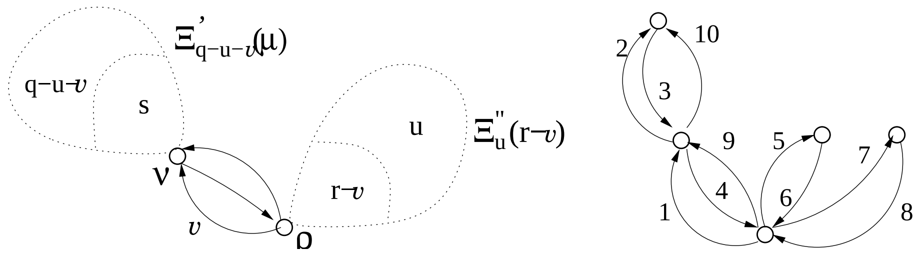


Figure 8: The first-passage decomposition and example of the walk

Let us consider the subset $\Xi_q(r, v, s) \subset \Xi_q(r)$ consisting of the walks that pass the edge $e_1 = (\rho, \nu)$ $v$ times and that have $s$ steps attached to $\nu$ that do not pass $e_1$ (see Figure 7). We denote the set of such steps $\nu \to \pi$ with $\pi \neq \rho$ by $\mathcal{S}_{\nu\to\bar{\rho}}$. Certainly, such a walk has $r - v$ steps that belong to the set $\mathcal{S}_{\rho\to\bar{\nu}}$.

Any walk of $\Xi_q(r, v, s)$ can be separated into three parts or in other words into the following three subwalks: the first one contains the steps $\rho \to \nu$ and $\nu \to \rho$ only, the second one is attached to $\nu$ with the elements of $\mathcal{S}_{\nu\to\bar{\rho}}$ (the left part of the walk) and the third one is attached to $\rho$ by the elements of $\mathcal{S}_{\rho\to\bar{\nu}}$ (the right part of the walk). We denote the left and the right parts by $\Xi'_{q-u-v}(s)$ and by $\Xi''_u(r - v)$, respectively, where $u$ denotes the number of steps in the right-hand part of the walk. We denote the by $\Xi_q(r, v, s, u)$ the set of elements of $\Xi_q(r, v, s)$ such that the right-hand part of the walk consists of $u$ steps.

Then we can write equality

$$|\Xi_q(r, v, s, u)| = \binom{[\frac{v-1}{2}] + [\frac{s}{2}]}{[\frac{v-1}{2}]} \binom{[\frac{r}{2}] - 1}{[\frac{v}{2}] - 1} f_{q-u-v}(s) f_u(r - v), \qquad (6.12)$$

where we have taken into account obvious equalities $|\Xi'_{q-u-v}(s)| = |\Xi_{q-u-v}(s)|$ and $|\Xi''_u(r - v)| = |\Xi_u(r - v)|$.

Let us explain the from of the combinatorial coefficients in the right-hand side of (6.12). If $r$ and $v$ are both even, then there are $v/2$ steps in $\mathcal{S}_{\nu\to\rho}$ and $r/2 - v/2$ steps in $\mathcal{S}_{\rho\to\bar{\pi}}$. Therefore there is a choice to perform each of the $r/2 - v/2$ steps after one of $v/2$ steps $\nu \to \rho$. This produces the second binomial coefficient of (6.12).

If $r$ is odd and $v$ is even, then the last step of the form $\rho \to \pi$ with $\pi \neq \rho$ is to be performed after that all the steps $\nu \to \rho$ are done. Then we get again the combinatorial coefficient as that given by (6.12). If $v$ is odd, then there are still $[v/2]$ steps $\nu \to \rho$.

Let us explain the first binomial coefficient of (6.12). If $s$ and $v$ are both even, then there are $\binom{v/2+s/2-1}{v/2-1}$ possibilities to perform $s/2$ steps of $\mathcal{S}_{\nu\to\bar\rho}$ after $v/2$ steps of $\mathcal{S}_{\rho\to\nu}$. If $s$ is odd, then the last step is performed after that all $v/2$ steps are performed. If $v$ is odd, then there are $[v/2]+1$ steps $\rho \to \nu$.

Summing (6.12) with respect to parameters $\alpha, u$, and $s$, and taking into account that $\mathbf{E}a = c/n$, we get relation (6.11) for the contribution to $F_q$ determined by the walks with $r, r \geq 2$ steps attached to $\rho$. If $r = 1$, then corresponding contribution is given by $cF_{q-1}$. Summing all contributions, we obtain formula (6.10). Lemma is proved.

*Remark.* It is easy to see from (6.12) that $f_3(2) = 0$. Moreover, it is not hard to show that $f_{2l+1}(2s) = 0$ for all $l, s \geq 1$. This is in agreement with the obvious observation that the set of walks $\Xi_{2l+1}(2, s)$ is empty.

## 6.4 Cumulants of dilute and sparse $X$-model

Analysis of cumulants of $X$-model in the dilute random graphs regime is very similar to that of the $Y$-model in the same regime.

**Lemma 6.4** *Relations (4.11) and (4.12) are true with*

$$\Phi_k^{(2q'+1)}(B) = (4q'+2)^{k-1} \quad \textit{and} \quad \Phi_k^{(2q')}(B) = \left[\frac{1}{q'+1}\binom{2q'}{q'}\right]^k 2^{k-1} d_k^{(q')}, \tag{6.13}$$

*where $d_k^{(q')}$ is determined by relations (5.1) and (5.2) with $q$ replaced by $q'$.*

*Proof.* According to Theorem 4.2, the leading contribution to (4.7) in the case of odd $q = 2q'+1$ is given by the diagrams $\delta_k$ such that the graph $G(\delta_k)$ is of maximal number of edges and contains one cycle only. Taking into account that each element $\lambda_j$ is represented by a cyclic graph with $2q'+1$ edges, the only possibility to get rid of the "extra" cycles in $\delta_k$ and is to glue the elements $\lambda_j$ all long one of each other. To get maximal number of edges in $\delta_k$, we keep the cyclic structure of $\lambda_j$. When doing this, we have to choose the direction and the edge of $\lambda_1$ to glue the first elements of $\lambda_j, j = 2, \dots, k$. The number of possibilities is given by $2^{k-1} \times (2q'+1)^{k-1}$ and we get the first equality of (6.13).

Regarding the case of even $q = 2q'$, we conclude that to construct a tree with the maximal number of edges we have first to construct a tree from each element $\lambda_j$ and then to draw $k-1$ arcs between these $k$ trees obtained. It is easy to see that the maximal size of the tree obtained by gluing the edges of $\lambda_j$ is $q'$ and the number of different trees is given by Catalan number $(q'+1)^{-1}\binom{2q'}{q'}$ [25]. Then we produce the connected diagrams exactly as in the case of dilute $Y$ model. This gives the second relation of (6.13). Lemma is proved.

Similarly to the $Y$-model, the sparse random graphs regime of $X$-model is more complicated than the dilute random graphs regime. The average value of random variable $X_n^{(2q')} = \mathrm{Tr}\, A^{2q'}$ is studied in [1, 13] (see also [16]). The limit $\Phi_1^{(2q')}(3) = \lim_{n\to\infty} \frac{1}{nc^{q'}} \mathbf{E} X_n$ is determined by a system of recurrent relations that have the form similar to that described in Lemma 6.2. We do not present these results.

The variance of $X_n^{(2q')}$ is studied in [26] in more general setting than that of this paper. The resulting expression is also determined by a system of recurrent relations. This system is much more complicated that that for the first moment of $X_n$. We refer the reader to the paper [26] for corresponding theorems.

### 6.5 Formal limit of the partition function

Returning to the normalized partition function (2.12) that describes the $Y$-model with $q = 2$, one can see that

$$\frac{1}{p_n n^2} \log \hat{Z}_n(\beta, tg_n) = \frac{n-1}{2p_n n} \log \left( \frac{1+e^{-2\beta'}}{1+e^{-2\beta}} \right) + \frac{1}{p_n n^2} \log \mathbf{E}_{\beta'} \left\{ e^{tg_n Y_n} \right\}, \tag{6.15}$$

where $\beta' = \beta - tg_n$ and $g_n = (p_n n)^{-1}$. It is clear that the three asymptotic regimes introduced in Section 4 are characterized by the corresponding behavior of the inverse "temperature" $\beta$ and elementary analysis shows that

$$\lim_{n\to\infty} \frac{n-1}{2p_n n} \log \left( \frac{1+e^{-2\beta'}}{1+e^{-2\beta}} \right) = \begin{cases} 0, & \text{if } \beta = Const; \\ 0, & \text{if } \beta = \frac{1}{2}\log\frac{n}{c},\ 1 \ll c \ll n; \\ \frac{\exp\{2tc^{-1}\}-1}{2}, & \text{if } \beta = \frac{1}{2}\log\frac{n}{c},\ c = Const, \end{cases}$$

where $p_n = e^{-2\beta}(1+e^{-2\beta})^{-1}$. Then, assuming that the limit of the last term of (6.15) exists, we conclude that (4.4) implies relation

$$\lim_{n\to\infty} \frac{1}{p_n n^2} \log \hat{Z}_n(\beta, tg_n) = \frac{\delta_{\omega,3}}{2} \left( \exp\left\{ \frac{2t}{c} \right\} - 1 \right) + \sum_{k=1}^{\infty} \frac{t^k F_k^{(2)}(\omega)}{k!}, \tag{6.16}$$

where $\omega$ indicates the full, the dilute, and the sparse random graphs regimes, and $\delta_{\omega,3}$ denotes the Kronecker $\delta$-symbol.

Comparing expressions (6.1), (6.2), and (6.4) for $F_k^{(2)}$ with $\omega = 1, 2, 3$, we see the difference between the limiting rate functions of (6.16). Indeed, $F_k^{(2)}$ does not depend on $c$ in dilute random graphs regime, $\omega = 2$, and is given by a polynomial in degrees of $1/c$ in the sparse random graph regime, $\omega = 3$. In particular, regarding the small-$t$ expansion of the right hand-side of (6.16) in the sparse random graphs limit and taking into account that $F_1^{(2)}(3) = 1 + c^{-1}$, we see that this approximation to the rate function is $t(1+\frac{2}{c})(1+o(1)), t \to 0$. The same difference between the rate functions is observed in the general case of $q$, with exponential term of (6.16) replaced by $\frac{1}{2}(\exp\{\frac{2t}{c^{q-1}}\} - 1)$.

Let us stress that the above reasoning relies strongly on the existence of the limit

$$\lim_{n\to\infty} \frac{1}{p_n n^2} \log \mathbf{E}_{\beta'} \left\{ e^{t g_n V_n} \right\} = f_q(t) \tag{6.18}$$

with $V_n = X_n^{(q)}$ or $Y_n^{(q)}$. This problem is difficult to solve by using the cumulant expansion (3.1). As it is pointed out many times (see for example [8]), the terms of the formal relation (3.1) with positive $g$ are used to enumerate combinatorial structures related to the matrix integrals. In the rigorous sense, the series (3.1) never converges in this case. Existence and analyticity in $g$ of the terms of asymptotic expansion (3.1) with respect to $n$ is proved for the matrix integrals just recently [3, 7] by using powerful techniques of the integrable models and the Riemann-Hilbert problem.

# 7 Summary

The Gibbs weight determined by the graph Laplacian generates a measure $\mu_n$ on the ensemble of $n$-dimensional adjacency matrices of simple non-oriented graphs. This measure is invariant with respect to the permutations of the basis vectors and determines the Erdős-Rényi ensemble of random graphs with the edge probability $p_n$.

Regarding the sum over the set of weighted adjacency matrices as the analog of the matrix integrals, we determine the discrete analog of matrix models with the quartic potential. In the general case of $q$-power potential, we distinguish two different families of discrete Erdős-Rényi matrix models. These are related with the numbers of $q$-step walks and $q$-step closed walks over the random graphs denoted by $Y_n^{(q)}$ and $X_n^{(q)}$, respectively.

The logarithm of the standard partition function of these models is determined by the cumulants of random variables $X_n^{(q)}$ and $Y_n^{(q)}$. We develop a diagram technique to study the limiting behavior of the cumulants in three major asymptotic regimes of full, dilute and sparse random graphs determined by the edge probability $p_n$ as $n \to \infty$. We prove that the limits of these cumulants, when properly normalized, exist in all of the three asymptotic regimes. As a consequence, the Central Limit Theorem is shown to be true for centered and normalized variables $X$ and $Y$. This implies CLT for the moments of the normalize spectral measure of the adjacency matrix of random Erdős-Rényi graphs.

We show that the limiting expressions of the cumulants are related with the number of non-rooted trees constructed with the help of labeled edges.

In the simplest case of the dilute random graphs regime, the exponential generating function of these numbers $H(z)$ verifies the Lagrange (or Pólya) equation. Passing to the sparse random graphs regime, we derive more general equations that determine $H(z)$ in this case.

These results imply an observation that the asymptotic regimes we consider are different not only with respect to the normalization factors of the cumulants, but also with the respect to the rate functions of large deviations formally determined as the limit $f_q(t)$ (6.18). It should be noted that we did not prove rigorously the existence of this limit because our results concern the leading terms of the cumulants only.

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