SCALING LIMIT OF LOOP ERASED RANDOM WALK — A NAIVE APPROACH

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ABSTRACT. We give an alternative proof of the existence of the scaling limit of loop-erased random walk which does not use Löwner's differential equation.

1. INTRODUCTION

Loop erased random walk is a process for creating a random simple path, which starts from a regular random walk and then removes all loops in a chronological order until a simple path is reached. In dimension 2, it is typical to stop the process on the boundary of some bounded domain $\mathcal{D}$, so the process creates a random simple path from the point of origin to $\partial\mathcal{D}$. Originally [L80] it was suggested as a model for investigating the self-avoiding random walk (i.e. a random walk conditioned not to hit itself) but it was found that these processes are cosingular. Notwithstanding, loop-erased random walk is still a useful model for a random simple path. See [D92] for connections with various physical models such as the "Q-states Potts model"[1] and polymer coalescence. Another connection to physics which is also interesting mathematically is the "Laplacian random walk," defined in [LEP86] and proved in [L87] to be identical to loop-erased random walk. The connection between loop-erased random walk and the "uniform random spanning tree" — the spanning tree of a graph chosen among all spanning trees with equal probabilities — has given thrust to the research of both. See [P91, W96].[2] The introduction to [S00] explains all these connections in a clear and concise way.

It is natural to assume that the distributions of loop-erased random walks on the graphs $\mathcal{D}\cap\delta\mathbb{Z}^2$ converge to a scaling limit as $\delta\to 0$ which would be a "loop-erased Brownian motion" though this term per se is meaningless as the process of loop erasure cannot be applied to Brownian motion: it has a dense set of loops which cannot be ordered chronologically. Like many similar processes, and in particular because regular random walk exhibits this phenomenon, one might expect the limit to be conformally invariant. As a rule of the thumb, conformal invariance can be expected for any process which is local and invariant to scaling and rotation, since a conformal map is, infinitesimally, just that, a rotation and

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[1] Loop-erased random walk is related to the case $Q=0$. It might be interesting to note that critical percolation is also a particular case, when $Q=1$.

[2] The strongest result in this direction, Wilson's algorithm [W96], is stated in lemma 1 below.

scaling.[3] This conjecture lay open for a long period, with the first important step done by Richard Kenyon [K00a, K00b] who proved the conformal invariance of certain measurables of loop-erased walk, as well as calculating explicit growth exponentials. Oded Schramm [S00] demonstrated how to describe the scaling limit of loop-erased random walk using Löwner's differential equation, assuming that the limit exists and is conformally invariant. Basically he showed that the generating function of Löwner's equation is distributed like $e^{iB(2t)}$ where B is a one dimensional Brownian motion (a good source[4] on Löwner's equation is [A73]). This result opened the road to the first proof of the conjecture [LSW02], and to additional exciting results that connect other random processes to SLE (stochastic Löwner equation) with only a different multiplicative parameter — see [S01, LSW02] for details. The aim of this paper is to give an alternative proof of the existence of loop-erased random walk.

Why give another proof of a known result, and a longer one to boot? Lawler-Schramm-Werner's proof is of the kind that "knows the answer". Very roughly, they started from the generating function of Löwner's equation for the discrete process (i.e. the loop-erased random walk, considered as a path in $\mathbb{C}$ from 0 to $\partial\mathcal{D}$), showed that its distribution converges to Brownian motion as $\delta \to 0$ and then used compactness arguments to get convergence in the stronger topology of simple paths in $\mathcal{D}$. My technique is "naive", it shows that loop-erased random walk converges without proving anything about the limit. Thus, for example, it does not really distinguish between simply connected and finitely connected domains.[5] Each approach can be extended in directions the other cannot. At the end of chapter 5 we discuss very briefly and without proofs some directions where this approach can be carried to.

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1.1. **About the proof.** Despite its length, in essence it is a simple proof, with the core argument being localization and symmetry. Let R be a random walk on $\mathbb{Z}^2$ from 0 stopped on $\partial\mathcal{D}$ for some domain $\mathcal{D}$. Let $S \subset \mathcal{D}$ be some (small) square. We write

$$\mathrm{LE}(R) = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

where γ_1 is the portion of $\mathrm{LE}(R)$ until the first time when $\mathrm{LE}(R)$ hits S. Notice that this is **not** the same as the loop-erasure of a random walk stopped on ∂S! γ_2 is the portion of $\mathrm{LE}(R)$ until the last time when $\mathrm{LE}(R)$ is inside S, and γ_3 is the reminder (the precise form of this division is in the main lemma, page 34). Tracing the process of loop-erasure in $\mathcal{D}$ one sees that γ_1 does not depend on anything that happens inside S: when one knows all entry and exit points of R from S, and all the trajectories that R does outside S, one can calculate γ_1. In particular, if we compare

[3] This might be the place to remark that loop-erased random walk is formally **not** a local process, which is a major obstacle to its analysis.

[4] Notwithstanding the fact that Ahlfors' use of Löwner's method for the proof of Biberbach's conjecture for the case $n = 3$ is a little outdated.

[5] For infinitely connected domains other factors are at work and the loop-erased random walk does not necessarily converge to a limit. See example on page 42.

random walks R_1 and R_2 on graphs G_1 and G_2, where $G_1 \setminus S = G_2 \setminus S$ and inside S we have some estimate of the sort

$$p_1(v) \simeq p_2(v) \tag{1}$$

where $p_i(v)$ is the probability of a random walk on G_i to exit S in a particular vertex v, then we should have that

$$\gamma_{1,1} \simeq \gamma_{1,2} \quad .$$

This argument and the precise meaning of "$\simeq$" are contained in lemma 17. To make this argument work for γ_3, we have to use the symmetry of loop-erased random walk (exact details in the main lemma). γ_2 describes what was coined in [S00] a "quasi-loop," and can be estimated using the methods ibid. (see lemma 18).

This concludes the main argument, and leaves us with the question: what are those mysterious graphs G_i which differ only on S and satisfy (1)? The answer here depends on the question asked. In this paper, we are trying to prove that the loop-erasure of random walks on $\delta\mathbb{Z}^2$ and $\frac{1}{2}\delta\mathbb{Z}^2$ are similar. Therefore we need the graphs G_i to be something that, on certain squares $d + [0,1]^2$, $d \in D \subset \mathbb{Z}^2$ is similar to $\frac{1}{2}\delta\mathbb{Z}^2$ and on others to $\delta\mathbb{Z}^2$. We call such graphs "hybrid." On a certain intuitive level, it seems obvious that when we construct this kind of graph the random walk on it will be similar to Brownian motion, for any defining set D (or in other words, for any dissection of $\mathbb{C}$ into squares of the two types). On a formal level, this requires delicately sawing together the transition areas (the "seams" in the terminology of this paper) and lots of technical details. This process is covered in chapter 3. It starts with the definition of a hybrid graph and the first step is showing the existence of a harmonic potential (lemma 6). Regrettably, this particular step requires some computer use, which is described in the appendix. With the harmonic potential defined, chapter 3 becomes a run-of-the-mill usage of comparisons of continuous and discrete harmonic functions, and culminates in lemma 14. (1) is a direct consequence of it, see lemma 16.

1.2. **Reading recommendations.** Chapter 2 contains various known or unsurprising facts about random walks and loop-erased random walks. Experts might want to skip or skim this part. Chapter 3, as explained above, develops the concept of a hybrid graph, a kind of interpolation between two different graphs, in particular between two grids of different step length, and shows that the random walk is not very different from the regular random walk. It is **highly** technical and can be skimmed by all. Read carefully the definition of a hybrid graph, and then the formulation of all lemmas but skip their proofs. This will not have a significant impact on your ability to understand later parts. The most interesting part is chapter 4, with the core being the main lemma, and, to a lesser extent, lemmas 16 and 17. I recommend to read it all, linearly, and take a breather after the main lemma. Starting from section 4.3, the proof gets "lighter" as there is no more need for the machinery of hybrid graphs. All notations are simpler and techniques are classical. In this part of the proof (section 4.3 and chapter 5) the only notable proof element is lemma 22. We wrap the proof up in chapter 5 which is a two-pages exercise in standard limit techniques that gives the classical formulation in terms of the weak limit. It features, though, an interesting example where loop-erased random walk does **not** converge (page 42) and the exact statement of the theorem (page 45).

2. Generalities

2.1. **Notations.** A weighted graph is a couple $G=(V,W)$ with V the set of vertices and $W : V\times V \to [0,\infty[$, $W(v,w)=W(w,v)$. Unflinchingly we shall confuse G with V, using set notations such as $v\in G$.

A path in G is a sequence $\gamma=\{\gamma_i\}$, $\gamma_i\in G$ with $W(\gamma_i,\gamma_{i+1})\neq 0$. A path is simple if $i\neq j$ implies $\gamma_i\neq\gamma_j$. The segment of a simple path γ between two points γ_i and γ_j is the subpath $\{\gamma_i,...,\gamma_j\}$ (or the reverse, if $j<i$). A subset $A\subset G$ is graph-connected if there is a path in A from every $v\in A$ to every $w\in A$.

For a finite path $\gamma=\{\gamma_i\}$ in a graph G we define its loop erasure, $\mathrm{LE}(\gamma)$, which is a simple path in G, by the consecutive removal of loops from γ. Formally,

$$\begin{aligned}\mathrm{LE}(\gamma)_1 &:= \gamma_1\\ \mathrm{LE}(\gamma)_{i+1} &:= \gamma_{j_i+1} \quad j_i:=\max\{j : \gamma_j=\mathrm{LE}(\gamma)_i\} \quad .\end{aligned}$$

It will be convenient to consider $\mathrm{LE}(\gamma)$ as a set of vertices and edges so that we can consider the reversal of $\mathrm{LE}(\gamma)$ as identical to $\mathrm{LE}(\gamma)$, and so that we can write $\mathrm{LE}(\gamma)\cup ...$

A random walk on a weighted graph G is a process R that moves at the nth step from $R(n)$ to $R(n+1)$ with the probability

$$\frac{W(R(n),R(n+1))}{\sum_v W(R(n),v)} \quad . \tag{2}$$

If $A\subset B\subset G$ and $v\in G$ then we denote by

$$q(v,A,B,G)$$

the probability of a random walk on G starting from v to hit B in A. A "hit" is only considered for $t\geq 1$ so that $v\in B$ does not imply a degenerate distribution. If $b\in B$ we shall write $q(v,b,B,G)$ as a short hand for $q(v,\{b\},B,G)$.

The Laplacian on a weighted graph G is an operator on functions $f : G\to\mathbb{R}$ (or to any linear space over $\mathbb{R}$),

$$(\Delta_G f)(v)=\sum_w W(v,w)(f(w)-f(v)) \quad .$$

Clearly, if T is a stopping time for R such that $R(0),...,R(T-1)\notin B\subset G$, and f is harmonic (i.e. $\Delta f\equiv 0$) on $G\setminus B$ then

$$\mathbb{E}\, f(R(T))=f(R(0)) \quad .$$

If $G\subset\mathbb{C}$ is a graph and $\mathcal{D}\subset\mathbb{C}$, we define

$$\begin{aligned}\partial_G\mathcal{D} &:= \{v\in\mathcal{D}\cap G : \exists w\in G,\, W(v,w)\neq 0 \ \wedge\ ]v,w[\not\subset\mathcal{D}\}\cup\\ &\quad\ \{w\in G\setminus\mathcal{D} : \exists v\in G,\, W(v,w)\neq 0\ \wedge\ ]v,w[\not\subset\mathbb{C}\setminus\mathcal{D}\}\\ \mathcal{D}^\circ &:= (G\cap\mathcal{D})\setminus\partial_G\mathcal{D}\end{aligned}$$

where $]v,w[$ is the open segment between v and w. We will hardly use the regular definitions of $\partial\mathcal{D}$ and $\mathcal{D}^\circ$ so there is little room for confusion. If $v\in\mathbb{C}$ we define a "random walk on G starting from v" as a random walk on G starting from the point of G closest to v. If more than one exist, choose the top-left point. This also applies to the notation $q(v,A,B,G)$.

When we say about a set $\mathcal{D}\subset\mathbb{C}$ that it is a polygon we mean that its boundary is a collection of linear segments of positive length, but not necessarily that it is simply connected. Punctures (i.e. holes of a single point), however, are not allowed.

For a compact metric space X, we denote by $\mathfrak{H}(X)$ the space of closed subsets of X with the Hausdorff metric,

$$d_{\mathfrak{H}}(A_0,A_1)=\max_{i=0,1}\sup_{a\in A_i}d(a,A_{1-i})$$

where as usual $d(b,A)=\inf_{a\in A}d(b,a)$. $\mathfrak{H}(X)$ is also a compact metric space. By $\mathfrak{M}(X)$ we denote the space of measures on X with the topology of weak convergence.

$\mathbb{N}$ denotes the natural integers (≥ 1). $\mathbb{Z}$ are all the integers. $\mathbb{D}$ will denote the disc $|z|<1$ and $\mathbb{T}$ is the circle $\partial\mathbb{D}$. When we write e.g. $z_0+R\mathbb{D}$ we mean the usual set addition and multiplication, so it evaluates to the set $\{z:|z-z_0|<R\}$. The only exception to this rule is that when $E\subset\mathbb{R}$ then the notation E^2 will be used as a short hand for $E+iE\subset\mathbb{C}$. In particular, $\mathbb{Z}^2$ will be considered as a subset of the complex plane $\mathbb{C}$ and also as a graph where

$$W(z,z')=\begin{cases}1 & |z-z'|=1\\ 0 & \text{otherwise}\end{cases}.$$

The notation $\mathbf{1}_A$ for a set A stands for the function which is one on A and zero outside A. The support of a function f, denoted by $\operatorname{supp} f$, is the set where $f(x)\neq 0$. The notations $\wedge$, $\vee$ and $\neg$ are used (somewhat informally) as shorts for "and", "or" and "not" respectively. The notation $P\sim Q$ means that the variables P and Q are identically distributed.

By C and c we denote constants, which could change from formula to formula (or even inside the same formula). C will usually pertain to constants "large enough" and c to constants "small enough". Occasionally we shall number them for clarity. The notation $x\approx y$ will be a shorthand for $cy\leq x\leq Cy$.

2.2. **Auxiliary results.**

Lemma 1. *(Wilson's algorithm) The uniform random spanning tree of a graph G can be constructed using the following inductive process: in the first step, the partially constructed tree will be one arbitrary vertex $v\in G$. On the nth step $(n>1)$, pick w_n not in the partially constructed tree and add to the latter a loop-erased random walk on G starting from w_n and stopped when first hitting the partially constructed tree. Continue until the tree spans all of G.*

We do not care what the "uniform random spanning tree of G" is (though it is what you would guess). Only that it does not depend on the algorithm for picking the v and the w_n's. This lemma allows to get all kinds of symmetries for loop-erased random walks, particularly that the loop-erased random walk from v to w is distributed identically to the loop-erased random walk from w to v (though that particular fact was known before).[6] The proof can be found in [W96].

Lemma 2. *Let $b_0, b_1 \in B \subset G$. Let R_i be a random walk starting at b_i, stopped at B and conditioned to hit b_{1-i}. Then*

$$\mathrm{LE}(R_0) \sim \mathrm{LE}(R_1)$$

Proof. Let R_i' be a random walk starting at b_i, stopped at B and conditioned to hit $\{b_0, b_1\}$. Clearly

$$R_i \sim R_i' \,|\, R_i' \text{ hits } b_{1-i} \quad .$$

Now let h be the solution of Dirichlet's problem on G, with the initial conditions

$$h(b_0) = h(b_1) = 1 \quad h(B \setminus \{b_0, b_1\}) = 0$$

and let G' be a weighted graph with $V' = V$ and $W'(v,w) = h(v)h(w)$. Let R_i'' be an (unconditioned) random walk on G' starting at b_i and stopped at $\{b_0, b_1\}$. It is easy to see that

$$R_i' \sim R_i''$$

so

$$R_i \sim R_i'' \,|\, R_i'' \text{ hits } b_{1-i} \quad .$$

Finally, denoting by R_i''' a random walk in G' from b_i to b_{1-i} we clearly get

$$\mathrm{LE}(R_i''') \sim \mathrm{LE}(R_i'' \,|\, R_i'' \text{ hits } b_{1-i})$$

and for $\mathrm{LE}(R_i''')$ we can use Wilson's algorithm to get

$$\mathrm{LE}(R_0''') \sim \mathrm{LE}(R_1''') \quad .$$

□

Lemma 3. *There exists a function a on $\mathbb{Z}^2$ such that*

$$\begin{aligned} \Delta a &= \delta_0 && (3)\\ a(z) &\geq a(0) \\ a(z) &= \frac{1}{2\pi}\log|z| + R(z), \quad |R(z)| \leq \frac{C_1}{|z|^2} && (4) \end{aligned}$$

A nice proof with a weaker estimate can be found in [S76, section 12.3]. The value of $a(0)$ is calculated in [S76, chapter 15] (note that Spitzer's a is $4(a + a(0))$ with respect to mine) and is $-\frac{\log 8 + 2\gamma}{4\pi}$. A proof that is missing only the actual calculation of C_1 can be found in [S49] (warning: 60 pages in German). Finally, see [KS] for a high-order expansion of a and an exact calculation of $C_1 = 0.017205...$ This function is called the (two dimensional) discrete harmonic potential.

Lemma 4. *Let $s := N - 1 + i(N - d)$ ($1 \leq d \leq N$) and let $I \subset \partial[-1,1]^2$ be a connected set with $\operatorname{diam} I > c_1$ and $d(I, (s+1)/N) > c_1$. Then*

$$q(s, NI \cap \mathbb{Z}^2, \partial[-N,N]^2, \mathbb{Z}^2) \approx dN^{-2}$$

provided that N is sufficiently large. The constants implicit in $\approx$ and the minimal N depend on c_1.

[6] See lemma 19 for a different use of Wilson's algorithm.

Sublemma 4.1. *Denote* $p_I(d,N) := q(s, NI\cap\mathbb{Z}^2, \partial[-N,N]^2, \mathbb{Z}^2)$*. Then*

$$p_I(d,N) \approx dN^{-2} + O\left(N^{-4}\log N\sum_{m=1}^{2N} mp'(m)\right)$$

where $p'(m) := \max_{d\le m} p_H(d,m)$ *and*

$$H := [-1-i, 1-i] \cup [-1-i, -1+i]$$

and the constants implicit in the $\approx$ *and in the* $O(\cdot)$ *above depend on* c_1*.*

Subproof. Denote $S = \,]-N,N[^2$ and denote the value inside the $O(\cdot)$ by E. Choose $J\subset\partial[-1,1]^2$ to be a connected set satisfying $I\subset J$ and

$$d(J,(s+1)/N) > \tfrac{1}{3}d(I,(s+1)/N) \qquad d(\partial[-1,1]^2\setminus J, I) > \tfrac{1}{3}d(I,(s+1)/N).$$

Our aim is to prove

$$p_I(d,N) \le CdN^{-2} + CE, \quad p_J(d,N) \ge cdN^{-2} - CE$$

which is enough, since we can then exchange the roles of I and J to get a lower estimate for p_I.

Let φ be the Riemann mapping of $[-1,1]^2$ on $\mathbb{D}$, $\varphi(0) = 0$, $\varphi'(0) > 0$. The reflection principle through the boundary (twice around the corners) for φ gives us that φ is analytic near every point of the boundary and in particular $\varphi_0'''' \le C$. From this and from the fact that φ preserves the angle near non-corners and doubles the angle near the corners we get that $\varphi'(b) = 0$ only if b is a corner, and at the corners $\varphi'(b) = 0$ and $\varphi''(b) \ne 0$ — these can be summed up as

$$\varphi'(b) \approx d(b,K) \tag{5}$$

where K is the set of corners, $\{-1,1\}+\{-i,i\}$.

Let f be a real 5 times differentiable function on $\mathbb{T}$ with $f(z) = 0$ for $\arg z \in \varphi\left(\partial[-1,1]^2\setminus J\right)$, $f(z) = 1$ for $\arg z\in\varphi(I)$ and $0\le f\le 1$ and with $f^{(k)}\le C$, $k = 0,\dots,5$ where C depends only on c_1. Extend f to a harmonic function on $\mathbb{D}$ (we will call the extended function f as well), let $\tilde f$ be the complex conjugate of f with $\tilde f(0) = 0$ and let $F = f + i\tilde f$. It is easy to see that $F^{(k)}\le C$ for $k = 0,1,2,3,4$. We define a function $\bar g$ on $G := \mathbb{Z}^2\cap\bar S$ by

$$\bar g(v) := f(\varphi(v/N)) \quad .$$

Expanding $F(\varphi(v/N))$ to a power series around v/N and using the fact that

$$(z+1)^k + (z-1)^k + (z+i)^k + (z-i)^k - 4z^k = 0 \quad k = 0,1,2,3 \tag{6}$$

we get (here we used the boundedness of $\varphi^{(k)}$ and $F^{(k)}$)

$$|\Delta_G\bar g(v)| \le CN^{-4} \quad . \tag{7}$$

We “fix” $\bar g$ on S° as follows:

$$g(z) := \bar g(z) - \sum_{x\in S^\circ}\Delta\bar g(x)\cdot l_x(z)$$

where $l_x(z) := a(z-x) - r_x(z)$, a is the harmonic potential from lemma 3, and r_x is the solution of Dirichlet's problem on G with the conditions

$$r_x(z) = a(z-x) \quad z\in\partial S.$$

It is clear from these that $g(z)$ is harmonic on S° and on ∂S we have

$$g(z) = \bar{g}(z)$$

Next we wish to estimate $g(s) - \bar{g}(s)$. Let R^s be a random walk starting from s and stopped when hitting $\partial S \cup \{x\}$ with some $x \notin \partial S$ (let t be the stopping time). (4) gives that $a(z-x) \le C \log N$ and the boundedness principle gives the same for $r_x(z)$. Because $l_x(z)$ is harmonic on $S^\circ \setminus \{x\}$ we get

$$l_x(s) = \mathbb{E}(l_x(R^s(t))) \le C \log N \cdot \mathbb{P}(R^s(t) = x) \quad . \tag{8}$$

Let $m := \max |\operatorname{Re}(s-x)|, |\operatorname{Im}(s-x)|$. If $m \ge d$ then clearly $\mathbb{P}(R^s(t) = x) \le p_H(d,m)$. If $m < d$ then a similar argument gives $\mathbb{P}(R^s(t) = x) \le 2p_H(m,m)$. So in both cases we have $l_x(s) \le Cp'(m) \log N$ and hence

$$|g(s) - \bar{g}(s)| \le CN^{-4} \log N \sum_{m=1}^{2N} mp'(m) = CE \tag{9}$$

$\bar{g}(s)$ is easy to estimate (using (5)) because we have

$$d(\varphi(s/N), \mathbb{T}) \approx N^{-1}\varphi'((s+1)/N) \approx dN^{-2}$$

and an estimate of f using the Poisson kernel and the fact that $d(\varphi((s+1)/N), \varphi(J)) > c$, gives

$$|\bar{g}(s)| = |f(\varphi(s/N))| \approx d(\varphi(s/N), \mathbb{T}) \approx dN^{-2} \quad . \tag{10}$$

Translating the estimates on g to an estimate on the probability $p(d,N)$ is done by again examining the random walk R^s starting from s but this time stopped on ∂S (let t be the stopping time). Now g is harmonic on S°, g is one on NI and on $\partial S \setminus NI$ we have, $0 \le g \le 1$. All these give

$$\begin{aligned} g(s) &= \mathbb{E}(g(R^s(t))) \\ &= p_I(d,N)\,\mathbb{E}(g(R^s(t)) \,|\, R^s(t) \in I) + (1 - p_I(d,N))\,\mathbb{E}(g(R^s(t)) \,|\, R^s(t) \notin I) \\ &\ge p_I(d,N) \end{aligned}$$

and similarly $g(s) \le p_J(d,N)$. With (9) and (10) the sublemma is proved. □

Proof of lemma 4. First we estimate p_H from above: we use the sublemma, plugging the estimate $p'(N) \le 1$ in the right hand side to get

$$p_H(d,N) \le CdN^{-2} + CN^{-2} \log N \quad , \tag{11}$$

and in particular $p'(N) \le CN^{-1}$. We now use the sublemma again, plugging this estimate into the right hand side, and we are done. □

3. The hybrid graph

The proof of the theorem (see page 33) requires some kind of interpolation between the grids $\frac{1}{N}\mathbb{Z}^2$ and $\frac{1}{N'}\mathbb{Z}^2$. Before describing the variant I am using, I wish to make an unusually vague comment. There seems to be some tradeoff between symmetry and analyticity, in the sense that there exist models for which it is **much** easier to prove that the hybrid process is a good approximation of a Brownian motion, but the symmetries necessary are not obvious. Being the analyst that I am, I chose a model for which the proof of lemma 6 below is long and technical, but all the symmetries are ready-made for me. Someone more inclined toward combinatorics might have produced a nicer proof.

Definition. For a set $D \subset \mathbb{Z}^2$, and an integer N, we define the **hybrid graph** $G(D,N)$, which is a weighted graph, as follows: The set of vertices V is a union of the following sets:

1. The set
$$V_0 := \left\{ Nz \in \mathbb{Z}^2 \,:\, \lfloor z \rfloor \notin D \right\}$$
where $\lfloor x + iy \rfloor := \lfloor x \rfloor + i \lfloor y \rfloor$ i.e. the vector composed of the two integer values of x and y.
2. The set V_1 which is defined by
$$V_1 := \left\{ 2Nz \in \mathbb{Z}^2 \,:\, \lfloor z \rfloor \in D,\, N^{-1} \le d(z, V_0) \right\} \quad .$$

As for the edges, if v_1 and $v_2 \in V_n$ then we put an edge connecting them $\Leftrightarrow$ $|v_1 - v_2| = 2^{-n}/N$ and make its weight 1. If $v_1 \in V_0$, $v_2 \in V_1$ and $|v_1 - v_2| = N^{-1}$ then we connect them by an edge with weight $\frac{1}{2}$ while if $|v_1 - v_2| = N^{-1}\sqrt{\frac{5}{4}}$ then we connect them by an edge with weight $\frac{1}{4}$. We denote this weight by W. See figure 1, left, on the following page.

The vertices where V_0 touches V_1 are called the **seams** and are denoted by $\bar{G}$:
$$\bar{G} := \bigcup_{n=0,1} \{ v \in V_n \,:\, \exists w \in G \setminus V_n \ \wedge \ W(v,w) \neq 0 \}$$
We note that $\#\bar{G} \le CN\#D$ where $\#X$ is the number of elements of a set X. The seams are relevant because we want to use an argument similar to (6)-(7) on our hybrid graph. Thus if f is an analytic function we can write
$$\Delta_G f(v) \le C \frac{\max\{ f^{(k)} z \,:\, |z - v| < N^{-1} \}}{N^k} \tag{12}$$
where $k = 4$ outside $\bar{G}$. On $\bar{G}$, outside the "seam-intersections" we still have (6) for $i = 0$ and 1, so we can take $k = 2$ (this is easy to verify). Thus we define the seam-intersections $\bar{\bar{G}}$ using
$$v \in \bar{\bar{G}} \Leftrightarrow \sum_w W(v,w)(v - w) \neq 0 \quad .$$
On $\bar{\bar{G}}$ we can only take $k = 1$, but luckily there are even less of these: $\#\bar{\bar{G}} \le 8\#D$. An example of $\bar{G}$ and $\bar{\bar{G}}$ illustrated is in figure 1, right.

Eventually, (see page 33) we shall examine random walks on a hybrid graph with a random set D, so this model is (locally) a variation on random walk in a random environment. This might lead the reader to assume that he is in for logarithmic drift and other cool effects. This is not so — the model G was constructed to avoid these effects, and in particular, for all A the drift is negligible as lemma 6 and later 7 will demonstrate.

Lemma 5. *The hybrid graph G is planar.*

This is easy to verify.

Lemma 6. *There exists a C_2 and a C_3 such that for any $D \subset [-M, M]^2$, for all $N > C_2 M^{C_3}$ the graph $G = G(D,N)$ satisfies that there exists functions $a_v : G \to \mathbb{R}$ with*
$$\Delta_G a_v(w) \quad = \quad \delta_v(w) \tag{13}$$

$$a_v(w) = K_v \log(N|v-w|) + O\left(N|v-w|\right)^{-c_2} \quad \forall v \neq w \tag{14}$$
$$a_v(v) = O(1) \tag{15}$$
$$K_v \approx 1 \quad . \tag{16}$$

c_2 *depends on* C_3*.*

The dependency above between N_0 (the minimal allowed N) and $\operatorname{diam} D$ is not the best possible. The proof can be refined, to work for certain infinite D's, though not to general ones — a checkerboard, i.e. $D = (2\mathbb{Z} + 2i\mathbb{Z}) \cup (2\mathbb{Z} + 2i\mathbb{Z} + 1 + i)$ seems to be a particularly bad example.

Sublemma 6.1. *Let* G *be a metric graph and let* b_v *be functions satisfying*

$$\Delta_G b_v = \delta_v + R_v$$

with

$$\sum_w |R_v(w)| \leq \beta, \quad \beta < 1$$

and $\bigcup_v \operatorname{supp} R_v$ *finite. Then there exists an* a_v *satisfying (13). Furthermore,*

1. *There exist coefficients* $\tau_{v,w}$ *such that*
$$a_v = \sum_w \tau_{v,w} b_w \tag{17}$$
with
$$\sum_w |\tau_{v,w}| \leq \frac{1}{1-\beta} \tag{18}$$
2. *If* $\beta < \frac{1}{2}$ *then*
$$\sum_w \tau_{v,w} > \frac{1-2\beta}{1-\beta} \tag{19}$$

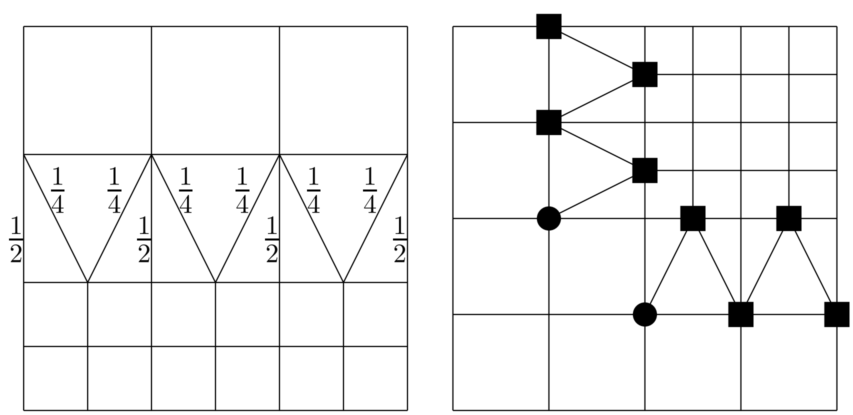

FIGURE 1. On the left, a hybrid graph near a seam with the weights marked (all unmarked edges have weight 1). On the right, a hybrid graph near a seam intersection. The set D is $\{0\}$ and 0 is the middle of the image. Vertices from $\bar{G}$ are marked with a square and vertices from $\bar{\bar{G}}$ are marked with a circle.

3. *If*

$$\sum_{w\,:\,|v-w|>r} |R_v(w)| \le Ar^{-\mu}$$

then

$$\sum_{w\,:\,|v-w|>r} |\tau_{v,w}| \le C(\beta)Ar^{-\mu} \quad . \tag{20}$$

Of course, the graph G is weighted, but these weights appear only in the definition of the Laplacian Δ_G. The proof below is a simple use of convolution on L^1 spaces.

Subproof. Define $a_v^1 = b_v$ and inductively

$$a_v^n = a_v^{n-1} - \sum_w (\Delta a_v^{n-1} - \delta_v)(w) a_w^1 \quad . \tag{21}$$

Notice that the fact that $\bigcup_v \operatorname{supp} R_v$ is finite gives that the sum is finite for all n. This allows to write

$$\begin{aligned} \sum_w |(\Delta a_v^n - \delta_v)(w)| &= \sum_w \left| \sum_x (\Delta a_v^{n-1} - \delta_v)(x) R_x(w) \right| \\ &\le \sum_x |(\Delta a_v^{n-1} - \delta_v)(x)| \sum_w |R_x(w)| \\ &\le \beta \sum_x |(\Delta a_v^{n-1} - \delta_v)(x)| \le \beta^n \end{aligned} \tag{22}$$

which gives that a_v^n converge and that $a_v := \lim_{n\to\infty} a_v^n$ satisfies $\Delta a_v(w) = \delta_v$. (17), (18) and (19) are also clear because defining

$$\tau_{v,w} := \delta_v(w) - \sum_{n=1}^{\infty} \Delta a_v^n(w) - \delta_v(w) \tag{23}$$

works. We are left therefore with (20). We wish to get for every n an estimate of the type

$$\sum_{w\,:\,|v-w|>r} |\Delta a_v^n(w)| \le A_n r^{-\mu} \quad .$$

Let d satisfy $\beta < d^\mu < 1$. In the second line of (22) we divide the sum over x as follows:

$$\sum_{w\,:\,|w-v|>r} |\Delta a_v^n(w)| \le \beta \left(\sum_{|x-v|\le dr} + \sum_{|x-v|>dr} \right)$$

The first sum can be estimated by

$$\sum_{|x-v|\le dr} |(\Delta a_v^{n-1} - \delta_v)(x)| \sum_{|w-v|>r} |R_x(w)| \le \beta^{n-1} A_1 (r-dr)^{-\mu}$$

while the second sum can be estimated by

$$\sum_{|x-v|>dr} |(\Delta a_v^{n-1}(x)| \sum_{|w-v|>r} |R_x(w)| \le A_{n-1}(dr)^{-\mu}\beta$$

so we can write

$$A_n \le \frac{\beta^{n-1}A_1}{(1-d)^\mu} + A_{n-1}\frac{\beta}{d^\mu} \quad . \tag{24}$$

The choice of d is now clear as it ensures that A_n converges exponentially to zero, and in particular $\sum A_n < \infty$. This finishes (20) and the sublemma. $\square$

Definition. If $s = s_1 + is_2$ and $s_j \in \frac{1}{2}\mathbb{Z}$ we define a function $A(s, \cdot)$ on $\mathbb{Z}^2$ as follows:

1. If s_1 and s_2 are integers, we take $A(s, v) = a(s - v)$ with a the harmonic potential on $\mathbb{Z}^2$ defined in lemma 3 above;
2. If $s_1 \notin \mathbb{Z}$ and $s_2 \in \mathbb{Z}$, we define
$$\begin{aligned} t^{\pm} &:= s \pm \tfrac{1}{2} \\ A(s) &:= \frac{1}{2}(A(t^-) + A(t^+)) \quad ; \end{aligned}$$
3. If $s_1 \in \mathbb{Z}$ and $s_2 \notin \mathbb{Z}$ we define $A(s)$ symmetrically;
4. If both $s_i \notin \mathbb{Z}$ we define
$$\begin{aligned} t^{\pm,\pm} &:= s \pm \tfrac{1}{2} \pm \tfrac{1}{2} i \\ A(s) &:= \frac{1}{4}(A(t^{-,-}) + A(t^{-,+}) + A(t^{+,-}) + A(t^{+,+})) \quad . \end{aligned}$$

Sublemma 6.2. *For all s, $\Delta A(s, \cdot)$ is zero except possibly at the four integer points nearest to s, and has the estimate*
$$A(s, v) = \frac{1}{2\pi} \log |s - v| + R_s(v) \tag{25}$$
$$|R_s(v)| \leq \frac{C}{|s - v|^2} \tag{26}$$

The proof is a simple verification of the 4 cases above and we shall omit it.

For the following two sublemmas it will be convenient to use the (somewhat non-standard) notation $\mathbb{Z}^+ = \{0, 1, ...\}$, $\mathbb{Z}^- = \mathbb{Z} \setminus \mathbb{Z}^+$.

Sublemma 6.3. *For $D = \mathbb{Z}^- + i\mathbb{Z}$ we have a function*
$$b_v(w) : G(D, N) \to \mathbb{R}$$
such that for every rectangle S,
$$\sum_{w \in S} |\Delta b_v(w) - \delta_v(w)| \leq \frac{C}{N d(v, S \cap \bar{G})} \tag{27}$$

Subproof. Define
$$b_v(w) = \begin{cases} A(Nv, Nw) - \frac{1}{2\pi} \log N & \lfloor w \rfloor \notin D \\ A(2Nv, 2Nw) - \frac{1}{2\pi} \log 2N & \lfloor w \rfloor \in D \end{cases} \tag{28}$$
which makes it clear that $\Delta b_v - \delta_v$ is different from zero only on $\bar{G}$. Since in our case $\bar{\bar{G}} = \emptyset$ we can use (12) with $k = 2$, to get for any analytic function f
$$|(\Delta_G f)(v)| \leq C N^{-2} \max_{|z - v| \leq N^{-1}} |f''(z)| \tag{29}$$
and since $\log |z|$ is the real part of such a function (and using (25)) we get
$$|\Delta b_v(w)| \leq \frac{C N^{-2}}{|v - w|^2} \quad .$$
This obviously gives (27). $\square$

Sublemma 6.4. *Let D be a union of $0-4$ of the quarter planes $\mathbb{Z}^\pm + i\mathbb{Z}^\pm$. Then we have a function $b_v(w) : G(D,N) \to \mathbb{R}$ with*

$$\sum_w |\Delta b_v(w) - \delta_v(w)| \le 0.4 \quad . \tag{30}$$

Further, if S is any rectangle then (27) also holds.

Subproof. We define b by (28). (27) follows easily from sublemma 6.3 and estimating the sum on $\bar{\bar{G}}$ by (12) with $k=1$ and by $\#\bar{\bar{G}} \le 2$. (30) was done numerically and is summed up in appendix A (page 46). □

Proof of lemma 6. The lemma will follow from sublemma 6.1 with the function $b_v(w)$ again defined by (28). Defining $r(w) = |\Delta b_v(w) - \delta_v|$ we need only estimate $\sum r(w)$, which is non-zero only on $\bar{G}$. Let now $v \in G$. Let q be the integer point closest to v (if more than one exists, choose any), and let $S(v) = q + [-0.9, 0.9]^2$ (we may assume $N > 10$). (27) outside S gives

$$\sum_{w \in \bar{G} \setminus S(v)} r(w) \le CM^2/N \quad . \tag{31}$$

and with (30),

$$\sum |(\Delta b_v - \delta_v)(w)| \le 0.45 + CM^2/N$$

so for $N > CM^2$ (i.e. $C_3 = 2$) we can use sublemma 6.1 and get (13). Further, taking $C_3 > 2$ we get from (31) and (27) that

$$s < 1 \quad \Rightarrow \quad \sum_{|w-v|>s} r(w) \le \frac{C}{Ns} + CN^{2C_3^{-1}-1} \le C\,(Ns)^{-c}$$

$$s \ge 1 \quad \Rightarrow \quad \sum_{|w-v|>s} r(w) \le \frac{CM^2}{Ns} \le C(Ns)^{-c}$$

from which (20) gives us the same estimate for the coefficients τ in (17). To use that, define

$$K_v := \sum_w \tau_{v,w} \quad .$$

First note that the requirement (16) follows from (18) and (19). Next, (17) gives

$$a_v(x) - \frac{1}{2\pi}\log|v-x| \sum_w \tau_{v,w} = \sum_w \tau_{v,w}\left(b_w(x) - \frac{1}{2\pi}\log|v-x|\right) = \Sigma_1 + \Sigma_2$$

where Σ_1 denotes the sum on w satisfying $N|v-w| \le (N|v-x|)^{1/2}$ and Σ_2 denotes the reminder. For Σ_1, (remember (25))

$$\begin{aligned} b_w(x) &= \frac{1}{2\pi}\log|w-x| + O\,(N|w-x|)^{-2} \\ &= \frac{1}{2\pi}\log|v-x| + O\left((N|v-x|)^{-1/2} + (N|v-x|)^{-2}\right) \end{aligned}$$

and since $\sum_w \tau_{v,w} \le C$ we get

$$\Sigma_1 = O\,(N|v-x|)^{-1/2} \tag{32}$$

To estimate Σ_2 we write

$$\Sigma_2 = \sum_{k=1}^{\infty} \Sigma_{2,k}$$

where $\Sigma_{2,k}$ is the sum on w satisfying $(N|v-x|)^{k/2} < N|v-w| \leq (N|v-x|)^{(k+1)/2}$. In this case we can estimate

$$\left| b_w(x) - \frac{1}{2\pi} \log|v-x| \right| \leq Ck \log N|v-x|$$

and with $\tau_{v,w} \leq C(N|v-w|)^{-c}$ we get

$$\Sigma_{2,k} \leq C \left((N|v-x|)^{k/2} \right)^{-c} Ck \log N|v-x|$$

which we sum over all k and get

$$\Sigma_2 \leq C\,(N|v-x|)^{-c} \quad . \tag{33}$$

(32) and (33) give (14). (15) is an immediate consequence of (13), (16) and (14). This finishes lemma 6. □

Definition. A hybrid graph for which $D \subset [-M, M]^2$ and $N > C_2 M^{C_3}$ is called **admissible**. We explicitly reiterate the requirement $C_3 > 2$ (which was also used in the proof of lemma 6).

3.1. **Global estimates.** In this section will shall prove some simple estimates of hitting probabilities of random walks on admissible hybrid graphs where the probability involved is (approximately) independent of N. These are much easier than, for example, estimates for the hitting probability of a single point, as in lemma 14 further on.

Lemma 7. *Let f be a harmonic function on a domain $E \subset \mathbb{C}$, $1 < \operatorname{diam} E < \infty$. Let $G = G(D, N)$, be an admissible hybrid graph. Then there exists a function f' on $G \cap E$, G-harmonic on E° with*

$$|f - f'| \leq CK(f)(\operatorname{diam} E)^2 N^{-1} \log N$$

with $K(f)$ the maximum on E of all partial derivatives of f up to and including order 4.

Here, and in other lemmas formulated similarly, we in effect fix the multiplicative constant in the requirement on N before everything else, i.e. the lemma should read "There exists some C such that for all f ...".

Proof. By locally adding to f the complex conjugate $\tilde{f}$ we can use (12) to get

$$|(\Delta_G f)(v)| \leq CK(f) N^{-k(v)}$$

where

$$k(v) := \begin{cases} 4 & v \notin \bar{G} \\ 2 & v \in \bar{G} \setminus \bar{\bar{G}} \\ 1 & v \in \bar{\bar{G}} \end{cases} \quad . \tag{34}$$

On the other hand,

$$\#\{v \in [-M, M]^2 \,:\, k(v) = k\} \leq \begin{cases} CM^2 & k = 1 \\ CNM^2 & k = 2 \\ CN^2M^2 & k = 4 \end{cases} \tag{35}$$

(see page 9 for these size estimates). This gives

$$\sum_{v\in E^\circ} |(\Delta_G f)(v)| \le CK(f)(\operatorname{diam} E)^2 N^{-1} \quad .$$

Defining

$$f' = f - \sum_{v\in E^\circ} (\Delta_G f)(v)\cdot a_v$$

we get the required result with (14). □

Remark. We shall typically use lemma 7 to show that if we have a random walk in a good domain $\mathcal{D}$ (smooth boundary) starting from a point v not too near the boundaries, then the probability to hit a sizable portion I of the boundary is $> c$. This is done, as in lemma 4, by taking the solution f of the (continuous) Dirichlet problem on $\mathcal{D}$ with $f = 1$ on I and 0 on $\mathcal{D}\setminus I$ (or a smooth approximation of that), approximating f with a G-harmonic f' and using $f' = \mathbb{E}f'(\text{hit point})$. The following lemma is an example.

Lemma 8. *Let $z\in\mathbb{C}$ and $r>1$ some number. Let R be a random walk on an admissible hybrid graph $G = G(D,N)$, starting from v where $\frac{3}{4}r < |v-z| < \frac{4}{3}r$. Let t be the stopping time when $|R(t)-z| < \frac{1}{2}r$ or $|R(t)-z| > 2r$. Then*

$$\mathbb{P}\{R|_{[0,t]} \text{ contains a loop around } z\} > c$$

Proof. Use the previous lemma a number of times (4 should do) to force R to turn 2π around z and then cross itself. If $N > Cr^{C_3}$ we can use lemma 7 directly. Otherwise we have $r > CM$ and then on the annulus $A := (z+2r\mathbb{D})\setminus(z+\frac{1}{2}r\mathbb{D})$ we have

$$G\cap A = \frac{1}{N}\mathbb{Z}^2\cap A$$

and the question is equivalent to taking $D' = \emptyset$, $r' = 1$ and $N' = Nr$, for which we can again use lemma 7. □

Lemma 9. *Let $G = G(D,N)$ be an admissible hybrid graph, let $K\subset G$ be a connected group of vertices, let $v\in G$, $d(v,K) > 1$, and let R be a random walk starting from v and stopped on $\partial_G(v+r\mathbb{D})$, $r > \max(1,\operatorname{diam} K)$. Then*

$$\mathbb{P}\{R\cap K = \emptyset\} \le C\left(\frac{\operatorname{diam} K}{d(v,K)}\right)^{-c_3} \quad .$$

Proof. Let $r_n = d(v,K)\cdot 2^n$, let T_k be the stopping times and n_k the numbers defined inductively by

$$T_k = \min\{t : \exists n_k \ne n_{k-1} \ \wedge \ R(t)\in\partial(v+r_{n_k}\mathbb{D})\} \quad ;$$

and let E_k be the events that R does a loop around v between T_{k-1} and T_k. The process does not stop before we have at least $L \ge c\log(\operatorname{diam} K/d(v,K))$ E_k's for which $n_k\ne 0$ and the previous lemma gives a lower bound for the probability of $E_k \,|\, \neg E_1,\dots,\neg E_{k-1}$. The fact that the graph is planar (lemma 5) means that the event E_k implies that R will necessarily intersect K between r_{n_k-1} and r_{n_k+1}. Therefore

$$\mathbb{P}\{R\cap K=\emptyset\} \le \prod_{k=1}^{L}\mathbb{P}\{\neg E_k \,|\, \neg E_1,\dots,\neg E_{k-1}\} \le c^L \quad .$$ □

Remark. For Brownian motion the constant c_3 is $\frac{1}{2}$ (this is not too difficult to see — for example, one can use Löwner's differential equation to prove that the minimal probability happens when K is a straight half line and then calculate the probability explicitly). $c_3 = \frac{1}{2}$ also for a simple random walks — see e.g. [K87] where an equivalent result is proved. I have no reason to assume this is not true in our case too, but we shall not need it. The simple proof above is taken from [S00, lemma 2.1].

3.2. **Local estimates.** The aim of this section is to prove lemma 14 (see also the simplified representation (63)) which describes the hitting probability of a point using the geometry of the domain combined with the local structure of the graph.

Lemma 10. *Let $G = G(D,N)$ be an admissible hybrid graph, let $r > 2/N$, let $v \in G$, let $E = \partial_G(v + r\mathbb{D}) \cup \{v\}$, and let R be a random walk starting from some $w \in G$,$c_4 r < |v - w| < \frac{1}{2}r$ and stopped on E. then the probability p that R hits E at v is*

$$\approx \frac{1}{\log rN} \quad .$$

The constants implicit in the $\approx$ depend on c_4.

Proof. Let t be the stopping time. Let $a_v(w)$ be the harmonic potential from lemma 6 with respect to the point v. Then

$$a_v(w) = \mathbb{E}\, a_v(R(t)) = (1-p)\,\mathbb{E}(a_v(R(t))\,|\,R(t) \neq v) + p a_v(v)$$

and plugging in (14) we get

$$p = \frac{K_v \log \frac{|v-w|}{r} + O(Nr)^{-c}}{a_v(v) - K_v \log rN + O(Nr)^{-c}}$$

and since we can assume $r > \frac{C}{N}$ (the possibility to hit v is always positive) and using (15) we get $p \approx \log^{-1} rN$. □

The following definition binds together a number of conditions that are not really essential but make calculations and proofs easier, hence the name.

Definition. Let $G = G(D,N)$ be an admissible hybrid graph. We say about a rectangle $S = [r,s] + i[t,u]$ that it is **easy** in G if $r,s,t,u \in \frac{1}{N}\mathbb{Z}$ and $\operatorname{str} S \cdot \operatorname{diam} S > \frac{1}{N}$ where $\operatorname{str} S$ is defined as the maximal number satisfying

$$\operatorname{str} S \leq \frac{r-s}{t-u} \leq \frac{1}{\operatorname{str} S} \tag{36}$$

and

$$d(K(S), \bar{G} \cap S^\circ) \quad \geq \quad \operatorname{str} S \cdot \operatorname{diam} S \tag{37}$$

$$d(\partial S, \bar{\bar{G}} \cap S^\circ) \quad \geq \quad \operatorname{str} S \cdot \operatorname{diam} S \tag{38}$$

and where $K(S) = \{r,s\} + i\{t,u\}$ is the set of corners of S.

We shall only be interested in rectangles for which $\operatorname{str} S$ is relatively large. Think about $\operatorname{str} S \geq 0.1$ if you want to get a good notion of what this definition is all about. Further, when we say "let S be an easy rectangle..."

we always mean in addition "with $\operatorname{str} S$ bounded below by a universal constant".

We note that it follows from (37) and (38) that a seam can only intersect the boundary of an easy rectangle perpendicularly (see figure 2 on page 24). As this feature seem to follow from local properties of G in a manner that looks a little random, we note it here. While it simplifies notations here and there, this is not a significant feature of this definition.

Lemma 11. *Let $G = G(D, N)$ be admissible and let S be an easy rectangle in G. Then for every $u \in S$, $d(u, \partial S) > c_5 \operatorname{diam} S$ and $b \in \partial S$ we have*

$$q(u, b, \partial S, G) \approx \frac{d(b, K(S))}{N \operatorname{diam}^2 S} \tag{39}$$

$$q(b, u, \partial S \cup \{u\}, G) \approx \frac{d(b, K(S))}{N \operatorname{diam}^2 S \log(N \operatorname{diam} S)} \tag{40}$$

provided that $N > C + C(\operatorname{diam} S)^{C_3}$. The constants implicit in the $\approx$ in (39) and (40) above depend on c_5 and on $\operatorname{str} S$.

Proof. Denote $m = N \operatorname{diam} S$ and $d = Nd(b, K(S))$ so that the right hand sides of (39) and (40) become d/m^2 and $d/(m^2 \log m)$ respectively. Since (37) implies that the hitting probabilities of the corners are always 0, we may assume that m is sufficiently large (the minimal m will depend on c_5 and $\operatorname{str} S$). We start with (40). Denote $p(b, u) := q(b, u, \partial S \cup \{u\}, G)$. We divide the b's into two cases:

case 1: $d < c_6 m$ (we shall fix c_6 later). Assume for simplicity that b is closest to the lower left corner, k — the other 3 corners are identical. In this case we define $\epsilon := c_6 \operatorname{diam} S$ and

$$\begin{aligned} Y_1 &:= k + [0, 2\epsilon]^2 \\ X_1 &:= \partial_G \left((k + 2\epsilon + i[\epsilon, 2\epsilon]) \cup (k + [\epsilon, 2\epsilon] + 2i\epsilon) \right) \end{aligned}$$

For $c_6 < \frac{1}{2} \operatorname{str} S$ we have that $G \cap Y_1$ is simply a regular grid and we can use lemma 4 to get

$$q(b, X_1, \partial Y_1, G) \geq C d m^{-2} \quad .$$

Next we define

$$X_2 := \partial_G \left(u + \epsilon \mathbb{D} \right) \quad . \tag{41}$$

For $c_6 < \frac{1}{4} c_5$ we get $d(X_1, X_2) > c \operatorname{diam} S$ and then we can use lemma 7 to get

$$q(x_1, X_2, \partial S \cup X_2, G) \approx 1 \quad \forall x_1 \in X_1.$$

Finally on X_2 we use lemma 10 to get

$$q(x_2, u, \partial S \cup \{u\}, G) \approx \frac{1}{\log m} \quad \forall x_2 \in X_2$$

— we use here

$$x_2 \in u + 4\epsilon\mathbb{D} \subset S \subset u + C(\operatorname{diam} S)\mathbb{D} \quad .$$

This gives $p(b, u) \geq c d m^{-2} \log^{-1} m$. The estimate of $p(b, u) \leq C d m^{-2} \log^{-1} m$ is identical, but uses a larger X_1, namely

$$X_1 := \partial_G Y_1 \setminus \partial_G S \quad .$$

Notice that we are now able to fix $c_6 = \min\left(\frac{1}{3} \operatorname{str} S, \frac{1}{5} c_5\right)$, for example.

case 2: $d \geq c_6 m$. This is only slightly more complicated. Again assume for simplicity that b is in the lower side of S. We start with

$$\begin{aligned} Y_1 &:= b + I + i\left[0, \frac{\operatorname{diam} S}{\log^2 m}\right] \\ X_1 &:= \partial_G\left(b + I + i\frac{\operatorname{diam} S}{\log^2 m}\right) \\ I &:= \left[-C_4\frac{\operatorname{diam} S}{\log m}, C_4\frac{\operatorname{diam} S}{\log m}\right] \end{aligned}$$

where C_4 will be fixed later. To estimate $q(b, X_1, \partial Y_1 \cup \partial S, G)$ examine the function $\operatorname{Im} v$. For m sufficiently large the condition (38) implies $\bar{\bar{G}} \cap Y_1 = \emptyset$, and thus $\operatorname{Im} v$ is G-harmonic on Y_1. Let $b' \in S^\circ$ be a neighbor of b, so $\operatorname{Im} b' - \operatorname{Im} b \approx N^{-1}$, R a random walk starting from b' and stopped on $\partial_G Y_1$ and let t be the stopping time. Then

$$\operatorname{Im} b' = \mathbb{E} \operatorname{Im} R(t)$$

so we get

$$q(b', X_1, \partial_G Y_1, G) \approx m^{-1} \log^2 m$$

provided we show that the probability to exit Y_1 on the “sides” (outside $X_1 \cup \partial S$) is small. But this probability is clearly (e.g. by the technique of lemma 7) exponential in the ratio of the length and width of Y_1 so by picking C_4 sufficiently large we can ignore it. Summing over all neighbors b' — usually there is only one but if $b \in \bar{G}$ there could be two[7] — we get

$$q(b, X_1, \partial_G Y_1 \cup \partial_G S, G) \approx m^{-1} \log^2 m \quad .$$

Next define

$$\begin{aligned} Y_2 &:= b + [-\epsilon, \epsilon] + i[0, 2\epsilon] \\ X_2 &:= \partial_G(b + [-\epsilon, \epsilon] + 2i\epsilon) \end{aligned}$$

where again $\epsilon := c_7 \operatorname{diam} S$ where c_7 is to be defined later, and use lemma 7 to show that, for m sufficiently large,

$$q(x_1, X_2, \partial Y_2, G) \approx \frac{1}{\log^2 m} \quad \forall x_1 \in X_1.$$

Finally define X_3 similarly to (41)

$$X_3 := \partial_G (u + \epsilon\mathbb{D})$$

and repeat the process of case 1 to get $p(b, u) > cm^{-1} \log^{-1} m$. As in case 1, the estimate $p(b, u) < Cm^{-1} \log^{-1} m$ follows by merely replacing X_2 with $\partial Y_2 \setminus \partial S$. We see that it is enough to pick $c_7 = \frac{1}{5} c_5$.

Thus (40) is finished. To get (39) we use the symmetry of random walk:

$$q(u, b, \partial S, G) = \frac{q(b, u, \partial S \cup \{u\}, G)}{q(u, \partial S, \partial S \cup \{u\}, G)} \cdot \frac{\sum_v W(b, v)}{\sum_v W(u, v)} \tag{42}$$

[7] b cannot have three neighbors in S° because the seams always intersect ∂S perpendicularly — see the comment just after the definition of an easy rectangle on the preceding page.

where W is the weight function of G. we need to estimate $q(u,\partial S,\partial S\cup\{u\},G)$ and we simply sum (42) over all b to get

$$q(u,\partial S,\partial S\cup\{u\},G)\approx\sum p(b,a)\approx\frac{1}{\log m}$$

and the lemma is finished. □

Lemma 12. *Let S be an easy rectangle in an admissible hybrid graph G. Let $u\in S^\circ$ and $b\in\partial S$ with $|u-b|>c_8\operatorname{diam}S$. Then*

$$q(u,b,\partial S,G)\approx\frac{d(u,\partial S)d(u,K(S))d(b,K(S))}{N\operatorname{diam}^4 S}$$

provided that $N>C+C(\operatorname{diam}S)^{C_3}$. The constants implicit in the $\approx$ depend on c_8 and on $\operatorname{str}S$.

The proof is an easy combination of the ideas of the previous proof (take a square around u, a square around b, etc) and we shall omit it.

Lemma 13. *In the previous lemma, without the assumption $|u-b|>c\operatorname{diam}S$, we get*

$$\begin{aligned} q(u,b,\partial S,G) &\approx \frac{d(u,\partial S)f(u)f(b)}{N|u-b|^2} \qquad (43)\\ f(v) &:= \min\left(\frac{d(v,K(S))}{|u-b|},1\right) \quad . \end{aligned}$$

Proof. Let S_i be a sequence of easy rectangles, $\operatorname{str}S_i\ge c$, with $u,b\in S_1\subset S_2\subset\cdots\subset S_k=S$,

$$2\operatorname{diam}S_i\le\operatorname{diam}S_{i+1}\le C\operatorname{diam}S_i \quad ,$$

and where S_1 satisfies $\operatorname{diam}S_1\le C|u-b|$, $d(u,\partial S_1)=d(u,\partial S)$, $d(u,K(S_1))\approx f(u)\operatorname{diam}S_1$ and $d(b,K(S_1))\approx f(b)\operatorname{diam}S_1$. Clearly for some choice of constants such a sequence can always be found. A little consideration will show that these conditions also imply $d(b,K(S_i))\le Cf(b)\operatorname{diam}S_i$ for all i. Define

$$p_i:=q(u,b,\partial S_i,G),\quad q_i:=q(u,\partial S_i\setminus\partial S,\partial S_i,G) \quad .$$

Now, lemma 12 on S_1 gives $p_1\approx E$ and $q_1\le CE_2$ where

$$E_2:=\frac{d(u,\partial S)f(u)}{|u-b|}\quad E:=E_2\frac{f(b)}{N|u-b|}$$

(E is of course also the right hand side of (43)). For the other p_i's we use lemma 12 on S_{i+1} (and $q_i\le q_1\le CE_2$) to get

$$\begin{aligned} p_{i+1}-p_i &= \sum_{v\in\partial S_i\setminus\partial S} q(u,v,\partial S_i,G)\cdot q(v,b,\partial S_{i+1},G)\\ &\le C\frac{d(b,K_i)}{N\operatorname{diam}^2 S_i}\sum_v q(u,v,\partial S_i,G)\le C\frac{f(b)}{N\operatorname{diam}S_i}E_2 \quad . \end{aligned}$$

which finishes the direction $\le$ since

$$p_k=p_1+\sum p_{i+1}-p_i\le CE\left(\sum 2^{-n}\right)=CE$$

The direction $\ge$ is immediate since $p_k\ge p_1\ge cE$. □

Lemma 14. *Let $G = G(D, N)$ be an admissible hybrid graph. Let S be an easy rectangle in G, with $\operatorname{diam} S \approx 1$. Let $u \in S^\circ$ satisfy $d(u, \partial S) > c_9$, let $b \in \partial S$, and let $p = q(u, b, \partial S, G)$. Then*

$$p = -\frac{1}{2\pi} \sum_{s \in S^\circ} W(b, s) \log |\varphi_u(s)| + O\left(pN^{-1/3} \log N\right) \tag{44}$$

where φ_u is the Riemann mapping taking S to $\mathbb{D}$, $\varphi_u(u) = 0$, $\varphi_u'(u) > 0$, and where W is the weight function of G.

We also assume $N > C(\operatorname{diam} S)^{C_3}$. The constant implicit in the O depends on c_9, $\operatorname{str} S$ and on the constants implicit in the condition $\operatorname{diam} S \approx 1$.

Proof. The proof is based on examining the (unique) solution l_v of the equation

$$\begin{aligned} l_v(b) &= 0 \qquad & b \in \partial S \\ \Delta_G l_v(z) &= \delta_v(z) \qquad & z \in S^\circ \end{aligned} \tag{45}$$

We notice that the maximum principle shows that $l \leq 0$.

Sublemma 14.1. $|l_v(z)| \leq C \log N$

Subproof. We can write $l_v(z) = a_v(z) - r_v(z)$ where $a_v(z)$ comes from lemma 6 and $r_v(z)$ is the solution of Dirichlet's problem on S with the conditions $r_v(z) = a_v(z)$ on ∂S. Lemma 6 gives that $a_v(z) \leq C \log N$ and the maximum principle gives the same for $r_v(z)$. □

Sublemma 14.2. *For s a neighbor of $b \in \partial S$*

$$|l_v(s)| \leq CN^{-1} \log N \min(|v - b|^{-1}, d(b, K(S))|v - b|^{-2}) \tag{46}$$

Subproof. Let R^s be a random walk starting from s and stopped when hitting $\partial S \cup \{v\}$ (let t be the stopping time). Because $l_v(z)$ is harmonic on $S \setminus (\partial S \cup \{v\})$ we get

$$-l_v(s) = -\mathbb{E}(l_v(R^s(t))) \leq C \log N \cdot \mathbb{P}(R^s(t) = v) \tag{47}$$

and $\mathbb{P}(R^s(t) = v) \leq Cq(b, v, \partial S \cup \{v\}, G)$ can be estimated using lemma 13 (use $d(v, \partial S) \leq |v - b|$ and $f(v) \leq 1$) and symmetry (like e.g. (42)). □

Next some basic facts about φ_u. Denote by M the middle of S. As in the proof of lemma 4, we start with $u = M$, and the reflection principle through the boundary (twice around the corners) gives us that φ_M is analytic near every point of the boundary and in particular $\varphi_M'''' \leq C$ (here we used the restrictions (36) on the geometry of S, and the continuity of the Riemann mapping in the domain[8]). For other u we may take

$$\varphi_u(x) = F(\varphi_M(z)), \quad F(x) = \frac{z - \mu}{1 - z\overline{\mu}} \quad . \tag{48}$$

where $\mu := \varphi_M(u)$. Explicit differentiation gives $F^{(n)} \leq C$ and hence

$$\varphi_u(u + z) = A_u z + O\left(|z|^2\right) \tag{49}$$

where $A_u := \varphi_u'(u) \approx 1$. Also

$$\left|(\log \varphi_u)^{(n)}(z)\right| \leq \frac{C}{|\varphi^n(z)|} \quad . \tag{50}$$

[8] In this case it is easiest to prove this using the Schwarz-Christoffel formula.

Sublemma 14.3. *Let s be a neighbor of b and $d = d(b, K(S))N$. Then*

$$l_u(s) = \frac{1}{2\pi} \log|\varphi_u(s)| + O\left(dN^{-2}E\right) \tag{51}$$

where

$$E := \frac{\log N}{Nr}, \qquad r := \min\left(d(u, \bar{G}) + \tfrac{1}{4}N^{-1}, N^{-2/3}\right)$$

Subproof. Start with the following function on G: let N' be N if $\lfloor u \rfloor \notin D$ and $2N$ if $\lfloor u \rfloor \in D$. Define

$$\bar{l}_u(v) := \begin{cases} a(N'(v-u)) + \frac{1}{2\pi}\log A_u/N' & |v-u| \le r \\ \frac{1}{2\pi}\log|\varphi_u(v)| & |v-u| > r \end{cases} \tag{52}$$

where a is the harmonic potential on $\mathbb{Z}$ — notice that on $\{|v-u| \le d(u, \bar{G})\}$ the hybrid graph is simply $\frac{1}{N'}\mathbb{Z}^2$ so the use of a makes sense. In the case $u \in \bar{G}$ (or $r < \frac{1}{N'}$ if you prefer) we simply define $\bar{l}_u(u)$ so as to satisfy $\Delta_G \bar{l}_u(u) = 1$. The uniqueness of l_u gives

$$l_u(w) = \bar{l}_u(w) - \sum_{v \in S^\circ \setminus \{u\}} \Delta \bar{l}_u(v) \cdot l_v(w) \quad .$$

since the right hand side clearly satisfies (45) (we shall only use that for $w = s$, though). To estimate $l_u - \bar{l}_u$ we use sublemma 14.2 for the l_u's appearing on the right hand side and $\Delta\bar{l}_u(v)$ is estimated as follows:

1. For $0 < |z| < r - N^{-1}$ we have $\Delta\bar{l}_u(z+u) = 0$.
2. At the transition annulus $|z| = r + O(N^{-1})$, (4) gives us (for $|z| \le r$)

$$\left|\bar{l}_u(z+u) - \frac{1}{2\pi}\log|A_u z|\right| \le \frac{C}{(Nr)^2} \tag{53}$$

   while using (49) together with the fact $A_u \ge c$ gives for $r < |z| \le r + \frac{1}{N}$,

$$\left|\bar{l}_u(z+u) - \frac{1}{2\pi}\log|A_u z|\right| \le Cr \le CN^{-2/3} \le \frac{C}{(Nr)^2} \quad . \tag{54}$$

3. Finally, for $|v-u| > r+1$ expand $\log|\varphi| = \operatorname{Re}\log\varphi(z)$ to a power series around z and (12) will give (using (50))

$$|\Delta\bar{l}_u(v)| \le \frac{C}{(N|\varphi(v)|)^{k(v)}}$$

   ($k(v)$ from (34)).

This division into three cases, combined with the different possibilities for $k(v)$ and $|v-b|$ is formalized by dividing $G \cap S = \cup_{i=0}^{6} F_i$ with the F_i's defined as follows:

$$F_0 := \left\{v \,:\, |v-u| < r - N^{-1}\right\}, \quad F_1 := \left\{v \,:\, \big||v-u| - r\big| \le N^{-1}\right\} \quad ;$$

F_2-F_4 are subsets of

$$V := \left\{v \in S^\circ \,:\, r + \frac{1}{N} < |v-u| \ \wedge\ |v-b| > \alpha\right\}, \quad \alpha := \tfrac{1}{2}\min \operatorname{str} S \cdot \operatorname{diam} S, c_9 \quad ,$$

$F_2 := V \setminus \bar{G}$, $F_3 := V \cap (\bar{G} \setminus \bar{\bar{G}})$ and $F_4 := V \cap \bar{\bar{G}}$; and finally F_5-F_6 are related similarly to $V' := \{|v-b| \leq \alpha\}$ — there is no F_7 because $V' \cap \bar{\bar{G}} = \emptyset$ due to (38). This gives

$$\begin{aligned} |l_u(s) - \log|\varphi_u s|| &\leq \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 \\ \Sigma_i &:= \sum_{v \in F_i} |\Delta \bar{l}_u(v)| \cdot |l_v(s)| \end{aligned}$$

(Σ_0 being 0). We now estimate them one by one.

For Σ_1 we use (53), (54), sublemma 14.2 and $\#F_1 \leq CNr$ to get

$$\Sigma_1 \leq \sum_{v \in F_1} \frac{C}{(Nr)^2} \cdot \frac{Cd \log N}{N^2 |v-b|^2} \leq CdN^{-2}E \quad .$$

For Σ_2 we use the easy fact that $|\varphi_u(v)| \geq c|v-u|$ to sum by distance from u and get

$$\begin{aligned} \Sigma_2 &\leq \sum_{v \in F_2} \frac{C}{(N|\varphi(v)|)^4} \cdot \frac{Cd \log N}{N^2 |v-s|^2} \leq \frac{Cd \log N}{N^2} \sum_{k=Nr}^{N} k \frac{C}{k^4} \leq \\ &\leq C \frac{d \log N}{N^2} \cdot \frac{1}{(Nr)^2} \leq CdN^{-2}E \quad . \end{aligned}$$

A similar estimate for Σ_3 and Σ_4 (remember (35)) gives

$$\Sigma_3, \Sigma_4 \leq CdN^{-2}E$$

Next we tackle Σ_5. This time we sum by distance from b:

$$\begin{aligned} \Sigma_5 &\leq \frac{C \log N}{N^4} \Bigg(\sum_{\substack{v \in F_5 \\ N|v-b| > d}} \frac{Cd/N^2}{|v-b|^2} + \sum_{\substack{v \in F_5 \\ N|v-b| \leq d}} \frac{C/N}{|v-b|} \Bigg) \\ &\leq \frac{C \log N}{N^4} \left(\sum_{k=d}^{cN} k \frac{Cd/N^2}{(k/N)^2} + \sum_{k=1}^{d} k \frac{C/N}{k/N} \right) \leq \frac{Cd \log^2 N}{N^4} \\ &\leq CdN^{-2}E \end{aligned}$$

and similarly

$$\Sigma_6 \leq \frac{C \log^2 N}{N^2}$$

and since (37) implies that Σ_6 is relevant only when $d > cN$ this is also $\leq CdN^{-2}E$. Summing the estimates for $\Sigma_1, \dots, \Sigma_6$ gives us (51). □

With sublemma 14.3 proved, we are capable of proving lemma 14 for the case $d(u, \bar{G}) > N^{-2/3}$, and to get some estimate for the other case. We again use the time-symmetry of random walks in the form (42). For the nominator of (42), examine the random walk R^s starting from s and stopped on $\partial S \cup \{u\}$ and t the stopping time and get

$$l_u(s) = \mathbb{E}(l_u(R^s(t))) = l_u(u) \cdot q(s, u, \partial S \cup \{u\}, G) \quad . \tag{55}$$

so remembering (2),

$$q(b, u, \partial S \cup \{u\}, G) = \frac{\sum_{s \in S^\circ} W(b,s) l_u(s)}{l_u(u) \cdot \sum_s W(b,s)}$$

For the denominator, we examine a random walk starting from v a neighbor of u and find in the same manner

$$q(v, \partial S, \partial S \cup \{u\}, G) = 1 - \frac{l_u(v)}{l_u(u)}$$

and summing over all v we get

$$q(u, \partial S, \partial S \cup \{u\}, G) = 1 - \sum_v \frac{W(u,v)}{\sum_w W(u,w)} \cdot \frac{l_u(v)}{l_u(u)} = \frac{-(\Delta l_u)(u)}{l_u(u) \cdot \sum_w W(u,w)} \tag{56}$$

so (55), (56) and (42) with (45) give

$$q(u, b, \partial S, G) = -\sum_s W(b,s) l_u(s) \tag{57}$$

and with (51) and $p \approx dN^{-2}$ we get

$$p = -\frac{1}{2\pi} \sum_{s \in S^\circ} W(b,s) \log |\varphi_u(s)| + O(pE) \quad . \tag{58}$$

This proves the lemma for the case $d(u, \bar{G}) > N^{-2/3}$.

For the other case, let u satisfy $d(u, \bar{G}) \le N^{-2/3}$. Let $m \approx N^{-1/2}$ satisfy that the square S' of side length m around u is easy (clearly such an m can be found). With this S' we can write

$$p = \sum_{v \in \partial S'} q(u, v, \partial S', G) \cdot q(v, b, \partial S, G) \quad ; \tag{59}$$

$q(v, b, \partial S, G)$ can be estimated by (58) to give

$$q(v, b, \partial S, G) = -\frac{1}{2\pi} \sum_{s \in S^\circ} W(b,s) \log |\varphi_v(s)| + O(pE_v) \quad ; \tag{60}$$

Thus we have to estimate $\varphi_u - \varphi_v$. But if $\nu := \varphi_u(v)$ then $|\nu| \le Cm$ and furthermore

$$\varphi_v = F(\varphi_u), \quad F(z) := \frac{z - \nu}{1 - z\bar{\nu}} \quad .$$

Writing the Taylor expansion of $\log |\varphi_u|$ near b and plugging in the derivatives of F will give

$$\begin{aligned}
&\big| |\log \varphi_v(s)| - |\log \varphi_u(s)| \big| \le \\
&\le \quad \left| (b-s) \operatorname{Re} \frac{\varphi_u'}{\varphi_u}(b) \right| \cdot O(|\nu|) + \left| \frac{(b-s)^2}{2} \operatorname{Re} \frac{\varphi_u'' \varphi_u - \varphi_u'^2}{\varphi_u^2}(b) \right| \cdot O(|\nu|) + O(N^{-3}) \\
&\le \quad O\left(\frac{|\nu| \cdot |\varphi'|}{N} + \frac{|\nu|}{N^2} + N^{-3} \right)
\end{aligned}$$

so we can replace φ_v with φ_u in (60) to get

$$q(v, b, \partial S, G) = -\frac{1}{2\pi} \sum_{s \in S^\circ} W(b,s) \log |\varphi_u(s)| + O\left(pE_v + dN^{-5/2} \right)$$

and summing over (59) we get

$$p = -\frac{1}{2\pi} \sum_{s \in S^\circ} W(b,s) \log |\varphi_u(s)| + O(\Sigma_1 + \Sigma_2) \tag{61}$$

with Σ_1 and Σ_2 defined by

$$\begin{aligned}\Sigma_1 &:= p\sum_{v\in\partial S'} q(u,v,\partial S',G)\cdot E_v \\ \Sigma_2 &:= \sum q(u,v,\partial S',G)\cdot dN^{-5/2} = dN^{-5/2} \leq CpN^{-1/2} \qquad (62)\end{aligned}$$

To estimate Σ_1 use lemma 11 and get

$$q(u,v,\partial S',G) \leq \frac{C}{m}$$

so summing by distance from $\bar{G}$ we get

$$\begin{aligned}\Sigma_1 &\leq CpN^{-1/3}\log N + p\sum_{k=1}^{N^{1/3}} \frac{C}{m}\cdot\frac{C\log N}{Nk} \\ &\leq CpN^{-1/3}\log N\end{aligned}$$

which with (61) and (62) proves the lemma. □

Remarks. 1. The following weaker form of the lemma will probably look more familiar:

$$p = \kappa_b\frac{|\varphi_u'(b)|}{2\pi N} + O\left(pN^{-1/3}\log N + N^{-2}\right) \qquad (63)$$

where the structure constant κ_b is defined by

$$\kappa_b := \begin{cases} \frac{1}{2} & b\in V_1 \\ 1 & b\in V_0\setminus\bar{G} \\ \frac{9}{8} & b\in V_0\cap\bar{G}\end{cases}, \qquad (64)$$

V_0 and V_1 from the definition of a hybrid graph. See figure 2. To get (63) just write a Taylor expansion of $\log|\varphi| = \operatorname{Re}\log\varphi$ near b and a few orientation arguments will allow to calculate $\arg(b-s)\varphi'/\varphi$. For example, for $b\in G\setminus\bar{G}$, we get $\arg(b-s)\varphi'/\varphi = 0$. As already remarked, the conditions (37) and (38) imply that ∂S can only intersect a seam perpendicularly — otherwise we would need a number of additional special values for κ_b. The additional error N^{-2} in (63) is the second term in the Taylor expansion. This error is of course meaningful only for b close to the boundaries.

2. The $\log$ factor can be removed. Basically one has to take the estimates for l given by sublemma 14.3 which, for $|u-s|>c$ are better than those of sublemma 14.2, and plug them right back into the estimates of Σ_1-Σ_4. Also the assumption $\operatorname{diam} S\approx 1$ is unnecessary — without it the lemma holds with $m := N\operatorname{diam} S$ instead of N.

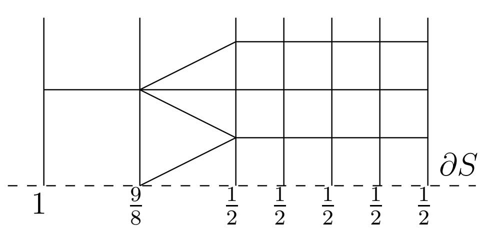

FIGURE 2. κ_b on the boundary of an easy rectangle.

3. The final statement of lemma 14 can be translated back into l_u terms using (57) to give an estimate in (51) that does not depend on $d(u, \bar{G})$.
4. The value of r is not the best for any u. For example, if S is a square around u then the symmetry of the situation shows that
$$\varphi(u+z) = -i\varphi(u+iz) = -\varphi(u-z) = i\varphi(u-iz)$$
from which we may conclude
$$\varphi(u+z) = A_u z + O(|z|^5) \quad . \tag{65}$$
The improved estimate in (54) allows to pick $r = N^{-1/3}$ (assume for simplicity $d(u, \bar{G}) > N^{-1/3}$) and to get in (44) the error estimate $O(pN^{-2/3} \log N)$.
5. This is actually a rather nice result even for random walks on $\frac{1}{N}\mathbb{Z}^2$ (which is of course a trivial hybrid graph — take $D = \emptyset$ and get $\kappa_b \equiv 1$). For example, it shows that for the square $[-1, 1]^2$ the hitting probability from 0 of b is $\frac{1}{2\pi N}|\varphi'(b)| + O(N^{-5/3} \log N)$. In comparison, the probability that a Brownian motion will hit an interval of length $\frac{1}{N}$ around b is $\frac{1}{2\pi}|\varphi'(b)| + O(N^{-2})$. In other words, because there are no "quantization effects" we get an error $N^{-2/3} \log N$ better than what we would expect for, say, a quantized circle.
6. Forgetting for the moment hybrid graphs we reread the proof for the case of u in the center of a square in $\frac{1}{N}\mathbb{Z}^2$. The role of the symmetries of the grid $\mathbb{Z}^2$ seems to suggest that an equivalent calculation for a random walk on a triangular grid will give stronger results. Let $p(b)$ be the probability that a random walk on a triangular grid with step length $\frac{1}{N}$ starting from 0 will hit a regular hexagon centered at 0 of side length 1 at the point b. We see that we get a better estimate in (65), better estimates in (4)[9] and $k(v) = 6$ for all v, so we should be able to get $p = \frac{1}{2\pi N}|\varphi'(b)| + O(N^{-2})$, no? However, for a hexagon, the inverse Schwarz-Christoffel function φ is no longer analytic around the corners, which gives an additional error term $O(d^{-1}N^{-5/2} \log N)$ where d is the distance to the nearest corner. Indeed, at the very corner $\varphi'(z)$ is 0, while the hitting probability is $\geq cN^{-3/2}$.
7. Trying the formulation $p = -\frac{1}{\pi\sqrt{12}} \sum W(b, s) \log |\varphi(s)|$ in the previous remark improves the error estimates in the hexagon's edges' middle parts but not near the corners. A careful calculation will give in this case that the best r is $N^{-0.4}$ and the error is
$$O(pN^{-1.8} + d^{-5/2}N^{-4})$$
and again, when $d < C/N$ the error becomes $O(N^{-3/2})$, which is exactly the magnitude of p.

[9] See [KS] for a proof that $a_T(z) = \frac{1}{\pi\sqrt{12}} \log |z| + \beta + O(|z|^{-4})$ where a_T is the harmonic potential of the triangular grid (notice that our a_T is $\frac{1}{6}$ of the a in [KS]) which can be normalized to get $\beta = 0$. Interestingly, the value $\frac{1}{\pi\sqrt{12}}$ may also be deduced from the proof above by summing (44) (with the factor $\frac{1}{2\pi}$ replaced with the value we are calculating) over all b and using the fact that $\int |\varphi'| = 2\pi$.

4. The proof core

For simplicity of notation, assume throughout this chapter that $M>1$ and $N>1$ (so we don't have to worry about $\log$'s being zero).

4.1. **Localization.**

Lemma 14 gave a relatively precise estimate of the difference between random walk on $\mathbb{Z}^2$ and on a hybrid graph. In this section we mold this general lemma into some corollaries in the form required for the proof of the theorem. Specifically, we estimate the amount a random walk changes when the graph is changed on one square from $\frac{1}{N}\mathbb{Z}^2$ to $\frac{1}{2N}\mathbb{Z}^2$, or, in our notation, when one $z\in\mathbb{Z}^2$ is moved into or out of D.

Lemma 15. *Assume for* $i=1,2$

$$Q_i=\prod_{k=1}^{K}M_{i,k}$$

and for all k, $|M_{1,k}-M_{2,k}|\leq\epsilon\min M_{1,k},M_{2,k}$. *Assume* $\epsilon K\leq\frac{1}{2}$. *Then*

$$|Q_1-Q_2|\leq C\epsilon K\min Q_1,Q_2 \quad .$$

This exercise is left for the reader.

Lemma 16. *Let* $D_1,D_2\subset\mathbb{Z}^2\cap[-M,M[^2$ *with* $D_1\triangle D_2=\{z\}$ *(*$\triangle$ *being the symmetric difference). Let* $S:=z+[-1,2]^2$. *Let* $G_i=G(D_i,N)$ *be admissible,* $N>CM^{C_3}$. *Let* $B\subset(G_1\setminus S^\circ)\cap[-M,M]^2=(G_2\setminus S^\circ)\cap[-M,M]^2$ *be a set containing a loop around a point* $a\in\mathbb{C}$ *and let* $b\in B$ *be some point. Then*

$$\begin{aligned}|p_1-p_2| &\leq CN^{-1/3}\log N\log^2 M\min(p_1,p_2)\\ p_i &:= q(a,b,B,G_i) \quad .\end{aligned}$$

Proof. Let $m_2=\frac{1}{N}\left\lfloor\frac{2}{3}N\right\rfloor$, $m_3=\frac{1}{N}\left\lfloor\frac{1}{3}N\right\rfloor$ and let $S_j=[-m_j,1+m_j]^2$ ($j=2,3$). We note that both squares are easy for both G_i.

Sublemma 16.1. *Let* $x\in S_3$ *and let* x_i *be a point of* G_i *closest to* x. *Let* $w\in\partial S_2$ *and let* $q_i=q(x_i,w,\partial S_2,G_i)$. *Then*

$$|q_1-q_2|\leq CN^{-1/3}\log N\min(q_1,q_2) \tag{66}$$

Subproof. If $x_1=x_2$ this follows immediately from lemma 14 because the list of neighbors s of w, $W(w,s)$ and φ_x do not depend on i. Otherwise we have to estimate $\log|\varphi_{x_1}|-\log|\varphi_{x_2}|$ using $|x_1-x_2|\leq 1/N$. For this we use the representation (coming from expanding the $\log\varphi$ in (44) into a Taylor series)

$$q_i=\kappa_w\frac{|\varphi'|}{N}+\sum_s W(w,s)\operatorname{Re}\frac{(s-w)^2}{2}\cdot\frac{\varphi''\varphi-\varphi'^2}{\varphi^2}+O(N^{-3}) \quad ,$$

where κ_w is defined in (64) (and is independent of i). $|x_1-x_2|\leq 1/N$ gives

$$\left|\varphi_{x_1}^{(j)}-\varphi_{x_2}^{(j)}\right|\leq CN^{-1}\min\left(\varphi_{x_1}^{(j)},\varphi_{x_2}^{(j)}\right) \qquad j=0,1,2$$

and the sublemma is proved. □

Sublemma 16.2. *Let* $v,w\in\partial S_2$, *then the probabilities* $q^v=q(v,b,\partial S_3\cup B,G_i)$ *(which obviously don't depend on* i*) satisfy*

$$q^v\approx q^w \tag{67}$$

Subproof. If $d(v,w) < \frac{1}{12}$ this is easy because one can find some $2d(v,w) < m < \frac{1}{3}$ such that $S_4 := v + [-m,m]^2$ is easy and write

$$q^x = \sum_{y \in \partial S_4} q(x,y,\partial S_4, G_i) \cdot q(y,b,\partial S_3 \cup B_i, G_i)$$

and since $q(x,y,\partial S_4,G_i) \approx d(y,K(S_4))/N$ for both $x = v$ and w, we are done. If $d(v,w) \geq \frac{1}{12}$ just write a chain $v_i \in \partial S_2$, $v_0 = v$, $v_n = w$ and $|v_i - v_{i+1}| < \frac{1}{12}$ and use the first case inductively ($n \leq 17$, for example). □

Sublemma 16.3. *Let $v \in \partial S_2$. then the probabilities $q_i = q(v,b,B,G_i)$ satisfy*

$$|q_1 - q_2| \leq CN^{-1/3} \log N \log^2 M \cdot \min(q_1, q_2) \tag{68}$$

Subproof. Let R_i^v be random walks on G_i starting from v. Define stopping times $t_i(0) = 0$ and

$$\begin{aligned} t_i(2k+1) &:= \min\{t > t_i(2k) \,:\, R_i(t) \in \partial S_3 \cup B\} \\ t_i(2k) &:= \min\{t > t_i(2k-1) \,:\, R_i(t) \in \partial S_2\} \end{aligned}$$

and let k_i be the first k such that $R_i^v(t_i(k)) \in B_i$ (k_i is always odd, of course). Let $p_{i,k}(w)$ be the probability that $k_i > k$ and $R_i^v(t_i(k)) = w$. Then we can write $p_{i,k}(w)$ as a sum

$$p_{i,k}(w) = \sum_{\vec{w}} \mathbb{P}\left(R_i^v(t_i(l)) = w_l,\ l = 0,\dots,k\right)$$

where the sum is over all vectors $\vec{w} = \{w_l\}_{l=0}^k$ with $w_0 = v$, $w_k = w$, $w_{2l} \in \partial S_2$ and $w_{2l+1} \in \partial S_3$. Lemma 15 and sublemma 16.1 will now give for every $k < K := \lfloor cN^{1/3} \log^{-1} N \rfloor$,

$$|p_{1,k}(w) - p_{2,k}(w)| < CkN^{-1/3} \log N \min(p_{1,k}(w), p_{2,k}(w)) \quad . \tag{69}$$

From these we derive an equivalent estimate for

$$q_{i,k} := \mathbb{P}\left(\{k_i = k\} \cap \{R_i^v(t_i(k)) = b\}\right)$$

since

$$q_{i,k+1} = \sum_{w \in \partial S_2} p_{i,k}(w) \cdot q(w,b,\partial S_3 \cup B, G_i) \tag{70}$$

and we can sum over w. On the other hand, summation over sublemma 16.2 shows that the probability to hit b is approximately independent from k, i.e.

$$q_i \approx \frac{q_{i,2k+1}}{\mathbb{P}(k_i > 2k)} \tag{71}$$

and a simple exit probability estimate shows that for k odd

$$\mathbb{P}\left(k_i = k \,|\, k_i \geq k\right) \geq \frac{c}{\log M} \quad . \tag{72}$$

To get (72) notice that the fact that B contains a loop around a allows to bound the hitting probability of B by the hitting probability of $\partial[-M,M]^2$ which can be estimated by $\frac{c}{\log M}$, e.g. using lemma 7. This gives

$$|q_1 - q_2| \quad \leq \quad \sum_{k=1}^{\infty} |q_{1,k} - q_{2,k}| \leq$$

$$\leq \sum_{k=1}^{K} CkN^{-1/3}\log N\left(1-\frac{c}{\log M}\right)^k \min q_1,q_2 + \\ + \sum_{k=K+1}^{\infty} C\left(1-\frac{c}{\log M}\right)^k \min q_1,q_2 \tag{73}$$

and we are done. □

The lemma is now easy: if $a \in S_3$ we write

$$q(a,b,B,G_i) = \sum_{v\in\partial S_2} q(a,v,\partial S_2,G_i)\cdot q(v,b,B,G_i)$$

and use sublemma 16.1 to show that $q_i := q(a,v,\partial S_2,G_i)$ satisfy

$$|q_1-q_2| \leq CN^{-1/3}\log N \min q_1.q_2$$

and get the result using sublemma 16.3. If $a \notin S_3$ we similarly write

$$q(a,b,B,G_i) = q(a,b,B\cup\partial S_3,G_i) + \sum_{v\in\partial S_3} q(a,v,B\cup\partial S_3,G_i)\cdot q(v,b,B,G_i)$$

and since both $q(a,b,B\cup\partial S_3,G_i)$ and $q(a,v,B\cup\partial S_3,G_i)$ are independent of i, the lemma follows from sublemma 16.3 like the previous case. □

Lemma 17. *With the notations of lemma 16 and $a \notin S$, let $\check{R}_i$ be a random walk on G_i starting from a and conditioned to hit B at b. Let $\check{\gamma}_i$ be the segment of $\mathrm{LE}(\check{R}_i)$ until S, or all of $\mathrm{LE}(\check{R}_i)$ if $\mathrm{LE}(\check{R}_i)\cap S=\emptyset$. Then*

$$\sum_{\gamma} |\mathbb{P}(\check{\gamma}_1=\gamma)-\mathbb{P}(\check{\gamma}_2=\gamma)| \leq CN^{-1/3}\log^2 M\log^3 N$$

where the sum is taken on all the simple paths γ in G_i from a to $S\cup b$.

We note that (due to $a\notin S$) we can assume $a\in G_1$ and get $a\in G_2$ too. Note also and that a path in G_1 from a to $S\cup b$ is also a path in G_2 from a to $S\cup b$.

Proof. We keep all notations from the proof of lemma 16. Denote by R_i^a a random walk on G_i starting from a and stopped on B and let k_i^a and t_i^a be the equivalents (for R_i^a) of k_i and t_i from sublemma 16.3. Combining (71) and (72) and summing over $v\in\partial S_2$ we get

$$\begin{aligned}\mathbb{P}\left(\{k_i^a>k\}\cap\{R_i^a \,\mathrm{hits}\, b\}\right) &\leq C\left(1-\frac{c}{\log M}\right)^k \mathbb{P}\left(\{k_i^a>1\}\cap\{R_i^a\,\mathrm{hits}\,b\}\right)\\ &\leq C\left(1-\frac{c}{\log M}\right)^k \mathbb{P}\left(\{R_i^a\,\mathrm{hits}\,b\}\right) \quad .\end{aligned}$$

This means that by taking $K=\lfloor C\log M\log N\rfloor$ we can write

$$\mathbb{P}(k_i^a>K\mid R_i^a\,\mathrm{hits}\,b) < CN^{-2} \quad . \tag{74}$$

Next, denote by γ_i the unconditioned version of $\check{\gamma}_i$ i.e. the segment of $\mathrm{LE}(R_i^a)$ until S. Since γ_i obviously depends only on the portions of R_i^a outside S_2 we can, as in sublemma 16.3, sum over all vectors $\vec{w}=\{w_j\}_{j=1}^{k-1}$, $w_{2l}\in\partial S_2$, $w_{2l+1}\in\partial S_3$ and get, using (66) and lemma 15,

$$|p_{1,k}-p_{2,k}| \leq CkN^{-1/3}\log N\min\left(p_{1,k},p_{2,k}\right)$$

$$p_{i,k} := \mathbb{P}\left(\{\gamma_i = \gamma\} \cap \{k_i^a = k\} \cap \{R_i^a(t_i^a(k)) = b_i\}\right)$$

and summing over k from 1 to K we get

$$\begin{aligned} |p_1 - p_2| &< CK^2 N^{-1/3} \log N \min p_1, p_2 + E_1 + E_2 \quad (75) \\ p_i &:= \mathbb{P}\left(\{\gamma_i = \gamma\} \cap \{R_i^a \,\text{hits}\, b_i\}\right) \\ E_i &:= \mathbb{P}\left(\{\gamma_i = \gamma\} \cap \{R_i^a \,\text{hits}\, b_i\} \cap \{k_i^a \geq K\}\right) \end{aligned}$$

which finishes the proof since $\check{p}_i := \mathbb{P}(\check{\gamma}_i = \gamma)$ is given by

$$\check{p}_i = \frac{p_i}{\mathbb{P}(R_i^a \,\text{hits}\, b)}$$

and the difference between the nominators can be estimated by lemma 16. The factors E_i are dealt with by (74) which gives $\sum_\gamma \frac{E_i}{\mathbb{P}(R_i^a \,\text{hits}\, b)} \leq CN^{-2}$. □

Definition. Let γ be a path in a metric graph G, let $z \in \mathbb{Z}^2$ and let $r > \varepsilon > 0$. An (r, ϵ, z)-**quasi loop** of γ are two points $v, w \in \gamma$, $|v - z| \leq \epsilon$, $|w - z| \leq \epsilon$ such that the section of γ between v and w has a diameter $> r$. We denote $\gamma \in \text{QL}(r, \epsilon, z)$.

Note that this is slightly different than the (r, ϵ, z)-quasi loops of [S00].

Lemma 18. *There exists a constant c_{10} such that for every $G = G(D, N)$ admissible hybrid graph; every $a \in G$; every $B \subset G \cap [-M, M]^2$ containing a loop around a; and every $\epsilon \geq 1$ and $r > M^{1-c_{10}}$ we have*

$$\mathbb{E}(\#\{z : \text{LE}(R^a) \in \text{QL}(r, \epsilon, z)\}) < CM^{-c}\epsilon^C \log N \quad (76)$$

where R^a is a random walk on G starting from a and stopped on B.

This lemma follows from lemma 9 like lemma 3.4 in [S00] follows from lemma 2.1 ibid. However, the differences (especially those resulting from the fact that B is not necessarily the boundary of a simply connected domain) seem to merit a reproduction of Schramm's proof.

Proof. We may assume $64\epsilon < r < \sqrt{2}M$. For $4\epsilon < s < \frac{1}{16}r$ and $z \in \mathbb{C}$ we define the sets $S_3 := z + (s + \epsilon)\mathbb{D}$ and $S_2 := z + 2s\mathbb{D}$. As in lemma 17, we define stopping times $t^a(k)$ by

$$\begin{aligned} t^a(2k+1) &:= \min\{t > t^a(2k) : R^a(t) \in \partial S_3 \cup B\} \\ t^a(2k) &:= \min\{t > t^a(2k-1) : R^a(t) \in \partial S_2 \cup B\} \quad (77) \end{aligned}$$

and k^a to be the first k such that $R^a(t^a(k)) \in B$. Next we define the variable

$$X := X(z) := \#\{x \in \mathbb{Z}^2 \cap (z + s\mathbb{D}) : \text{LE}(R^a[0, t^a(k^a)]) \in \text{QL}(r, \epsilon, x)\} \quad .$$

With these notations we can write

$$\mathbb{E}\, X \leq M^2\, \mathbb{P}(k^a > K) + \sum_{k=1}^{K} \mathbb{E}(X \cdot \mathbf{1}_{\{k^a = k\}}) \quad . \quad (78)$$

and as above (72) holds for k odd, so for $K > C \log^2 M$ for some C sufficiently large we get

$$\mathbb{P}(k^a > K) \leq \left(1 - \frac{c}{\log M}\right)^{K/2} \leq M^{-C}$$

for any desired constant in the exponent of M. In other words, with this K we can ignore the first summand in (78). Let us therefore define the number of quasi-loops up to the kth time

$$X_k := \#\{x \in \mathbb{Z}^2 \cap (z + s\mathbb{D}) : \mathrm{LE}(R^a[0, t^a(k)]) \in \mathrm{QL}(r, \epsilon, x)\} \quad .$$

The process of loop-erasing between $t^a(2k)$ and $t^a(2k+1)$ can only destroy (r, ϵ, x)-quasi loops for $x \in z + s\mathbb{D}$ so we get

$$X \cdot \mathbf{1}_{\{k^a=2k+1\}} \le X_{2k} \cdot \mathbf{1}_{\{k^a=2k+1\}} \le X_{2k} \cdot \mathbf{1}_{\{k^a>2k\}}$$

With this in mind we define

$$\Delta_k := (X_{2k+2} - X_{2k}) \cdot \mathbf{1}_{\{k^a>2k+2\}} \quad .$$

Sublemma 18.1. *Let $v \in \partial S_3$ be some vertex; and let γ be a simple path on G starting and ending on ∂S_2; let R^v be a random walk starting from v and stopped at $\partial S_2 \cup B$; and define*

$$\begin{aligned} \delta'(v,\gamma) &:= \#\{x \in \mathbb{Z}^2 \cap (z + s\mathbb{D}) : d(x, \gamma) \le \epsilon \ \wedge \ d(x, R^v) \le \epsilon\} \cdot \mathbf{1}_{\{R^v \cap \gamma = \emptyset\}} \quad , \\ \delta(v,\gamma) &:= \delta'(v,\gamma) \cdot \mathbf{1}_Y \end{aligned}$$

where Y is the event that R^v stops on ∂S_2. Then

$$\mathbb{E}\,\delta'(v,\gamma) \le C\epsilon^2 \qquad \mathbb{E}\,\delta(v,\gamma) \le Cs^{-c}\epsilon^C.$$

Subproof. Define the event $E(t)$ by

$$E(t) := \{\exists x \in \mathbb{Z}^2 \cap (z + s\mathbb{D}) : d(x,\gamma) \le \epsilon \ \wedge \ d(x, R^v(t)) \le \epsilon\}$$

and use E to define stopping times $s_1, \dots$ by

$$s_1 := \min\{t : E(t) \ \vee \ R^v(t) \in \partial S_2 \cup B\}$$

and for $i > 1$

$$s_i := \min\{t > s_{i-1} : (E(t) \ \wedge \ |R^v(t) - R^v(s_{i-1})| > C_5\epsilon) \ \vee \ R^v(t) \in \partial S_2 \cup B\}$$

where C_5 will be defined promptly. Let l be the number of s_i's defined before the process is stopped i.e. $s_{l+1} \in \partial S_2 \cup B$. Lemma 9 gives for $i < l$ that the probability not to intersect γ between s_i and s_{i+1} is $\le c < 1$ assuming C_5 is large enough — fix C_5 to satisfy that. This gives $\mathbb{P}(l > A) \le c^A$ for any value of A. Further, for the time period between s_A and s_{l+1} lemma 9 gives that

$$\mathbb{P}(Y \cdot \mathbf{1}_{\{R^v \cap \gamma = \emptyset\}} \,|\, l \ge A) \le q(R^v(s_A), \partial S_2, \partial S_2 \cup \gamma, G) \le C(s/\epsilon)^{-c} \quad .$$

Finally, it is clear that $\delta(v,\gamma) > CA\epsilon^2$ implies $l > A$ therefore

$$\begin{aligned} \mathbb{E}\,\delta(v,\gamma) \le \sum_{A=1}^{\infty} \mathbb{E}(\delta(v,\gamma) \cdot \mathbf{1}_{\{l=A\}}) \le& \\ \le C\epsilon^2 \sum_{A=1}^{\infty} A\,\mathbb{P}(Y \cap \{R^v \cap \gamma = \emptyset\} \cap \{l > A-1\}) \le& \\ \le C\epsilon^2 \sum_{A=1}^{\infty} Ac^{A-1} \cdot C(s/\epsilon)^{-c} \le Cs^{-c}\epsilon^C& \quad . \end{aligned} \tag{79}$$

The inequality for $\mathbb{E}\,\delta'(v,\gamma)$ follows similarly. □

Returning to the estimate of Δ_k we notice that in order to have $X_{2k+2} > X_{2k}$ we need to have for some $x \in \mathbb{Z}^2 \cap (z + s\mathbb{D})$ that $\mathrm{LE}(R^a[0, t^a(2k+2)])$ contains an (r, ϵ, x)-quasi loop and $\mathrm{LE}(R^a[0, t^a(2k)])$ doesn't contain one. This requires at least that

1. x is ϵ-near at least one segment γ of $\mathrm{LE}(R^a([0, t^a(2k)])) \cap S_2$.
2. $R_k := R^a([t^a(2k+1), t^a(2k+2)])$ gets ϵ-near x and then fails to intersect at least one of the segments γ from 1.

In other words, the number of such x's can be estimated by $\delta(R^a(t^a(2k+1)), \gamma)$. With this in mind we denote by $\Gamma(t, u)$ the collection of connected components γ of $\mathrm{LE}(R^a([0, t])) \cap (z + u\mathbb{D})$ satisfying $\gamma \cap \partial S_3 \neq \emptyset$ and $R^a(t) \notin \gamma$ and get

$$\mathbb{E}\,\Delta_k \leq \sum_{\gamma \in \Gamma_k} \max_{v \in \partial S_3} \mathbb{E}\,\delta(v, \gamma) \leq C(\#\Gamma_k) s^{-c} \epsilon^C$$

where $\Gamma_k := \Gamma(t^a(2k+1), 2s)$. It easy to see that $\#\Gamma_k \leq k$, and summing up to k we get

$$\mathbb{E}\, X_{2k} \cdot \mathbf{1}_{\{k^a > 2k\}} \leq C s^{-c} k^2 \epsilon^C \quad .$$

Another summation, up to K, will give us

$$\begin{aligned} \mathbb{E}\, X &\leq M^{-C} + C s^{-c} \epsilon^C \log^6 M + \mathbb{E}\, X' \\ X' &:= \Delta'_{(k^a-2)/2} \cdot \mathbf{1}_{\{k^a \text{ even}\}} \\ \Delta'_k &:= (X_{2k+2} - X_{2k}) \cdot \mathbf{1}_{\{k^a > 2k+1\}} \quad . \end{aligned} \tag{80}$$

Thus we are left with the estimate of $\mathbb{E}\, X'$, which is the behavior near the boundary — if $B \cap S_2 = \emptyset$ then of course k^a is always odd and we get $X' \equiv 0$. It is at this point that we utilize the difference between r and s. Further, it will be easier to use entry probabilities rather than exit probabilities. Thus the first step will be a time-reversed lemma 9.

Sublemma 18.2. *Let $S_0 := z + \frac{1}{2} r\mathbb{D}$ and $S_1 := z + \frac{1}{4} r\mathbb{D}$. Let γ be a simple path between ∂S_0 and ∂S_3. Then*

$$q(v, \partial S_3, \partial S_3 \cup \partial S_0 \cup \gamma, G) \leq C(r/s)^{-c} \quad \forall v \in \partial S_1. \tag{81}$$

Subproof. Let $\gamma' := \gamma \cap \left(\left(z + \frac{1}{8} r\mathbb{D} \right) \setminus S_2 \right)$ (remember the condition $s < \frac{1}{16} r$) and let w be an arbitrary point with $\frac{1}{4} s < |w - z| < \frac{3}{4} s$. Lemma 9 shows that

$$q(w, \partial \left(z + \tfrac{1}{8} r\mathbb{D} \right), \partial \left(z + \tfrac{1}{8} r\mathbb{D} \right) \cup \gamma' \cup \partial(z + \tfrac{1}{4} s\mathbb{D}), G) \leq C(r/s)^{-c} \quad .$$

It is easy to get from that, using lemmas 7 and 10 as in the proof of lemma 11 that for $w \in \partial(z + \frac{1}{2} s\mathbb{D})$

$$q(w, v, \partial S_0 \cup \gamma' \cup \{w, v\} \cup \partial(z + \tfrac{1}{4} s\mathbb{D}), G) \leq C \frac{(r/s)^{-c}}{\log sN \log rN} \leq C \frac{(r/s)^{-c}}{\log^2 N} \tag{82}$$

and the symmetry of random walk (in the form (42)) gives the same estimate for $q(v, w, \partial S_0 \cup \gamma' \cup \{w, v\} \cup \partial(z + \frac{1}{4} s\mathbb{D}), G)$. Reversing the argument used to get (82) we get

$$q(v, \partial S_3, \partial S_3 \cup \partial S_0 \cup \gamma', G) \leq C(r/s)^{-c} \quad \forall v \in \partial S_1.$$

and then of course it holds for γ as well. □

Sublemma 18.3. *For every z, s and r,*

$$\mathbb{E}\,X' \le C\epsilon^2 \log^4 M \log N \left(\frac{r}{s}\right)^{-c} q(a, B\cap(z+4s\mathbb{D}), B, G) \tag{83}$$

Subproof. Let $k>0$ be some integer, and, with the same S_0 and S_1 as above, define times s_1 and s_2 by

$$\begin{aligned} s_1 &:= \max\{t^a(1) < t < t^a(2k+1) : R^a(t) \in \partial S_0\} \\ s_2 &:= \min\{t > s_1 : R^a(t) \in \partial S_1\} \end{aligned}$$

— if $R^a([t^a(1), t^a(2k+1)]) \cap \partial S_0 = \emptyset$ define both to be $t^a(2k+1)$. We notice that R^a from s_1 (and therefore from s_2 as well) to $t^a(2k+1)$ is a random walk conditioned not to hit ∂S_0. Lemma 7 gives

$$q(v, \partial S_3, \partial S_0 \cup \partial S_3, G) \ge \frac{c}{\log r/s} \quad \forall v \in \partial S_1$$

so (81) gives for any path γ from ∂S_0 to ∂S_3

$$\mathbb{P}(R^a([s_2, t^a(2k+1)]) \cap \gamma = \emptyset \,|\, R^a([0,s_2])) \le C(r/s)^{-c}\log r/s \le C(r/s)^{-c}$$

and in particular this is true for $\gamma \in \Gamma(s_2, \frac{1}{2}r)$ so

$$\mathbb{E}\left(\#\Gamma\left(t^a(2k+1), \tfrac{1}{2}r\right) \,\middle|\, k^a > 0\right) \le C(r/s)^{-c} \cdot \max \#\Gamma(s_2, \tfrac{1}{2}r) \le Ck(r/s)^{-c} \quad .$$

Denoting $\Gamma_k' := \Gamma(t^a(2k+1), \frac{1}{2}r)$ we get

$$\mathbb{E}\,\Delta_k' \le \mathbb{E}\sum_{\gamma\in\Gamma_k'} \max_{v\in\partial S_3} \delta_k'(v,\gamma) \le C\epsilon^2 k \left(\frac{r}{s}\right)^{-c} \mathbb{P}(k^a > 0)$$

and summing over k we get

$$\mathbb{E}\,X' \le \sum_{k=1}^{\infty} \mathbb{E}\,\Delta_k' \le M^{-C} + C\epsilon^2 \log^4 M \left(\frac{r}{s}\right)^{-c} \mathbb{P}(k^a > 0) \quad .$$

The only thing left is to notice that (83) is obvious when $B \cap S_2 = \emptyset$ so we can assume it has at least one point. This implies that the probability to hit $B\cap(z+4s\mathbb{D})$ when starting from an arbitrary point in ∂S_3 before exiting from $z+4s\mathbb{D}$ is $\ge \frac{c}{\log sN} \ge \frac{c}{\log N}$. Therefore $\mathbb{P}(k^a>0) \le C\log N \cdot q(a, B\cap(z+4s\mathbb{D}), B, G)$ and the sublemma is proved. □

Lemma 18 now follows by summing over z. Let $z_1, \dots, z_l$ be the points of $s\mathbb{Z}^2 \cap [-M,M]^2$, so $l \le C(M/s)^2$. Then $[-M,M]^2 \subset \bigcup z_i + s\mathbb{D}$ and therefore (80) gives

$$\begin{aligned} \mathbb{E}(\#\{z : \mathrm{LE}(R^a) \in \mathrm{QL}(r,\epsilon,z)\}) &\le \sum_{i=1}^{l} \mathbb{E}\,X(z) \le \\ &\le C\left(\frac{M}{s}\right)^2 s^{-c}\epsilon^C \log^6 M + \sum_{i=1}^{l} \mathbb{E}\,X'(z) \end{aligned}$$

which makes it clear that for some c_{11}, taking $s = M^{1-c_{11}}$ would make the first summand $\le C\epsilon^C M^{-c}$. For the sum on $\mathbb{E}\,X'$ we use (83) to get

$$\sum_{i=1}^{l} \mathbb{E}\,X'(z) \quad \le \quad C\epsilon^2 \log^4 M \log N \left(\frac{r}{s}\right)^{-c} \sum_{i=1}^{l} q(a, B\cap(z_i+4s\mathbb{D}), B, G)$$

$$\leq \quad C\epsilon^2 \log^4 M \log N \left(\frac{r}{s}\right)^{-c} \cdot C$$

so by taking $c_{10} = \frac{1}{2}c_{11}$ we get that the second summand is $\leq C\epsilon^C M^{-c} \log N$ and the lemma is finished. □

Remark. As in [S00], this result (in the case $D = \emptyset$, i.e. a regular random walk) implies that for every open bounded set $\mathcal{D}$, every subsequence limit of the random walks on $G := \delta\mathbb{Z}^2$ starting from some $a \in \mathcal{D}$ and stopped on $\partial_G \mathcal{D}$ is supported on the set of simple paths (this follows from lemma 18 exactly like theorem 1.1 in [S00] follows from lemma 3.4 ibid.). Thus we get a strengthening of the second statement of the above mentioned theorem 1.1 — it is now true for any open set $\mathcal{D}$, without the restriction that the diameter of every component of $\partial\mathcal{D}$ is positive. The example on page 42 shows that in this setting the formulation using subsequence limits is necessary as the limit does not necessarily exists.

4.2. **Symmetry.**

Main lemma. *Let $\mathcal{D} \subset [-1,1]$ be an open polygon; let $\delta > 0$ and for $i = 1,2$ let $G_i = 2^{1-i}\delta\mathbb{Z}^2$. Let $a \in \mathcal{D}$ and let a_i be the point of G_i closest to a. Let R_i be a random walk in G_i from a_i to $\partial_{G_i}\mathcal{D}$. Let $\mathcal{E} \subset \mathcal{D}$ be some open set, $a \in \mathcal{E}$. Then*

$$\mathbb{P}(\mathrm{LE}(R_2) \subset \mathcal{E} + C_6\delta^{c_{12}}\mathbb{D}) \quad > \quad \mathbb{P}(\mathrm{LE}(R_1) \subset \mathcal{E}) - C\delta^c \tag{84}$$
$$\mathbb{P}(\mathrm{LE}(R_1) \subset \mathcal{E} + C_6\delta^{c_{12}}\mathbb{D}) \quad > \quad \mathbb{P}(\mathrm{LE}(R_2) \subset \mathcal{E}) - C\delta^c \tag{85}$$

for $\delta < \delta_0(\mathcal{D})$. The constants C_6 and c_{12} are independent of $\mathcal{D}$, $\mathcal{E}$, δ and a.

Proof. We begin with the proof of (84). Let M be some value — we shall fix the best value for M later. Let $N = \frac{1}{M\delta}$. One of the conditions on M will be that $N \in \mathbb{N}$. Define

$$\mu := N^{-1/3} \log^3 N \log^2 M$$

and

$$p_1 := \mathbb{P}(\mathrm{LE}(R_1) \subset \mathcal{E}) \quad .$$

step 1: Define subsets $Y \subset X \subset \mathbb{Z}^2$ as follows:

$$X := \{x : x + [0,1]^2 \cap M\mathcal{D} \neq \emptyset\}$$

$$Y := \left\{x \in X \,:\, d(x, M(\partial\mathcal{E} \setminus \partial\mathcal{D})) > M^{1-c_{10}} + 3\right\} \cap \{d(x, \partial_{\mathbb{Z}^2} M\mathcal{D}) > 3\} \quad .$$

where c_{10} is taken from lemma 18. For every $0 \leq k \leq \#Y$, define $H_{1,k} := G(D_k, N)$ to be a hybrid graph with D_k a random subset of Y of size k. Let $a_{1,k}$ be the point of $H_{1,k}$ closest to Ma. Let $S_{1,k}$ be a random walk on $H_{1,k}$ starting from $a_{1,k}$ and stopped on $B_{1,k} := \partial_{H_{1,k}} M\mathcal{D}$. Let

$$p_{2,k} := \mathbb{P}(\mathrm{LE}(S_{1,k}) \subset M\mathcal{E})$$

(notice that $p_1 = p_{2,0}$).

Sublemma M.1. *With the definitions above*

$$|p_{2,k} - p_{2,k+1}| \leq CM^{-c}\left(\frac{1}{k+1} + \frac{1}{\#Y - k}\right) + C\mu \quad . \tag{86}$$

Subproof. We may couple D_k and D_{k+1} and assume that $D_k \subset D_{k+1}$ (one may either think about D_{k+1} as D_k with a random point from $Y \setminus D_k$ added or about D_k as D_{k+1} with a random point removed). Let $\{z\} := D_{k+1} \setminus D_k$. Let $Z = z + [-1, 2]^2$. We construct $\mathrm{LE}(S_{1,k})$ as follows:

- let b_k be a random point on $B_{1,k}$ chosen with the hitting probabilities of $S_{1,k}$.
- Let $\check{S}_k$ be a random walk from $a_{1,k}$ to $B_{1,k}$ conditioned to hit b_k.
- Let $\check{\gamma}_k$ be random simple path from $a_{1,k}$ to $\big(\partial_{H_{1,k}} Z\big) \cup \{b_k\}$, which has the same distribution as the segment of $\mathrm{LE}(\check{S}_k)$ until Z, or all of $\mathrm{LE}(\check{S}_k)$ if $\mathrm{LE}(\check{S}_k) \cap \partial Z = \emptyset$. In particular, if $a_{1,k} \in Z$, then $\check{\gamma}_k = \{a_{1,k}\}$.
- Let c_k be the point where $\check{\gamma}_k$ hits Z if it does. If $a_{1,k} \in Z$ let $c_k = a_{1,k}$.
- Let $\check{T}_k$ be a random walk on $H_{1,k}$ starting from b_k and conditioned to hit $B_{1,k} \cup \check{\gamma}_k$ in c_k, or $\emptyset$ if $\check{\gamma}_k$ never hits Z.
- Let $\gamma_k = \check{\gamma}_k \cup \mathrm{LE}(\check{T}_k)$.

An easy application of lemma 2 (symmetry of conditioned loop-erased random walk) shows that $\gamma_k \sim \mathrm{LE}(S_{1,k})$. Lemma 16 shows that, if $N > CM^{C_3}$,

$$\begin{aligned} \sum_b |q_k^1 - q_{k+1}^1| &\leq CN^{-1/3} \log N \log^2 M \leq C\mu \\ q_k^1(b) &:= \mathbb{P}(b_k = b) \quad . \end{aligned} \tag{87}$$

Next we use lemma 17 for the random walk on $H_{1,k}$ starting from $a_{1,k}$, stopped on $B_{1,k}$, and conditioned to hit b_k (notice that the condition $d(z, \partial_{\mathbb{Z}^2} M\mathcal{D}) > 3$ for all $z \in Y$ ensures the condition $B_{1,k} \cap (z + [-1, 2]^2) = \emptyset$ required by lemma 17). This shows that

$$\begin{aligned} \sum_\gamma |q_k^2 - q_{k+1}^2| &\leq C\mu \qquad \forall b \in B_{1,k} \\ q_k^2(b, \gamma) &:= \mathbb{P}(\check{\gamma}_k = \gamma \,|\, b_k = b) \quad . \end{aligned} \tag{88}$$

Thirdly, we use lemma 17, this time for a random walk starting from b_k, stopped on $B_{1,k} \cup \check{\gamma}_k$ and conditioned to hit $\check{c}_k$ to show that, when $\check{\gamma}'_k$ is the portion of $\mathrm{LE}(\check{T}_k)$ up to Z,

$$\begin{aligned} |q_k^3 - q_{k+1}^3| &\leq C\mu \qquad \forall b, \gamma \\ q_k^3(b, \gamma) &:= \mathbb{P}(\check{\gamma}'_k \subset M\mathcal{E} \,|\, b_k = b \ \wedge \ \gamma_k = \gamma) \quad . \end{aligned} \tag{89}$$

Summing (87), (88) and (89) gives

$$\begin{aligned} &|\,\mathbb{P}((\check{\gamma}_k \cup \check{\gamma}'_k) \subset M\mathcal{E}) - \mathbb{P}((\check{\gamma}_{k+1} \cup \check{\gamma}'_{k+1}) \subset M\mathcal{E})| = \\ &\qquad = \left| \sum_{b,\gamma} q_k^1 q_k^2 q_k^3 - q_{k+1}^1 q_{k+1}^2 q_{k+1}^3 \right| \\ &\qquad \leq C\mu + \left| \sum_{b,\gamma} (q_k^1 q_k^2 - q_{k+1}^1 q_{k+1}^2) q_k^3 \right| \\ &\qquad \leq C\mu \end{aligned} \tag{90}$$

In other words, we have proved that the probabilities (for k and $k+1$) that both segments of $\mathrm{LE}(S_{1,k})$, leading up to Z and from Z to $B_{1,k}$ to be in $\mathcal{E}$ are close. Thus the only case we haven't covered is of $\mathrm{LE}(S_{1,k})$ doing a loop inside Z. This would be a quasi-loop with $\epsilon \le d(z, \partial_{H_{1,k}} Z) < 3$. Since $d(Z, M(\partial\mathcal{E} \setminus \partial\mathcal{D})) > M^{1-c_{10}}$ we can denote

$$q_k^4(s) := \mathbb{P}(\mathrm{LE}(S_{1,k}) \in \mathrm{QL}(M^{1-c_{10}}, 3, s))$$

and then write (90) as

$$|p_{2,k} - p_{2,k+1}| \le C\mu + q_k^4(z) + q_{k+1}^4(z) \tag{91}$$

The estimate of (91) is where the random choice of z plays its part. Lemma 18 gives us that

$$\sum_{s \in Y \setminus D_k} q_k^4(s) \le \sum_{s \in Y} q_k^4(s) \le CM^{-c}$$

and since z is chosen randomly from $Y \setminus D_k$ we have

$$\mathbb{E}\, q_k(z) \le \frac{CM^{-c}}{\#(Y \setminus D_k)} = \frac{CM^{-c}}{\#Y - k}$$

and therefore

$$\mathbb{P}(\mathrm{LE}(S_{1,k}) \in \mathrm{QL}(M^{1-c_{10}}, 3, z)) = \mathbb{E}\, q_k(z) \le \frac{CM^{-c}}{\#Y - k} \quad .$$

For $q_{k+1}(z)$ we similarly have

$$\sum_{z \in D_{k+1}} q_{k+1}(z) \le CM^{-c}$$

and since z can also be thought of as being chosen randomly from D_{k+1} we get

$$\mathbb{E}\, q_{k+1}(z) \le \frac{CM^{-c}}{k+1}$$

and the sublemma is proved. □

step 2: Define

$$\begin{aligned} p_3 &:= \mathbb{P}(\mathrm{LE}(S_{1,\#Y}) \subset M\mathcal{E}_2) \\ \mathcal{E}_2 &:= \mathcal{E} + (2M^{-c_{10}} + 6M^{-1})\mathbb{D} \quad . \end{aligned}$$

Clearly this gives

$$p_3 > p_{2,\#Y} \tag{92}$$

and

$$d(M(\partial\mathcal{E}_2 \setminus \partial\mathcal{D}), Y') > M^{1-c_{10}} + 3$$

where

$$Y' := X \setminus (Y \cup \{z : d(z, \partial_{\mathbb{Z}^2} M\mathcal{D}) > 3) \quad .$$

step 3: As in step 1, for every $0 \le k \le \#Y'$, define $H_{2,k} = G(D_{2,k}, N)$ to be a hybrid graph with $D_{2,k} = Y \cup D'_k$ and D'_k a random subset of Y' of size k. Again, let $a_{2,k}$ be the point of $H_{2,k}$ closest to Ma, let $S_{2,k}$ be a random walk on $H_{2,k}$ starting from $a_{2,k}$ and stopped on $\partial_{H_{2,k}} M\mathcal{D}$, and let

$$p_{4,k} := \mathbb{P}(\mathrm{LE}(S_{2,k}) \subset M\mathcal{E}_2) \quad .$$

Again notice that $p_{4,0} = p_3$.

Sublemma M.2. *With the definitions above*

$$|p_{4,k}-p_{4,k+1}|\leq CM^{-c}\left(\frac{1}{k+1}+\frac{1}{\#Y'-k}\right)+C\mu \quad . \tag{93}$$

The proof of this sublemma is identical to that of sublemma M.1 and we shall omit it.

step 4: Define $H_3=H_{2,\#Y'}$, $a_3=a_{2,\#Y'}$ and S_3 a random walk on H_3 starting from a_3 and stopped on $\partial_{H_3}M\mathcal{D}_2$ where

$$\mathcal{D}_2:=\mathbb{C}\setminus\overline{(\mathbb{C}\setminus\mathcal{D})+5M^{-1}\mathbb{D}} \quad .$$

If $a_3\notin M\mathcal{D}_2$, let S_3 be the trivial path $\{a_3\}$. Let

$$p_5:=\mathbb{P}(\mathrm{LE}(S_3)\subset M\mathcal{E}_3)$$

where

$$\mathcal{E}_3:=\mathcal{E}_2+M^{-c_{10}}\mathbb{D}$$

Sublemma M.3. *For M sufficiently large*

$$p_5>p_{4,\#Y'}-CM^{-c} \quad . \tag{94}$$

Subproof. $\mathbb{C}\setminus\mathcal{D}$ has a finite number of connected components, $\{T_i\}$. The quantity that interests us is

$$\tau:=\min_{T_i}\{\operatorname{diam}T_i\} \quad .$$

Now the walks $S_{2,\#Y'}$ and S_3 are walks on the same graph stopped at $\partial M\mathcal{D}$ and $\partial M\mathcal{D}_2$ respectively. Therefore if we define t_1 and t_2 to be the stopping times of S_3 on $\partial M\mathcal{D}$ and $\partial M\mathcal{D}_2$ (define $t_2=0$ if $a\notin M\mathcal{D}_2$) then the question reduces to an estimate of

$$\mathbb{P}(\mathrm{LE}(S_3([0,t_2]))\not\subset M\mathcal{E}_3\ \wedge\ \mathrm{LE}(S_3([0,t_1]))\subset M\mathcal{E}_2) \quad . \tag{95}$$

Let T be the graph-connected-component of $H_3\setminus(M\mathcal{D})^\circ$ closest to $S_3(t_2)$ — the definition of $\mathcal{D}_2$ gives that $d(T,S_3(t_2))\leq 6$. It's easy to see that $\operatorname{diam}T\geq\tau M-3$, and then get from lemma 9 that

$$\mathbb{P}(S_3([t_2,t_1])\text{ exits }S_3(t_2)+\epsilon\mathbb{D})\leq C\left(\min(\tau M-3,\epsilon)\right)^{-c}\quad\forall\epsilon\geq 1. \tag{96}$$

On the other hand, if $S_3([t_2,t_1])\subset S_3(t_2)+\epsilon\mathbb{D}\subset S_3(t_1)+2\epsilon\mathbb{D}$ and in addition the event of (95) hold then we can conclude that $\mathrm{LE}(S_3[0,t_2])\in\mathrm{QL}(M^{1-c_{10}},2\epsilon,S_3(t_1))$, and lemma 18 gives the bound

$$\mathbb{P}\left(\mathrm{LE}(S_3[0,t_2])\in\mathrm{QL}\left(M^{1-c_{10}},2\epsilon,S_3(t_1)\right)\right)\leq CM^{-c}\epsilon^C \quad . \tag{97}$$

We choose $\epsilon=M^c$ with c sufficiently small and combine (96) and (97) to get the required estimate of (95) which holds whenever $\epsilon\leq\tau M-3$ or equivalently

$$M>\tau^{-c}+C \quad . \tag{98}$$

□

step 5: Define $H_4=G([-M,M]^2\cap\mathbb{Z}^2,N)$, a_4 the point of H_4 closest to a and S_4 a random walk on H_4 starting from a_4 and stopped on $\partial_{H_4}M\mathcal{D}$. Let

$$p_6:=\mathbb{P}(\mathrm{LE}(S_4)\subset M\mathcal{E}_4)$$

where

$$\mathcal{E}_4:=\mathcal{E}_3+M^{-1/2}\mathbb{D}$$

Sublemma M.4. *With the definitions above*

$$p_6 > p_5 - CM^{-c} \quad . \tag{99}$$

Subproof. As in sublemma M.3, we need to show that

$$\mathbb{P}(\mathrm{LE}(S_4([0,t_1]) \not\subset M\mathcal{E}_4 \ \wedge\ \mathrm{LE}(S_4([0,t_2]) \subset M\mathcal{E}_3) \le CM^{-c}$$

with the same t_1 and t_2. Unlike in sublemma M.3, this requires no recourse to lemma 18 but rather follows directly from lemma 9 since this event implies that $S_4[t_2,t_1] \not\subset S_4(t_2) + M^{1/2}\mathbb{D}$ whose probability can be bounded by

$$C\left(\min(\tau M - 3, M^{1/2})\right)^{-c}$$

and if (98) is fulfilled then this is $\le CM^{-c}$. □

final step: At this point our environment is no longer hybrid[10] — in effect $H_4 \cap \mathcal{D} \equiv (\frac{1}{2}M\delta)\mathbb{Z}^2 \cap \mathcal{D}$. Thus we can return to the notations of G_i, R_i etc. and get

$$p_6 = \mathbb{P}(\mathrm{LE}(R_2) \subset \mathcal{E}_4)$$

Summing up (86), (92), (93), (94) and (99) we get

$$p_6 > p_1 - CM^{-c}\log M + CM^2\mu \quad . \tag{100}$$

The only thing left now is to choose M. The following conditions must be met:

1. $CM^2\mu \le CM^{-c}$;
2. $N > CM^{C_3}$ — this will also give that $H_{1,k}$ and $H_{2,k}$ are admissible;
3. $N \in \mathbb{N}$;
4. $M > \tau^{-1/2} + C$ (that's (98) on the preceding page).

For some c_{13} sufficiently small, if we choose $M \approx \delta^{-c_{13}}$ then we will have $N \approx M^{\frac{1-c_{13}}{c_{13}}}$ and therefore (say take $c_{13} < \frac{1}{7}$) that $M^2\mu < CM^{-c}$. Requirement 2 will also follow if c_{13} is sufficiently small — this depends on the constant C_3 that appears in lemma 6. Since C_3 can be chosen to be any value > 2 then the restriction on our c_{13} is in effect only the weaker $c_{13} < \frac{1}{3}$. To fulfill condition 4, we need some assumption on δ: $\delta < \delta_0(\mathcal{D}) = c\tau^C$ will be enough. Clearly condition 3 is no obstacle. Plugging this into (100) will give

$$p_6 > p_1 - CM^{-c} > p_1 - C\delta^c \quad .$$

On the other hand, for an appropriate C_6 and c_{12}, $\mathcal{E}_4 \subset \mathcal{E} + C_6\delta^{c_{12}}\mathbb{D}$. This finishes (84).

The proof of (85) is identical, with D and $([-M,M] \cap \mathbb{Z}^2) \setminus D$ replaced everywhere. Thus ends the main lemma. □

Remarks. 1. The requirement from $\mathcal{D}$ to be a polygon was rather excessive. In effect we used it only in steps 4 and 5 to show $\tau > 0$. Therefore the main lemma holds, for example, for any bounded domain $\mathcal{D}$ with no punctures (here we mean punctures in the sense of connected components of $\mathbb{C} \setminus \mathcal{D}$ with only one point, but not necessarily isolated). It is not difficult to see that punctures in $\mathcal{D}$ would

[10]Actually, it was already true in step 5.

require to reformulate the main lemma so as to take into consideration the distance between a and the nearest puncture. See also the example on page 42 for the problems punctures could bring about.
2. The division into z's close to $\partial M\mathcal{D}$ and far from it is not really necessary — it is possible to extend lemmas 16 and 17 to work when $B \cap Z \neq \emptyset$ and thus save steps 4 and 5 in the main lemma. However, with this extension the formulation of lemmas 16 and 17 is very awkward. We would need two B_i which are "almost similar", two b_i's, and make provisions for the cases when $\kappa_{b_1} \neq \kappa_{b_2}$ since the probability to hit b_i depends on κ_{b_i} (see (64)). The proofs (especially that of lemma 16) would also suffer from a canworm of geometric issues.
3. An alternative to the use of random hybrid graphs, is to randomize the starting point a. This would give similar results (especially with results of the next section).
4. I am happy to promise to my readers that this is the last time the term "hybrid graph" is mentioned in this paper. Or, to be more precise, we will still refer to some lemmas formulated using hybrid graphs — particularly to the ubiquitous lemma 9 — but only for the non-hybrid case i.e. $D = \emptyset$.

4.3. **Continuity.** In this section we prove some simple estimates that show that the probability of a loop-erased random walk to be in a set is continuous in the point of departure, the set and the environment.

Lemma 19. *Let $\mathcal{E}$ and $\mathcal{D}$ be open sets. Let $v, w \in \mathcal{E} \cap \mathcal{D} \cap \mathbb{Z}^2$. Let R^x be a random walk on $\mathbb{Z}^2$ started from x and stopped on $\partial_{\mathbb{Z}^2}\mathcal{D}$. Then*

$$|\,\mathbb{P}(\mathrm{LE}(R^v) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R^w) \subset \mathcal{E})| \leq C \left(\frac{d(v, \partial\mathcal{E} \cup \partial\mathcal{D})}{d(v,w)} \right)^{-c}$$

Proof. Denote $\mu = d(v, \partial\mathcal{E} \cup \partial\mathcal{D})/d(v,w)$. Let S^w be a random walk started from w and stopped on $\mathrm{LE}(R^v) \cup \partial G$. Lemma 9 says that the probability of S^w to hit R^v before exiting $\mathcal{E} \cap \mathcal{D}$ is $\leq C\mu^{-c}$. But since Wilson's algorithm says that $\mathrm{LE}(R^w)$ has the same distribution as $\mathrm{LE}(S^w)$ unioned with the segment of $\mathrm{LE}(R^v)$ from $S^w \cap \mathrm{LE}(R^v)$ to ∂G, the lemma is finished. □

Definition. For $\mathcal{E}, \mathcal{D}$ open and for $r > 0$ we define $X_1(r; \mathcal{D}) \subset \mathbb{Z}^2 \setminus \mathcal{D}^\circ$ to be the union of all graph-connected-components of $\mathbb{Z}^2 \setminus \mathcal{D}^\circ$ of diameter $< r$. Next we define

$$X_2(r; \mathcal{E}, \mathcal{D}) := \{x \in \partial_{\mathbb{Z}^2}\mathcal{D} \,:\, d(x, \partial\mathcal{E}) < r\} \tag{101}$$

and thirdly $X_3 := X_1 \cup X_2$. Next, for $a \in \mathcal{D} \cap \mathbb{Z}^2$ and for $i = 1, 2, 3$, define

$$\rho_i(r; a, \mathcal{E}, \mathcal{D}) := q(a, X_i, \partial\mathcal{D}, \mathbb{Z}^2) \quad .$$

and

$$\rho_i(r, \delta; a, \mathcal{E}, \mathcal{D}) := \rho_i(\delta^{-1}r; \lfloor \delta^{-1}a \rfloor, \delta^{-1}\mathcal{E}, \delta^{-1}\mathcal{D}) \quad .$$

The "good" sets (or rather triplets a, $\mathcal{E}$, $\mathcal{D}$) are the ones satisfying

$$\lim_{(r,\delta)\to(0,0)} \rho_3(r, \delta) = 0 \quad .$$

There are counter example, though. In the example on page 42, with $\mathcal{E} =]-2,2[^2$, we have $\limsup \rho_1 > 0$. It is also possible to construct non-trivial examples where the culprit is ρ_2.

Lemma 20. *Let $0 < s < r$. Let $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{E}$ be open sets and assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are similar in the following sense:*

$$d(v, \partial\mathcal{D}_1 \setminus X_3(r; \mathcal{E}, \mathcal{D}_1)) \leq s \quad \forall v \in \partial\mathcal{D}_2 \setminus X_3(r+s; \mathcal{E}, \mathcal{D}_2) \tag{102}$$

and similarly with $\mathcal{D}_1$ and $\mathcal{D}_2$ replaced. Let $a \in \mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{E} \cap \mathbb{Z}^2$. Let R_i be random walks on $\mathbb{Z}^2$ started from a and stopped on $\partial_{\mathbb{Z}^2}\mathcal{D}_i$. Then

$$\begin{aligned} |p_1 - p_2| &\leq \rho_3(r+s; a, \mathcal{E}, \mathcal{D}_1) + \rho_3(r+s; a, \mathcal{E}, \mathcal{D}_2) + C\left(\frac{r}{s}\right)^{-c} \\ p_i &:= \mathbb{P}(\mathrm{LE}(R_i) \subset \mathcal{E}) \quad . \end{aligned}$$

If $\delta \neq 1$ this is true if (102) holds with a δ version of X_3: $X_3(r, \delta; \mathcal{E}, \mathcal{D}) := \delta X_3(\delta^{-1}r; \delta^{-1}\mathcal{E}, \delta^{-1}\mathcal{D})$.

Proof. This follows from lemmas 9 and 18 — see steps 4 and 5 of the main lemma on pages 36–37 for a more detailed version of this argument. □

Lemma 21. *Let $\mathcal{E}, \mathcal{D}$ be open sets. Let $v \in \mathcal{D} \cap \mathbb{Z}^2$. Let R_δ be a random walk on $\delta\mathbb{Z}^2$ started from v and stopped on $\partial\mathcal{D}$. Then*

$$\limsup_{\substack{(\tau,\delta)\to(0,0) \\ \tau\in\delta\mathbb{Z}^2}} |\,\mathbb{P}(\mathrm{LE}(R_\delta) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R_\delta) \subset \mathcal{E} + \tau)| \leq 3 \limsup_{(r,\delta)\to(0,0)} \rho_3(r, \delta; a, \mathcal{E}, \mathcal{D})$$

Proof. Let R'_δ be a random walk stopped on $\partial\mathcal{D} - \tau$. We use lemma 20 with $s = |\tau|$ and some $r > |\tau|$ to get

$$\begin{aligned} &|\,\mathbb{P}(\mathrm{LE}(R_\delta) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R'_\delta) \subset \mathcal{E})| \leq \\ &\qquad \leq \rho_3(r + |\tau|, \delta; \mathcal{D}) + \rho_3(r + |\tau|, \delta; \mathcal{D} - \tau) + C\left(\frac{r}{|\tau|}\right)^{-c} \quad . \end{aligned} \tag{103}$$

Now to estimate $\rho_3(\mathcal{D} - \tau)$, we write $\rho_3 \leq \rho_1 + \rho_2$ and an argument like lemma 19 shows

$$\rho_1(r, \delta; a, \mathcal{D} - \tau) = \rho_1(r, \delta; a + \tau, \mathcal{D}) \leq \rho_1(r, \delta; a, \mathcal{D}) + C\left(\frac{d(a, \partial\mathcal{D})}{\tau}\right)^{-c} \tag{104}$$

and similarly

$$\begin{aligned} \rho_2(r, \delta; a, \mathcal{D} - \tau, \mathcal{E}) &\leq \rho_2(r + |\tau|, \delta; a + \tau, \mathcal{D}, \mathcal{E}) \leq \\ &\leq \rho_2(r + |\tau|, \delta; a, \mathcal{D}, \mathcal{E}) + C\left(\frac{d(a, \partial\mathcal{D} \cup \partial\mathcal{E})}{\tau}\right)^{-c} \end{aligned} \tag{105}$$

Next we define R''_δ to be a random walk starting from $a - \tau$ stopped on $\mathcal{D} - \tau$ and again use lemma 19 to get

$$|\,\mathbb{P}(\mathrm{LE}(R'_\delta) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R''_\delta) \subset \mathcal{E})| \leq C\left(\frac{d(a, \partial\mathcal{D} \cup \partial\mathcal{E})}{|\tau|}\right)^{-c} \quad . \tag{106}$$

Summing (103), (104), (105) and (106) and estimating $\rho_1, \rho_2 \leq \rho_3$ we get

$$\begin{aligned} &|\,\mathbb{P}(\mathrm{LE}(R_\delta) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R_\delta) \subset \mathcal{E} + \tau)| = \\ &\qquad = |\,\mathbb{P}(\mathrm{LE}(R_\delta) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R''_\delta) \subset \mathcal{E})| \leq \end{aligned} \tag{107}$$

$$\leq 3\rho_3(r+2|\tau|,\delta;a,\mathcal{E},\mathcal{D}) + C\left(\frac{d(a,\partial\mathcal{D}\cup\partial\mathcal{E})}{|\tau|}\right)^{-c} + C\left(\frac{r}{|\tau|}\right)^{-c}$$

and choosing $r = \sqrt{|\tau|}$ will make the two summands on the right of (107) converge to 0 when $(\tau,\delta)\to(0,0)$. □

Lemma 22. *Let $a \in \mathcal{E}\cap\mathcal{D}$ where $\mathcal{E}$ is a polygon, and assume*

$$\lim_{(r,\delta)\to(0,0)} \rho_3(r,\delta;a,\mathcal{E},\mathcal{D}) = 0 \quad . \tag{108}$$

Let R_δ be a random walk on $G := \delta\mathbb{Z}^2$ starting from a and stopped on $\partial_G\mathcal{D}$. Then

$$\lim_{(\epsilon,\delta)\to(0,0)} |\,\mathbb{P}(\mathrm{LE}(R_\delta)\subset\mathcal{E}) - \mathbb{P}(\mathrm{LE}(R_\delta)\subset\mathcal{E}+\epsilon\mathbb{D})| = 0 \quad . \tag{109}$$

Proof. For A, B satisfying $A\cap\mathcal{D}\subset B\cap\mathcal{D}$ we denote

$$F(A,B) := \mathbb{P}(\emptyset \neq (\mathrm{LE}(R_\delta)\cap B)\subset A)$$

so (109) is equivalent to

$$\lim_{(\epsilon,\delta)\to(0,0)} F\left((\mathcal{E}+\epsilon\mathbb{D})\setminus\mathcal{E}, \mathcal{D}\setminus\mathcal{E}\right) = 0 \quad .$$

Let $\{\xi_i\}_{i=1}^n$ be the vertices of $\mathcal{E}$ inside $\mathcal{D}$, let $\{I_i\}_{i=1}^{n+1}$ be the segments and let $I_i^\perp$ be orthogonal segments oriented to the exterior of $\mathcal{E}$. Denote $I_i^\perp = [0,\nu_i[$, $|\nu_i| = 1$. For each ξ_i we use a simple hitting probability estimate to get

$$\begin{aligned}\mathbb{P}(\mathrm{LE}(R_\delta)\cap(\xi_i+\epsilon\mathbb{D})\neq\emptyset) &\leq \mathbb{P}(R_\delta\cap(\xi_i+\epsilon\mathbb{D})\neq\emptyset)\\ &\leq C\frac{\log(d(a,\xi_i)/\operatorname{diam}\mathcal{D})}{\log(\epsilon/\operatorname{diam}\mathcal{D})}\end{aligned}$$

which converges to zero (in effect, much more precise estimates are known — see e.g. [K00a]). Let therefore $\epsilon_1(\mu)$ and $\delta_1(\mu)$ satisfy that

$$\sum_i \mathbb{P}(\mathrm{LE}(R_\delta)\cap(\xi_i+\epsilon\mathbb{D})\neq\emptyset) < \mu \quad \forall\delta<\delta_1(\mu),\ \epsilon<\epsilon_1(\mu)$$

which gives us in F notation

$$F((\mathcal{E}+\epsilon\mathbb{D})\setminus\mathcal{E},\mathcal{D}\setminus\mathcal{E}) \leq \mu + F\left(\bigcup P_i(\epsilon), \mathcal{D}\setminus\mathcal{E}\right)$$

where $P_i(\epsilon) := \left(I_i+\epsilon I_i^\perp\right)\setminus\bigcup(\xi_i+\epsilon_1(\mu)\mathbb{D})$. Clearly, the event $\emptyset\neq\mathrm{LE}(R_\delta)\cap(\mathcal{D}\setminus\mathcal{E})\subset\bigcup P_i(\epsilon)$ implies that for at least one i, $\emptyset\neq\mathrm{LE}(R_\delta)\cap P_i(\epsilon)$. If in addition the different $P_i(\epsilon)$ are disjoint, which happens for $\epsilon<\epsilon_2(\mu)$, then we have

$$\leq \mu + \sum_i F\left(P_i(\epsilon), \mathcal{D}\setminus\left(\mathcal{E}\cup\bigcup_{j\neq i}P_j(\epsilon)\right)\right) \quad .$$

The next step are some F calculations, based on the following easy inequalities which we call the monotonicity of F:

$$\begin{aligned}F(A,B) &\leq F(A\cup C, B\cup C)\\ F(A,B) &\geq F(A, B\cup C) \quad .\end{aligned} \tag{110}$$

For $r = 1,2,3$ we define

$$J_i^r := I_i \setminus \bigcup_j \left(\xi_j + \tfrac{1}{4}r\epsilon_1(\mu)\mathbb{D}\right) \quad .$$

For a suitable $\epsilon_3(\mu)$ we shall get $P_i(\epsilon) \subset J_i^3 + \epsilon I_i^\perp$ whenever $\epsilon < \epsilon_3(\mu)$, and $(P_i(\epsilon) \cup \mathcal{E}) \cap (J_j^1 + dI_j^\perp) = \emptyset$ whenever $i \neq j$ and $\epsilon < d < \epsilon_3(\mu)$. This allows to write

$$F\left(P_i(\epsilon), \mathcal{D} \setminus \left(\mathcal{E} \cup \bigcup_{j \neq i} P_j(\epsilon)\right)\right) \leq F(P_i(\epsilon), J_i^1 + dI_i^\perp) \leq F(J_i^3 + \epsilon I_i^\perp, J_i^1 + dI_i^\perp)$$

From this point on we shall drop the notation i from $I_i^\perp$, J_i^r, and ν_i. Now, lemma 21 with (108) and the obvious

$$\rho_3(\epsilon; J^1 + dI^\perp) \leq \rho_3(\epsilon + d; \mathcal{E})$$

give that for any $\epsilon < d < \epsilon_4(\mu)$, $\delta < \delta_2(\mu)$ and for any $\lambda \in \delta\mathbb{Z}^2$, $|\lambda| < \lambda_1(\mu)$,

$$|F(J^3 + \epsilon I^\perp, J^1 + dI^\perp) - F(J^3 + \lambda + \epsilon I^\perp, J^1 + \lambda + dI^\perp)| \leq \mu \quad . \tag{111}$$

Next, if in addition $\epsilon < \frac{d}{2m}, \frac{\lambda_1(\mu)}{2m}$ and $\delta < \delta_3(m, \epsilon)$ we can pick $\lambda_1, ..., \lambda_{m-1} \in \delta\mathbb{Z}^2$, $|\lambda_i| \leq \lambda_1(\mu)$ that will satisfy

$$\begin{aligned} J^3 + \lambda_j + \epsilon I^\perp &\subset J^2 + 2j\epsilon\nu + 2\epsilon I^\perp \\ J^1 + \lambda_j + dI^\perp &\supset J^2 + 2(j+1)\epsilon\nu + 2(m-j-1)\epsilon I^\perp \end{aligned}$$

(just take λ_j similar to $(2j + \frac{1}{2})\epsilon\nu$). See figure 3 below. From these and (110) we get

$$\begin{aligned} &F(J^3 + \lambda_j + \epsilon I^\perp, J^1 + \lambda_j + dI^\perp) \leq \\ &\quad \leq F(J^3 + \lambda_j + \epsilon I^\perp, (J^1 + \lambda_j + dI^\perp) \cap (J^2 + 2j\epsilon\nu + 2(m-j)\epsilon I^\perp) \\ &\quad \leq F(J^2 + 2j\epsilon\nu + 2\epsilon I^\perp, J^2 + 2j\epsilon\nu + 2(m-j)\epsilon I^\perp) \quad . \end{aligned} \tag{112}$$

The same holds for $j = 0$ with $\lambda_0 = 0$. But clearly

$$\begin{aligned} &F(J^2 + 2m\epsilon I^\perp, J^2 + dI^\perp) = \\ &\qquad \sum_{j=0}^{m} F(J^2 + 2j\epsilon\nu + 2\epsilon I^\perp, J^2 + 2j\epsilon\nu + (d - 2j\epsilon)I^\perp) \end{aligned} \tag{113}$$

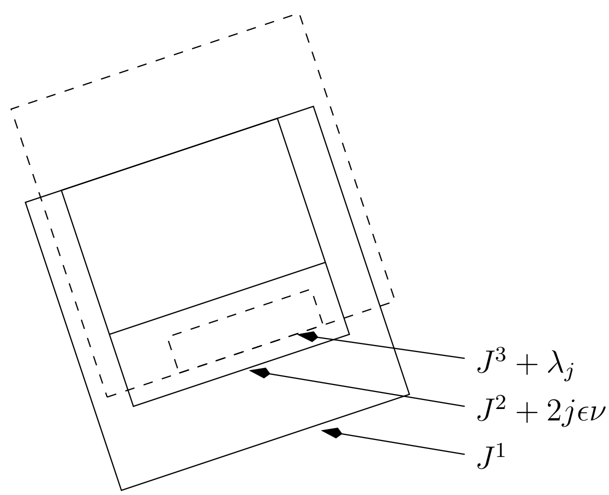

FIGURE 3. Segments shifted by multiples of $\epsilon\nu$ are in solid lines; segments shifted by λ_j are in dashed lines.

since the event $\emptyset \neq \mathrm{LE}(R_\delta) \cap J^2 + dI^\perp \subset J^2 + 2m\epsilon I^\perp$ whose probability is measured on the left can be divided according to $\max\{j : R_\delta \subset J^2 + 2j\epsilon I^\perp\}$. Summing (111), (112) and (113) we get

$$1 \geq F(J^2 + 2m\epsilon I^\perp, J^2 + dI^\perp) \geq mF(J^3 + \epsilon I^\perp, J^1 + dI^\perp) - m\mu \quad .$$

This finishes the lemma — we pick m large, then pick $\mu < \frac{1}{m}$, then $d < \epsilon_4(\mu), \lambda_1(\mu)$, then $\epsilon_{\min} := \min \epsilon_1(\mu), \epsilon_2(\mu), \epsilon_3(\mu), \frac{d}{2m}$ and for every $\delta < \delta_1(\mu)$, $\delta_2(\mu), \delta_3(m, \epsilon_{\min})$ we get

$$F(J^1 + \epsilon_{\min} I^\perp, J^3 + dI^\perp) \leq \frac{C}{m}$$

and since $F(J^1 + \epsilon I^\perp, J^3 + dI^\perp)$ is decreasing in ϵ and since $F((\mathcal{E} + \epsilon\mathbb{D}) \setminus \mathcal{E}, \mathcal{D} \setminus \mathcal{E}) \leq$ a finite sum of those, we are done. □

Lemma 23. *Let $a \in \mathcal{E} \cap \mathcal{D}$ where $\mathcal{E}$ and $\mathcal{D}$ are polygons. Let R_n be a random walk on $G := 2^{-n}\mathbb{Z}$ starting from a and stopped on $\partial_G \mathcal{D}$. Then $\mathbb{P}(\mathrm{LE}(R_n) \subset \mathcal{E})$ converges to a limit as $n \to \infty$.*

Proof. Use the main lemma (shrink $\mathcal{D}$ to fit $[-1, 1]^2$ if necessary) repeatedly to get $\forall n, m > N_0$

$$\mathbb{P}(\mathrm{LE}(R_n) \subset \mathcal{E}) \leq \mathbb{P}(\mathrm{LE}(R_m) \subset \mathcal{E} + C2^{-cN_0}\mathbb{D}) + C2^{-cN_0} \quad .$$

Since it is obvious that for $\mathcal{D}$ and $\mathcal{E}$ polygons $\rho(\epsilon) \to 0$ (if $\partial\mathcal{D} \cap \partial\mathcal{E}$ contains a segment, just extend $\mathcal{E}$ a bit so as to make $\partial\mathcal{D} \cap \partial\mathcal{E}$ finite), lemma 22 gives

$$\mathbb{P}(\mathrm{LE}(R_n) \subset \mathcal{E} + C2^{-cN_0}\mathbb{D}) \leq \mathbb{P}(\mathrm{LE}(R_n) \subset \mathcal{E}) + \mu(N_0)$$

where $\mu(N_0) \to 0$, which gives

$$|\,\mathbb{P}(\mathrm{LE}(R_n) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R_m) \subset \mathcal{E})| \leq C2^{-cN_0} + \mu(N_0) \quad .$$ □

Remark. My intuition would have been that lemma 23 — while probably being worthy of proof in its own right — wouldn't be necessary for the proof of the theorem since the weak limit is insensitive to small inflations. However, I was not able to surmount certain technical difficulties in translating this intuition into a proper proof.

5. The limit process

In this section we shall conclude the theorem from lemma 23. This is a standard limit process, so we shall explain it only briefly. We start with an example that will justify our choice of acceptable $\mathcal{D}$'s.

Example. An open set $\mathcal{D} \subset \mathbb{C}$ such that the loop-erased random walk (and even the regular random walk) from a point a to $\partial_{2^{-n}\mathbb{Z}^2}\mathcal{D}$ does not converge as $n \to \infty$ in $\mathfrak{M}(\mathfrak{H}(\mathcal{D}))$.

- Let $E' \subset \left]-\frac{1}{3}, \frac{1}{3}\right[$ be a closed set with $mE' \geq \frac{1}{2}$ and $E' \cap \mathbb{Q} = \emptyset$, $\mathbb{Q}$ being the rationales so E' is a Cantor-like set. Let
$$E = \left(E' \pm i\tfrac{1}{3}\right) \cup \left(iE' \pm \tfrac{1}{3}\right) \quad .$$
- Let n_k be a sequence converging to ∞ sufficiently fast (we shall specify them later).
- Let $r_k \in 2^{-n_k}\mathbb{Z} \setminus 2^{-n_k+1}\mathbb{Z}$ satisfy $|r_k - \frac{1}{3}| \leq 2^{-n_k+1}$.

- Let
$$P'_k := \left(2^{-n_k}\mathbb{Z} \setminus 2^{-n_k+1}\mathbb{Z}\right) \cap \left\{x \,:\, d(x, E') < \frac{1}{k}\right\}$$
$$P_k := (P'_k \pm ir_k) \cup (iP'_k \pm r_k)$$
- Let
$$\mathcal{D} = [-1,1]^2 \setminus \left(E \cup \bigcup_{k=1}^{\infty} P_k\right)$$

It is easy to see that $\mathcal{D}$ is an open set.

Lemma 24. *With an appropriate choice of n_k, the probabilities*
$$q(0, \partial[-1,1]^2, \partial\mathcal{D}, 2^{-n}\mathbb{Z}^2)$$
do not converge.

This of course implies that the distributions of the regular random walk do not converge in $\mathfrak{M}(C([0,\infty[\to \mathcal{D}))$, the distributions of the loop-erased random walk do not converge in $\mathfrak{M}(\mathfrak{H}(\mathcal{D}))$ and would probably exclude convergence in any reasonable topology.

Proof. Denote $\mathcal{D}_k = [0,1]^2 \setminus \bigcup_{j=1}^{k} P_j$. Because P'_k, r_k and E' all avoid $2^{-n}\mathbb{Z}$ for all $n < n_1$, and moreover, avoid all edges of this graph (when viewed as line segments in $\mathbb{C}$), we get that R_n is identical to a walk on $2^{-n}\mathbb{Z}^2 \cap \mathcal{D}_0 \equiv 2^{-n}\mathbb{Z}^2 \cap [-1,1]^2$. At n_1 we have that $\partial\mathcal{D} \subset \partial[-r_k, r_k]^2$ satisfies that $\#\partial\mathcal{D} \geq c\#\partial[-r_k,r_k]^2$ and in particular has a hitting probability $\geq c$. For $n_1 \leq n < n_2$, however, $\partial\mathcal{D} = \partial\mathcal{D}_1$ and since it is just a finite set of points, for n_2 sufficiently large the hitting probability of $\partial\mathcal{D}_1$ can be made as small as desired. For $n = n_2$ we have $\partial\mathcal{D} = \partial\mathcal{D}_2$ and again has hitting probability $\geq c$, etc. To sum it all up, if n_k increases sufficiently fast, we have that the probability of R_n to hit $\partial[-1,1]^2$ fluctuates between 1 and c. This c can be chosen arbitrarily close to 0 merely by changing E'. □

In view of this example, we must somehow restrict the $\mathcal{D}$'s we talk about. We shall examine the class of bounded finitely-connected open sets.

Lemma 25. *The conclusion of lemma 23 holds for any open, bounded and finitely connected $\mathcal{D}$ if*
$$\lim_{(\epsilon,n)\to(0,\infty)} \rho_3(\epsilon, 2^{-n}; a, \mathcal{D}, \mathcal{E}) \to 0 \quad .$$

Proof. Let $\{H_i\}_{i=0}^m$ be the connected components of $\mathbb{C} \setminus \mathcal{D}$, H_0 being the unbounded one. Let $\mu > 0$ be some parameter. Let P_i be simply connected polygons with $d_{\mathfrak{H}}(\partial H_i, \partial P_i) \leq \mu$. Assume μ is sufficiently small as to make the ∂P_i pairwise disjoint, and define $P = P_0 \setminus \cup_{i>0} P_i$. Our goal is to use lemma 20 for $\mathcal{D}$ and P, and we need to estimate the effects of discretization. This is easy to do, since for any X simply connected,
$$\partial_{\delta\mathbb{Z}^2} X \neq \emptyset \Rightarrow d_{\mathfrak{H}}(\partial X, \partial_{\delta\mathbb{Z}^2} X) \leq \tfrac{1}{\sqrt{2}}\delta$$
and if $Y = \left(X \cap \delta\mathbb{Z}^2\right) \cup \partial_{\delta\mathbb{Z}^2} X$ then
$$|\operatorname{diam} X - \operatorname{diam} Y| \leq \sqrt{2}\delta \quad .$$

These two imply that, when $2^{-n} < \mu$, the requirements of lemma 20 will be satisfied when $s > C\mu$ and as a result we will get,

$$|\,\mathbb{P}(\mathrm{LE}(R_n) \subset \mathcal{E}) - \mathbb{P}(\mathrm{LE}(R'_n) \subset \mathcal{E})| \leq$$
$$\leq C\left(\frac{\epsilon}{s}\right)^{-c} + \rho_3(\epsilon + s, 2^{-n}; a, \mathcal{E}, \mathcal{D}) + \rho_3(\epsilon + s, 2^{-n}; a, \mathcal{E}, P)$$

where R_n is as in lemma 23 and R'_n is a random walk stopped on ∂P. $\rho_3(P)$ can be estimated as in lemma 21 to give

$$\rho_3(\epsilon + s, 2^{-n}; a, \mathcal{E}, P) \leq 2\rho_3(\epsilon + 2s, 2^{-n}; a, \mathcal{E}, \mathcal{D}) + C\left(\frac{d(a, \partial\mathcal{D} \cup \partial\mathcal{E})}{s}\right)^{-c} \quad .$$

This — with lemma 23 — reduces our lemma to an exercise in calculus. □

Lemma 26. *For any $\mathcal{D} \subset \mathbb{C}$ open, bounded and finitely connected, and for any $a \in \mathcal{D}$,*

$$\lim_{(\epsilon,\delta)\to(0,0)} \rho_1(\epsilon, \delta; a, \mathcal{D}) = 0$$

Proof. Let $\{H_i\}_{i=1}^m$ be the connected components of $\mathbb{C} \setminus \mathcal{D}$. Let

$$\tau := \min\{\operatorname{diam} H_i \,:\, \operatorname{diam} H_i \neq 0\} \quad .$$

A simple discretization estimates shows that for $\epsilon < \tau$ and δ sufficiently small we can ignore the holes with positive diameter. For the punctures, let H be the set of punctures and let $d = d(a, H)$. Then

$$q(a, \partial H, \partial\mathcal{D}, \delta\mathbb{Z}^2) \leq Cm\frac{\log\operatorname{diam}\mathcal{D} - \log d}{\log\delta^{-1} + \log\operatorname{diam}\mathcal{D}}$$

which clearly converge to 0 as $\delta \to 0$. □

Definition. The family of polygons $\mathcal{E}$ such that

$$\rho(\epsilon, 2^{-n}; a, \mathcal{E}, \mathcal{D}) \to 0 \tag{114}$$

will be denoted by $\mathcal{X}$.

Note that $\mathcal{X}$ is closed to finite unions and intersections.

Lemma 27. *$\mathcal{X}$ is dense in the sense that for any bounded open O and for any ϵ there exists a set $V \in \mathcal{X}$ such that $O \subset V \subset O + \epsilon\mathbb{D}$.*

Proof. In view of lemma 26 we need only estimate ρ_2. Assume $\mathcal{E}$ is a polygon that satisfies the additional requirement that each final segment of $\mathcal{E}$ before meeting $\partial\mathcal{D}$ is a segment from a point ξ to the point of $\partial\mathcal{D}$ closest to ξ. Then we have, for r sufficiently small, that X_2 (from the definition of ρ_2, (101)) is contained in a finite union of balls of radius $2r$. This obviously gives that $\rho_2 \to 0$. Finally, it is a fun topological exercise to see that the additional condition above does not interfere with the density of the family of polygons (in the same sense as above). □

Definition. In $\mathfrak{H}(\bar{\mathcal{D}})$ define

$$Y(E_0; E_1, ..., E_n) := \{F \in \mathfrak{H}(\bar{\mathcal{D}}) \,:\, F \subset E_0 \;\wedge\; (F \not\subset E_i \;\forall i \geq 1)\}$$

and

$$\mathcal{Y} := \left\{\bigcup_{i=1}^{k} Y(E_0^i; E_1^i, ..., E_{n_i}^i) \,:\, E_j^i \in \mathcal{X}\right\} \quad .$$

Lemma 28. *$\mathcal{Y}$ is dense in the sense that for every open set $O \subset \mathfrak{H}(\bar{\mathcal{D}})$ and for every ϵ there exists $V \in \mathcal{Y}$ such that*

$$O \subset V \subset B(O, \epsilon)$$

where $B(O, \epsilon) := \{F \in \mathfrak{H}(\bar{\mathcal{D}}) \,:\, d(F, O) < \epsilon\}$.

The proof follows easily from lemma 27 and the existence of ϵ-nets of finite sets in $\mathfrak{H}(\bar{\mathcal{D}})$ and we shall omit it.

Lemma 29. *With the notations of lemma 25, for every $Y \in \mathcal{Y}$*

$$\mathbb{P}(\mathrm{LE}(R_n) \in Y)$$

converges.

Proof. This follows from lemma 25, the inclusion-exclusion principle and some set algebra. □

Theorem. *Let $\mathcal{D} \subset \mathbb{C}$ be an open bounded finitely connected set, let $a \in \mathcal{D}$ and let R_n be a random walk on $G := 2^{-n}\mathbb{Z}^2$ starting from a and stopped on $\partial_G \mathcal{D}$. Let μ_n be the distribution measures of $\mathrm{LE}(R_n)$. Then μ_n converge in the weak-* topology of $\mathfrak{M}(\mathfrak{H}(\bar{\mathcal{D}}))$.*

Proof. It is enough to show that $\mu_n(f)$ converges for every continuous f since this proves that μ_n converges to an arbitrarily chosen sub-sequence limit. Let f be a continuous function on $\mathfrak{H}(\bar{\mathcal{D}})$. Since $\mathfrak{H}(\bar{\mathcal{D}})$ is compact f is bounded and uniformly continuous and we may write, for every N, and for $M < \min f$,

$$f = M + \sum_{k=MN}^{\infty} \frac{1}{N} \mathbf{1}_{O_k} + E$$

where $O_k := f^{-1}\left(\left[\frac{k}{N}, \infty\right[\right)$, the sum is finite and $|E| \leq \frac{1}{N}$. Further, the uniform continuity of f gives that for some ϵ, $B(O_{k+1}, \epsilon) \subset O_k$ for all k. Lemma 28 allows us to take $V_k \in \mathcal{Y}$ satisfying

$$O_k \subset V_k \subset B(O_k, \epsilon)$$

and get from lemma 29 that $\mu_n(M + \sum \mathbf{1}_{V_k})$ converges and

$$\left| f - M - \sum \mathbf{1}_{V_k} \right| \leq \frac{2}{N}$$

so

$$\limsup \mu_n(f) - \liminf \mu_n(f) \leq \frac{2}{N}$$

and since this is true for all N and for all f the theorem is proved. □

5.1. **Extensions.** The technique demonstrated in this paper is quite flexible. The only property of loop-erased random walk crucially used is symmetry. Below are a few possible future directions.

- A hybrid graph that interpolates between $\frac{1}{2N}\mathbb{Z}^2$ and $\frac{1}{3N}\mathbb{Z}^2$ can be used to show that the scaling limit is invariant to multiplication by $\frac{2}{3}$ (i.e. if $L(\mathcal{D})$ is the scaling limit on $\mathcal{D}$ then $L(\mathcal{D}) \sim \frac{2}{3}L(\frac{3}{2}\mathcal{D})$. This will easily give that the limit of loop-erased random walks on $\mathcal{D} \cap \delta\mathbb{Z}^2$ converges to a weak limit as $\delta \to 0$ continuously.

- It is possible to use this technique to show that the scaling limit is invariant to conformal maps. Very roughly, the proof is as follows: it is only necessary to define a hybrid graph that interpolates between $\frac{1}{N}\mathbb{Z}^2$ and $\varphi\left(\frac{1}{N}\mathbb{Z}^2\right)$ where φ is the conformal map. If φ is close to 1 in the sense that $|\varphi' - 1| \leq \epsilon$ and $|\varphi''| \leq \epsilon$ then it is possible to construct the graph by linking points on the seams to the closest points on the other part of the graph. The requirement that $\bar{\bar{G}}$ is only within $O(\frac{1}{N})$ distance from $\mathbb{Z}^2$ gives linear equations for the weights of these links which can always be solved and the solution is bounded. This reduces the calculation of β (i.e. the proof of lemma 6) to a few specific graphs.
- I believe this technique might work in 3 dimensions as well. We are now working on the details.
- On the other hand, it is hard to image how this technique might be used for percolation, the UST Peano curve, or any other process where quasi-loops do exist (in other words, where the limit is SLE$_\kappa$ with $\kappa > 4$).

Appendix A. Proof of (30)

There is nothing much to say here, really. Clearly we can assume $N = 1$. The values of the harmonic potential of $\mathbb{Z}^2$ at specific points can be calculated by McCrea-Whipple's algorithm. This algorithm basically uses the fact that there is a close formula for $a(n + in)$, namely

$$a(n+in) = \frac{1}{\pi}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right) \quad .$$

With these values at hand, the value at any other point can be calculated using a line-by-line recursion which uses only the harmonicity and symmetry to $\frac{\pi}{4}$ rotations. See [S76, chapter 15] for a more detailed exposition. This allows to calculate $\sum \Delta b_v$ on a finite rectangle (200 was used for the results below). To estimate the error outside this rectangle one needs to explicitize the constants in the proofs of sublemmas 6.2-6.4. It must be noted, though, that this requires to know a value for C_1, the constant in the estimate (4) of the harmonic potential on $\mathbb{Z}^2$. This is done in [KS, section 4], and the value is

$$C_1 = \frac{9}{4}\left(17 - \frac{48 + \log 72 + 2\gamma}{\pi}\right) = 0.0172... \tag{115}$$

The table on the next page summarizes the values of the maximal $\beta_v := \sum |\Delta b_v - \delta_v|$ and the v where they occur for all configurations of D in sublemma 6.4. All numerical results are with an error of ± 0.02. All programs used are available upon demand.

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D	$\max \beta_v$	happens at
$\mathbb{Z}^2$ or $\emptyset$	0	everywhere
$\mathbb{Z}^2 \setminus (\mathbb{Z}^-)^2$	0.31	$-1-i$
$\mathbb{Z}+i\mathbb{Z}^+$	0.19	$\mathbb{Z}+\frac{1}{2}-i$
$(\mathbb{Z}^+)^2 \cup (\mathbb{Z}^-)^2$	0.34	$-\frac{3}{2}, -\frac{3}{2}i, \frac{1}{2}-i, -1+\frac{1}{2}i$
$(\mathbb{Z}^+)^2$	0.39	$-\frac{1}{2}-i, -1-\frac{1}{2}i$

TABLE 1. β for simple hybrid graphs

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