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HISTORY OF THE PARALLEL POSTULATE.¹

By FLORENCE P. LEWIS, Goucher College.

Like the famous problems of construction, Euclid's postulate concerning parallels is a thought that links the ages. Its history is a long story with dramatic climax and far-reaching influence on modern mathematical and general scientific thought. I wish to recall briefly the salient features of the story, and to state what seem to me its suggestions in regard to the teaching of elementary geometry.

Euclid's fifth postulate (called also the eleventh or twelfth axiom) states: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which are the angles less than two right angles." The earliest commentators found fault with this statement as being not self-evident. Concerning the meaning of *axiom*, Aristotle says: "That which it is necessary for anyone to hold who is to learn anything at all is an axiom;" and "It is ignorance alone that could lead anyone to try to prove the axiom." Without going into the difficult question of the precise distinction to the Greek mind between axiom and postulate, we may take it that the character of being indisputable pertained to each. Postulates stating that a straight line joining any two points can be drawn, that a circle can be drawn with given center and radius, or that all right angles are equal, were accepted, while the postulate of parallels was scrutinized and admitted at best with reluctance.

Proclus, writing in the fifth century A. D., gives some of the reasons for this attitude, and we may surmise others. The postulate makes a positive statement about a region beyond the reach of possible observation or geometrical intuition. Proclus insinuates that those who "suppose they have ground for instantaneous belief" are "yielding to mere plausible imaginings"; the conclusion is "plausible but not necessary."² The converse is proved in Proposition 27, book I of Euclid's *Elements*, and there seems to be no reason why this proposition should be more or less self-evident than its converse. The fact that the two lines continually approach each other³ was not a convincing argument to the Greek geometer who was acquainted with the relation of the hyperbola to its asymptote. The form of statement of the postulate is long and awkward compared with that of the others, and its obviousness thereby lessened. There is evidence that Euclid himself endeavored to prove the statement before putting it down as a postulate; for in some manuscripts it appears not with the others but only just before Proposition 29, where it is indispensable to the proof. If the order is significant, it indicates that the author did not at first intend to include this among the postulates, and that he finally did so only when he found that he could neither prove it nor proceed without it.

¹ Read before the Association of Teachers of Mathematics in New England, May 3, 1919.

² Cf. Heath's *Euclid*, Vol. I, and Bonola's *Non-Euclidean Geometry*, Chicago, Open Court, 1912, to which reference is made throughout this paper.

³ Even the meaning of this phrase requires further elucidation.

Most of the early geometers appear to have attacked the problem. Proclus quotes and criticizes several proofs, and gives one of his own. He instances one writer who even attempted to prove the falsity of the statement, the argument being similar to those used in Zeno's paradoxes. The common opinion, however, seems to have been that the postulate stated a truth, but that it ought to be proved. Euclid had proved two sides of a triangle greater than the third, which is far more obvious than this. If the statement was true it should be proved in order to convince the doubters; if false, it should be removed. In no case should it be retained among the fundamental presuppositions. Sir Henry Savile (1621) and the Italian Saccheri (1733) refer to it as a blot or blemish on a work that is otherwise perfect, and this expresses the common attitude of mathematicians until the first quarter of the nineteenth century.

Early attempts at proof usually took the form of a change in the definition of parallels, or the substitution, conscious or unconscious, of a new assumption. Neither of these methods resulted in satisfaction to any but their inventors; for the definitions usually concealed an assumption, and the new postulates were no more obvious than the old. Posidonius, quoted by Proclus, defines parallels as lines everywhere equidistant. This begs the question; surely such parallels do not meet, but may there not be in the same plane other lines, not equidistant, which also do not meet? The definition involves also the assumption, that the locus of points in a plane at a given distance from a straight line is a straight line, and this was not self-evident.¹ Ptolemy says that two lines on one side of a transversal are *no more parallel* than their extensions on the other side; hence if the two angles on one side are together less than two right angles, so also are the two angles on the other side, which is impossible since the sum of the four angles is four right angles. This is another way of saying that through a point but one parallel to a given line can be drawn, which is exactly Euclid's postulate. Proclus himself assumes (with some concealment) that if a line cuts one of two parallels it cuts the other, which is again postulate 5. Even as late as the close of the eighteenth century we find this argument advanced by one Thibault, and attributed also to Playfair: Let a line segment with one end *A* at a vertex of a triangle be rotated through the exterior angle. Translate it along the side until *A* comes to the next vertex and repeat the process. We finally arrive at the original position and must therefore have rotated through 360° . Hence the sum of the interior angles of a triangle is 180° ; and, since Legendre had satisfactorily proved that this proposition entails Euclid's postulate of parallels, the latter is at last demonstrated. The fact that the same process could equally well be carried out with a spherical triangle, in which the angle-sum is not 180° , might have given him pause. The assumption that translation and rotation are independent operations is in fact equivalent to Euclid's postulate.

¹ It should be noted that even the meaning of the criterion suggests several questions of logic. If two lines are so placed that perpendiculars to one of them from points on the other are equal, will the same statement hold when the rôles of the two lines are reversed? Will a perpendicular to one of two non-intersecting lines necessarily be perpendicular to the other? Of course the answers to these questions are closely bound up with the very postulate under discussion.—EDITH.

Heath gives (l. c.) a long and instructive list of these substitutes. In the course of centuries the minds of those interested became clear on one point: they did not wish merely to know whether it was possible to substitute some other assumption for Euclid's, though this question has its interest; they wished to know primarily whether exactly his form of the postulate was logically deducible from his other postulates and established theorems. To change the postulate was merely to re-state the problem.

After certain Arabs and Persians had had their say in their day, the curtain rises on the Italian Renaissance of the sixteenth century, where the problem was attacked with great vigor. French and British assailants were not lacking. The first modern work devoted entirely to the subject was published by Cataldi in 1603. When the eighteenth century took up the unfinished business of proving the parallel postulate, we find most of the giants of those days attacking the enemy of geometers with an even keener sense that without victory there could be no peace. Yet d'Alembert toward the close of the century could still refer to the state of the theory of parallels as "the scandal of elementary geometry." Klügel in 1763 examined thirty demonstrations of the postulate. He was perhaps the first to express doubt of its demonstrability. Lagrange, according to De Morgan, in about 1800, when in the act of presenting to the French Academy a prepared memoir on parallels, interrupted his reading with the exclamation, "Il faut que j'y songe encore," and withdrew his manuscript.

While the results of these investigations were on the whole negative, certain positive and valuable results were nevertheless obtained. The relation between the parallel postulate and the angle sum of a triangle was clearly brought out. Legendre proved that if in a single triangle the angle sum is two right angles, the postulate holds. Other equivalents are of interest. John Wallis and Laplace wished to assume: There exists a figure of arbitrary size similar to any given figure. Gauss could proceed rigorously provided he could prove the existence of a rectilinear triangle whose area is greater than any previously assigned area. W. Bolyai could have succeeded with the assumption that a circle can be passed through any three points not in a straight line. It must be borne in mind, moreover, that few mathematical questions have served so well as whetstones on which to sharpen the critical powers of mankind.

The work of the Italian priest Saccheri deserves notice because his method is that which finally brought the discussion to a close. Though published in 1733 his results did not become well known until after 1880, and therefore had little influence on other investigations. Legendre's *Réflexions*, published a hundred years later, covered much of the same ground without advancing quite so far. The title of Saccheri's work is *Euclides ab omni Naevo Vindicatus*, Euclid Vindicated of every Flaw. His plan was to prove the postulate by assuming its contradictory and showing that an inconsistency followed. He succeeded in proving that, according as in one triangle the angle sum is greater than, equal to, or less than two right angles, the same holds in every triangle, and that accordingly Euclid's postulate or one of its contradictories will hold.

He makes three hypotheses which were recognized later to correspond to the elliptic, Euclidean and hyperbolic geometries. But at the end of his work, in order to exhibit a contradiction when Euclid's postulate is denied, he is forced to make use of a somewhat vague and unacceptable assumption about "the nature of a straight line."

Gauss's activity in connection with the parallel postulate is of especial interest because of its psychologic aspect. It is difficult for us to picture a mathematician hesitating to publish a discovery for fear of the outcry that its publication might produce—perhaps not many would be displeased to awaken an echo; yet this is believed by some to have been the attitude of Gauss. Though he was keenly interested and thought deeply on the subject of parallels, he published nothing; he feared, as he said, "the clamor of the Bœotians." When forced to write a letter on the subject, he begs his correspondent to keep silence as to the information imparted. In 1831 he writes in a letter: "In the last few weeks I have begun to put down a few of my Meditations [on parallels] which are already to some extent forty years old. These I had never put in writing, so that I have been compelled three or four times to go over the whole matter afresh in my head. Also I wished that it should not perish with me." It is only when we call to mind the unrivalled place of honor held by Euclidean geometry among branches of human knowledge—a respect no doubt enhanced by the prominence given it in Kant's *Critique of Pure Reason*—that we realize the uncomfortable position of one who even appeared to attack its validity. Gauss's meditations were leading him through tedious and painstaking labors to the conclusion that Euclid's fifth postulate was not deducible from his other postulates. The minds of those not conversant with the intricacies of the problem might easily rush to the conclusion that Euclid's geometry was therefore untrue, and feel the whole structure of human learning crashing about their ears.

Between 1820 and 1830, the conclusion toward which Gauss tended was finally made sure by the invention of the hyperbolic non-Euclidean geometry by Lobachevsky and Johann Bolyai, working simultaneously and independently. The question, Is Euclid's fifth postulate logically deducible from his other postulates? is answered by showing that the denial of this postulate while all the others are retained leads to a geometry as consistent as Euclid's own. The method, we recall, was that used by Saccheri, whose intellectual conservatism alone prevented his reaching the same result. The famous postulate is only one of three mutually exclusive hypotheses which are logically on the same footing. Thus was Euclid "vindicated" in an unexpected manner. Knowingly or not, the wise Greek had stated the case correctly, and only his followers had been at fault in their efforts for improvement. To quote Heath: "We cannot but admire the genius of the man who concluded that such an hypothesis, which he found necessary to the validity of his whole system, was really indemonstrable."

Thus in some sense the problem of the parallel postulate was laid to rest, but its spirit marches on. If the fifth postulate could without logical error be replaced by its contradictory, could the other postulates be similarly treated? What is

the nature of a postulate or axiom? What requirements should a satisfactory system of axioms fulfill? Are we sure that accepted proofs will bear as keen scrutiny as that to which proofs of the postulate have been subjected? The facing of these questions has brought us to the modern critical study of the foundations of geometry. It has been realized that if geometry is to continue to enjoy its reputation for logical perfection, it should at least try to deserve it. The edge of criticism, sharpened on the parallel postulate, is turned against the whole structure. Out of this movement has grown the critical examination of the foundations of algebra, of projective geometry, of mechanics, of logic itself; and the end is not in sight.

One obvious result of this critical study is that geometrical axioms are not necessary truths, but merely presuppositions: they are the hypotheses on which the whole body of theorems rests. It is essential that a system of axioms should be consistent with each other, and desirable that they be non-redundant, and complete. No one has found Euclid's system inconsistent,¹ and redundancy would be a crime against elegance rather than against logic. But on the score of completeness Euclid is far from giving satisfaction. He not infrequently states conclusions which could be arrived at only by looking at a figure, *i.e.*, by space intuition; but we are all familiar with cases where space intuition misleads (for example in the fallacious proof that all triangles are isosceles), and if we accept it as a guide how can we be sure that our intuitions will always agree? In constructing an equilateral triangle Euclid says, "From point *C* where the two circles meet, draw . . ." Perhaps they do meet—but not on the basis of anything previously stated. In dropping a perpendicular from a point to a line, he says a certain circle will meet the line twice. Why should it not cut thrice or not at all? In another proof he says that a certain line will lie within a certain angle. I *see* that it does, but I do not see it proved. We are told in the midst of a proof to bisect an angle of a triangle and produce the bisector to meet the opposite side. How do we know it will meet? Because it is not a parallel. And probably it is not parallel because it is inside the triangle. How do we know it is inside; or, being inside, that it must get outside? When have these terms been defined? You may answer: It is not necessary to define them because everyone with common sense knows inside from outside without being told. "Who is so dull as not to perceive . . .?" says Simson, one of Euclid's apologists. This may be granted. But it must be pointed out that common sense knows that two straight lines cannot enclose space, yet this is given prominence as an axiom; or that a straight line is the shortest distance between two points, yet this is proved as a proposition. To state in words what distinguishes the inside from the outside of a polygon is not easy. The word "between" is likewise difficult of definition. But the modern geometer imbued with the critical spirit feels it necessary to define such terms, and what is more, he finds a way of doing it.

¹ It is in fact possible to show that the system is consistent, provided we agree to accept the axioms of arithmetic as consistent. While this is only a transformation of the problem, it is logically important to recognize the possibility of such a transformation.—EDITOR.

Hilbert's *Grundlagen der Geometrie*, published in 1899, is a classic product of this movement. It presupposes no space concepts at all, but only such general logical terms as "corresponds to," "associated with," "determined by." Contrary to tradition, it does not begin by defining terms. The first sentence is: "I think of three systems of things which I call points, lines and planes." Note the unadorned simplicity of the concept *things*. The axioms serve as definitions. They state, in non-spatial terms, relations between these "things"; that is to say, the points, lines and planes are such things as have such and such relations. "That is all ye know on earth, and all ye need to know." Twenty-one axioms are found necessary, as against Euclid's meager five. The whole work could be read and comprehended by a being with no space intuitions whatever. We could substitute the names of colors or sounds for points, lines and planes, and get on equally well. The ideal of making a thing "so plain that a blind man could see it" is literally realized. And the age-old ideal of a body of proved propositions, close-knit together by unassailable logic, is immeasurably nearer realization.

Although to the best of my knowledge no one has yet had the hardihood to invite a child of fourteen to consider "three systems of things,"¹ the modern critical movement is not without bearing on problems of teaching. I wish, with proper humility, to put forward a few ideas on this subject.

If it is true that our traditional formal geometry, taken directly or indirectly from Euclid, is not the logically perfect thing we had imagined, and if its modern perfected descendant is so abstract that not even the most rationalistic of us would venture to force it on beginners, why not acknowledge these facts and bravely face anew the question of how we can best make the study of elementary geometry serve its proclaimed purpose of training the mind? I would suggest two lines along which progress might be made. First, by sacrificing the ideal of non-redundancy in our underlying assumptions we could save time and stimulate interest by arriving more quickly at propositions whose truth is not immediately evident and which could be presented as subjects for investigation. Must we, because Euclid did it, prove that the base angles of an isosceles triangle are equal? A child that has cut the triangle out of paper and folded it over knows as much as any proof can teach him. If to treat the proposition in this way is repugnant to the teacher's logical conscience, let him privately label it "axiom" or "postulate," and proceed, even though this proposition could have been proved. *The place to begin producing arguments is the place where the truth of the proposition is even momentarily in doubt.* One statement which presents itself with a question mark and is found after investigation to be true or to be false is worth ten obviously true statements proved with all the paraphernalia of hypothesis, conclusion, step one and step two, with references. The only apparent reason for proving in the traditional way the theorems on the isosceles triangle and congruent triangles is in order to familiarize the student with the above-mentioned paraphernalia. This brings me to my second suggestion.

Formal geometry has been looked upon as a complete and perfect thing to

¹ It is possible that Halsted's *Rational Geometry* has a somewhat similar purpose.—EDITOR.

which the learner can with profit play the sedulous ape. Yet I sometimes think that by emphasizing too early the traditional form of presentation of geometrical argument, and paying too little attention to the psychology of the learner, we may have corrupted some very good minds. "I wish to prove . . .," says the student; meaning, I wish to prove something stated and accepted as true in advance of argument. Should we not prefer to have our students say, "I wish to examine . . ., to understand, to find out whether . . ., to discover a relation between . . ., to invent a means of doing . . ."? What better slogan could prejudice desire than "I wish to prove"? The conscienceless way in which college debaters collect and enumerate arguments regardless of the issues involved is another aspect of the same evil. A student said not long ago, "The study of mathematics would be good fun if we did not have to learn proofs." It had never been brought home to her that mathematical reasoning is not a thing to be acquired, like a knowledge of Latin verbs, but a thing to be participated in like any other form of exercise. Another said, "I cannot apply my geometry because all we did in school was to learn the proofs and pass the examinations."

In the midst of a proof the student hesitates and says, "I am sure this is the next step, but I cannot recall the reason for it." The step and its reason would occur simultaneously to a mind that had faced the proposition as a problem and thought it through. I should like to see in every text an occasional page of exercises to prove or disprove. And if formal proofs must be printed in full, by all means let some of the proofs be wrong.

When the student has thus halted with one foot in the air in this progress from step to step down the printed page, on what does the ability to proceed depend? On the ability to quote something: to quote, usually, a single statement—compact, authoritative, triumphantly produced. Surely it is bad training that leads the mind to *expect* to find support for its surmises in a form so simple; and the temptation to substitute ability to quote in place of the labor of finding out the truth may be a real danger. What wonder if the mind so trained quotes Washington's Farewell Address or the Monroe Doctrine and feels that its work is done?

I do not mean that formal proof should never be given. It has its place as an exercise in literary composition; for it deals with the form in which thought is expressed. We should, however, take every possible precaution to see that the thought is first there to be expressed, lest the form be mistaken for the substance. Just how this is to be brought about I am not prepared to discuss, although I suspect that drawing and measurement in the early stages of study, problems of construction and investigation, and the total absence of complete proofs from the printed page, would help. I wish merely to state my belief that only in so far as we succeed in these aims shall we succeed in making geometry really train the mind.

It can be done, said the butcher, I *think*;
It must be done, I am sure.

One point further. Perhaps we are a little too modest about the importance

of having our students retain something of the subject-matter of the courses we teach. Evidently it is here that memory, based on understanding, should rightly be used. I sometimes think we might in some way collectively take out insurance against a student's arriving at the junior year in college in the belief that two triangles are similar whenever they have a side in common.

RECENT PUBLICATIONS.

REVIEW.

Introductory Mathematical Analysis. By W. PAUL WEBBER and LOUIS C. PLANT.
New York, Wiley, 1919. 13 + 304 pages. Price \$2.00.

This combination book for freshmen covers plane trigonometry and topics from advanced algebra, analytic geometry, and the calculus, which are treated in the order named.

The first seven chapters (70 pages) may be said to form an introduction containing a review of algebra and geometry, and such general topics as computations, including logarithms and the slide rule, rectangular coördinates with graphical representations of statistics and equations, classes of numbers, variables, functions, and limits. Interpolation in tables involving two independent variables is given, and simple problems to determine the form of an equation representing a given empirical table are solved. In a later chapter some problems of the latter sort are solved by using logarithmic and semi-logarithmic paper.

Chapter VIII (50 pages) gives the usual course in plane trigonometry in condensed form. In defining the six functions a tabular form is used in which a row is given to each function and a column to an angle in each of the four quadrants, a point $P_1(x_1, y_1)$ being chosen on the terminal line of an angle α_1 in the first quadrant, a point $P_2(x_2, y_2)$ on the terminal line of an angle α_2 in the second quadrant, etc. The table includes, in parentheses, the sign of each function for each quadrant. A radian "is defined by the equation $\text{Arc } AB = r \cdot \theta$, where r is the radius of the arc AB and θ is the angle in radians." Portions of a traverse table and of a table of haversines, and some problems in indirect fire, appear among the applications.

The chapter on the circular functions is followed by one on polar coördinates, complex numbers, and vectors, which connects trigonometry with algebra. Vector and scalar products are given considerable attention, with application to equilibrium of particles and rigid bodies. A short chapter on equations gives a few of the theorems from the theory of equations, but the opportunity to tie it to the preceding chapter by the theorem that complex roots occur in conjugate pairs was not improved. The reviewer holds no brief for this theorem, especially as he thinks other material more suitable for and valuable to freshmen than complex numbers, but the omission is noticeable in a text which considers the derivative of u^n for a complex value of n , which treats series with complex terms, and which expresses the sine and cosine in the well-known exponential forms in order to