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Decay of correlations

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**Dedicated to Micheline Ishay*

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Introduction

The decay of correlations or, alternatively, the rate of approach of some initial distribution to an invariant one, is a problem addressed in many fields. In statistical mechanics such a problem is discussed via the study of the transfer operator (also called Ruelle-Perron-Frobenius operator) [33]. In the field of stochastic differential equations it is possible to obtain general results by PDE techniques [19]. For dynamical systems the widest used approach is by Markov partitions [5].

The Markov partition is a tool whereby a hyperbolic dynamical system can be reduced to (or, better, coded into) a statistical mechanical-like system: the "symbolic dynamics." At the moment, the main limitations of this approach come from the need for uniform hyperbolicity and from the necessity to use infinite Markov partitions for discontinuous systems. These limitations obstruct the extension of this technique to more general systems, with some notable exceptions ([6], [25]).

As already mentioned, in statistical mechanics the desired results are obtained by studying the transfer operator. Given that the Perron-Frobenius operator is a natural object in the context of dynamical systems, it seems plausible that some advantages could be gained by investigating directly the Perron-Frobenius operator, without previously coding the dynamics. Moreover, adopting such a point of view, the investigation of dynamical systems would be more similar to that of random systems, and the study of random perturbations of dynamical systems could be facilitated.

Results in the direction of a direct study of the Perron-Frobenius operator already exist ([29], [3], [34], [35], [20]), but they are virtually limited to one dimensional systems. Particularly relevant are the investigations of one-dimensional quadratic families of maps and their random perturbations in [42], [12]; the study of non-Markov maps, since Markov partitions are not well suited to study such examples; and [13], [15], where one-dimensional expansive Markov maps are studied with an approach similar to the one proposed here.

In this paper I describe a technique, originally due to G. Birkhoff ([9], [10]), that permits a direct study of the Perron-Frobenius operator, and I show that its field of applicability is wider than that of Markov partitions. In essence, it is possible to construct, systematically, metrics (Hilbert metrics) with respect to which the Perron-Frobenius operator is a contraction. Such a contraction allows us to obtain the invariant measure (if not already known) by an elementary, and constructive, fixed point theorem, rather than by some compactness argument (this may please some idiosyncratic people, myself included), and automatically implies an exponential rate for the decay of correlations.

I illustrate such an approach by applying it to several examples. For the sake of brevity and clarity the results are not presented in their full generality. In particular, all the arguments used for two-dimensional smooth maps can be extended to the n-dimensional case. Results concerning more general systems (notably billiards and non-uniformly hyperbolic maps) will be published in separate papers. I also hope that the present exposition will prompt others to try to apply this method to the many cases where it could yield new results (e.g. dissipative systems, flows, etc.).

The structure of the paper is as follows: Section 1 describes the Hilbert metric and its properties. It is a brief review of the subject, intended to provide an easy reference for the reader. Section 2 shows how the technique works in the simplest example: a one-dimensional, uniformly hyperbolic map. It also mentions other consequences that can be obtained (e.g. central limit theorem-type results). In Section 3, I show how to extend the approach to the multi-dimensional case—the smooth case is treated. Section 4 is the most technical; in it I show that even the presence of singularities can be accommodated, provided they are not too wild. I do not claim to have the weakest possible conditions on the singularities; more work in this direction is needed. The paper includes two appendices. The first presents a well-known result in the theory of maps with singularities. I do not refer to the literature because the form in which the result is needed here is nonstandard and (misleadingly) appears to be stronger than the usual statement. In the second appendix, I prove a result concerning the absolute continuity of the stable foliation for discontinuous systems. More precisely, I show that the Jacobian of the canonical isomorphism between two stable manifolds is Hölder continuous. This result is not surprising but, to my knowledge, is not in the literature.

A final word: the reader should be aware that, in order to simplify the formulas, no attempt has been made to optimize the constants involved. Many times the value of a constant used in an inequality reflects only my predilection for the numbers 2 and 3.

1. Operators and invariant cones

This section illustrates some results in lattice theory originally due to Garrett Birkhoff. For more details see [10], and [31] for a recent overview of the field. Consider a topological vector space $\mathbb{V}$ with a partial ordering "$\preceq$," that is a vector lattice.[1] We require the partial order to be "continuous," i.e. given $\{f_n\} \in \mathbb{V}$, $\lim_{n\to\infty} f_n = f$, if $f_n \succeq g$ for each n, then $f \succeq g$. We call such vector lattices "integrally closed."[2]

We define the closed convex cone[3] $\mathcal{C} = \{f \in \mathbb{V} \mid f \neq 0,\, f \succeq 0\}$ (hereafter, the term "closed cone" $\mathcal{C}$ will mean that $\mathcal{C} \cup \{0\}$ is closed), and the equivalence relation "$\sim$": $f \sim g$ if and only if there exists $\lambda \in \mathbb{R}^+ \backslash \{0\}$ such that $f = \lambda g$. If we call $\widetilde{\mathcal{C}}$ the quotient of $\mathcal{C}$ with respect to $\sim$, then $\widetilde{\mathcal{C}}$ is a closed convex set. Conversely, given a closed convex cone $\mathcal{C} \subset \mathbb{V}$, enjoying the property $\mathcal{C} \cap -\mathcal{C} = \emptyset$, we can define an order relation by

$$f \preceq g \iff g - f \in \mathcal{C} \cup \{0\}.$$

Henceforth, each time that we specify a convex cone we will assume the corresponding order relation and vice versa. The reader must therefore be advised that "$\preceq$" will mean different things in different contexts.

It is then possible to define a projective metric Θ (Hilbert metric),[4] in $\mathcal{C}$, by the construction:

$$\begin{aligned} \alpha(f, g) &= \sup\{\lambda \in \mathbb{R}^+ \mid \lambda f \preceq g\}, \\ \beta(f, g) &= \inf\{\mu \in \mathbb{R}^+ \mid g \preceq \mu f\}, \\ \Theta(f, g) &= \log\left[\frac{\beta(f, g)}{\alpha(f, g)}\right], \end{aligned}$$

where we take $\alpha = 0$ and $\beta = \infty$ if the corresponding sets are empty.

The importance of the previous constructions is due, in our context, to the following theorem.

THEOREM 1.1. *Let $\mathbb{V}_1$, and $\mathbb{V}_2$ be two integrally closed vector lattices; $\widetilde{T}:\mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $\widetilde{T}(\mathcal{C}_1) \subset \mathcal{C}_2$, for two closed convex cones $\mathcal{C}_1 \subset$*

[1] We are assuming the partial order to be well-behaved with respect to the algebraic structure: for each $f, g \in \mathbb{V}$, $f \succeq g \iff f - g \succeq 0$; for each $f \in \mathbb{V}$, $\lambda \in \mathbb{R}^+\backslash\{0\}$, $f \succeq 0 \implies \lambda f \succeq 0$; for each $f \in \mathbb{V}$, $f \succeq 0$ and $f \preceq 0$ imply $f = 0$ (antisymmetry of the order relation).

[2] To be precise, in the literature "integrally closed" is used in a weaker sense. First, $\mathbb{V}$ does not need a topology. Second, it suffices that for $\{\alpha_n\} \in \mathbb{R}$, $\alpha_n \to \alpha$; $f, g \in \mathbb{V}$, if $\alpha_n f \succeq g$, then $\alpha f \succeq g$. Here we will ignore these and other subtleties: our task is limited to a brief account of the results relevant to the present context.

[3] Here, by "cone," we mean any set such that, if f belongs to the set, then λf belongs to it as well, for each $\lambda > 0$.

[4] In fact, we define a semi–metric, since $f \sim g \Rightarrow \Theta(f, g) = 0$. The metric we describe corresponds to the conventional Hilbert metric on $\widetilde{\mathcal{C}}$.

$\mathbb{V}_1$ and $\mathcal{C}_2 \subset \mathbb{V}_2$ with $\mathcal{C}_i \cap -\mathcal{C}_i = \emptyset$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathcal{C}_i$. Setting $\Delta = \sup_{f,g \in T(\mathcal{C}_1)} \Theta_2(f,g)$ we have

$$\Theta_2(\widetilde{T}f, \widetilde{T}g) \leq \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f,g) \qquad \forall f, g \in \mathcal{C}_1$$

$(\tanh(\infty) \equiv 1)$.

Proof. The proof is provided for the reader's convenience.

Let $f,\, g \in \mathcal{C}_1$; on the one hand if $\alpha = 0$ or $\beta = \infty$, then the inequality is obviously satisfied. On the other hand, if $\alpha \neq 0$ and $\beta \neq \infty$, then

$$\Theta_1(f,\, g) = \ln \frac{\beta}{\alpha}$$

where $\alpha f \preceq g$ and $\beta f \succeq g$, since $\mathbb{V}_1$ is integrally closed. Notice that $\alpha \geq 0$, and $\beta \geq 0$ since $f \succeq 0$, $g \succeq 0$. If $\Delta = \infty$, then the result follows from the facts that $\alpha \widetilde{T} f \preceq \widetilde{T} g$ and $\beta \widetilde{T} f \succeq \widetilde{T} g$. If $\Delta < \infty$, then, by hypothesis,

$$\Theta_2\left(\widetilde{T}(g - \alpha f),\, \widetilde{T}(\beta f - g)\right) \leq \Delta$$

which means that there exist $\lambda,\, \mu \geq 0$ such that

$$\begin{aligned} \lambda \widetilde{T}(g - \alpha f) &\preceq \widetilde{T}(\beta f - g), \\ \mu \widetilde{T}(g - \alpha f) &\succeq \widetilde{T}(\beta f - g) \end{aligned}$$

with $\ln \frac{\mu}{\lambda} \leq \Delta$. The previous inequalities imply

$$\begin{aligned} \frac{\beta + \lambda\alpha}{1+\lambda} \widetilde{T} f &\succeq \widetilde{T} g, \\ \frac{\mu\alpha + \beta}{1+\mu} \widetilde{T} f &\preceq \widetilde{T} g. \end{aligned}$$

Accordingly,

$$\begin{aligned} \Theta_2(\widetilde{T}f,\, \widetilde{T}g) &\leq \ln \frac{(\beta + \lambda\alpha)(1+\mu)}{(1+\lambda)(\mu\alpha + \beta)} = \ln \frac{e^{\Theta_1(f,g)} + \lambda}{e^{\Theta_1(f,g)} + \mu} - \ln \frac{1+\lambda}{1+\mu} \\ &= \int_0^{\Theta_1(f,g)} \frac{(\mu - \lambda) e^{\xi}}{(e^{\xi} + \lambda)(e^{\xi} + \mu)} d\xi \leq \Theta_1(f,\, g) \frac{1 - \frac{\lambda}{\mu}}{\left(1 + \sqrt{\frac{\lambda}{\mu}}\right)^2} \\ &\leq \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f,\, g). \end{aligned}$$

□

Remark 1.2. If $\widetilde{T}(\mathcal{C}_1) \subset \mathcal{C}_2$, then it follows that $\Theta_2(\widetilde{T}f,\, \widetilde{T}g) \leq \Theta_1(f,\, g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of $\widetilde{T}$ (see [9] for details). The relevance of this theorem for the study of invariant measures and their ergodic properties is obvious.

It is natural to wonder about the strength of the Hilbert metric compared to other, more usual, metrics. While, in general, the answer depends on the cone, it is nevertheless possible to state an interesting result.

LEMMA 1.3. *Let $\|\cdot\|$ be a norm on the vector lattice $\mathbb{V}$, and suppose that, for each $f, g \in \mathbb{V}$,*

$$-f \preceq g \preceq f \Longrightarrow \|f\| \geq \|g\|.$$

Then, given $f, g \in C \subset \mathbb{V}$ for which $\|f\| = \|g\|$,

$$\|f - g\| \leq \left(e^{\Theta(f,g)} - 1\right) \|f\|.$$

Proof. We know that $\Theta(f, g) = \ln \frac{\beta}{\alpha}$, where $\alpha f \preceq g$, $\beta f \succeq g$. This implies that $-g \preceq 0 \preceq \alpha f \preceq g$, i.e. $\|g\| \geq \alpha\|f\|$, or $\alpha \leq 1$. In the same manner it follows that $\beta \geq 1$. Hence,

$$\begin{aligned} g - f &\preceq (\beta - 1)f \preceq (\beta - \alpha)f, \\ g - f &\succeq (\alpha - 1)f \succeq -(\beta - \alpha)f \end{aligned}$$

which implies

$$\|g - f\| \leq (\beta - \alpha)\|f\| \leq \frac{\beta - \alpha}{\alpha}\|f\| = \left(e^{\Theta(f,g)} - 1\right) \|f\|. \qquad \square$$

Many normed vector lattices satisfy the hypothesis of Lemma 1.3 (e.g. Banach lattices[5]); nevertheless, we will see that some important examples treated in this paper do not.

2. One-dimensional maps

In this section we study a simple class of uniformly hyperbolic maps using the general approach described in the previous section. The results discussed here are well-known [33], but they are an ideal starting point for the task at hand.

[5] A Banach lattice $\mathbb{V}$ is a vector lattice equipped with a norm satisfying the property $\| \, |f| \, \| = \|f\|$ for each $f \in \mathbb{V}$, where $|f|$ is the least upper bound of f and $-f$. For this definition to make sense it is necessary to require that $\mathbb{V}$ be "directed," i.e. any two elements have an upper bound.

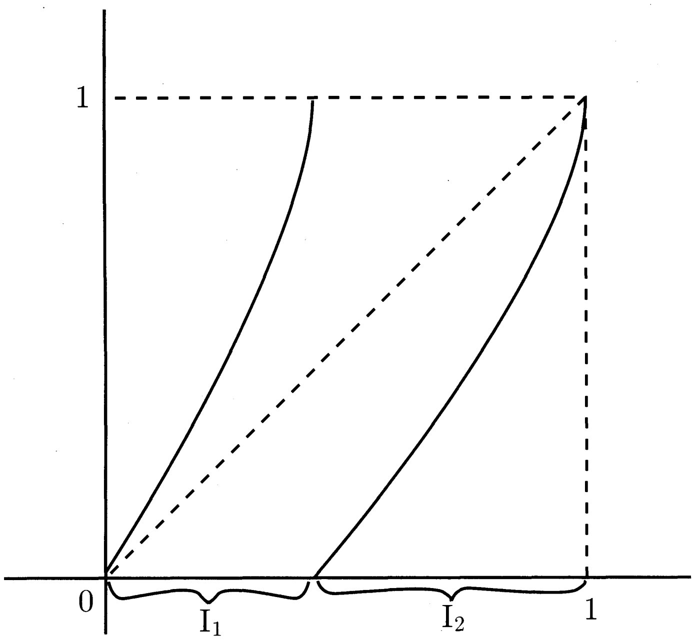

FIGURE 1. The map.

We consider a special one-dimensional map (see Figure 1). Let I_1, I_2 be two closed intervals such that $I_1 \cup I_2 = [0, 1]$, $I_1 \cap I_2 = \partial I_1 \cap \partial I_2$, and $T_i : I_i \to [0, 1]$, one-to-one and onto, such that $T_i \in C^{(2)}(I_i)$ and $DT_i \geq \lambda > 1$. Now we consider the map[6]

$$T(x) = \begin{cases} T_1(x) & \text{if } x \in I_1 \\ T_2(x) & \text{if } x \in I_2. \end{cases} \tag{2.1}$$

We will see how the results in Section 1 can be used in order to show that the map (2.1) has a unique invariant measure that is absolutely continuous with respect to the Lebesgue measure. In addition, any absolutely continuous measure (with a sufficiently smooth density), if iterated by the map, will converge to the invariant one at an exponential rate (implying the exponential decay of correlations).

Let us see how the map can be lifted to the measures. Let $f \in L^\infty([0, 1])$, $g \in L^1([0, 1])$, and

$$\int_0^1 f \circ T(x) g(x) dx = \int_0^1 f(x) \left[\sum_{i=1}^{2} g \circ T_i^{-1}(x) |D_x T_i^{-1}| \right] dx = \int_0^1 f \widetilde{T} g.$$

[6]The present technique applies to a much wider class of maps: for example any expansive Markov map and a wide class of expansive non-Markov maps [26]. Nevertheless, since this section has a purely propaedeutic and didactic aim, it would be inappropriate to present the results in their full generality.

The operator $\widetilde{T} : L^1([0, 1]) \to L^1([0, 1])$, which describes how the density of the measure $g(x)dx$ is transformed under the iteration of the map T, is called the Perron-Frobenius operator.[7]

The result will follow from the study of the semigroup generated by the operator $\widetilde{T}$. Such a semigroup maps a cone of functions strictly inside itself, and therefore it has a fixed point that corresponds to the density of the unique absolutely continuous invariant measure.

Define the following cone:[8]

$$\mathcal{C}_a = \left\{ g \in C^{(0)} \;\middle|\; \forall x,\, y \in [0,\, 1] \;\; g(x) > 0;\; \frac{g(x)}{g(y)} \le e^{a|x-y|} \right\}.$$

LEMMA 2.1. *Given $\sigma \in (\lambda^{-1}, 1]$, $\widetilde{T}\mathcal{C}_a \subset \mathcal{C}_{\sigma a}$ provided $a \ge \frac{D}{\sigma - \lambda^{-1}}$, where*

$$D = \sup_{\substack{x \in [0,\, 1] \\ i \in \{1,\, 2\}}} \left| \frac{D_x^2 T_i^{-1}}{D_x T_i^{-1}} \right|.$$

Proof. Let $g \in \mathcal{C}_a$, $x \le y \in [0,\, 1]$, and

$$\begin{aligned}
\widetilde{T}g(x) &\le \sum_{i=1}^{2} g(T_i^{-1}(y))|D_x T_i^{-1}| e^{a|T_i^{-1}(x) - T_i^{-1}(y)|} \\
&\le \sum_{i=1}^{2} g \circ T_i^{-1}(y) |D_y T_i^{-1}| e^{a|T_i^{-1}(x) - T_i^{-1}(y)| + \left|\ln(|D_x T_i^{-1}|) - \ln(|D_y T_i^{-1}|)\right|} \\
&\le \sum_{i=1}^{2} g \circ T_i^{-1}(y) |D_y T_i^{-1}| e^{a \int_x^y |D_\xi T_i^{-1}| d\xi + \int_x^y \left| \frac{D_\xi^2 T_i^{-1}}{D_\xi T_i^{-1}} \right| d\xi} \\
&\le \widetilde{T}g(y) e^{a\lambda^{-1}|x-y| + D|x-y|} \le \widetilde{T}g(y) e^{\sigma a |x-y|}.
\end{aligned}$$

□

In order to see if we can apply Theorem 1.1, we have to derive the Hilbert metric associated with our cone.

LEMMA 2.2. *Let Θ be the Hilbert metric associated to $\mathcal{C}_a$; when $f, g \in \mathcal{C}_a$,*

$$\Theta(f, g) = \ln \left[\sup_{\substack{x,\, y \in [0,\, 1] \\ u,\, v \in [0,\, 1]}} \frac{\left(e^{a|x-y|} g(y) - g(x)\right) \left(e^{a|u-v|} f(v) - f(u)\right)}{\left(e^{a|x-y|} f(y) - f(x)\right) \left(e^{a|u-v|} g(v) - g(u)\right)} \right].$$

[7] Other definitions of the Perron-Frobenius operator can be found in the literature, but the differences with the one adopted here are immaterial.

[8] This choice is made in order to simplify the exposition. Essentially, we require the logarithm of the functions in the cone to be Lipschitz, with a Lipschitz constant less than a. We could have imposed a condition on the Hölder continuity or the variation of the functions in the cone; this would have produced an estimate for the decay of correlations that is valid for a wider class of functions, and allowed us to treat maps that are $C^{(\alpha)}$, $\alpha > 1$, but not $C^{(2)}$.

Proof. We need to compute α and β. From the definitions, it follows that $\alpha f \preceq g$ if and only if

$$g(x) - \alpha f(x) \geq 0 \qquad \text{for all } x \in [0,\, 1]$$

and

$$g(x) - \alpha f(x) \leq e^{a|x-y|}\left(g(y) - \alpha f(y)\right) \text{ for all } x,\, y \in [0,\, 1].$$

Accordingly,

$$\alpha = \min\left\{ \inf_{x\in[0,\,1]} \frac{g(x)}{f(x)};\ \inf_{\substack{x,\,y\in[0,\,1]\\ x\neq y}} \frac{e^{a|x-y|}g(y) - g(x)}{e^{a|x-y|}f(y) - f(x)} \right\}.$$

However, choosing $x_0 \in [0,\, 1]$ such that

$$\frac{g(x_0)}{f(x_0)} = \inf_{x\in[0,\,1]} \frac{g(x)}{f(x)},$$

for each $x \neq x_0$, we have

$$\frac{e^{a|x_0-x|}g(x_0) - g(x)}{e^{a|x_0-x|}f(x_0) - f(x)} = \frac{e^{a|x_0-x|}\frac{g(x_0)}{f(x_0)}f(x_0) - \frac{g(x)}{f(x)}f(x)}{e^{a|x_0-x|}f(x_0) - f(x)} \leq \frac{g(x)}{f(x)}.$$

This means that

$$\alpha = \inf_{\substack{x,\,y\in[0,\,1]\\ x\neq y}} \frac{e^{a|x-y|}g(y) - g(x)}{e^{a|x-y|}f(y) - f(x)}.$$

Analogously,

$$\beta \quad = \quad \sup_{\substack{x,\,y\in[0,\,1]\\ x\neq y}} \frac{e^{a|x-y|}g(y) - g(x)}{e^{a|x-y|}f(y) - f(x)}. \qquad \square$$

LEMMA 2.3.

$$\Delta = \operatorname{diam}(\mathcal{C}_{\sigma a}) \leq 2\ln\frac{1+\sigma}{1-\sigma} + 2\sigma a.$$

Proof. For each $f,\, g \in \mathcal{C}_{\sigma a}$ we have

$$\begin{aligned} \Theta(f,\, g) \quad &= \quad \ln \sup_{\substack{x,\,y\in[0,\,1]\\ u,\,v\in[0,\,1]}} \frac{\left(e^{a|x-y|} - e^{-\sigma a|x-y|}\right)\left(e^{a|u-v|} - e^{-\sigma a|u-v|}\right) g(y)f(v)}{\left(e^{a|x-y|} - e^{\sigma a|x-y|}\right)\left(e^{a|u-v|} - e^{\sigma a|u-v|}\right) f(y)g(v)} \\ &\leq \quad \ln\frac{(1+\sigma)^2}{(1-\sigma)^2}e^{2\sigma a}. \qquad \square \end{aligned}$$

Lemmas 2.1–2.3, together with Theorem 1.1, imply that for all $f,\, g \in \mathcal{C}_a$

$$\Theta(\widetilde{T}f,\, \widetilde{T}g) \leq \Lambda\Theta(f,\, g)$$

with

$$\Lambda = \tanh\left(\frac{\Delta}{4}\right).$$

Remark 2.4. In order to have Δ as small as possible it is convenient to choose $a = \frac{D}{\sigma - \lambda^{-1}}$. It is then possible to choose the $\sigma = \sigma_*$ that minimizes Δ. Hence, the bound on the contraction rate Λ depends only on λ and D.

Note that, for each $g_1, g_2 \in C^{(0)}([0, 1])$, $-g_1 \preceq g_2 \preceq g_1$ implies $-g_1(x) \leq g_2(x) \leq g_1(x)$; that is $|g_2(x)| \leq g_1(x)$, for each $x \in [0, 1]$. Consequently, for each L^p norm, $\|g_2\|_p \leq \|g_1\|_p$, and Lemma 1.3 applies.

Let $g \in C^{(0)}([0, 1])$, $\int_0^1 g = 1$, and $g \in \mathcal{C}_{a_*}$ with $a_* = \frac{D}{\sigma_* - \lambda^{-1}}$. Since $\int_0^1 \widetilde{T}^n g = \int_0^1 g = 1$,

$$\|\widetilde{T}^{n+m}g - \widetilde{T}^n g\|_1 \leq e^{\Theta(\widetilde{T}^{n+m}g, \widetilde{T}^n g)} - 1 \leq e^{\Lambda^{n-1}\Theta(\widetilde{T}(\widetilde{T}^m g), \widetilde{T}g)} - 1 \leq e^{\Delta\Lambda^{-1}}\Delta\Lambda^{n-1}.$$

This means that $\{\widetilde{T}^n g\}$ is a Cauchy sequence in L^1; in addition, $\{\widetilde{T}^n g\}$ are equicontinuous. Let $\varphi_* = \lim_{n\to\infty} \widetilde{T}^n g$; clearly $\widetilde{T}\varphi_* = \varphi_*$, $\varphi_* \in \mathcal{C}_{\sigma_* a_*}$, and φ_* is not dependent on g; i.e., φ_* is the density of the unique invariant measure absolutely continuous with respect to Lebesgue.

Up to now, we have shown the existence of the invariant measure, obtained some of its regularity properties, and been able to estimate the decay of correlations for all functions $f \in L^\infty([0, 1])$, $g \in \mathcal{C}_{a_*}$. The next theorem gives a slightly stronger result, and an alternative to using Lemma 1.3.

THEOREM 2.5. *There exist $K, r \in \mathbb{R}^+$ such that, for each $f \in L^1([0,1])$, $g \in C^{(1)}([0,1])$ with $\int_0^1 g = 1$,*

$$\left|\int_0^1 f \circ T^n g dx - \int_0^1 f\varphi_* dx\right| \leq K\|f\|_1(\|g\|_1 + r\|g'\|_\infty)\Lambda^n.$$

Proof. Consider $g \in \mathcal{C}_{a_*}$, normalized so that $\int_0^1 g = 1$ (i.e., g can be thought as the density of a measure). Then,

$$\left|\int_0^1 f(\widetilde{T}^n g - \varphi_*)dx\right| \leq \|f\|_1 \left\|\frac{\widetilde{T}^n g}{\varphi_*} - 1\right\|_\infty \|\varphi_*\|_\infty.$$

LEMMA 2.6. *If $\mathcal{C}_+ = \{g \in C^{(0)}([0,1]) \mid g(x) \geq 0$, for all $x \in [0,1]\}$, and Θ_+ is the corresponding Hilbert metric, then, for each $g_1, g_2 \in \mathcal{C}_a$,*

$$\Theta_+(g_1, g_2) \leq \Theta(g_1, g_2).$$

Proof. Since $\mathcal{C}_+ \supset \mathcal{C}_a$, the identity is a map from $C^{(0)}([0, 1])$ to itself that maps $\mathcal{C}_a$ into $\mathcal{C}_+$. The result follows then from Theorem 1.1. □

A simple computation yields

$$\Theta_+(g_1,\, g_2) = \ln \sup_{x,\, y \in [0,\, 1]} \frac{g_1(x) g_2(y)}{g_1(y) g_2(x)}.$$

Using the previous facts, and the trivial equality

$$\frac{(\widetilde{T}^n g)(x)}{\varphi_*(x)} = \frac{\widetilde{T}^n g(x) \varphi_*(y)}{\widetilde{T}^n g(y) \varphi_*(x)} \frac{\widetilde{T}^n g(y)}{\varphi_*(y)},$$

we have

$$e^{-\Theta_+(\widetilde{T}^n g,\, \varphi_*)} \frac{\widetilde{T}^n g(y)}{\varphi_*(y)} \le \frac{\widetilde{T}^n g(x)}{\varphi_*(x)} \le e^{\Theta_+(\widetilde{T}^n g,\, \varphi_*)} \frac{\widetilde{T}^n g(y)}{\varphi_*(y)}$$

for each $x,\, y \in [0,\, 1]$. Because $\int_0^1 (\widetilde{T}^n g - \varphi_*) = 0$, and the fact that both functions are continuous, there exists $y_n \in [0,\, 1]$: $\widetilde{T}^n g(y_n) = \varphi_*(y_n)$. Using the previous inequalities, with $y = y_n$, we obtain, for each $x \in [0,\, 1]$,

$$e^{-\Theta_+(\widetilde{T}^n g,\, \varphi_*)} \le \frac{\widetilde{T}^n g(x)}{\varphi_*(x)} \le e^{\Theta_+(\widetilde{T}^n g,\, \varphi_*)}$$

and

$$\left\| \frac{\widetilde{T}^n g}{\varphi_*} - 1 \right\|_\infty \le e^{\Theta_+(\widetilde{T}^n g,\, \varphi_*)} - 1 \le e^{\Lambda^{n-1} \Theta(\widetilde{T} g,\, \varphi_*)} - 1 \le e^{\Lambda^{-1} \Delta} \Delta \Lambda^{n-1}.$$

This estimate shows that for each $f \in L^1([0,\, 1])$, $g \in \mathcal{C}_{a_*}$,

$$\begin{aligned} \left| \int_0^1 f \circ T^n g dx - \int_0^1 f \varphi_* dx \int_0^1 g dx \right| \quad &\le \quad K \|f\|_1 \|g\|_1 \Lambda^n, \\ K = e^{\Lambda^{-1} \Delta} \Delta \Lambda^{-1} \|\varphi_*\|_\infty &. \end{aligned}$$

Let us now consider $g \in C^{(0)}([0,\, 1])$, $g \ge 0$, and piecewise differentiable with $\|g'\|_\infty < \infty$. If we define $g_\rho = g + \rho \varphi_*$, $\rho \in \mathbb{R}^+$, then for each $x > y$ we have

$$\begin{aligned} g_\rho(x) \quad &\le \quad e^{\ln[g(x) + e^{\sigma_* a_* (x-y)} \rho \varphi_*(y)] - \ln[g(y) + \rho \varphi_*(y)]} g_\rho(y) \\ &\le \quad e^{\int_y^x \frac{|g'(\xi) + \rho \sigma_* a_* e^{\sigma_* a_* (\xi - y)} \varphi_*(y)|}{g(\xi) + \rho e^{\sigma_* a_* (\xi - y)} \varphi_*(y)} d\xi} g_\rho(y) \le e^{\left[\frac{\|g'\|_\infty}{\varphi_*(y)} \rho^{-1} + \sigma_* a_* \right] (x-y)} g_\rho(y). \end{aligned}$$

Choosing

$$\rho = \sup_{x \in [0,\, 1]} \frac{\|g'\|_\infty}{\varphi_*(x) a_* (1 - \sigma_*)}$$

we have $g_\rho \in \mathcal{C}_{a_*}$; hence, the decay of correlations for g_ρ implies the decay of correlations for g. The result for arbitrary $g \in C^{(1)}([0,\, 1])$ follows, since any smooth function can be written as the difference of two piecewise differentiable positive functions. $\square$

Theorem 2.5 immediately yields:

COROLLARY 1 (mixing). *For each* $f \in L^\infty([0,1])$, $g \in L^1([0,1])$, $\int_0^1 g = 1$,

$$\lim_{n\to\infty} \left| \int_0^1 (f \circ T^n)\, g - \int_0^1 f\varphi_* \right| = 0.$$

Another corollary of Theorem 2.5 is the so-called Central Limit Theorem [21].

COROLLARY 2 (CLT). *For each* $f \in C^1([0,1])$, $\int_0^1 f\varphi_* = 0$, *the distribution of the random variable*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f \circ T^n$$

converges to a Gaussian law.

3. Two dimensions (the smooth uniformly hyperbolic case)

In the rest of the paper we will see how the approach presented in Section 2 can be implemented in more general situations.

Clearly the first possible generalization is the multi-dimensional case. For simplicity, we will confine ourselves to the two-dimensional area-preserving case. Hence, this section is dedicated to symplectic smooth uniformly hyperbolic two-dimensional maps.

The symplectic nature of the maps implies the existence of a special invariant measure (the symplectic volume), which simplifies our approach. The higher dimensional conservative case can be treated in the same fashion. On the contrary, in the case of dissipative systems, the present technique should still yield results, but more care would be needed in its application.

The main idea is that in one direction the P-F (Perron-Frobenius) operator acts as in the one-dimensional case, i.e. it makes functions smoother, but in the other direction the P-F operator increases oscillations. To overcome this phenomenon we integrate the functions on short curves in the latter direction. Under the action of the P-F operator, this is equivalent to averaging the functions on longer and longer curves. Hence, also in the second direction, a regularization is taking place.[9]

The previous argument will be made precise by Lemma 3.2. The Hilbert metric used in this section is weaker than any L^p norm (in particular Lemma 1.3

[9]The necessity to take averages comes from the fact that, in the multi-dimensional case, measures converge to the invariant one only weakly, contrary to what we have seen in the one-dimensional case.

does not hold); nonetheless, according to Theorem 3.9, it suffices to control the decay of correlation for Hölder continuous functions.

Starting by describing the class of maps under consideration and by recalling some standard constructions, I will then introduce the cones of functions used in the proof.

3.1. *The map.* Let $\mathcal{M}$ be a compact smooth symplectic manifold and $T : \mathcal{M} \to \mathcal{M}$ a symplectomorphism. Let there exist a bundle of cones $\{\mathcal{C}(x)\}_{x\in\mathcal{M}}$ in the tangent bundle (i.e., $\mathcal{C}(x) \subset \mathcal{T}_x\mathcal{M}$) strictly invariant ($D_xT^{-1}\mathcal{C}(x) \subset \operatorname{int} \mathcal{C}(T^{-1}x) \cup \{0\}$) and continuous: the "edges" of the cones vary with continuity. Then the map T is uniformly hyperbolic and its stable and unstable foliations are continuous (conversely, for such maps, it is easy to construct a family of continuous strictly invariant cones with the required properties; for more details see [27]).

It follows that there exists a smooth metric on $\mathcal{M}$ such that, with $v^{\mathrm{s}}(x)$ and $v^{\mathrm{u}}(x)$ the stable and unstable normalized vectors at x,

$$\begin{aligned} \|D_xTv^{\mathrm{s}}(x)\| &\leq \lambda^{-1}, \\ \|D_xTv^{\mathrm{u}}(x)\| &\geq \lambda \end{aligned}$$

for some $\lambda > 1$. Furthermore, the Riemannian volume is equal to the symplectic one. We will, from now on, use this metric. In addition, we will suppose that

$$\begin{aligned} \inf_{v\in\mathcal{C}(x)} \|D_xT^{-1}v\| &\geq \lambda > 1, \\ \sup_{\substack{v,\, w\in\mathcal{C}(x)\\ \|v\|=\|w\|=1}} \|v-w\| &\leq \frac{1}{2}. \end{aligned}$$

(This can always be achieved, eventually by choosing a smaller $\lambda > 1$ and by replacing $\mathcal{C}(x)$ by $D_{T^nx}T^{-n}\mathcal{C}(T^nx)$ for a large enough n.)

Further, we assume that the map enjoys the mixing property.[10] In the present case the Perron-Frobenius operator $\widetilde{T} : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ is given by

$$\widetilde{T}g = g \circ T^{-1}.$$

3.2. *Cones of functions.* Due to the presence of both contracting and expanding directions the approach of the previous section must be slightly modified. To overcome this difficulty we will, in some sense, quotient away (or, better yet, "integrate away") the unwanted direction through the introduction of suitable test functions. This is achieved by introducing a set of curves, and

[10] The previous assumptions imply that the map is locally ergodic [4], [27], [22], and mixing on each ergodic component [17], [32], but several ergodic components could still be present.

considering averages of a function taken along these curves with respect to appropriate measures. Consider the following class of closed, connected and differentiable curves

$$\Gamma_\delta = \{\gamma \subset \mathcal{M} \mid \delta \leq \text{length}(\gamma) \leq 2\delta;\ \forall x \in \gamma,\ \gamma'(x) \subset \mathcal{C}(x);\ |\kappa_\gamma(x)| \leq H\}$$

where $\gamma'(x)$ is the unitary tangent vector to γ at x, $\kappa_\gamma(x)$ is the curvature of γ at x, and $\delta \in (0, 1)$.[11] On each of these curves, the measure m_γ, induced by the Riemannian structure (arc-length), is naturally defined. The plan is to perform integrals on γ, with respect to a large class of measures. We proceed to characterize such classes by specifying the properties of their densities with respect to m_γ.

Let us define the cones of "density" functions. Given any connected smooth curve γ, let

$$\widetilde{\mathcal{D}}_a(\gamma) = \left\{ f \in C^{(0)}(\gamma) \;\middle|\; f > 0;\ \frac{f(x)}{f(y)} \leq e^{ad(x,\,y)^\nu} \right\}$$

where d means the distance along γ, and $\nu \in (0, \frac{1}{2}]$. In other words, we are considering positive functions whose logarithms have a bounded modulus of ν-Hölder continuity.

The reader will certainly notice the similarity to the cones used in the one-dimensional case. Let us remember that

$$\rho_\gamma(f_1, f_2) = \ln \left[\sup_{x,\,y,\,u,\,v \in \gamma} \frac{\left[e^{ad(x,\,y)^\nu} f_1(x) - f_1(y)\right] \left[e^{ad(u,\,v)^\nu} f_2(u) - f_2(v)\right]}{\left[e^{ad(x,\,y)^\nu} f_2(x) - f_2(y)\right] \left[e^{ad(u,\,v)^\nu} f_1(u) - f_1(v)\right]} \right]$$

is the Hilbert metric in $\widetilde{\mathcal{D}}_a(\gamma)$.

The set of functions we will consider is

$$\mathcal{D}_a(\gamma) = \{f \in \widetilde{\mathcal{D}}_a(\gamma) \mid \rho_\gamma(1, f) \leq R\}.$$

For later purposes we also define

$$\mathcal{D}_+(\gamma) = \{f \in C^{(0)}(\gamma) \mid f \geq 0\},$$

together with its Hilbert metric

$$\rho_+(f_1, f_2) = \ln \left[\sup_{x,\,y \in \gamma} \frac{f_1(x) f_2(y)}{f_1(y) f_2(x)} \right].$$

Of course, by Theorem 1.1, $\rho_+ \leq \rho_\gamma$ (since the inclusion of $\widetilde{\mathcal{D}}_a(\gamma)$ in $\mathcal{D}_+(\gamma)$ is a contraction).

[11]To fix a length scale, let us assume $\text{diam}(\mathcal{M}) = 1$.

It is now possible to introduce the cone of functions that will be used in the proof (here, as in the following, the integration is done with respect to the measure m_γ):

$$\mathcal{C}_{b,\,c} = \left\{ g \in C^{(0)}(\mathcal{M}) \,\middle|\, \forall \gamma \in \Gamma_\delta,\ \forall x \in \gamma,\ \forall f,\ f_1,\ f_2 \in \mathcal{D}_a(\gamma)\ \int_\gamma gf > 0; \right.$$
$$\left. \frac{f_2(x) \int_\gamma g f_1}{f_1(x) \int_\gamma g f_2} \le e^{b\rho_\gamma(f_1, f_2)};\ \|D^{\mathrm{u}} g\|_\infty \le c \inf_{\substack{\gamma \in \Gamma_\delta \\ f \in \mathcal{D}_a(\gamma)}} \frac{\int_\gamma fg}{\int_\gamma f} \right\};$$

by D^{u} we mean the derivative in the unstable direction. We require that $D^{\mathrm{u}} g$ be well-defined, apart from, at most, countably many points, on each unstable manifold (the reader may feel free to think that g is everywhere differentiable: I will use our weaker assumption only at the very end of Theorem 3.9, and not in an essential way). The choice of piecewise smooth functions, in the unstable direction, is done only to simplify the exposition. More general functions can easily be considered by defining larger cones.[12]

As anticipated, once we integrate along the "problematic" direction, we can consider a cone quite similar to the previous ones. In essence the main idea is to forget about the actual value of a function and consider instead only averages values. It is then natural to require that averages done with respect to similar measures (that is, measures whose densities are close in the ρ_γ metric) are almost equal.

For later use it is useful to define two quantities relative to a given function (the maximal and minimal average): For $g \in C^{(0)}(\mathcal{M})$,

$$|||g|||_+ = \sup_{\substack{\gamma \in \Gamma_\delta, \\ f \in \mathcal{D}_a(\gamma)}} \frac{\int_\gamma fg}{\int_\gamma f},$$

$$|||g|||_- = \inf_{\substack{\gamma \in \Gamma_\delta \\ f \in \mathcal{D}_a(\gamma)}} \frac{\int_\gamma fg}{\int_\gamma f}.$$

3.3. *Contraction of the Hilbert metric.* The reader may wonder about how big the cone $\mathcal{C}_{b,\,c}$ is. The next lemma shows that it is far from empty.

---

[12]If γ and $\widetilde{\gamma}$ are two curves in Γ_δ, obtained one from the other by "translation" along the unstable direction and ϕ is the canonical isomorphism from $\widetilde{\gamma}$ to γ along the unstable direction, one can replace the last condition in $\mathcal{C}_{b,c}$ by

$$\frac{\int_\gamma gf}{\int_{\widetilde{\gamma}} gf \circ \phi} \le e^{c\tau^\sigma}$$

for some $\sigma \in (0,\, 1)$ and where τ is the distance between γ and $\widetilde{\gamma}$. (Not by chance $\mathcal{D}_a(\gamma)$ is chosen so that if $f \in \mathcal{D}_a(\gamma)$, then $f \circ \phi \in \mathcal{D}_{a'}(\gamma)$ for some $a' \ge a$.)

LEMMA 3.1. *For each $b > 1$ the cone $\mathcal{C}_{b,c}$ is a closed convex cone with nonempty interior in the $C^{(1)}$ topology.*

Proof. The closure and the convexity can easily be checked by direct computation. The third property follows since the open set $\{g \in C^{(1)}(\mathcal{M}) \mid g > p > 0;\ \|Dg\|_\infty < cp\}$ is contained in $\mathcal{C}_{b,c}$. In fact,

$$\begin{aligned} f_2(x)\int_\gamma gf_1 &= f_1(x)\int_\gamma g(y)f_2(y)\frac{f_2(x)f_1(y)}{f_1(x)f_2(y)}dm_\gamma(y) \\ &\leq e^{\rho_+(f_1,f_2)}f_1(x)\int_\gamma gf_2 \\ &\leq e^{\rho_\gamma(f_1,f_2)}f_1(x)\int_\gamma gf_2. \end{aligned}$$

□

By now, it is clear that we need the cone to be invariant under the action of the P-F operator. Lemma 3.2 shows that this is the case.

LEMMA 3.2. *For a, b and H (see the definition of Γ_δ) large enough, $\widetilde{T}\mathcal{C}_{b,c} \subset \mathcal{C}_{\chi b,\chi c}$ for some $\chi < 1$.*

Proof. We check the first property first. For $\gamma \in \Gamma_\delta$, $x \in \gamma$, f_1, $f_2 \in \mathcal{D}_a(\gamma)$ and for each $g \in \mathcal{C}_{b,c}$,

$$\int_\gamma (\widetilde{T}g)f_1 = \int_\gamma g\circ T^{-1}f_1 = \int_{T^{-1}(\gamma)} gf_1\circ T\left|\det(D_xT|_{T^{-1}\gamma})\right|.$$

By $D_xT|_{T^{-1}\gamma}$ we mean the restriction of D_xT to a map from the direction $(T^{-1}\gamma)'$ to the direction γ'. Moreover, we use $\det(\cdot)$, in spite of the fact that in our case it is superfluous, to emphasize the multi-dimensional nature of the argument.

In order to proceed it is necessary to estimate the latter quantity in terms of the averages used in the definition of the cone; the next result is the first step in such a direction.

SUBLEMMA 3.3. *For each $\gamma \in \Gamma_\delta$, there exists a collections of curves $\{\gamma_i\} \in \Gamma_\delta$, such that $\gamma_i \cap \gamma_j = \partial\gamma_i \cap \partial\gamma_j$ and $\bigcup_i \gamma_i = T^{-1}\gamma$, provided H has been chosen large enough.*

Proof. We start by subdividing $T^{-1}(\gamma)$ in connected pieces of length δ plus, at most, one piece longer than δ but shorter than 2δ. Since $\gamma'(x) \in \mathcal{C}(x)$, and $D_xT^{-1}\mathcal{C}(x) \subset \mathcal{C}(T^{-1}x)$, we have $(T^{-1}\gamma)'(T^{-1}x) \in \mathcal{C}(T^{-1}x)$. We are left with the condition on the curvature.

Let $\widetilde{\gamma}(s)$ and $\widetilde{\gamma}_1(s_1)$ be the curves γ and $T^{-1}\gamma$ parametrized by arc-length. If η is the unit vector in the direction $\widetilde{\gamma}_1''$, then $\langle \eta, DT^{-1}\gamma' \rangle = 0$ and $|\kappa_{T^{-1}\gamma}(s_1)| = |\langle \eta, \frac{d}{ds_1}\widetilde{\gamma}_1'(s_1)\rangle|$. Since $\frac{ds_1}{ds} = \|DT^{-1}\widetilde{\gamma}'\|$, it follows that

$$\begin{aligned} |\kappa_{T^{-1}\gamma}(s_1)| &\le \lambda^{-1}\left|\left\langle \eta, \frac{d}{ds}\frac{D_{\widetilde{\gamma}(s)}T^{-1}\widetilde{\gamma}'(s)}{\|D_{\widetilde{\gamma}(s)}T^{-1}\widetilde{\gamma}'(s)\|}\right\rangle\right| \\ &\le \lambda^{-2}\left|\left\langle \eta, \frac{d}{ds}D_{\widetilde{\gamma}(s)}T^{-1}\widetilde{\gamma}'(s)\right\rangle\right| \\ &\le \lambda^{-2}(c_2 + |\langle \eta, D_{\widetilde{\gamma}(s)}T^{-1}\widetilde{\gamma}''(s)\rangle|) \end{aligned}$$

where c_2 depends only on the second derivative of T. To conclude the estimate, let us fix s and choose explicit orthogonal coordinates at $\widetilde{\gamma}(s)$ and $T^{-1}\widetilde{\gamma}(s)$. That is, we take the axes in the directions $\widetilde{\gamma}'$, $\widetilde{\gamma}''$, and $\widetilde{\gamma}_1'$, $\widetilde{\gamma}_1''$, respectively. In these coordinates DT^{-1} has the form

$$DT^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with $a = \|D_{\widetilde{\gamma}(s)}T^{-1}\widetilde{\gamma}'(s)\| \ge \lambda$. In addition, since the Riemannian volume equals the symplectic one, $\det(DT^{-1}) = 1$, i.e., $a = c^{-1}$. Using such a representation yields

$$|\langle \eta, D_{\widetilde{\gamma}(s)}T^{-1}\widetilde{\gamma}''(s)\rangle| \le (0\ 1)\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ |\kappa_\gamma(s)| \end{pmatrix} = |\kappa_\gamma(s)|a^{-1},$$

or

$$|\kappa_{T^{-1}\gamma}(s_1)| \le \lambda^{-2}(c_2 + \lambda^{-1}H) \le H. \qquad \square$$

It is then natural to write

$$\int_\gamma (\widetilde{T}g)f_1 = \sum_i \int_{\gamma_i} gf_1 \circ T|\det(DT|_{\gamma_i})|.$$

We have obtained a sum of integrals taken over allowed curves; thus the question of the densities remains.

To continue we need to analyze the maps $\widehat{T}_i : C^{(0)}(\gamma) \to C^{(0)}(\gamma_i)$ defined by

$$(\widehat{T}_i f)(x) = f \circ T(x)\left|\det(D_xT|_{\gamma_i})\right| \qquad \text{for all } x \in \gamma_i. \tag{3.1}$$

From now on, with a slight abuse of notation, we will drop the subscript "i", since the range of $\widehat{T}$ is always clear from the context.

SUBLEMMA 3.4. *$\widehat{T}(\widetilde{\mathcal{D}}_a(\gamma)) \subset \widetilde{\mathcal{D}}_{\sigma a}(\gamma_i)$ for some $\sigma < 1$, provided $a > c_3\delta^{1-\nu}(1-\lambda^{-\nu})^{-1}$, where c_3 depends only on the system and H.*

Proof. For each $f \in \widetilde{\mathcal{D}}_a(\gamma)$, $x, y \in \gamma_i \in \Gamma_\delta$,

$$\frac{(\widehat{T}f)(x)}{(\widehat{T}f)(y)} \le e^{ad(Tx,\,Ty)^\nu + \mu(c_2+\mu H)d(x,\,y)} \le e^{(a\lambda^{-\nu}+c_3\delta^{1-\nu})d(x,\,y)^\nu},$$

where μ is the maximal expansion coefficient. □

SUBLEMMA 3.5. *If R (see the definition of $\mathcal{D}_a(\gamma)$) is sufficiently large and $a > c_3\delta^{1-\nu}(1-\lambda^{-\nu})^{-1}$, then $\widehat{T}(\mathcal{D}_a(\gamma)) \subset \mathcal{D}_a(\gamma_i)$; in addition, for each $f_1, f_2 \in \mathcal{D}_a(\gamma)$,*

$$\rho_{\gamma_i}(\widehat{T}f_1, \widehat{T}f_2) \le \xi\rho_\gamma(f_1, f_2)$$

for some $\xi < 1$.

Proof. The constant function 1 is an element of $\mathcal{D}_a(\gamma_i)$; for each $f \in \widetilde{\mathcal{D}}_{\sigma a}(\gamma_i)$ we have

$$\begin{aligned}
\rho_{\gamma_i}(1,\, f) &= \ln\left[\sup_{x,\,y,\,u,\,v\in\gamma_i} \frac{\left[e^{ad(x,\,y)^\nu}-1\right]\left[e^{ad(u,\,v)^\nu}f(u)-f(v)\right]}{\left[e^{ad(x,\,y)^\nu}f(x)-f(y)\right]\left[e^{ad(u,\,v)^\nu}-1\right]}\right] \\
&\le \ln\left[\sup_{x,\,y,\,u,\,v\in\gamma_i} \frac{\left[e^{ad(x,\,y)^\nu}-1\right]\left[e^{a(1+\sigma)d(u,\,v)^\nu}-1\right]f(v)}{\left[e^{a(1-\sigma)d(x,\,y)^\nu}-1\right]\left[e^{ad(u,\,v)^\nu}-1\right]f(y)}\right] \\
&\le \ln\left[\frac{(1+\sigma)}{(1-\sigma)}e^{a\sigma 2^\nu\delta^\nu}\right] \le R.
\end{aligned}$$

Hence, $\widehat{T}(\mathcal{D}_a(\gamma)) \subset \mathcal{D}_a(\gamma_i)$. In addition, since the diameter of $\widehat{T}_i(\widetilde{\mathcal{D}}_a(\gamma))$ in $\widetilde{\mathcal{D}}_a(\gamma_i)$ is less than R, the second part of the result follows from Theorem 1.1. □

Accordingly,

$$\int_\gamma \widetilde{T}gf = \sum_i \int_{\gamma_i} g\widehat{T}f \ge 0.$$

Next, let $x \in \gamma$, and choose points $z_i \in T\gamma_i$;

$$\begin{aligned}
f_2(x)\int_\gamma(\widetilde{T}g)f_1 &= \sum_i \frac{f_2(x)}{\widehat{T}f_2(z_i)}\widehat{T}f_2(z_i)\int_{\gamma_i} g\widehat{T}f_1 \\
&\le \sum_i e^{b\rho_{\gamma_i}(\widehat{T}f_1,\,\widehat{T}f_2)}\frac{f_2(x)}{\widehat{T}f_2(z_i)}\widehat{T}f_1(z_i)\int_{\gamma_i} g\widehat{T}f_2 \\
&\le e^{\xi b\rho_\gamma(f_1,\,f_2)}\sup_{z,\,z'\in\gamma}\frac{f_2(z)f_1(z')}{f_1(z)f_2(z')}f_1(x)\sum_i\int_{\gamma_i} g\widehat{T}f_2 \\
&\le e^{[\xi b\rho_\gamma(f_1,\,f_2)+\rho_+(f_1,\,f_2)]}f_1(x)\sum_i\int_{T\gamma_i}\widetilde{T}gf_2 \\
&\le e^{(\xi b+1)\rho_\gamma(f_1,\,f_2)}f_1(x)\int_\gamma(\widetilde{T}g)f_2
\end{aligned}$$

which yields the wanted result for b large enough. The third of the conditions is easily checked after we notice that

$$\frac{\int_\gamma (\widetilde{T}g)f}{\int_\gamma f} \geq \sum_i \frac{\int_{\gamma_i} \widehat{T}_i f}{\int_\gamma f} |||g|||_- = |||g|||_-$$

implies $|||\widetilde{T}g|||_- \geq |||g|||_-$. □

By now we know that the cone is mapped inside itself, yet this does not necessarily means that the diameter of the image is finite. To check this last property we start with a preliminary result.

LEMMA 3.6. *With* $K = \frac{(1+\chi)(1+b)}{(1-\chi)(b-1)}$, *for each* $g \in \mathcal{C}_{\chi b, \chi c}$,

$$\Theta(g, 1) \leq \ln\left[K \frac{|||g|||_+}{|||g|||_-}\right]$$

(*from now on* Θ *designates the Hilbert metric in* $\mathcal{C}_{b,c}$).

Proof. For each $\lambda > 0$, such that $g - \lambda \in \mathcal{C}_{b,c}$, a direct computation shows

$$\begin{aligned}
\lambda &\leq \inf_{\substack{\gamma\in\Gamma_\delta \\ f\in\mathcal{D}_a(\gamma)}} \frac{\int_\gamma gf}{\int_\gamma f} \equiv \alpha_0, \\
\lambda &\leq \inf_{\substack{\gamma\in\Gamma_\delta \\ f_1, f_2\in\mathcal{D}_a(\gamma)}} \frac{e^{b\rho_\gamma(f_1,f_2)} f_1 \int_\gamma g f_2 - f_2 \int_\gamma g f_1}{e^{b\rho_\gamma(f_1,f_2)} f_1 \int_\gamma f_2 - f_2 \int_\gamma f_1} \equiv \alpha_1, \\
\lambda &\leq \frac{c|||g|||_- - \|D^{\mathrm{u}} g\|_\infty}{c} \equiv \alpha_2;
\end{aligned}$$

conversely, if $\lambda \leq \min\{\alpha_1, \alpha_2, \alpha_3\}$, then $g - \lambda \in \mathcal{C}_{b,c}$.

Accordingly, $\alpha = \min\{\alpha_0, \alpha_1, \alpha_2\}$ (see §1 for a definition of α in this context). In addition,

$$\begin{aligned}
\alpha_0 &= |||g|||_-, \\
\alpha_1 &\geq \inf_{\substack{\gamma\in\Gamma_\delta \\ f_1, f_2\in\mathcal{D}_a(\gamma)}} \frac{\left[e^{b\rho_\gamma(f_1,f_2)} - e^{b\chi\rho_\gamma(f_1,f_2)}\right] f_1 \int_\gamma g f_2}{\left[e^{b\rho_\gamma(f_1,f_2)} - e^{-\rho_\gamma(f_1,f_2)}\right] f_1 \int_\gamma f_2} \\
&\geq |||g|||_- \inf_{t\in\mathbb{R}^+} \frac{e^{bt} - e^{\chi bt}}{e^{bt} - e^{-t}} \geq \frac{(1-\chi)b}{b+1} |||g|||_-,
\end{aligned}$$

and

$$\alpha_2 \geq (1-\chi)|||g|||_-.$$

This shows that $\alpha \geq \frac{(1-\chi)b}{b+1}|||g|||_-$. Moreover, a similar computation shows that $\beta \leq \frac{(1+\chi)b}{b-1}|||g|||_+$ and concludes the proof, since $\Theta(g, 1) = \ln\frac{\beta}{\alpha}$. □

It is interesting (and helpful) to notice that, if we consider the cone

$$\mathcal{C}_+ = \left\{ g \in C^{(0)}(\mathcal{M}) \;\middle|\; g \not\equiv 0;\ \int_\gamma fg \geq 0 \ \ \forall \gamma \in \Gamma_\delta,\ f \in \mathcal{D}_a(\gamma) \right\}$$

and its Hilbert metric Θ_+, then

$$\Theta_+(g,\,1) = \ln \frac{|||g|||_+}{|||g|||_-}.$$

In this language, Lemma 3.6 states: for each $g \in \mathcal{C}_{\chi b,\,\chi c}$, $\Theta(g,\,1) \leq \ln K + \Theta_+(g,\,1)$.

Up to now we have let the scale δ be arbitrary. In the next step it is instead essential to compare averages on curves having some fixed scale. In the present context we could simply fix δ to be some appropriate value δ_0. Nevertheless, it is inspiring to observe that one can relate averages on any small scale δ with averages on some fixed scale δ_0.

Let $\delta_0 > 0$ be fixed, define

$$\begin{aligned} |||g|||_+^0 &= \sup_{\substack{\gamma \in \Gamma_{\delta_0} \\ f \in \mathcal{D}_a(\gamma)}} \frac{\int_\gamma gf}{\int_\gamma f}, \\ |||g|||_-^0 &= \inf_{\substack{\gamma \in \Gamma_{\delta_0} \\ f \in \mathcal{D}_a(\gamma)}} \frac{\int_\gamma gf}{\int_\gamma f}. \end{aligned}$$

LEMMA 3.7. *For each $\delta \leq \delta_0$, there exists $N_0 \in \mathbb{N}$ such that, for each $g \in \mathcal{C}_{b,c}$, $n \geq N_0$,*

$$\begin{aligned} |||\widetilde{T}^n g|||_+ &\leq |||g|||_+^0, \\ |||\widetilde{T}^n g|||_- &\geq |||g|||_-^0, \end{aligned}$$

provided $a > \frac{c_3}{1-\lambda^{-1}} \delta_0^{1-\nu}$.

Proof. For each $\gamma \in \Gamma_\delta$, $f \in \mathcal{D}_a(\gamma)$ and $n \geq \frac{\ln(2\delta_0\delta^{-1})}{\ln \lambda} = N_0$ we have $|T^{-n}\gamma| > 2\delta_0$; we can then divide $T^{-n}\gamma$ in curves $\{\gamma_i\} \subset \Gamma_{\delta_0}$. The condition on a insures that $\widehat{T}^n f \in \mathcal{D}_a(\gamma_i)$. Now

$$\frac{1}{\int_\gamma f} \int_\gamma \widetilde{T}^n g f = \frac{1}{\int_\gamma f} \sum_i \int_{\gamma_i} g \widehat{T}^n f \leq \frac{1}{\int_\gamma f} \sum_i \int_{\gamma_i} \widehat{T}^n f |||g|||_+^0 = |||g|||_+^0;$$

that is, $|||\widetilde{T}^n g|||_+ \leq |||g|||_+^0$. The other inequality is proved in the same way. □

3.4. *Diameter of the image.* While up to now we did not use the mixing property, it will be necessary to do so in what follows. The main implication of mixing, in our context, is of a geometric nature: it asserts that given a finite number of sets of positive measure, some image of each one of them will

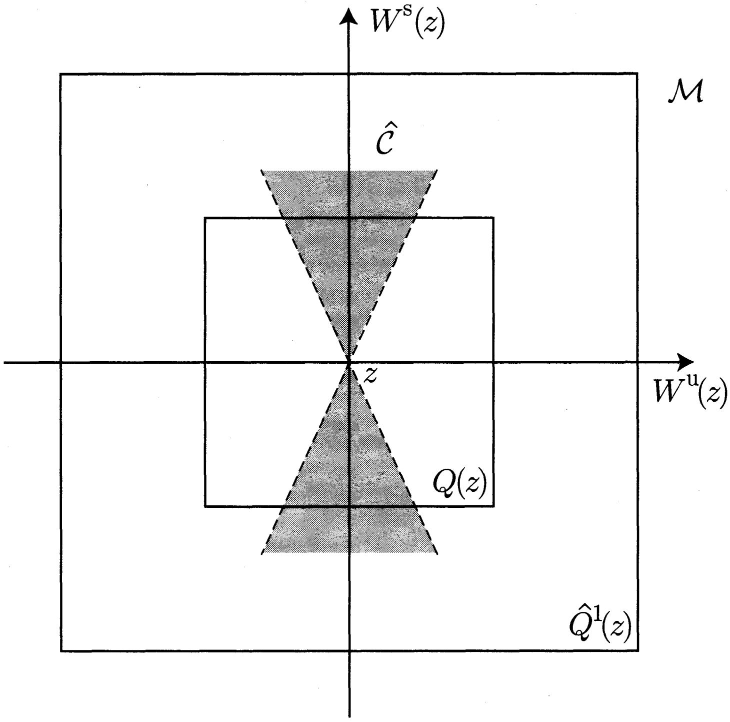

FIGURE 2. Square neighborhood.

intersect all the others. Consequently, given two curves that intersect such sets, the images of one curve will eventually get close to the other.

To make this argument precise we will construct a covering of $\mathcal{M}$, made up of sets that are well-behaved with respect to the curves in Γ_{δ_0}. The construction of the covering may seem a bit technical, but its main point is to insure that each curve in Γ_{δ_0} is contained in some element of the covering, and that the relevant geometric properties of the iterates of any curve can be described solely in terms of the covering.

Consider $z \in \mathcal{M}$. There exists a Cartesian system of coordinates in a neighborhood of z such that the axes are parallel to $W^s(z)$ and $W^u(z)$, respectively. By the continuity of the cone family, there exists a smaller neighborhood of z where each curve parallel to one of the axes is either in $\mathcal{C}$ (in the stable direction) or in its complement (unstable direction). Let $\widehat{Q}(z)$ be a square (in the Cartesian coordinates), with center in z, enjoying the above properties.

We can use the Cartesian structure to identify all the tangent spaces at $w \in \widehat{Q}(z)$ with the tangent space at z, and use the Riemannian metric at z to define a Cartesian metric in $\widehat{Q}(z)$. We then consider some smaller neighborhood $\widehat{Q}_1(z)$ such that, calling d_c the Cartesian distance and d the Riemannian one,

$\frac{3}{4}d \le d_c \le \frac{5}{4}d$, and such that $\bigcup_{w\in\widehat{Q}_1(z)} \mathcal{C}(w) \subset \{|\xi| \le \frac{1}{2}|\eta|\} = \widehat{\mathcal{C}}$; this means that we can consider a unique cone $\widehat{\mathcal{C}}$ in the neighborhood $\widehat{Q}_1(z)$. We finally consider the square neighborhood $Q(z)$, centered at z, but with size half the size of $\widehat{Q}_1(z)$ (see [27] for more detail on such a construction). Since $\mathcal{M}$ is compact, there are finitely many $z_i \in \mathcal{M}$ such that $\bigcup_i Q(z_i) = \mathcal{M}$.

We choose δ_0 to be $\frac{1}{4}$ the size of the smallest of the $Q(z_i)$.

Given a square $G \subset \widehat{Q}_1(z_i)$, for some i, of size $4\delta_0$, it is easy to see that every curve in Γ_{δ_0} that intersects a concentric square of size $\frac{\delta_0}{4}$ will be completely contained in G. We will call such a concentric square the "core" of G (see [27] (§1-3) for a more extensive discussion of the concept of a core and its use). We also define the "central strip" of a square, as a strip, parallel to $W^{\mathrm{u}}(z)$, of size $\frac{\delta_0}{4}$, and containing the core.

In each $\widehat{Q}_1(z_i)$ we construct squares G of size $4\delta_0$ such that the union of their cores covers $Q(z_i)$; the collection $\mathcal{G}$, made by such squares, is a covering of $\mathcal{M}$ with the following properties

(1) $\bigcup_{G\in\mathcal{G}} \operatorname{core}(G) = \mathcal{M}$;

(2) for all $\gamma \in \Gamma_{\delta_0}$, $G \in \mathcal{G}$, if $\gamma \cap \operatorname{core}(G) \ne \emptyset$, then $\gamma \subset G$;

(3) for all $\gamma \in \Gamma_{\delta_0}$, there exists $G \in \mathcal{G}$: $\gamma \cap \operatorname{core}(G) \ne \emptyset$, and the part of γ that intersects the central strip of G consists of γ minus two pieces, each of length larger than $\frac{\delta_0}{4}$.

We will say that γ intersects G "properly" if the last of the latter properties is satisfied. Such a covering is the main ingredient in the following fundamental lemma.

LEMMA 3.8. *When $\delta \le \delta_0$, then there exist $N_* \in \mathbb{N}$ and $\Delta \in \mathbb{R}^+$ such that*

$$\operatorname{diam}(\widetilde{T}^{N_*}\mathcal{C}_{b,c}) \le \Delta < \infty,$$

$N_* \le N + N_0$, where N depends only on T and $\mathcal{G}$, and N_0 is defined as in Lemma 3.7.

Proof. Since the map under consideration is mixing there exists $N \in \mathbb{N}$ such that, for each G, $G' \in \mathcal{G}$,

$$T^{-N}\operatorname{core}(G) \cap \operatorname{core}(G') \ne \emptyset.$$

Without loss of generality we can impose $N > n_1$ (n_1 to be chosen later). Given any function $g \in \mathcal{C}_{b,c}$, we choose $\gamma_* \in \Gamma_{\delta_0}$ and $f_* \in \mathcal{D}_a(\gamma_*)$ such that

$$\frac{\int_{\gamma_*} g f_*}{\int_{\gamma_*} f_*} \ge \frac{1}{2}|||g|||_+^0.$$

Let us call G_* an element of $\mathcal{G}$ properly intersected by γ_*. Given any $\gamma \in \widehat{\Gamma}_{\delta_0}$, there will be an element G of $\mathcal{G}$ that is properly intersected by γ. This implies that there exists $w \in \operatorname{core}(G) \cap T^N \operatorname{core}(G_*)$; by construction, $W^{\mathrm{u}}_{\mathrm{loc}}(w) \cap \gamma \neq \emptyset$, and consists of only one point w_1; moreover $d(T^N(w_1), T^N(w)) \leq \lambda^{-N} \frac{\delta_0}{4}$. Hence, there exists $\gamma_1 \subset T^{-N}\gamma$ such that it is the "translate",[13] along the unstable direction, of γ_*. In fact, by construction, $W^{\mathrm{u}}(T^{-N}w) \cap \gamma_* \neq \emptyset$, and, if we cut $T^{-N}\gamma$ at $T^{-N}w_1$, each of the two resulting pieces is longer than $\lambda^N \frac{\delta_0}{8} \geq \lambda^{n_1} \frac{\delta_0}{8} \geq 3\delta_0$ (where we have chosen n_1 such that $\lambda^{n_1} \geq 24$). Hence, $T^{-N}\gamma \backslash \gamma_1$ can be divided into pieces belonging to Γ_{δ_0}, and

$$\frac{1}{\int_\gamma f} \int_\gamma \widetilde{T}^N g f \geq \frac{1}{\int_\gamma f} \int_{\gamma_1} g \widehat{T}^N f.$$

Let ϕ be the canonical isomorphism (see Appendix II) from γ_* to γ_1 and J_ϕ its Jacobian

$$\frac{1}{\int_\gamma f} \int_\gamma \widetilde{T}^N g f \geq \frac{1}{\int_\gamma f} \int_{\gamma_*} g \circ \phi \left[(\widehat{T}^N f) \circ \phi \right] J_{\phi^{-1}}.$$

From the results recalled in Appendix II it follows that, for R, a, N large and δ_0 small enough, $\widetilde{f} = (\widehat{T}^N f) \circ \phi \, J_{\phi^{-1}} \in \mathcal{D}_a(\gamma_*)$. Hence, for some $z \in \gamma_*$,

$$\begin{aligned}
\frac{1}{\int_\gamma f} \int_\gamma \widetilde{T}^N g f &\geq \frac{1}{\int_\gamma f} \int_{\gamma_*} g \widetilde{f} - \frac{\delta_0 \|D^{\mathrm{u}} g\|_\infty}{\int_\gamma f} \int_{\gamma_*} \widetilde{f} \\
&\geq \frac{1}{\int_\gamma f} \left[\frac{\widetilde{f}(z)}{f_*(z)} \int_{\gamma_*} f_* g e^{-2bR} - c\delta_0 |||g|||_- \int_{\gamma_*} \widetilde{f} \right] \\
&\geq \frac{1}{\int_\gamma f} \left[\frac{\widetilde{f}(z)}{2 f_*(z)} |||g|||_+^0 e^{-2bR} \int_{\gamma_*} f_* - c\delta_0 |||g|||_- \int_{\gamma_*} \widetilde{f} \right] \\
&\geq \left[\frac{e^{-2(b+1)R}}{2} |||g|||_+^0 - c\delta_0 |||g|||_- \right] \frac{\int_{\gamma_1} \widehat{T}^N f}{\int_\gamma f}.
\end{aligned}$$

On the one hand, if $\frac{e^{-2(b+1)R}}{4} |||g|||_+^0 \leq c\delta_0 |||g|||_-$, then we have immediately

$$\frac{|||\widetilde{T}^{N+N_0} g|||_+}{|||\widetilde{T}^{N+N_0} g|||_-} \leq 4c\delta_0 e^{2(b+1)R}.$$

On the other hand, if $\frac{e^{-2(b+1)R}}{4} |||g|||_+^0 \geq c\delta_0 |||g|||_-$, then we choose an arbitrary

[13]By "translate" we mean that, for each $z \in \gamma_*$, $W^{\mathrm{u}}_{\mathrm{loc}}(z) \cap \gamma_1 \neq \emptyset$ and consists of a single point; in addition, for each $w \in \gamma_1$, $W^{\mathrm{u}}_{\mathrm{loc}}(w) \cap \gamma_* \neq \emptyset$ and consists of a single point. See Appendix II for more details.

point $\zeta \in T^N\gamma_1 \subset \gamma$, and we have

$$\begin{aligned}\frac{1}{\int_\gamma f}\int_\gamma \widetilde{T}^N g f &\geq \frac{\int_{T^N\gamma_1}\frac{f(y)}{f(\zeta)}dm(y)}{\int_\gamma \frac{f(y')}{f(\zeta)}dm(y')}\frac{e^{-2(b+1)R}}{4}|||g|||_+^0 \\ &\geq e^{-2^\nu a\delta_0^\nu}\frac{|T^N\gamma_1|}{|\gamma|}\frac{e^{-2(b+1)R}}{4}|||g|||_+^0 \\ &\geq e^{-2^\nu a\delta_0^\nu}\mu^{-N}\frac{e^{-2(b+1)R}}{4}|||g|||_+^0\end{aligned}$$

where $\|DT\|_\infty \leq \mu$. From the above chain of inequalities follows:

$$\frac{1}{\int_\gamma f}\int_\gamma \widetilde{T}^N g f \geq e^{-2^\nu a\delta_0^\nu}\mu^{-N}\frac{e^{-2(b+1)R}}{4}|||\widetilde{T}^N g|||_+^0.$$

This last inequality yields

$$|||\widetilde{T}^N g|||_-^0 \geq \frac{e^{-2^\nu a\delta_0^\nu}\mu^{-N}e^{-2(b+1)R}}{4}|||\widetilde{T}^N g|||_+^0,$$

which implies

$$\frac{|||\widetilde{T}^{N+N_0}g|||_+}{|||\widetilde{T}^{N+N_0}g|||_-} \leq 4e^{2^\nu a\delta_0^\nu}\mu^N e^{2(b+1)R}. \qquad \square$$

Thanks to Lemma 3.8 we can apply Theorem 1.1.

Summarizing, there exists $\Lambda_0 \in (0, 1)$ such that, for each $g_1, g_2 \in \mathcal{C}_{b,c}$

$$\Theta(\widetilde{T}^n g_1, \widetilde{T}^n g_2) \leq \Lambda_0^{\left[\frac{n}{N+N_0}\right]}\Theta(g_1, g_2).$$

Let $\Lambda = \Lambda_0^{[N+N_0+1]^{-1}}$.

3.5. *Decay of correlations.* We have thus obtained a contraction with respect to the Hilbert metric; it remains to verify if this suffices to estimate the decay of correlations. As already remarked, Lemma 1.3 does not apply to this case, yet it is still possible to prove the wanted result.

THEOREM 3.9. *There exist $K > 0$, and $r > 0$ such that, for each $g, f \in C^{(1)}(\mathcal{M})$*

$$\left|\int_{\mathcal{M}} f\widetilde{T}^n g - \int_{\mathcal{M}} f\int_{\mathcal{M}} g\right| \leq K\|g\|_*\|f\|_*\Lambda^n$$

with $\|h\|_* = \int_{\mathcal{M}}|h| + r\|h'\|_\infty$.

Proof. The basic idea behind this result is to partition $\mathcal{M}$ into curves belonging to Γ_δ. Unfortunately, such a simple idea it is not trivially implementable. Of course, such partitions exist (e.g. Markov partitions) but it is not immediately obvious which constraints they force on δ. In order to avoid the problem entirely, we content ourselves with a much coarser partition and use a dynamical argument to refine it.

We have already constructed a covering $\{Q(z_i)\}$ of $\mathcal{M}$. From such covering we can obtain a partition $\mathcal{P} = \{P_i\}$ of $\mathcal{M}$, such that $Q(z_i) \supset P_i$ and ∂P_i is piecewise smooth.

Next, for each $\delta_1 < \delta$, we can partition a neighborhood of $Q(z_i)$ into squares of size δ_1, with sides parallel to the axes. We will collect into $\mathcal{P}^i_+(\delta_1)$ all the squares p such that $p \cap P_i \neq \emptyset$, and into $\mathcal{P}^i_-(\delta_1)$ all the squares p such that $p \subset P_i$; let $\mathcal{P}_+(\delta_1) = \bigcup_i \mathcal{P}^i_+(\delta_1)$, $\mathcal{P}_+(\delta_1) = \bigcup_i \mathcal{P}^i_+(\delta_1)$. Clearly, for each $\varepsilon > 0$, there exists $\delta_1 \in \mathbb{R}^+$ such that ($m(\cdot)$ stands for the invariant measure; $m(\mathcal{M}) = 1$),

$$\begin{aligned} \sum_{p\in\mathcal{P}_+} m(p) &\leq 1+\varepsilon, \\ \sum_{p\in\mathcal{P}_-} m(p) &\geq 1-\varepsilon. \end{aligned} \tag{3.2}$$

These two series of "quasi-partitions" of $\mathcal{M}$ will be sufficient to prove:

LEMMA 3.10. *If $f \in C^{(0)}(\mathcal{M})$, $f \in \widetilde{\mathcal{D}}_{\sigma a/2}(\gamma)$ for each $\gamma \in \Gamma(\delta)$, and $g \in \mathcal{C}_{b,c}$, $g \geq 0$, then*

$$\int_{\mathcal{M}} f|||g|||_- \leq \int_{\mathcal{M}} fg \leq \int_{\mathcal{M}} f|||g|||_+.$$

Proof. For each $\varepsilon > 0$, choose δ_1 such that (3.2) holds, then, for each $n_1 > 0$,

$$\int_{\mathcal{M}} fg = \int_{\mathcal{M}} \widetilde{T}^{n_1}(fg) \geq \sum_{p\in\mathcal{P}_-} \int_p \widetilde{T}^{n_1}(fg).$$

By construction, each p is naturally partitioned into curves belonging to Γ_{δ_1}. If $\widetilde{h}_p$ is the smooth density of the invariant measure expressed in the local Cartesian coordinate $(\eta,\, \xi)$ of the square $Q(z_i)$ and $\gamma_\xi \subset \{(s,\, \xi)\}_{s\in\mathbb{R}}$, then, by application of Fubini's theorem,

$$\int_{\mathcal{M}} fg \geq \sum_{p\in\mathcal{P}_-} \int d\xi \int_{\gamma_\xi} d\eta \widetilde{h}_p \widetilde{T}^{n_1}(fg) = \sum_{p\in\mathcal{P}_-} \int d\xi \int_{T^{-n_1}\gamma_\xi} \widehat{T}^{n_1}(\widetilde{h}_p) fg.$$

We choose n_1 large enough such that, for each p, $f\widetilde{T}^{n_1}\widetilde{h}_p \in \widetilde{\mathcal{D}}_{\sigma a}(\gamma) \subset \mathcal{D}_a(\gamma)$, and $|T^{-n_1}\gamma_\xi| > \delta$ for each ξ,

$$\begin{aligned}\int_{\mathcal{M}} fg &\geq \sum_{p\in\mathcal{P}_-}\int d\xi \int_{T^{-n_1}\gamma_\xi} \widehat{T}^{n_1}(\widetilde{h}_p) f |||g|||_- = \sum_{p\in\mathcal{P}_-}\int d\xi \int_{\gamma_\xi} \widetilde{h}_p \widetilde{T}^{n_1} f |||g|||_- \\ &\geq \left(\int_{\mathcal{M}} f - \int_{\mathcal{M}\setminus \bigcup\limits_{p\in\mathcal{P}_-} p} \widetilde{T}^{n_1} f\right) |||g|||_- \geq \left(\int_{\mathcal{M}} f - \varepsilon\|f\|_\infty\right)|||g|||_- .\end{aligned}$$

This yields the first inequality, since ε is arbitrary. The second inequality is proved in the same way. □

The previous lemma implies

$$\int_{\mathcal{M}} f\widetilde{T}^n g \leq \frac{|||\widetilde{T}^n g|||_+}{|||\widetilde{T}^n g|||_-}\int_{\mathcal{M}} f \int_{\mathcal{M}} g \leq e^{\Theta(\widetilde{T}^n g,\,1)}\int_{\mathcal{M}} f\int_{\mathcal{M}} g$$

and

$$\int_{\mathcal{M}} f\widetilde{T}^n g \geq e^{-\Theta(\widetilde{T}^n g,\,1)}\int_{\mathcal{M}} f\int_{\mathcal{M}} g.$$

The preceding inequalities show that the Theorem holds for functions that satisfy the hypotheses of Lemma 3.10.

Next, we will consider $f,\, g \in C^{(0)}(\mathcal{M})$; $f \geq 0$, $g \geq 0$, f, g almost everywhere differentiable with $\|f'\|_\infty < \infty$, and $\|g'\|_\infty < \infty$; in addition, for each stable curve γ we assume f to be piecewise differentiable along γ, and g to be piecewise differentiable along each unstable manifold. Let $f_\rho = f + \rho$, and $g_\xi = g + \xi$ ($\rho,\, \xi \in \mathbb{R}^+$); then

$$\frac{f(x)}{f(y)} \leq e^{\int_x^y \frac{\|f'\|_\infty}{\rho}} \leq e^{\frac{\|f'\|_\infty}{\rho} d(x,y)^\nu \delta^{1-\nu}}.$$

It follows then that, choosing $\rho = \frac{2\|f'\|_\infty \delta^{1-\nu}}{\sigma a}$, we have $f \in \mathcal{D}_{\frac{\sigma a}{2}}(\gamma)$ for each $\gamma \in \Gamma_\delta$. We can perform a similar computation for g_ξ; it yields $g_\xi \in \mathcal{C}_{b,\,c}$, provided $\xi = \frac{\|g'\|_\infty}{c}$. This implies

$$\begin{aligned}\left|\int_{\mathcal{M}} f\widetilde{T}^n g - \int_{\mathcal{M}} f\int_{\mathcal{M}} g\right| &= \left|\int_{\mathcal{M}} f_\rho \widetilde{T}^n g_\xi - \int_{\mathcal{M}} f_\rho \int_{\mathcal{M}} g_\xi\right| \leq K\Lambda^n \int_{\mathcal{M}} f_\rho \int_{\mathcal{M}} g_\xi \\ &\leq K\Lambda^n \left[\int_{\mathcal{M}} f + \delta^{1-\nu}\frac{2\|f'\|_\infty}{a\sigma}\right]\left[\int_{\mathcal{M}} g + \frac{\|g'\|_\infty}{c}\right].\end{aligned}$$

Again, the proof is concluded since any smooth function can be written as the difference of positive functions with the above properties. □

Remark 3.11. An interesting feature of the proof presented in this section is the possibility of computing explicitly the constants K and Λ by only looking at a finite number of iterations of the map. It would be interesting to perform such computations (eventually by some numerical method) in specific examples to verify if the present strategy can produce realistic estimates.

Review. Let us review the strategy of the proof before going to the next section. The proof consists, essentially, of five steps. The first is the specification of an invariant cone of functions in terms of smoothness in the unstable direction and averages in the stable direction. The second is the reduction of the estimate of the diameter of the images to the estimate of the ratio of the maximal and minimal average. The third (not particularly essential in this context but essential in the following) is the reduction of the estimate of the averages on the scale δ to averages on some fixed scale δ_0. The fourth is the estimate of the averages on the scale δ_0 by using mixing first and the absolute continuity second. The fifth (and last) step consists in showing that the contraction in the Hilbert metric suffices to imply the exponential decay of correlation.

In the next section we will carry out the same five steps in a more general context. I advise the reader to refer to this section if some confusion arises in the next, more technical, one.

4. Two dimensions (the nonsmooth uniformly hyperbolic case)

In this section we will consider a class of maps that is similar to, but more restrictive than, the one considered in [27] and [23].

More precisely, let $\mathcal{M} \subset \mathbb{R}^2$ be a connected compact set on which the map T is defined: $T : \mathcal{M} \to \mathcal{M}$.[14] For simplicity, we will assume $\mathcal{M}$ is a square, but we could as well require that $\partial\mathcal{M}$ consist of a finite number of smooth lines intersecting transversally. We will call a set with the above more general property a "symplectic box" (see [27] for a general definition). The symplectic box $\mathcal{M}$ is partitioned in two ways into unions of an equal number of symplectic boxes

$$\mathcal{M} = \mathcal{M}_1^+ \bigcup \cdots \bigcup \mathcal{M}_m^+ = \mathcal{M}_1^- \bigcup \cdots \bigcup \mathcal{M}_m^-.$$

Two boxes of one partition can overlap at most on their boundaries, i.e.,

$$\mathcal{M}_i^\pm \bigcap \mathcal{M}_j^\pm \subset \partial\mathcal{M}_i^\pm \bigcap \partial\mathcal{M}_j^\pm, \quad i,j = 1,\ldots,m.$$

The map T is defined separately on each of the symplectic boxes $\mathcal{M}_i^+$, $i = 1,\ldots,m$. It is a symplectomorphism (i.e. it preserves the area and it is smooth)

[14]The case in which the map is defined on finitely many disconnected sets, and each set is a two-dimensional symplectic manifold with boundary, can be treated in the same way.

of the interior of each $\mathcal{M}_i^+$ onto the interior $\mathcal{M}_i^-$, $i = 1, \dots, m$ and a homomorphism of $\mathcal{M}_i^+$ onto $\mathcal{M}_i^-$, $i = 1, \dots, m$. We will assume that T is $C^{(2)}$ in each box up to and including the boundary.[15]

We will say that T is a (discontinuous) symplectic map on $\mathcal{M}$. Formally T is not well-defined on the set of points which belong to the boundaries of several plus-boxes: it has several values. We adopt the convention that the image of a subset of $\mathcal{M}$ under T contains all such values.

Let us introduce the singularity sets $\mathcal{S}^+$ and $\mathcal{S}^-$:

$$\mathcal{S}^\pm = \{p \in \mathcal{M} \mid p \text{ belongs to at least two of the boxes } \mathcal{M}_i^\pm, i = 1, \dots, m\}.$$

The plus-singularity set $\mathcal{S}^+$ is a closed subset, and T is continuous on its complement. Similarly T^{-1} is continuous on the complement of $\mathcal{S}^-$.

Now $\mathcal{S}^+ \bigcup \partial\mathcal{M}$ is the union of all the boundaries of the plus-boxes and $\mathcal{S}^- \bigcup \partial\mathcal{M}$ is the union of all the boundaries of the minus-boxes, i.e.,

$$\mathcal{S}^\pm \bigcup \partial\mathcal{M} = \bigcup_{i=1}^{m} \partial\mathcal{M}_i^\pm.$$

Note that most of the points in the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ do not belong to $\mathcal{S}^-$ or $\mathcal{S}^+$.

Our assumptions imply that the singularity sets $\mathcal{S}^\pm$ and the union of boundaries $\bigcup_{i=1}^m \partial\mathcal{M}_i^\pm$ is the finite union of smooth lines intersecting transversally.

We assume T is uniformly hyperbolic, and that at each point $x \in \partial\mathcal{M}_i^\pm$ the stable and unstable directions are uniformly transversal to $\partial\mathcal{M}_i^\pm$.

We assume T enjoys the mixing property.[16]

In addition, we chose $\zeta \in [\sqrt{3} - 1, 1)$ such that

$$\lambda^{-\zeta}\mu^{1-\zeta} < 1 \tag{4.1}$$

where λ and μ are the minimal and maximal expansion coefficients of T, respectively (see §4 for a more precise definition of λ and μ).

Next, we have a condition specific to the present context, which does not play any role in the study of weaker statistical properties (like mixing and ergodicity). We assume that there exists $n_0 \in \mathbb{N}$, such that

$$M(n_0)\lambda^{-\zeta n_0} < \frac{1}{33}, \tag{4.2}$$

$M(n)$ being the maximal number of smooth lines belonging to $\bigcup_{i=0}^n T^i\mathcal{S}^-$ that intersect at one point.[17]

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[15]This is a serious restriction: in particular, billiards do not satisfy it.

[16]The observations in footnote 10 apply here as well.

[17]The choice of 33 is completely arbitrary. The point is that if $M(n)\lambda^{-\zeta n} < 1$, for some n, then there exists n_0 that satisfies (4.2).

Remark 4.1. Note that condition (4.2) can always be fulfilled if $M(n)$ has polynomial growth. But it is not necessary to be able to control $M(n)$ for each n, since (4.2) is a condition concerning some fixed power of T.

Henceforth, we will let $n_0 = 1$ and $M(1) \equiv M$, since this can always be achieved, eventually replacing T by T^{n_0}.

Finally, we assume that there exists a family of cones in the tangent of $\mathcal{M}$ (not to be confused with the cones of functions used later). First we have a cone family $\mathcal{C}_-(x) \in \mathcal{T}_x\mathcal{M}$, continuous on $\mathcal{M}$ and nondegenerate (i.e., $\operatorname{int}(\mathcal{C}_-(x)) \neq \emptyset$) in $\operatorname{int}(\mathcal{M})$. We assume that $\mathcal{C}_-(x)$ contains the stable, but not the unstable, direction, and that if $x \in \mathcal{S}^-$, then the tangent to $\mathcal{S}^-$ is not contained in $\mathcal{C}_-(x)$, together with the corresponding requirement for $\mathcal{S}^+$.[18]

We define $\mathcal{C}_+(x)$ to be the complement of $\mathcal{C}_-(x)$. In addition, we will consider $\widehat{\mathcal{C}}_-(x) = DT^{-l}\mathcal{C}_-(T^l x)$, where l is chosen so that the width of $\widehat{\mathcal{C}}_-(x)$ is less than some $\varepsilon \in \mathbb{R}^+$, and $DT^{-1}\widehat{\mathcal{C}}_-(x) \subset \mathcal{C}_-(T^{-1}x)$ (which implies $DT^{-l-1}\widehat{\mathcal{C}}_-(x) \subset \widehat{\mathcal{C}}_-(T^{-l-1}x)$; i.e., $\widehat{\mathcal{C}}_-$ is a strictly invariant cone family for some fixed power of T).

If the reader wishes to keep in mind concrete examples of the type of systems the results of this section apply to, she can consider the following: $\mathcal{M} = \mathbb{T}^2$, $T(x, y) = F(A(x, y))$ where

$$A = \begin{pmatrix} 1 & a \\ a & 1 + a^2 \end{pmatrix}$$

with $a \notin \mathbb{N}$ (which implies that the linear part is discontinuous on the torus); T is area-preserving; $F : \mathbb{T}^2 \to \mathbb{T}^2$ is a C^2 function sufficiently close to the identity so that the hyperbolicity is not spoiled (see [27] for a discussion of the symplectic boxes' formalism, when applied to this case).

Another possibility are the "piecewise linear standard maps" introduced in [40], [41]. They are maps $T : \mathbb{T}^2 \to \mathbb{T}^2$ defined by

$$T(x_1, x_2) = (x_1 + x_2 + Af(x_1), x_2 + Af(x_1))$$

where (x_1, x_2) are taken modulo 1, f is a periodic function

$$f(t) = |t| - \frac{1}{2}, \quad \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2},$$

and A is a real parameter. The mapping T preserves the Lebesgue measure. For $A = 1$ there is a simple invariant domain $\mathcal{D}$ in the torus. It was proved in

[18] These conditions amount to little more than transversality of $\mathcal{S}^{\pm}$ with respect to the unstable and stable directions, together with uniform transversality of the stable and unstable directions at each point at which they are both defined. It would be possible to further weaken our requirement, but it is not clear how useful such a generalization would be. In fact, in almost all the known examples the proof of the hyperbolicity of the system starts exactly with such a cone family.

[40] that the Lyapunov exponents are different from zero almost everywhere in $\mathcal{D}$, while in [27] it was proved that T is a Bernoulli system in $\mathcal{D}$. From the results of this section it follows that, for functions supported in $\mathcal{D}$, the correlations decay exponentially.

4.1. *Cones of functions.* We will use the same notation of the previous section.[19] The only difference is due to the presence of the discontinuities, which forces us to choose a more complicated cone other than $\mathcal{C}_{bc}$:

$$\begin{aligned}\mathcal{C}_{b,c,d,L}(\delta) = \Bigg\{ g \in C^{(0)}\left(\mathcal{M}\setminus \bigcup_{n=0}^{\infty} T^n \mathcal{S}^-\right) \bigcap L^1(\mathcal{M}) \,\Bigg|\, &\text{for all } \gamma \in \Gamma_\delta; \\ f,\, f_1,\, f_2 \in \mathcal{D}_a(\gamma);\; x \in \gamma;\; I \in \Gamma_{|I|},\, |I| < \delta;\; &f_3,\, f_4,\, f_5 \in \mathcal{D}_a(I); \\ y \in I \text{ holds}, \int_\gamma gf > 0;\; \frac{f_2(x)\int_\gamma f_1 g}{f_1(x)\int_\gamma f_2 g} \le e^{b\rho_\gamma(f_1,\, f_2)};& \\ \frac{\|D^{\mathrm{u}} g\|_\infty}{|||g|||_-} \le c;\; \frac{|\int_I g f_3|}{f_3(y)} \le d\delta^{1-\zeta}|I|^\zeta |||g|||_-;\; \frac{|||g|||_+}{|||g|||_-} \le L;& \\ \frac{\int_I g f_4}{f_4(y)} \le e^{b\rho_I(f_4,\, f_5)} \frac{\int_I g f_5}{f_5(y)} + 2e^{b\rho_I(f_4,\, f_5)}\rho_I(f_4,\, f_5)\delta b d |||g|||_- \Bigg\}.&\end{aligned}$$

It was necessary to weaken the regularity requirements on g since, given $g \in C^{(0)}$, $g \circ T^{-1}$ may be discontinuous; our requirement is the maximal invariant regularity one can ask for. In addition, although g may be discontinuous on a dense set, it makes sense to talk about the derivative in the unstable direction. In fact, the local unstable manifold $W^{\mathrm{u}}_{\mathrm{loc}}(x)$ is well-defined at almost every point [23], and, by definition, enjoys the property $W^{\mathrm{u}}_{\mathrm{loc}}(x) \cap \bigcup_{n=0}^{\infty} T^n\mathcal{S}^- = \emptyset$. Hence, we can define $D^{\mathrm{u}}_x g$ to be the derivative along $W^{\mathrm{u}}_{\mathrm{loc}}(x)$. Finally, we have conditions concerning integrals on arbitrarily short curves; since such curves are unavoidable, some conditions of this type seem necessary; the condition on the rate between the maximal and minimal average turns out to be needed for technical reasons.

4.2. *Contraction of the Hilbert metric.* Of course, our aim is to prove the following:

THEOREM 4.2. *There exist $b, c, d, L \in \mathbb{R}^+$, $\delta_s \in \mathbb{R}^+$, $\chi \in (0,1)$, and a function $m\colon\mathbb{R}^+ \to \mathbb{N}$ such that, for $\delta \le \delta_s$,*

$$\widetilde{T}^{m(\delta)}\mathcal{C}_{b,c,d,L}(\delta) \subset \mathcal{C}_{\chi b, \chi c, \chi d, \chi L}(\delta).$$

---

[19] In the definition of Γ_δ (see §3.2) we use the cone family $\mathcal{C}_-$; the cone family $\widehat{\mathcal{C}}_-$ will also be used in the following. We will call $\widehat{\Gamma}_\delta$ the set of curves that have direction in such a narrower cone family.

Indeed, we shall see that this is sufficient to obtain a contraction of the Hilbert metric (Lemma 4.14). Nevertheless, the route to Theorem 4.2 (at least the one found by the author) is not very direct. Several intermediate steps are needed.

The next lemma is analogous to Lemma 3.2. It is however less satisfactory: it does not provide us with an invariant cone. The condition that fails pertains to the ratio of the averages. Yet, in the previous section we were able to control effectively this ratio only at the very end, by an independent argument. It is then reasonable to content oneself temporarily with the following result, and to proceed with crossed fingers.

LEMMA 4.3. *There exist $N(\delta)$, $\lim_{\delta\to 0} N(\delta)=\infty$ and $A\in\mathbb{R}^+$, such that, if $L>1$, $8L<d$, a and b are large enough* (*see Sublemma* 3.5 *and* (4.8)), *and $2^\nu\delta^\nu a<\ln 2$, then, for each $n\in\{A\ln d,\ldots,N(\delta)\}$,*

$$\widetilde{T}^n\mathcal{C}_{b,\,c,\,d,\,L}(\delta)\subset\mathcal{C}_{\chi b,\,\chi c,\,\chi d,\,\tau_n^2 L}(\delta)$$

where $\chi<1$, and $\tau_n^{-1}=1-3d\lambda^{-n}M^n\geq 1/\sqrt{2}$.

Proof. Not surprisingly all the difficulties arise because $T^{-n}\gamma$ may contain many, eventually very short, connected pieces. The main tool to overcome such problems follows:

SUBLEMMA 4.4. *There are $\delta_0\in\mathbb{R}^+$ and $N:\mathbb{R}^+\to\mathbb{N}$, such that, for each $\delta\leq\delta_0$, $n\leq N(\delta)$, and $\gamma\in\Gamma_\delta$, $T^{-n}\gamma$ consist, at most, of M^n pieces.*

Proof. Property (4.2) implies that there exists δ_0 such that $\mathcal{S}^-$ intersects at most M times a segment in the stable direction of length δ_0, and $M\lambda^{-\varsigma}<1$. Given $\delta\leq\delta_0$ and $\gamma\in\Gamma_\delta$, $T^{-1}\gamma$ will consist, at most, of M pieces. If $\mu=\|DT^{-1}\|_\infty$, then the length of such pieces will be, at most, $\mu\delta$. The result follows then by choice of $N(\delta)=\left[\frac{\ln\delta_0-\ln\delta}{\ln\mu}\right]$. □

The first step consists in comparing $|||g|||_-$ and $|||\widetilde{T}^n g|||_-$, for $n\leq N(\delta)$. Given $\gamma\in\Gamma_\delta$, $T^{-n}\gamma$ consists, at most, of M^n pieces $\{\widetilde{\gamma}_i\}$. We divide such pieces into two collections. The first, $\mathcal{J}_-$, will contain all the maximal connected pieces of length less than δ. The second collection, $\mathcal{J}_+$, will consist of connected pieces of $T^{-n}\gamma$ belonging to Γ_δ. In addition, we require $\left(\bigcup_{\widetilde{\gamma}\in\mathcal{J}_-}\widetilde{\gamma}\right)\cup\left(\bigcup_{\widetilde{\gamma}\in\mathcal{J}_+}\widetilde{\gamma}\right)=T^{-n}\gamma$. Given $\widetilde{\gamma}\in\mathcal{J}_-$ we choose some $x_{\widetilde{\gamma}}\in\widetilde{\gamma}$. Using such notation we can write

$$\begin{aligned}\frac{1}{\int_\gamma f}\int_\gamma\widetilde{T}^n gf &= \frac{1}{\int_\gamma f}\int_{\bigcup\limits_{\widetilde{\gamma}\in\mathcal{J}_+}\widetilde{\gamma}} g\widehat{T}^n f+\frac{1}{\int_\gamma f}\int_{\bigcup\limits_{\widetilde{\gamma}\in\mathcal{J}_-}\widetilde{\gamma}} g\widehat{T}^n f\\ &\geq \frac{1}{\int_\gamma f}\sum_{\widetilde{\gamma}\in\mathcal{J}_+}\int_{\widetilde{\gamma}} g\widehat{T}^n f-\frac{1}{\int_\gamma f}\sum_{\widetilde{\gamma}\in\mathcal{J}_-} f(T^n x_{\widetilde{\gamma}})d\lambda^{-n}\delta^{1-\varsigma}|\widetilde{\gamma}|^\varsigma|||g|||_-\end{aligned}$$

$$
\begin{aligned}
&\geq \left\{\left(1-\sum_{\widetilde{\gamma}\in\mathcal{J}_-}\frac{\int_{\widetilde{\gamma}}\widehat{T}^n f}{\int_\gamma f}\right) - dM^n\lambda^{-n}e^{a2^\nu\delta^\nu}\right\}|||g|||_- \\
&\geq \{(1-2\lambda^{-n}M^n) - 2dM^n\lambda^{-n}\}\,|||g|||_- \\
&\geq \left(1-3d(M\lambda^{-1})^n\right)|||g|||_- \equiv \tau_n|||g|||_-.
\end{aligned}
$$

Hence, if $n \geq \frac{2\ln d}{\ln M^{-1}\lambda} > \frac{\ln 3d}{\ln M^{-1}\lambda}$,

$$\int_\gamma \widetilde{T}^n g f \geq 0.$$

In addition, for $A > 0$ sufficiently large, given $n \in \{A\ln d, \dots, N(\delta)\}$, we get

$$|||\widetilde{T}^n g|||_- \geq \tau_n^{-1}|||g|||_- \geq \frac{1}{\sqrt{2}}|||g|||_-. \tag{4.3}$$

A similar computation yields

$$|||\widetilde{T}^n g|||_+ \leq \tau_n|||g|||_+. \tag{4.4}$$

We are now ready to verify the remaining conditions. We start by analyzing integrals over short curves.

Let $n \leq N(\delta)$ and consider $I \in \Gamma_{|I|}$, $|I| < \delta$. The image $T^{-n}I$ can be divided into many connected pieces that are either in Γ_δ or are shorter than δ; we call $\mathcal{J}_+$ the collection of the first and $\mathcal{J}_-$ the collection of the second. For each $f \in \mathcal{D}_a(\gamma)$ and $x \in I$,

$$\frac{\left|\int_I \widetilde{T}^n g f\right|}{f(x)} \leq \frac{1}{f(x)}\left\{\sum_{J\in\mathcal{J}_-}\left|\int_J g\widehat{T}^n f\right| + \sum_{J\in\mathcal{J}_+}\int_J g\widehat{T}^n f\right\}.$$

Given $J \in \mathcal{J}_-$, choose $x_J \in J$ such that $\left|\det(D_{x_J}T^n|_J)\right| = \inf_{z\in J}\left|\det(D_z T^n|_J)\right|$; then

$$
\begin{aligned}
\frac{\left|\int_I \widetilde{T}^n g f\right|}{f(x)} \leq\ & \frac{1}{f(x)}\left\{\sum_{J\in\mathcal{J}_-} f(T^n x_J)\left|\det(D_{x_J}T^n|_J)\right| d|J|^\varsigma\delta^{1-\varsigma}|||g|||_-\right. \\
&\left. + \sum_{J\in\mathcal{J}_+}\int_J \widehat{T}^n f|||g|||_+\right\}.
\end{aligned}
$$

Since,

$$|J|\left|\det(D_{x_J}T^n|_J)\right| \leq \int_{T^nJ}\frac{\left|\det(D_{x_J}T^n|_J)\right|}{\left|\det(D_{T^{-n}y}T^n|_J)\right|}dm(y) \leq |T^nJ|, \tag{4.5}$$

it follows that

$$\frac{\left|\int_I \widetilde{T}^n gf\right|}{f(x)} \le \left\{\sum_{J\in\mathcal{J}_-} d\frac{f(T^n x_J)}{f(x)}\lambda^{(\zeta-1)n}|T^n J|^\zeta \delta^{1-\zeta} + L\sum_{J\in\mathcal{J}_+}\int_{T^n J}\frac{f(y)}{f(x)}dm(y)\right\}$$

$$\times |||g|||_- \le de^{a2^\nu\delta^\nu}\left\{\lambda^{(\zeta-1)n}\delta^{1-\zeta}\left[\sum_{J\in\mathcal{J}_-}|T^n J|\right]^\zeta\left[\sum_{J\in\mathcal{J}_-}1\right]^{1-\zeta}\right.$$

$$\left. + Ld^{-1}\left|\bigcup_{J\in\mathcal{J}_+} T^n J\right|^\zeta \delta^{1-\zeta}\right\}|||g|||_-,$$

which implies:

$$\frac{\left|\int_I \widetilde{T}^n gf\right|}{f(x)} \le dK_1\delta^{1-\zeta}|I|^\zeta |||g|||_- \tag{4.6}$$

where K_1 can be taken to be $4\max\{(\lambda^{-1}M)^{(1-\zeta)n},\, Ld^{-1}\}$, provided δ is sufficiently small.

Choosing δ small and A large enough, it is possible to have $K_1 \le 1/2$, provided that

$$L \le \frac{d}{8}. \tag{4.7}$$

Collecting 4.3 and 4.6 we get

$$\frac{\left|\int_I \widetilde{T}^n gf\right|}{f(x)} \le \chi d\delta^{1-\zeta}|I|^\zeta |||\widetilde{T}^n g|||_-,$$

provided $n \in \{A\ln d, \ldots, N(\delta)\}$.

Moreover,

$$0 \le \frac{\|D^{\mathrm{u}}\widetilde{T}^n g\|_\infty}{|||\widetilde{T}^n g|||_-} \le \chi c.$$

Finally, when we look at the ratios of different averages, difficulties again originate from the possible existence of short pieces.

Let $\gamma \in \Gamma_\delta$ and $\mathcal{J}$ be the collection of the connected pieces of $T^{-n}\gamma$. In $\mathcal{J}_-$ we put the pieces shorter than δ, and we collect the others (after sub-dividing them in pieces belonging to Γ_δ) in $\mathcal{J}_+$.

$$\int_\gamma \widetilde{T}^n g f_1 \le \sum_{\widetilde{\gamma}\in\mathcal{J}} \frac{\widehat{T}^n f_1(x_{\widetilde{\gamma}})}{\widehat{T}^n f_2(x_{\widetilde{\gamma}})}\int_{\widetilde{\gamma}} g\widehat{T}^n f_2 e^{\xi b\rho_\gamma(f_1,\, f_2)}$$

$$+ \sum_{\widetilde{\gamma}\in\mathcal{J}_-} \xi\widehat{T}^n f_1(x_{\widetilde{\gamma}})2\delta e^{\xi b\rho_\gamma(f_1,\, f_2)} bd\rho_\gamma(f_1,\, f_2)|||g|||_-,$$

where we have used the fact that the analogues of Sublemmas 3.4 and 3.5 hold in this case as well (providing us with $\xi < 1$).

$$
\begin{aligned}
\frac{\int_\gamma \widetilde{T}^n g f_1}{f_1(x)} \leq & \Bigg[\sum_{\widetilde{\gamma}\in\mathcal{J}} \frac{f_1(T^n x_{\widetilde{\gamma}}) f_2(x)}{f_2(T^n x_{\widetilde{\gamma}}) f_1(x)} \int_{\widetilde{\gamma}} g\widehat{T}^n f_2 \\
& + 4\lambda^{-n}\delta b d\rho_\gamma(f_1, f_2) |||\widetilde{T}^n g|||_- \sum_{\widetilde{\gamma}\in\mathcal{J}_-} \frac{f_2(x) f_1(T^n x_{\widetilde{\gamma}})}{f_1(x)}\Bigg] \frac{e^{\xi b\rho_\gamma(f_1, f_2)}}{f_2(x)}.
\end{aligned}
$$

Let us call $\widetilde{\mathcal{J}}_- \subset \mathcal{J}_-$ the set of curves $\widetilde{\gamma}$ such that $\int_{\widetilde{\gamma}} g\widehat{T}^n f_2 < 0$; then

$$
\begin{aligned}
\frac{\int_\gamma \widetilde{T}^n g f_1}{f_1(x)} \leq & \Bigg[\sum_{\widetilde{\gamma}\in\mathcal{J}} \int_{\widetilde{\gamma}} g\widehat{T}^n f_2 + \sum_{\widetilde{\gamma}\in\widetilde{\mathcal{J}}_-} \left[1 - e^{-2\rho_\gamma(f_1, f_2)}\right] \left|\int_{\widetilde{\gamma}} g\widehat{T}^n f_2\right| \\
& + 4\lambda^{-n}\delta b d\rho_\gamma(f_1, f_2) |||\widetilde{T}^n g|||_- \sum_{\widetilde{\gamma}\in\mathcal{J}_-} f_2(T^n x_{\widetilde{\gamma}})\Bigg] \frac{e^{(1+\xi b)\rho_\gamma(f_1, f_2)}}{f_2(x)} \\
\leq & \Bigg[\int_\gamma \widetilde{T}^n g f_2 + 4\rho_\gamma(f_1, f_2) |||\widetilde{T}^n g|||_- d\delta\lambda^{-n} \sum_{\widetilde{\gamma}\in\widetilde{\mathcal{J}}_-} f_2(T^n x_{\widetilde{\gamma}}) \\
& + 4\lambda^{-n}\delta b d\rho_\gamma(f_1, f_2) |||\widetilde{T}^n g|||_- \sum_{\widetilde{\gamma}\in\mathcal{J}_-} f_2(T^n x_{\widetilde{\gamma}})\Bigg] \frac{e^{(\xi b+1)\rho_\gamma(f_1, f_2)}}{f_2(x)} \\
\leq & \left[1 + 16 b d\rho_\gamma(f_1, f_2) M^n \lambda^{-n}\right] e^{(\xi b+1)\rho_\gamma(f_1, f_2)} \frac{\int_\gamma \widetilde{T}^n g f_2}{f_2(x)} \\
\leq & \, e^{\left[\xi + b^{-1} + 16 M^n \lambda^{-n} d\right] b\rho_\gamma(f_1, f_2)} \frac{\int_\gamma \widetilde{T}^n g f_2}{f_2(x)} \\
\leq & \, e^{\chi b\rho_\gamma(f_1, f_2)} \frac{\int_\gamma \widetilde{T}^n g f_2}{f_2(x)}
\end{aligned}
$$

where we have set b and A large enough; more precisely

$$
b^{-1} + \xi + 16(M\lambda^{-1})^n d \leq \chi < 1. \tag{4.8}
$$

We are left to check the last condition in our definition of the cone. Let $I \in \Gamma_{|I|}$, with $|I| < \delta$, collect in $\mathcal{J}_+$ the pieces of $T^{-n}I$ longer than δ and in $\mathcal{J}_-$ the shorter ones.

$$\begin{aligned}
\frac{\int_I \widetilde{T}^n g f_1}{f_1(x)} &= \frac{1}{f_1(x)}\left\{\sum_{\gamma\in\mathcal{J}_+}\int_\gamma g\widehat{T}^n f_1 + \sum_{J\in\mathcal{J}_-}\int_J g\widehat{T}^n f_1\right\} \\
&\leq \frac{1}{f_1(x)}\left\{\sum_{\gamma\in\mathcal{J}_+}\frac{\widehat{T}^n f_1(x_\gamma)}{\widehat{T}^n f_2(x_\gamma)}\int_\gamma g\widehat{T}^n f_2 e^{\xi b\rho_I(f_1,f_2)}\right. \\
&\quad + \sum_{J\in\mathcal{J}_-}\left[\frac{\widehat{T}^n f(x_J)}{\widehat{T}^n f_2(x_J)}\int_J g\widehat{T}^n f_2 e^{\xi b\rho_I(f_1,f_2)}\right. \\
&\quad + \left.\left. 2\widehat{T}^n f_1(x_J)\xi e^{\xi b\rho_I(f_1,f_2)}\delta d b\rho_I(f_1,\, f_2)|||g|||_-\right]\right\}.
\end{aligned}$$

Again, we collect in $\widetilde{\mathcal{J}}_-$ the $J\in\mathcal{J}_-$ such that $\int_J g\widehat{T}^n f_2 < 0$; accordingly

$$\begin{aligned}
\frac{\int_I \widetilde{T}^n g f_1}{f_1(x)} &\leq e^{(\xi b+1)\rho_I(f_1,f_2)}\frac{\int_I \widetilde{T}^n g f_2}{f_2(x)} \\
&\quad + \frac{8\xi M^n}{\lambda^n}e^{\xi b\rho_I(f_1,f_2)}\delta d b\rho_I(f_1,\, f_2)|||\widetilde{T}^n g|||_- \\
&\quad + \sum_{J\in\widetilde{\mathcal{J}}_-} 2e^{(b\xi+1)\rho_I(f_1,f_2)}\rho_I(f_1,\, f_2)f_2(x)^{-1}\left|\int_J g\widehat{T}^n f_2\right| \\
&\leq e^{\chi b\rho_I(f_1,f_2)}\frac{\int_I \widetilde{T}^n g f_2}{f_2(x)} + 2\chi^2 e^{\chi b\rho_I(f_1,f_2)}\delta d b\rho_I(f_1,\, f_2)|||\widetilde{T}^n g|||_-,
\end{aligned}$$

provided

$$8\lambda^{-n}M^n \leq \chi^2. \qquad \square$$

Let us choose $n_* = [A\ln d]+1$, and δ so small that $N(\delta)\geq 2A\ln d$.

4.3. *Averages on different scales.* Next, we need estimates that allow us to compare the average of $\widetilde{T}^n g$ on the scale δ with the average of g on a fixed scale δ_0; for the time being, the only assumption on δ_0 is

$$a \in \left(\frac{c_4\delta_0^{1-\nu}}{1-\lambda^{-\nu}},\, 2^{-\nu}\ln 2\delta_0^{-\nu}\right).$$

We define[20]

$$|||g|||_+^0 = \sup_{\substack{\gamma\in\widehat{\Gamma}_{\delta_0}\\ f\in\mathcal{D}_a(\gamma)}}\frac{\int_\gamma gf}{\int_\gamma f},$$

[20]As already mentioned, $\widehat{\Gamma}$ is defined as in §3.2 by using $\widehat{\mathcal{C}}_-$ as a cone family.

$$|||g|||_-^0 = \inf_{\substack{\gamma\in\widehat{\Gamma}_{\delta_0}\\ f\in\mathcal{D}_a(\gamma)}} \frac{\int_\gamma gf}{\int_\gamma f}.$$

LEMMA 4.5. *There exist $\rho\in\mathbb{R}^+$ and $N_-(\delta)\le\frac{N(\delta)\delta_0^\rho}{16d\delta^\rho}=N_+(\delta)$ such that, for each $g\in\widetilde{T}^{n*}\mathcal{C}_{b,c,d,L}(\delta)$, δ sufficiently small, and $n\in\{N_-(\delta),\dots,N_+(\delta)\}$, either*

$$\frac{|||\widetilde{T}^n g|||_+}{|||\widetilde{T}^n g|||_-}\le\frac{3}{8}\frac{|||g|||_+}{|||g|||_-},$$

or

$$\begin{aligned}|||\widetilde{T}^n g|||_+ &\le 4|||g|||_+^0,\\ |||\widetilde{T}^n g|||_- &\ge \frac{1}{2}|||g|||_-^0.\end{aligned}$$

Proof. We start by noticing that, although Lemma 4.3 did not provide us with an invariant cone, we can control a huge number of images.

SUBLEMMA 4.6. *There exists $\rho\in\mathbb{R}^+$ such that*

$$n\in\left\{A\ln d,\,\dots,\,\frac{N(\delta)\delta_0^\rho}{16d\delta^\rho}\right\}\Longrightarrow\widetilde{T}^n\mathcal{C}_{b,\,c,\,d,\,L}\subset\mathcal{C}_{\chi b,\,\chi c,\,\chi d,\,4L}$$

provided that $32L\le d$. In addition, for each $g\in\mathcal{C}_{b,c,d,L}$

$$\begin{aligned}|||\widetilde{T}^n g|||_+ &\le 2|||g|||_+,\\ |||\widetilde{T}^n g|||_- &\ge \frac{1}{2}|||g|||_-.\end{aligned}$$

Proof. For each $n\in\{A\ln d,\,\dots,\,\left(\frac{\delta_0}{\delta}\right)^\rho\frac{N(\delta)}{16d}\}$, we write $n=\frac{kN(\delta)}{2}+m$, where we require $m\in\{A\ln d,\,\dots,\,N(\delta)\}$. According to Lemma 4.3 (in eqs. 4.3, 4.4), for $l\in\{A\ln d,\,\dots,\,N(\delta)\}$,

$$\begin{aligned}|||\widetilde{T}^l g|||_- &\ge (1-3d\lambda^{-l}M^l)|||g|||_-=\tau_l^{-1}|||g|||_-,\\ |||\widetilde{T}^l g|||_+ &\le \tau_l|||g|||_+.\end{aligned}$$

Moreover, A has been chosen such that $\tau_{A\ln d}^{-1}\ge\frac{1}{\sqrt{2}}$. Setting $\tau=\tau_{N(\delta)/2}$, all we need to check is that

$$\tau^{2(k-1)}\le 2.$$

The result follows, since

$$\tau^2\le e^{6d\lambda^{-\frac{N(\delta)}{2}}M^{\frac{N(\delta)}{2}}},$$

and, with $\rho = \frac{\ln \lambda M^{-1}}{2 \ln \mu}$,

$$\tau^2 \leq e^{6d\left(\frac{\delta}{\delta_0}\right)^\rho}$$

which implies the lemma. □

The idea of the proof is to iterate $\gamma \in \Gamma_\delta$ until the majority of the pieces produced are longer than δ_0. To be more precise we will iterate pieces of the images in order to obtain long connected pieces, but we will stop iterating them once they become longer than δ_0.

Let $\gamma_0 = \gamma$; let $n_i \in \mathbb{N}$ be the first integer for which, given $\gamma_i \subset \gamma$, $T^{-n_i}\gamma_i$ contains a connected piece longer than δ_0.[21] We will collect such long pieces in $\mathcal{J}_i$, after subdividing them in pieces belonging to $\widehat{\Gamma}_{\delta_0}$.[22] The sets $\gamma_i \subset \gamma$ are defined by induction: $\gamma_{i+1} = \gamma_i \setminus \left(\bigcup_{\widetilde{\gamma} \in \mathcal{J}_i} T^{n_i}\widetilde{\gamma}\right)$. In addition, we will collect all the connected pieces belonging to $T^{-n_i}\gamma_{i+1}$ (by construction such pieces are all shorter than δ_0) into two groups: $\mathcal{J}_i^+$ will contain the pieces longer than δ, and $\mathcal{J}_i^-$ the shorter pieces; $\widetilde{\mathcal{J}}_i = \mathcal{J}_i^+ \cup \mathcal{J}_i^-$.

By construction, $n_{i+1} > n_i$, and $T^{-n_i}\gamma_i$ consists, at most, of M^{n_i} connected pieces. For each $i_* > 0$, and $m \geq n_{i_*}$, provided $\left(\frac{\delta}{\delta_0}\right)^{-\rho} \frac{N(\delta)}{16d} \geq m$,

$$\begin{aligned}
\frac{\int_\gamma \widetilde{T}^m g f}{\int_\gamma f} &= \sum_{i=0}^{i_*} \sum_{\widetilde{\gamma} \in \mathcal{J}_i} \frac{\int_{\widetilde{\gamma}} \widetilde{T}^{m-n_i} g \widehat{T}^{n_i} f}{\int_\gamma f} + \sum_{\widetilde{\gamma} \in \widetilde{\mathcal{J}}_{i_*}} \frac{\int_{\widetilde{\gamma}} \widetilde{T}^{m-n_{i_*}} g \widehat{T}^{n_{i_*}} f}{\int_\gamma f} \\
&\leq \sup_{n \in \{0,\dots,m\}} |||\widetilde{T}^n g|||_+^0 + \frac{2}{|\gamma|} |||\widetilde{T}^{m-n_{i_*}} g|||_+ \sum_{\widetilde{\gamma} \in \mathcal{J}_{i_*}^+} |T^{n_{i_*}}\widetilde{\gamma}| \\
&\quad + 2d(\lambda^{-1}M)^{n_{i_*}} |||\widetilde{T}^{m-n_{i_*}} g|||_- .
\end{aligned}$$

To conclude the argument, we have to choose i_* large enough so that the unwanted terms in the previous equation are negligible but not beyond the applicability of Sublemma 4.6 $\left(\text{i.e., } n_{i_*} \leq \left(\frac{\delta}{\delta_0}\right)^{-\rho} \frac{N(\delta)}{16d}\right)$.

Since $n_{i_*} \geq N(\delta)$, it follows that $2d(\lambda^{-1}M)^{n_{i_*}} \leq \frac{1}{8}$, provided δ is sufficiently small. Let i_* be the first integer i for which $\sum_{\widetilde{\gamma} \in \widetilde{\mathcal{J}}_i} |T^{n_i}\widetilde{\gamma}| = |\gamma_{i+1}| \leq \frac{|\gamma|}{128L}$. This means that $|\gamma_{i_*}| > \frac{\delta}{128L}$; in addition, $T^{-k}\gamma_{i_*}$ will consist, at most, of M^k pieces, shorter than δ_0, for each $k < n_{i_*}$. Let us call $\mathcal{J}_k^*$ the collection of such pieces. Now,

$$\frac{\delta}{128L} \leq \sum_{\widetilde{\gamma} \in \mathcal{J}_k^*} \frac{|T^k\widetilde{\gamma}|}{|\widetilde{\gamma}|} |\widetilde{\gamma}| \leq \sum_{\widetilde{\gamma} \in \mathcal{J}_k^*} \frac{|T^k\widetilde{\gamma}|}{|\widetilde{\gamma}|} \delta_0 \leq \left(\lambda^{-1}M\right)^k \delta_0.$$

---

[21] It is easy to convince oneself that, if γ_i contains an interval, then n_i is finite.

[22] Since, n_1 must be larger than $N(\delta)$, choosing δ small enough, we can insure $N(\delta) \geq l$ (see the definition of $\widehat{\mathcal{C}}_-$ at the beginning of the section).

The previous computation shows that, when $k_0 = \frac{\ln(128L\delta_0\delta^{-1})}{\ln \lambda M^{-1}}$, $n_{i_*} \le k_0 + 1$.

Using our choice of i_* and the previous estimates, we conclude that, for each $m \in \{k_0+1, \ldots, N_+(\delta)\}$,

$$\frac{\int_\gamma \widetilde{T}^m g f}{\int_\gamma f} \le \sup_{n\in\{0,\ldots,m\}} |||\widetilde{T}^n g|||_+^0 + \frac{1}{4}|||\widetilde{T}^m g|||_- .$$

In the same manner one can prove:

$$\frac{\int_\gamma \widetilde{T}^m g f}{\int_\gamma f} \ge (1-2\lambda^{-1}M) \inf_{n\in\{0,\ldots,m\}} |||\widetilde{T}^n g|||_-^0 - \frac{1}{4}|||\widetilde{T}^m g|||_- .$$

To obtain a useful inequality it is necessary to estimate $|||\widetilde{T}^n g|||_\pm^0$. This is done by the following.

SUBLEMMA 4.7. *For each $g \in \widetilde{T}^{n_*}\mathcal{C}_{b,c,d,L}(\delta)$, and $n \le N_+(\delta)$,*[23]

$$\begin{aligned} |||\widetilde{T}^n g|||_+^0 &\le |||g|||_+^0 + \frac{3}{32}|||g|||_+, \\ |||\widetilde{T}^n g|||_-^0 &\ge \frac{3}{4}|||g|||_-^0 . \end{aligned}$$

Proof. The proof is by induction. Let $\gamma \in \widehat{\Gamma}_{\delta_0}$; if $\mathcal{J}_1$ contains the connected pieces of $T^{-1}\gamma$ longer than δ_0, $\mathcal{J}_2$ contains the ones shorter than δ_0 but longer than δ, $\mathcal{J}_3$ contains the shorter pieces, and $x_{\widetilde{\gamma}} \in \widetilde{\gamma}$ is an arbitrary point, then

$$\begin{aligned} \frac{\int_\gamma \widetilde{T} g f}{\int_\gamma f} &\le \sum_{\widetilde{\gamma}\in\mathcal{J}_1} \frac{\int_\gamma g \widehat{T} f}{\int_\gamma f} + \sum_{\widetilde{\gamma}\in\mathcal{J}_2} \frac{\int_{\widetilde{\gamma}} \widehat{T} f}{\int_\gamma f}|||g|||_+ + \sum_{\widetilde{\gamma}\in\mathcal{J}_3} \frac{\widehat{T} f(x_{\widetilde{\gamma}})}{\int_\gamma f}\delta d |||g|||_- \\ &\le |||g|||_+^0 + (\lambda^{-1}M)\left[2 + 2d\frac{\delta}{\delta_0}\right]|||g|||_+ \le |||g|||_+^0 + 3(\lambda^{-1}M)|||g|||_+ \end{aligned}$$

provided δ is sufficiently small. Our induction hypothesis is

$$|||\widetilde{T}^n g|||_+^0 \le |||g|||_+^0 + 3|||g|||_+ \sum_{i=1}^{n}(\lambda^{-1}M)^i . \tag{4.9}$$

We use a construction similar to the one used previously. We collect in $\mathcal{J}_1$ all the pieces of $T^{-1}\gamma$ longer than δ_0, in $\mathcal{J}_2$ all the pieces $\widetilde{\gamma}$ of $T^{-2}\gamma$ longer than δ_0, provided $T\widetilde{\gamma}$ was not contained in an element of the collection $\mathcal{J}_1$, and so on. The connected pieces $\widetilde{\gamma}$ of $T^{-n}\gamma$ that are shorter than δ_0, and such that $T^i\widetilde{\gamma}$, $i \in \{1, \ldots, n\}$, was always contained in maximal connected pieces shorter

[23] n_* is defined at the end of §4.2.

than δ_0, are collected in two sets: $\mathcal{J}_+$ contains the pieces longer than δ, and $\mathcal{J}_-$ the shorter ones. Assume (4.9) for all $j \leq n$, then

$$\begin{aligned}
\frac{\int_\gamma \widetilde{T}^{n+1} g f}{\int_\gamma f} &\leq \sum_{i=0}^{n} \sum_{\widetilde{\gamma} \in \mathcal{J}_{i+1}} \frac{\int_{\widetilde{\gamma}} \widetilde{T}^{n-i} g \widehat{T}^{i+1} f}{\int_\gamma f} \\
&\quad + \sum_{\widetilde{\gamma} \in \mathcal{J}_+} \frac{\int_{\widetilde{\gamma}} g \widehat{T}^{n+1} f}{\int_\gamma f} + \frac{2M^{n+1} d\delta}{\lambda^{n+1}\delta_0} |||g|||_- \\
&\leq \sum_{i=0}^{n} \sum_{\widetilde{\gamma} \in \mathcal{J}_{i+1}} \frac{\int_{\widetilde{\gamma}} \widehat{T}^{i+1} f}{\int_\gamma f} \left\{ |||g|||_+^0 + 3|||g|||_+ \sum_{j=1}^{n} (\lambda^{-1}M)^j \right\} \\
&\quad + 3(\lambda^{-1}M)^{n+1} |||g|||_+ \leq |||g|||_+^0 + 3|||g|||_+ \sum_{i=1}^{n+1} (\lambda^{-1}M)^i.
\end{aligned}$$

By (4.9) and (4.2),

$$|||\widetilde{T}^n g|||_+^0 \leq |||g|||_+^0 + \frac{3(\lambda^{-1}M)}{1 - \lambda^{-1}M} |||g|||_+ \leq |||g|||_+^0 + \frac{3}{32} |||g|||_+.$$

The second inequality is proved in a similar way. In fact,

$$\begin{aligned}
\frac{\int_\gamma \widetilde{T} g f}{\int_\gamma f} &\geq \sum_{\widetilde{\gamma} \in \mathcal{J}_1} \frac{\int_\gamma g \widehat{T} f}{\int_\gamma f} - \frac{2dM\delta}{\lambda\delta_0} |||g|||_- \\
&\geq (1 - 2\lambda^{-1}M) |||g|||_+^0 - \frac{M}{2\lambda} |||g|||_- \\
&\geq (1 - 2\lambda^{-1}M) |||g|||_+^0 - M\lambda^{-1} |||\widetilde{T} g|||_-,
\end{aligned}$$

since, for each function $\widetilde{g} \in \mathcal{C}_{b,c,d,L}$, $|||\widetilde{g}|||_- \leq |||\widetilde{g}|||_-^0$, we have

$$\frac{\int_\gamma \widetilde{T} g f}{\int_\gamma f} \geq \frac{1 - 2\lambda^{-1}M}{1 + \lambda^{-1}M} |||g|||_+^0.$$

This time, we will use the following induction hypothesis:

$$|||\widetilde{T}^n g|||_-^0 \geq h_n |||g|||_-^0$$

where $h_1 = \frac{1-2\lambda^{-1}M}{1+\lambda^{-1}M}$ and $h_{n+1} = \frac{1-2(\lambda^{-1}M)^{n+1}}{1+(\lambda^{-1}M)^{n+1}} h_n$.

$$\frac{\int_\gamma \widetilde{T}^{n+1} g f}{\int_\gamma f} \geq (1 - 2(\lambda^{-1}M)^{n+1}) h_n |||g|||_-^0 - \frac{(\lambda^{-1}M)^{n+1}}{2} |||g|||_-.$$

This concludes the argument, since

$$h_n = \prod_{i=1}^{n} \frac{1 - 2(\lambda^{-1}M)^i}{1 + (\lambda^{-1}M)^i} \geq e^{-4\sum_{i=1}^{n} (\lambda^{-1}M)^i} \geq \frac{3}{4}. \qquad \square$$

From the previous lemma we obtain immediately

$$\frac{5}{4}|||\widetilde{T}^n g|||_- \geq \left(1 - \frac{2}{33}\right)\frac{3}{4}|||g|||_-^0$$

which implies the last inequality in the statement of the lemma.

To conclude the argument let us consider two cases. First suppose that

$$|||\widetilde{T}^n g|||_+ \leq \frac{3}{16}|||g|||_+.$$

This would immediately imply the first possibility in the statement. Second, suppose that

$$|||\widetilde{T}^n g|||_+ \geq \frac{3}{16}|||g|||_+.$$

The above inequality, together with the inequalities obtained before, yields

$$|||\widetilde{T}^n g|||_+ \leq |||g|||_+^0 + \frac{3}{4}|||\widetilde{T}^n g|||_+,$$

which conclude the proof. □

4.4. *Comparison of averages.* The only task left is to compare the value of the integral on different γ's. Due to the presence of the singularities, this is much more complicated that in the smooth case. Nonetheless, the basic idea is the same. The only difference is that we will break up a curve in $\widehat{\Gamma}_{\delta_0}$ into pieces belonging to $\widehat{\Gamma}_{\delta_1}$, with $\delta_0 \geq \delta_1 \geq \delta$, and we will compare such pieces. The essential feature of this decomposition is that δ_1 can be chosen smaller than δ_0 but fixed, depending on the parameters of the cone, while still leaving the possibility of choosing δ as small as needed. To carry out this plan we need to introduce appropriate neighborhoods of $\mathcal{S}^-$.

Definition 4.8.

$$\begin{aligned} \mathcal{S}_\varepsilon^\pm &= \{z \in \mathcal{M} \mid \operatorname{dist}(z, \mathcal{S}^\pm) \leq \varepsilon\}, \\ \Sigma_n^\pm(c, \delta_1) &= \bigcup_{i=0}^{n} T^{\mp i}\left(\mathcal{S}^\pm_{c\lambda^{-i}\delta_1}\right). \end{aligned}$$

Moreover, we will say that two curves $\gamma_1, \gamma_2 \in \widehat{\Gamma}_\varepsilon$, $\sup_{x\in\gamma_1}\inf_{y\in\gamma_2} d(x, y) \leq \sigma^{-1}\delta_1$,[24] are the translates of each other in the unstable direction if each sufficiently long (but shorter than $\sigma^{-2}\delta_1$) unstable manifold intersecting one curve intersects the other as well.

We are now able to state our main comparison lemma.

[24]See the proof of Sublemma 4.13 for the definition of σ.

LEMMA 4.9. *There exists $m_* \in \mathbb{N}$ such that, if two curves $\gamma_1, \gamma_2 \in \widehat{\Gamma}_{\delta_1}$ are the translates of each other in the unstable direction, their distance is less than $\delta_1/2$, and $\gamma_1 \bigcap \Sigma_{m_*} = \gamma_2 \bigcap \Sigma_{m_*} = \emptyset$, then*

$$\frac{1}{2}\int_{\gamma_2} g - \frac{e^{-bR}}{256}|||g|||_-\delta_1 \le \int_{\gamma_1} g \le 2\int_{\gamma_2} g + \frac{e^{-bR}}{256}|||g|||_-\delta_1.$$

Proof. The lemma is a consequence of the results stated in Appendix II.

$$\begin{aligned}\int_{\gamma_1} g &= \int_{D(\phi)} g + \int_{\gamma_1\setminus D(\phi)} g \le \int_{R(\phi)} g\circ\phi^{-1}J_\phi + D\delta_1(\lambda^{-\varsigma}\mu^{1-\varsigma})^{m_*}|||g|||_- \\ &\le \int_{R(\phi)} gJ_\phi + \|D^{\mathrm{u}}g\|_\infty \frac{\delta_1}{2}\int_{R(\phi)} J_\phi + D\delta_1(\lambda^{-\varsigma}\mu^{1-\varsigma})^{m_*}|||g|||_-.\end{aligned}$$

Here we have used Lemma II.3. To use effectively the properties of g it is necessary for the domain of integration to be connected, while $R(\phi)$ is not. But, from Lemma II.5 and II.7 it follows that both ϕ and J_ϕ can be extended to continuous functions defined on all γ_2,[25] and the extensions satisfy the same bounds (i.e., the extension of ϕ is Lipschitz and the extension of J_ϕ is Hölder continuous). Let us use a tilde to identify such extensions:

$$\int_{\gamma_1} g \le \int_{\gamma_2} g\widetilde{J}_\phi + c|||g|||_-\delta_1^2 + 2D\delta_1(\lambda^{-\varsigma}\mu^{1-\varsigma})^{m_*}|||g|||_-.$$

Lemma II.7 shows that for a large and δ_1 small enough, $\widetilde{J}_\phi \in \mathcal{D}_a(\gamma_2)$; therefore

$$\int_{\gamma_1} g \le e^{b\rho_{\gamma_2}(1,\widetilde{J}_\phi)}\int_{\gamma_2} g + \left(2D(\lambda^{-\varsigma}\mu^{1-\varsigma})^{m_*} + c\delta_1\right)|||g|||_-\delta_1.$$

The result follows by choosing δ_1 sufficiently small, and m_* sufficiently large, with respect to b and c. □

To conclude this part of the argument, as we did in Section 3, we need to construct an appropriate covering of $\mathcal{M}$. We will use such a covering to control the images of a given curve.

We start with the same collection of squares $Q(z_i)$ constructed in Subsection 3.4.[26] We choose δ_0 smaller than $\frac{1}{4}$ the minimal size of the $Q(z_i)$, and such that the hypothesis of Sublemma 4.4 is satisfied. As in the previous section, given any square we have a notion of "core," although a more stringent one. In the previous section, a square of size $4\delta_0$ was called a "core of a square" a concentric square of size $\frac{\delta_0}{4}$; here we will call a "weak-core" a concentric square

[25] For example, one can take the extension to be linear with respect to arc-length where the functions are not defined.

[26] The presence of singularities restricts the possible points that can be chosen as the center of a square: only points with well defined stable and unstable directions are allowed. Nevertheless, almost all points are available, and the squares can still be chosen of uniform size.

with size $\frac{1}{4t}$ the size of the containing square ($t > 1$ will be chosen during the proof of Sublemma 4.13), while by "core" of a square of size $t\varepsilon$ (for some $\varepsilon > 0$), we mean the set of points in the weak-core that have a stable and unstable manifold longer than $4\varepsilon t$.[27] In addition, we will call a square "α–connecting" (for some $\alpha < 1$) if the measure of its core is at least α times the measure of the square.[28]

The key ingredient in the construction of our covering is a result very similar to the "Sinai Theorem," which is the cornerstone of the proof of ergodicity for the systems under consideration. In fact, the proof of the following theorem is almost indistinguishable from the proof of the Sinai Theorem in the simple uniformly hyperbolic case (I refer to the version in [27, §3]). However, since the statement looks stronger, a sketch of a proof is provided in Appendix I.

SINAI THEOREM 4.10. *For each square Q of size δ_0, and $t > 2$, there exist $\alpha_Q < 1$, $\delta_Q \leq \delta_0$, and $m_Q \in \mathbb{N}$ such that, for each $\alpha \leq \alpha_Q$, and $\delta_1 \leq \delta_Q$ a covering $\mathcal{G}(Q, \delta_1)$, of Q, may be constructed, made of squares of size $t\delta_1$ with the following properties*:

(i) $\bigcup_{G \in \mathcal{G}(Q,\delta_1)}$ *weak-core*$(G) \supset Q$;

(ii) *the number of squares G for which* $\bigcup_{|n| \leq m_Q} T^n \mathcal{S} \cap$ *weak-core*$(G) = \emptyset$, *and G is not α-connecting, is less than* $\frac{1}{256} K_Q^{-1} \frac{\delta_0}{\delta_1}$ (K_Q *being the maximal number of overlaps in the covering*);

(iii) *for each $\gamma \in \widehat{\Gamma}_{\delta_1}$, $\gamma \subset Q$, there exists $G \in \mathcal{G}(Q, \delta_1, m)$ such that $\gamma \subset G$ and γ intersects properly* (*see* §3.4) *G.*

In other words, a square Q can be covered by the weak–core of sufficiently small squares whose overwhelming majority contains an abundance of stable fibers.

According to Theorem 4.10, for every square $Q \in \{Q(z_i)\}$ we can construct coverings $\mathcal{G}(Q, \delta_1)$. Let us choose $m_0 = \max\{\{m_Q\}_{Q \in \mathcal{Q}}, m_*\}$, and $\delta_1 \leq \min\{\{\delta_Q\}_{Q\in\mathcal{Q}}, \beta^{-2}\mu^{-m_0}\delta_0\}$ (where $\beta > 1$ will be chosen in Sublemma 4.12, and depends only on m_0), and $\alpha = \min\{\alpha_Q\}$; then $\mathcal{G} = \bigcup_{Q\in\mathcal{Q}} \mathcal{G}(Q, \delta_1)$ is the covering of $\mathcal{M}$ that will be used in the following.

We are now positioned to make the last step: estimate the ratio of the averages; as in Subsection 3.4, a new element, the mixing property, allows us to close the argument.

[27] To be more precise, we will mean a slightly smaller set: the set of points constructed in Appendix I; cf. Sublemma 4.13 as well.

[28] The definition of core differs from the one given in [27, Def. 11.1]: there the requirement on the size of the manifolds is absent. Nevertheless, the definition of α-connecting is essentially the same.

LEMMA 4.11. *There exists $N \in \mathbb{N}$ such that, for $m \in \{N_-(\delta), \ldots, N_+(\delta)\}$, and for each $g \in \mathcal{C}_{b,c,d,L}(\delta)$,*

$$|||\widetilde{T}^m g|||_+ \leq \max\left\{1536 e^{2bR}\mu^N, \frac{3L}{4}\right\} |||\widetilde{T}^m g|||_-.$$

Proof. We have assumed the map to be mixing. Hence, there exists $N \in \mathbb{N}$ such that, given any two α-connecting G, $G' \in \mathcal{G}$,

$$T^{-N}(\mathrm{core}(G)) \cap \mathrm{core}(G') \neq \emptyset.$$

The strategy of the proof is as follows: first we will show that it is possible to control the average on any curve of size δ_0 via the average on a shorter curve contained in a connecting square (Sublemma 4.12). Then we will see that the mixing property implies that a connected piece belonging to some image of such a shorter curve is δ_1-close to any given curve. We will then be in a position to compare any two curves.

SUBLEMMA 4.12. *For each $\delta \leq \delta_1 d^{-1}$ small enough, $g \in \mathcal{C}_{b,c,d,4L}(\delta)$, $\gamma \in \widehat{\Gamma}_{\delta_0}$, $f \in \mathcal{D}_a(\gamma)$, there exist γ^+, $\gamma^- \in \widehat{\Gamma}_{\delta_1}$ ($\gamma^+, \gamma^- \subset \gamma$), such that*

$$\frac{1}{2}\frac{\int_{\gamma^-} fg}{\int_{\gamma^-} f} \leq \frac{\int_\gamma fg}{\int_\gamma f} \leq \frac{\int_{\gamma^+} fg}{\int_{\gamma^+} f} + \frac{1}{64}|||g|||_+.$$

In addition, both γ^+ and γ^- intersect properly an α-connecting square belonging to $\mathcal{G}$, and are disjoint from $\bigcup_{|n|\leq m_0} T^n\mathcal{S}$.

Proof. Consider the set $\gamma \backslash (\bigcup_{|n|\leq m_0} T^n\mathcal{S})$. Since $\delta_1 \leq \beta^{-2}\delta_0\mu^{-m_0}$ (for some β to be chosen later), one can divide γ into $\delta_0\delta_1^{-1}\beta^{-2}$ pieces, each of length $\delta_1\beta^2$. When we iterate each one of these pieces, for at most m_0 times, they remain shorter than δ_0; hence, they can be cut by $\mathcal{S}$ at most M^{m_0} times in each direction. This argument shows that $\gamma \setminus (\bigcup_{n\leq|m_0|} T^n\mathcal{S})$ consists, at most, of $2\frac{\delta_0}{\delta_1}\beta^{-2}M^{m_0}$ pieces. We collect in $\mathcal{J}_1$ the ones that are longer than $\beta\delta_1$; in $\mathcal{J}_2$ we collect the curves shorter than $\beta\delta_1$, but longer than δ; finally in $\mathcal{J}_3$ we collect the shorter ones.

$$\frac{\int_\gamma fg}{\int_\gamma f} \leq \sum_{\widetilde{\gamma}\in\mathcal{J}_1} \frac{\int_{\widetilde{\gamma}} fg}{\int_\gamma f} + \sum_{\widetilde{\gamma}\in\mathcal{J}_2} \frac{\int_{\widetilde{\gamma}} f}{\int_\gamma f}|||g|||_+ + \sum_{\widetilde{\gamma}\in\mathcal{J}_3} \frac{f(x_{\widetilde{\gamma}})}{\int_\gamma f} d\delta |||g|||_-.$$

Next, we sub-divide the curves belonging to $\mathcal{J}_1$ into curves belonging to $\widehat{\Gamma}_{\delta_1}$ and we split the resulting collection of curves into three parts: the curves intersecting properly a connecting square, collected in $\mathcal{J}_0$; the curves adjacent to a segment in $\bigcup_{|n|\leq m_0} T^n\mathcal{S}$, collected in $\mathcal{J}_4$; and the rest, collected in $\mathcal{J}_5$. We have assumed the uniform transversality between $\mathcal{S}^\pm$ and the stable and unstable directions; this implies the uniform transversality for $T^n\mathcal{S}$, $|n| \leq m_0$. Since the direction of γ', for $\gamma \in \widehat{\Gamma}_\delta$, may differ from the stable direction by, at most, some fixed arbitrary ε, we can assume, without loss of generality, that the curves in $\widehat{\Gamma}_\delta$ are uniformly transversal to $T^n\mathcal{S}^\pm$, $|n| \leq m_0$. We can therefore choose β so large that, if $z \in \widetilde{\gamma} \in \widehat{\Gamma}_\delta$ and the distance of z from $T^n\mathcal{S}$, $|n| \leq m_0$, along γ, is larger than $\beta\delta_1$, it follows $d(z, \mathcal{S}^\pm) \geq t\delta_1$ (t will be chosen at the end of Sublemma 4.13). Hence, by construction, the curves in $\mathcal{J}_5$ belong to squares that do not intersect $\bigcup_{|n|\leq m_0} T^n\mathcal{S}$. The Sinai Theorem implies that the total length of the curves in $\mathcal{J}_5$ must be less than $\frac{\delta_0}{256}$. Among the curves in $\mathcal{J}_0$ there exists one on which the average is larger; let us call it γ^+; then

$$\frac{\int_\gamma fg}{\int_\gamma f} \leq \frac{\int_{\gamma^+} fg}{\int_{\gamma^+} f} + 2\left(\frac{1}{256} + 4\beta^{-1}M^{m_0}\right)|||g|||_+ + 4dM^{m_0}\beta^{-2}\frac{\delta}{\delta_1}|||g|||_-.$$

Similarly, we have

$$\frac{\int_\gamma fg}{\int_\gamma f} \geq \sum_{\widetilde{\gamma}\in\mathcal{J}_0} \frac{\int_{\widetilde{\gamma}} fg}{\int_\gamma f} - \sum_{\widetilde{\gamma}\in\mathcal{J}_3} \frac{\left|\int_{\widetilde{\gamma}} gf\right|}{\int_\gamma f}.$$

Let γ^- be the curve in $\mathcal{J}_0$ on which the average is smaller; then

$$\frac{\int_\gamma fg}{\int_\gamma f} \geq \frac{3}{4}\frac{\int_{\gamma^-} fg}{\int_{\gamma^-} f} - 4dM^{m_0}\beta^{-2}\frac{\delta}{\delta_1}|||g|||_- \geq \frac{3}{4}\frac{\int_{\gamma^-} fg}{\int_{\gamma^-} f} - 4dM^{m_0}\beta^{-2}\frac{\delta}{\delta_1}\frac{\int_\gamma fg}{\int_\gamma f}.$$

The lemma follows when β is sufficiently large. □

Given $g \in \mathcal{C}_{b,c,d,4L}(\delta)$, choose $\gamma^* \in \widehat{\Gamma}_{\delta_0}$, and $f_* \in \mathcal{D}_a(\gamma^*)$, such that

$$|||g|||_+^0 \leq 2\frac{\int_{\gamma^*} gf_*}{\int_{\gamma^*} f_*}.$$

Then, there exist $\gamma_+^* \in \widehat{\Gamma}_{\delta_1}$, and an α-connecting square $G_* \in \mathcal{G}$ containing it, such that

$$|||g|||_+^0 \leq 2\frac{\int_{\gamma_+^*} gf_*}{\int_{\gamma_+^*} f_*} + \frac{1}{32}|||g|||_+.$$

Furthermore, for each $g \in \mathcal{C}_{b,c,d,4L}(\delta)$, $\gamma \in \widehat{\Gamma}_{\delta_0}$, and $f \in \mathcal{D}_a(\gamma)$, there exists $\gamma_- \in \widehat{\Gamma}_{\delta_1}$, and an α-connecting square $G_\gamma \in \mathcal{G}$ such that $\gamma_- \subset G_\gamma$, and

$$\frac{\int_\gamma fg}{\int_\gamma f} \geq \frac{1}{2}\frac{\int_{\gamma_-} fg}{\int_{\gamma_-} f}.$$

To compare the averages we need to know that, for each γ, a preimage of γ_- contains a translate of γ_+^*. The next lemma states that this is indeed the case.

SUBLEMMA 4.13. *$T^{-N}\gamma_-$ contains a translate of γ_+^*.*

Proof. Let $w \in \operatorname{core}(G_\gamma) \cap T^N \operatorname{core}(G_*)$; then, by definition, there exists a stable manifold W^{s} and an unstable manifold W^{u} of size $a\delta_1 t$ (consequently intersecting G_γ completely), and $W^{\mathrm{s}} \cap W^{\mathrm{u}} = \{w\}$. This implies $W^{\mathrm{u}} \cap \gamma_- \neq \emptyset$. Let $y = W^{\mathrm{u}} \cap \gamma_-$, and let $\widetilde{\gamma}_n$ be the smooth connected component of $T^{-n}\gamma_-$ containing $T^{-n}y$. To prove the lemma we have to estimate the size of $\widetilde{\gamma}_n$; this depends on the distance of $T^{-n}y$ from the singularities, which will be estimated in (4.11) and (4.12). Without loss of generality, we can assume that there exists $m \in \mathbb{N}$ such that G_γ intersects at most one smooth segment (call it I) belonging to $\bigcup_{n=0}^m T^n\mathcal{S}^-$ and $w \notin \Sigma_\infty^-(t,\,\delta_1) \setminus \Sigma_m^-(t,\,\delta_1)$ (cf. Appendix I); since a smooth segment can end only by intersecting another segment, the segment I cannot have its end points inside G_γ; moreover, we can assume the same properties for G_* (with respect to $\bigcup_{n=0}^m T^{-n}\mathcal{S}^+$ and Σ^+).

Since $T^{-n}W^{\mathrm{u}} \cap \mathcal{S}^- = \emptyset$, and since $\mathcal{S}^-$ is uniformly transversal to W^{u}, there exists $\sigma \in (0,\,1)$ such that, for each $z \in T^{-n}W^{\mathrm{u}}$,

$$d(z,\,\mathcal{S}^-) \geq \sigma \min\{d(z,\,T^{-n}w_1),\,d(z,\,T^{-n}w_2)\} \tag{4.10}$$

where $\{w_1,\,w_2\} = \partial(W^{\mathrm{u}} \cap G_\gamma)$. Let w_1 be the point of $\partial(W^{\mathrm{u}} \cap G_\gamma)$ closest to y, and $\widetilde{w}_1$ the point of $\partial(W^{\mathrm{s}}(T^{-N}w) \cap G_*)$ closest to $T^{-N}y$. Then, setting $W_1^{\mathrm{s}} = W^{\mathrm{s}}(T^{-N}w) \cap G_*$, we have

$$d(T^{-n}y,\,\mathcal{S}^-) \geq \sigma d(T^{-n}y,\,T^{-n}w_1) \quad \text{for all } n \in \{0,\ldots,N\}. \tag{4.11}$$

In fact, if $n \leq m$, then (4.11) is a consequence of (4.10) and the fact that $G_\gamma \cap \bigcup_{n=0}^m T^n\mathcal{S}^-$ contains at most a segment that intersects G_γ completely. If $n > m$, since $d(T^{-n}y,\,T^{-n}w) \leq \frac{\delta_1}{4}\lambda^{-n}$ and $w \notin \Sigma_\infty^-(t,\,\delta_1) \setminus \Sigma_m^-(t,\,\delta_1)$

$$d(T^{-n}y,\,\mathcal{S}^-) \geq \frac{t}{2}\delta_1\lambda^{-n} \geq \sigma d(T^{-n}y,\,T^{-n}w_1).$$

Analogously, one can take advantage of $T^{-N}w \in \operatorname{core}(G_*)$. Given two curves γ_1 and γ_2, $\ell(\gamma_1,\gamma_2)$ is defined as $\sup_{x\in\gamma_1}\inf_{y\in\gamma_2} d(x,\,y)$,

$$d(T^{-N+n}y,\,\mathcal{S}^-) \geq \sigma d(T^{-N+n}w,\,T^n\widetilde{w}_1) - \sigma^{-1}\ell(\widetilde{\gamma}_{N-n},\,T^nW_1^{\mathrm{s}}) \quad \text{if } n \leq m;$$

here we have used (4.10) and the uniform transversality between stable and unstable curves that allows us to choose σ such that $d(T^{-N+n}y,\,T^{-N+n}w) \leq \sigma^{-1} \inf_{z\in T^nW_1^{\mathrm{s}}} d(y,\,z)$. In addition,

$$\begin{aligned} d(T^{-N+n}y,\,\mathcal{S}^-) &\geq \frac{t}{2}\lambda^{-n}\delta_1 - d(T^{-N+n}y,\,T^{-N+n}w) \\ &\geq \sigma d(T^{-N+n}w,\,T^n\widetilde{w}_1) - \sigma^{-1}\ell(\widetilde{\gamma}_{N-n},\,T^nW_1^{\mathrm{s}}) \quad \text{if } n > m. \end{aligned}$$

Consequently, by choosing $\overline{n} \in \{0,\,\ldots,\,N\}$ such that, for the first time,

$$d(T^{-\overline{n}}w,\,T^{N-\overline{n}}\widetilde{w}_1) \geq 4\sigma^{-2}\ell(\widetilde{\gamma}_{\overline{n}},\,T^{N-\overline{n}}W_1^{\mathrm{s}})$$

and collecting the last three inequalities we have

$$d(T^{-n}y,\,\mathcal{S}^-) \geq \frac{\sigma}{2}d(T^{-n}w,\,T^{N-n}\widetilde{w}_1) \quad \text{for all } n \geq \overline{n}. \tag{4.12}$$

Then, for each $n < \overline{n}$,

$$|\widetilde{\gamma}_n| \geq \sigma d(T^{-n}y, T^{-n}w_1).$$

In fact, the latter relation is true for $n = 0$ because of (4.10). Now suppose that it is true for $n - 1$ and let $\widehat{\gamma}_n$ be the connected component of $\widetilde{\gamma}_{n-1} \setminus \mathcal{S}^-$ containing $T^{-n+1}y$; then

$$|\widehat{\gamma}_n| \geq \sigma d(T^{-n+1}y, T^{-n+1}w_1)$$

and

$$|\widetilde{\gamma}_n| = |T^{-1}\widehat{\gamma}_n| \geq \lambda^2 \sigma d(T^{-n}y, T^{-n}w_1).$$

Next, since γ intersects the weak-core of G_γ, $d(y, w_1) \geq td(y, w)$ and (see Sublemma II.2-c) $d(T^{-\overline{n}+1}y, T^{-\overline{n}+1}w_1) \geq d(T^{-\overline{n}+1}y, T^{-\overline{n}+1}w)$; hence

$$\begin{aligned}
|\widehat{\gamma}_{\overline{n}}| &\geq \sigma d(T^{-\overline{n}+1}y, T^{-\overline{n}+1}w_1) \geq \sigma^2 \ell(\widetilde{\gamma}_{\overline{n}-1}, T^{N-\overline{n}+1}W_1^{\mathrm{s}}) \\
&\geq \frac{\sigma^4}{4} d(T^{-\overline{n}+1}w, T^{N-\overline{n}+1}\widetilde{w}_1)
\end{aligned}$$

and

$$|\widetilde{\gamma}_{\overline{n}}| \geq \frac{\sigma^4}{4} e^{-c_4 \lambda^{-\overline{n}} \delta_1 t} d(T^{-\overline{n}}w, T^{N-\overline{n}}\widetilde{w}_1),$$

where we have used the distortion estimates of Lemma II.2 and c_4 is some fixed constant. For $n > \overline{n}$, by (4.12), it is possible to show, by induction, that

$$|\widetilde{\gamma}_n| \geq \frac{\sigma^4}{4} e^{-c_4 \lambda^{-\overline{n}} \sum_{i=0}^{n-\overline{n}} \lambda^i \delta_1 t} d(T^{-n}w, T^{N-n}\widetilde{w}_1).$$

That is

$$|\widetilde{\gamma}_N| \geq \frac{\sigma^4}{4} e^{-c_4 \delta_1 t} \delta_1 t \geq 6\delta_1$$

where we have chosen t sufficiently large ($t \geq 48\sigma^{-4}$) and δ_1 sufficiently small ($e^{-c_4\delta_1 t} \geq \frac{1}{2}$). Moreover, from the argument it is clear that after dividing $\widetilde{\gamma}_N$ at $T^{-N}y$ each of the two pieces is long, at least, $3\delta_1$. From this follows that $\widetilde{\gamma}_N$ must contain a translate of γ_+^*. □

We conclude the argument.

Consider an arbitrary $g \in \widetilde{T}^{n_*}\mathcal{C}_{b,c,d,L}(\delta)$. Let $\gamma_0 \subset T^{-N}\gamma_-$ be a translate of γ_*^+; the curve $\gamma_- \setminus T^N\gamma_0$ consists, at most, of two connected pieces γ_-^1, γ_-^2; hence

$$\frac{\int_\gamma f\widetilde{T}^N g}{\int_\gamma f} \geq \frac{1}{2}\frac{\int_{\gamma^-} \widetilde{T}^N gf}{\int_{\gamma^-} f} = \frac{1}{2}\frac{\int_{\gamma_0} g\widehat{T}^N f}{\int_{\gamma^-} f} + \frac{1}{2}\sum_{i=1}^{2}\frac{\int_{\gamma_-^i} \widetilde{T}^N gf}{\int_{\gamma^-} f}.$$

We would like to eliminate the integrals on the two curves γ_-^i, i.e., to show that their sum is a positive quantity. To do so, it suffices to notice that one of the two curves, let us say γ_-^1, must be longer than δ; if also the other curve is longer than δ, then both integrals are positive. Otherwise

$$\begin{aligned}\sum_{i=1}^{2}\int_{\gamma_-^i}\widetilde{T}^N gf &\geq \int_{\gamma_-^1} f|||\widetilde{T}^N g|||_- - f(x_{\gamma_-^2})d\delta|||\widetilde{T}^N g|||_- \\ &\geq f(x_{\gamma_-^2})\left[\frac{\delta_1}{4}-d\delta\right]|||\widetilde{T}^N g|||_- \geq 0\end{aligned}$$

provided $\delta \leq \frac{\delta_1}{4d}$. The above estimate allows us to write

$$\begin{aligned}\frac{\int_\gamma f\widetilde{T}^N g}{\int_\gamma f} &\geq \frac{e^{-bR}}{8\delta_1}\mu^{-N}\int_{\gamma_0} g \\ &\geq \frac{e^{-bR}}{16\delta_1}\mu^{-N}\int_{\gamma_*^+} g - \frac{e^{-2bR}\mu^{-N}}{2048}|||g|||_+ \\ &\geq \frac{e^{-2bR}}{32}\mu^{-N}\frac{\int_{\gamma_+^*} gf_*}{\int_{\gamma_+^*} f_*} - \frac{e^{-2bR}\mu^{-N}}{2048\delta_1}|||g|||_+ \\ &\geq \frac{e^{-2bR}}{64\mu^N}\left(|||g|||_+^0 - \frac{1}{16}|||g|||_+\right) \geq \frac{e^{-2bR}}{64\mu^N}\left(|||g|||_+^0 - \frac{L}{16}|||g|||_-\right).\end{aligned}$$

Up to now we have shown that, for each $g \in \widetilde{T}^{n_*}\mathcal{C}_{b,c,d,L}(\delta)$,

$$|||\widetilde{T}^N g|||_-^0 \geq \frac{e^{-2bR}\mu^{-N}}{64}\left(|||g|||_+^0 - \frac{L}{16}|||g|||_-\right). \tag{4.13}$$

The next, and final, step is to consider any $g \in \mathcal{C}_{b,c,d,L}(\delta)$; the lemma will then follow from the estimate (4.13) and Lemma 4.5.

In fact, Lemma 4.5 implies that, for each $m \in \{N_-(\delta), \ldots, N_+(\delta)\}$, calling $\widetilde{g} = \widetilde{T}^{n_*}g$, we have either

$$\frac{|||\widetilde{T}^{m+N}\widetilde{g}|||_+}{|||\widetilde{T}^{m+N}\widetilde{g}|||_-} \leq \frac{3|||\widetilde{g}|||_+}{8|||\widetilde{g}|||_-} \leq \frac{3L}{4}$$

which would conclude the proof, or

$$|||\widetilde{T}^{m+N}\widetilde{g}|||_- \geq \frac{1}{2}|||\widetilde{T}^N\widetilde{g}|||_-^0 \geq \frac{e^{-2bR}\mu^{-N}}{128}\left(|||\widetilde{g}|||_+^0 - \frac{L}{16}|||\widetilde{g}|||_-\right).$$

Now, we consider again two possibilities: if $|||\widetilde{g}|||_+^0 \leq \frac{3L}{32}|||\widetilde{g}|||_-$, then

$$|||\widetilde{T}^{m+N}\widetilde{g}|||_+ \leq 4|||\widetilde{g}|||_+^0 \leq \frac{3L}{8}|||\widetilde{g}|||_- \leq \frac{3L}{4}|||\widetilde{T}^{m+N}\widetilde{g}|||_-;$$

otherwise (again by the second alternative of Lemma 4.5)

$$|||\widetilde{T}^{m+N}\widetilde{g}|||_- \geq \frac{e^{-2bR}\mu^{-N}}{128\times 12}|||\widetilde{T}^{m+N}\widetilde{g}|||_+. \qquad \square$$

4.5. *Diameter of the image.* Lemma 4.11 was the last piece in the proof of Theorem 4.2: it suffices to choose

$$L > 1536\chi^{-1}\mu^N e^{2bR},$$

and

$$m(\delta) = A\ln d + N_-(\delta) + N,$$

in order to close the argument.[29]

We announced at the beginning of the section that Theorem 4.2 was sufficient for our purposes; this is shown by the next lemma.

LEMMA 4.14. *There exists $\Delta \in \mathbb{R}^+$:*

$$\operatorname{diam}(\mathcal{C}_{\chi b,\, \chi c,\, \chi d,\, \chi L}(\delta)) \le \Delta$$

where the diameter is computed with respect to the Hilbert metric Θ of $\mathcal{C}_{b,\,c,\,d,\,L}(\delta)$.

Proof. This lemma is similar to Lemma 3.6, only more computations are needed since we are now dealing with a more complex geometry.

Let $g \in \mathcal{C}_{\chi b,\, \chi c,\, \chi d,\, \chi L}$; if $\lambda \preceq g \preceq \mu$, then, in analogy with Lemma 3.6, we have

$$\begin{aligned}
\lambda &\le \inf_{\substack{\gamma\in\Gamma_\delta \\ f\in\mathcal{D}_a(\gamma)}} \frac{\int_\gamma gf}{\int_\gamma f} \equiv \alpha_0, \\
\lambda &\le \inf_{\substack{\gamma\in\Gamma_\delta \\ f_1,\, f_2\in\mathcal{D}_a(\gamma)}} \frac{e^{b\rho_\gamma(f_1, f_2)} f_1 \int_\gamma g f_2 - f_2 \int_\gamma g f_1}{e^{b\rho_\gamma(f_1, f_2)} f_1 \int_\gamma f_2 - f_2 \int_\gamma f_1} \equiv \alpha_1, \\
\lambda &\le \frac{c|||g|||_- - \|D^{\mathrm{u}} g\|_\infty}{c} \equiv \alpha_2.
\end{aligned}$$

In addition, in the present context, we have three extra conditions defining the cone that produces the following constraints: from the condition on the size of the integral over short pieces, it follows that

$$\lambda \le \inf \frac{d\delta^{1-\varsigma}|I|^\varsigma |||g|||_- - \frac{|\int_I gf|}{f(x)}}{d\delta^{1-\varsigma}|I|^\varsigma + \frac{\int_I f}{f(x)}} \equiv \alpha_3.$$

The bound on the ratio of the averages implies

$$\lambda \le \frac{L|||g|||_- - |||g|||_+}{L-1} \equiv \alpha_4,$$

[29] At this point the reader may wonder if such a choice is possible. In fact, N depends on δ_1, but not on δ. In turn, the choice of δ_1 depends on c and b, but not on d and L. It is therefore possible to choose all the parameters involved but L, d and δ. The above relations force a choice of L (and, consequently, of d); nonetheless we are still free to choose δ sufficiently small to satisfy all the necessary requirements.

and the last condition forces[30]

$$\lambda \le \inf \frac{e^{b\rho_I(f_1,\, f_2)} \frac{\int_I g f_2}{f_2(y)} - \frac{\int_I g f_1}{f_1(y)} + 2e^{b\rho_I(f_1,\, f_2)} \rho_I(f_1,\, f_2) b d\delta |||g|||_-}{e^{b\rho_I(f_1,\, f_2)} \frac{\int_I f_2}{f_2(y)} - \frac{\int_I f_1}{f_1(y)} + 2e^{b\rho_I(f_1,\, f_2)} \rho_I(f_1,\, f_2) b d\delta} \equiv \alpha_5.$$

Again $\alpha = \min\{\alpha_i\}$ (i.e., if $\lambda \le \alpha$, then $\lambda \prec g$). According to Lemma 3.6, $\min\{\alpha_0,\, \alpha_1,\, \alpha_2\} \ge \frac{(1-\chi)b}{b+1} |||g|||_-$; moreover, a direct computation, using explicitly the condition $g \in \mathcal{C}_{\chi b, \chi c, \chi d, \chi L}(\delta)$, shows that

$$\begin{aligned}
\alpha_3 &\ge \inf \frac{(1-\chi) d\delta^{1-\varsigma} |I|^\varsigma}{d\delta^{1-\varsigma}|I|^\varsigma - 2|I|} |||g|||_- = \frac{d(1-\chi)}{d+2} |||g|||_-, \\
\alpha_4 &\ge \frac{(1-\chi)L}{L-1} |||g|||_-, \\
\alpha_5 &\ge \inf \frac{\left[1 - e^{(\chi-1)b\rho_I(f_1,\, f_2)}\right] \frac{\int_I g f_2}{f_2(y)} + 2\rho_I(f_1,\, f_2)(1-\chi^2) b d\delta |||g|||_-}{\left[1 - e^{-(b+1)\rho_I(f_1,\, f_2)}\right] \frac{\int_I f_2}{f_2(y)} + 2\rho_I(f_1,\, f_2) b d\delta} \\
&\ge \inf \frac{\rho_I(f_1,\, f_2) b \left[2(1-\chi^2) d\delta |||g|||_- - (1-\chi) \left| \frac{\int_I g f_2}{f_2(y)} \right| \right]}{2\left[bd + (b+1)\right] \rho_I(f_1,\, f_2)\delta} \\
&\ge \chi(1-\chi) |||g|||_-.
\end{aligned}$$

These inequalities, together with the analogous ones for β_i, show that there exists a constant C, depending only on b, c, d, L, and χ, such that

$$\Theta(g,\, 1) \;\le\; \ln \frac{\beta}{\alpha} \le \ln C \frac{|||g|||_+}{|||g|||_-} \le \ln CL. \qquad \square$$

4.6. *Decay of correlations.*

THEOREM 4.15. *There exist $D, r > 0$ and $\Lambda < 1$ such that, for each $f, g \in C^1(\mathcal{M})$,*

$$\left| \int_{\mathcal{M}} (f \circ T^n) g - \int_{\mathcal{M}} f \int_{\mathcal{M}} g \right| \le D\Lambda^n \|f\|_* \|g\|_*,$$

where, given $h \in C^1$, $\|h\|_* = \|h\|_1 + r\|h'\|_\infty$.

Proof. The key idea is to introduce a smooth foliation on the phase space made of stable curves of uniform size. Once this is done we can decompose

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[30]In the following formula the numerator is positive since $\frac{\int_I g f_1}{f_1(y)} \le e^{\rho_+(f_1,\, f_2)} \frac{\int_I g f_2}{f_2(y)} \le e^{\rho_I(f_1,\, f_2)} \frac{\int_I g f_2}{f_2(y)}$, and $b > 1$.

the integral according to such a foliation and the result follows from the corresponding result on the decomposed integrals. In fact, our previous results show that the correlations, over stable curves, converge exponentially fast to the same value.

The required foliation can be accomplished in various ways and, in each concrete example, there may be a particularly convenient way to perform it; in the following I propose a general construction, only to convince the reader that foliations with the wanted properties exist.

We start with the same coarse partition $\mathcal{P} = \{P_i\}$ used in the smooth case (notice that, in the present context, the P_i can be chosen to be polygons). It would be possible to perform the same dynamical argument, only the presence of the singularities would make it much more complicated. Yet, since in the present case it has been necessary to introduce a very small scale δ, nothing much is lost by placing some further restrictions on it. We further refine the partitions until it consists only of convex polygons.

We can assume, without loss of generality, that each element of the new partition has the boundary naturally divided into, at most, four connected parts: two made up by stable lines and two made up by unstable lines. Such polygons can be foliated in many ways. For example, choose two stable segments I_1, I_2 belonging to the "opposite" pieces of stable boundary, with $\text{dist}(I_1, I_2) > 0$, and consider $x_i \in I_i$. Let $\zeta(s) = (1-s)x_1 + sx_2$ and associate to each $\zeta(s)$ a line passing by $\zeta(s)$ such that the line associated to $\zeta(0)$ contains I_1, the one associated to $\zeta(1)$ contains I_2 and the line bundle is smooth. The intersection of such a line bundle with the element of the partition constitutes the wanted "regular" partition.[31] By construction, all the stable curves so generated are longer than some $\delta_* \leq \delta_0$. Using such a partition we see that the analogue of Lemma 3.10 follows immediately, and the proof of the theorem is concluded by the same argument used in Theorem 3.9. □

Appendix I. The Sinai Theorem

This appendix is dedicated to the proof of Theorem 4.10. Since the argument is reasonably standard, I will provide only a sketch, and refer to [27] for more details.

We start by introducing a coarse partition of Q via $\mathcal{S}_{m_Q} = \bigcup_{n=0}^{m_Q}(T^n\mathcal{S}^-) \cup (T^{-n}\mathcal{S}^+)$. Clearly $Q \backslash \mathcal{S}_{m_Q}$ is partitioned into finitely many open domains $\{R_i\}$. Since $\widetilde{\mathcal{C}}_-(z) = DT^{-m_Q}\mathcal{C}_-(T^{m_Q}z)$ and $\widetilde{\mathcal{C}}_+(z) = DT^{m_Q}\mathcal{C}_+(T^{-m_Q}z)$ are uni-

[31] By "regular" we mean that when s and t (the distance from $\zeta(s)$ along the associated line) are new coordinates the Jacobian of the change of coordinate, restricted to the element of the partition γ, is in $\mathcal{D}_a(\gamma)$. See the proof of Lemma 3.10 for details.

formly continuous in each R_i, it is useful to further restrict the domains. We consider coverings $\{R_i^j\}$ of R_i made of elements of uniform size on which the cone families are almost constant. Next, we choose a point z_i^j in each R_i^j such that $z_i^j \notin \bigcup_{n\in\mathbb{Z}} T^n\mathcal{S}$, and we introduce in each R_j^i a cartesian system of coordinates with axes parallel to the stable and unstable directions at z_j^i. For each δ_1 sufficiently small, we can then introduce in R_j^i a family of squares of size $t\delta_1$ with sides parallel to the axes, such that their weak-core covers R_j^i. The collection of squares just described is the covering $\mathcal{G}(Q,\, \delta_1)$.

To prove the statement of Theorem 4.10, we need only consider the squares whose weak-cores do not intersect the boundaries of R^i. The set $\mathcal{S}_m = \bigcup_{|n|\le m} T^n\mathcal{S}$ consists of a finite collection of lines. Let us call $\mathcal{R}_m$ the set of points that belong to more than one line. By hypothesis $\mathcal{R}_m$ is a discrete set containing a finite number of points. At each point of $\mathcal{R}_m$ the lines in $\mathcal{S}_m$ are transversal; let θ_m be the minimal angle among them, and let N_m be the number of points belonging to $\mathcal{R}_m$.

We choose δ_1 smaller than $\frac{4}{t}$ the smallest distance among points in $\mathcal{R}_m$. If $\widetilde{\mathcal{R}}_m$ is the $\frac{2t\delta_1}{\theta_m}$ neighborhood of $\mathcal{R}_m$, then $\mathcal{S}_m\backslash\widetilde{\mathcal{R}}_m$ consists of a set of disjoint lines, at least $2t\delta_1$ apart. The total measure of the squares that intersect $\widetilde{\mathcal{R}}_m$ is less than $\pi N_m\left(\frac{2t\delta_1}{\theta_m}\right)^2$; by choosing δ_1 small enough,

$$\pi N_m\left(\frac{2t\delta_1}{\theta_m}\right)^2 \le \frac{t^2\delta_1\delta_0}{512}K_Q^{-2}.$$

Notice that the remaining squares can contain, at most, one line belonging to $\mathcal{S}_m\backslash\mathcal{S}_{m_Q}$. Moreover, choosing m_Q large and α small enough, it is possible to insure that a line belonging to $\mathcal{S}_m\backslash\mathcal{S}_{m_Q}$ is not sufficient to make a square not 2α-connecting (meaning that it can cut only fibers belonging to, at most, $1-2\alpha$ part of the weak-core). Consequently, if a square is not α-connecting (i.e., more than $1-\alpha$ part of its weak-core is covered by "short" fibers) it must contain at least a set of measure $\frac{\alpha}{8}\delta_1^2$ belonging to the tail $\Sigma_\infty^\pm(t,\, \delta_1)\backslash\Sigma_m^\pm(t,\, \delta_1)$. It is easy to see that the total measure of the tail is less than $Ct\lambda^{-m}\delta_1$ (for some fixed constant $C>0$), so the maximal number of squares that contain such a proportion of the tail cannot be more than $K_QCt\alpha^{-1}8\lambda^{-m}\delta_1^{-1}$, where K_Q is the maximal number of overlaps of the covering (clearly K_Q depends on t). One can then choose m so large that the total number of the the squares under discussion is less than $\frac{t^2\delta_1\delta_0}{512}K_Q^{-1}$.

The above estimates are all that are needed to prove Theorem 4.10-(ii).

Appendix II. Absolute continuity

This section provides some result that are either not readily available in the literature or new.

The main focus is the map induced by the unstable foliation between two nearby stable curves. It is well known that such a map is absolutely continuous, with Jacobian close to the identity, if the two curves are sufficiently close and "parallel" [23]; I will show that the map is Hölder continuous together with its Jacobian. This last property is well-known in the case of smooth maps [32], [30] but, to my knowledge, it is a new result when the maps are discontinuous (although some partial results along these lines exist in the case of billiards [6]).

We will restrict ourselves to the two-dimensional case; this will allow easier and more transparent proofs with respect to [23]. An extension to the multi-dimensional situation should not be out of the reach of the technique used here, but the author feels that it would be out of the scope of his present effort.

We deal with the class of maps described at the beginning of Section 4.

Consider curves $\gamma_1, \gamma_2 \in \widehat{\Gamma}_{\delta_2}$, such that $d(\gamma_1, \gamma_2) \leq c_1\delta_2$ for some $c_1 < 1$; from now on, we will take δ_2 smaller than some appropriate δ_0 (cf. §4). Note that, by the continuity of the cone field and the uniform transversality of the stable and the unstable directions, there exists a $c_0 < 1$ such that every unstable curve starting from γ_1, in the direction γ_2 and longer than $c_0^{-1}c_1\delta_2$ intersects all curves $\gamma_3 \in \widehat{\Gamma}_{3c_0^{-1}\delta_2}$ such that $\gamma_2 \subset \gamma_3$, and $\gamma_3 \setminus \gamma_2$ consists of two pieces of the same length. Let us define $D(\phi) = \{z \in \gamma_1 \mid W^{\mathrm{u}}(z) \cap \gamma_2 \neq \emptyset\}$, where $W^{\mathrm{u}}(z)$ is the local unstable manifold of z with size[32] less than or equal to $c_1 c_0^{-1}\delta_2$,[33] and the map $\phi : D(\phi) \subset \gamma_1 \to \gamma_2$ by

$$\phi(x) = W^{\mathrm{u}}(x) \cap \gamma_2.$$

If $R(\phi)$ is the range of the map (and the domain of ϕ^{-1}) and $m(\cdot)$ is the measure induced by arc-length, then we define a measure μ on $R(\phi)$ by the relation

$$\int_{R(\phi)} f d\mu = \int_{D(\phi)} f \circ \phi dm \qquad \text{for all } f \in C^{(0)}(\gamma_2).$$

From the general theory developed in [23], it follows[34] that:

THEOREM II.1. *The map ϕ is absolutely continuous. In addition, there exists $D_0 \in \mathbb{R}^+$ such that, with J_ϕ the Jacobian of ϕ, for each density point z of $R(\phi)$ the following holds:*

$$\frac{d\mu}{dm}(z) = J_{\phi^{-1}}(z) = \prod_{j=1}^{\infty} \left| \frac{D_{T^{-j}\phi^{-1}(z)}T\big|_{T^{-j}\gamma_1}}{D_{T^{-j}z}T\big|_{T^{-j}\gamma_2}} \right| \leq D_0.$$

[32]By "size" we mean the distance, along the manifold, between z and the furthest end-point.

[33]$W^{\mathrm{u}}(z) = \{z\}$ if z does not have an unstable manifold.

[34]The following formula was presented in [1] (see formula (5.3), page 151).

In order to use effectively the map ϕ in our context, we need more information: an estimate of the measures of $\gamma_1\backslash D(\phi)$ and $\gamma_2\backslash R(\phi)$; an estimate on the smoothness of ϕ; and an estimate on the smoothness of J_ϕ.

Let $\Sigma_n^-(c, \delta_2) = \bigcup_{i=0}^n T^i \mathcal{S}^-_{c\lambda^{-i}\delta_2}$.[35] From the construction of the unstable manifold (see [23]) it follows that, for $c > c_1 c_0^{-1}$, $\gamma_1\backslash\Sigma_\infty^-(c, \delta_2) \subset D(\phi)$. The main tool used in the following discussion is a rather detailed distortion lemma.

LEMMA II.2. *For δ_0 small enough there exists $D \in \mathbb{R}^+$ such that given any $z, y \in \mathcal{M}$, $n \in \mathbb{N}$, any $v \in \mathcal{T}_z\mathcal{M}$, $w \in \mathcal{T}_y\mathcal{M}$ such that $v \in DT^{-1}\mathcal{C}_-(Tz)$, $w \in DT^{-1}\mathcal{C}_-(Ty)$, and $\|v\| = \|w\| = 1$, with $v_i = \frac{D_zT^{-i}v}{\|D_zT^{-i}v\|}$, $w_i = \frac{D_yT^{-i}w}{\|D_yT^{-i}w\|}$, $z_i = T^{-i}z$ and $w_i = T^{-i}w$ for $i \in \{0, \dots, n\}$, the following hold:*[36]

$$\text{(a)}\quad \begin{cases} z, y \notin \Sigma_\infty^-(2c, \delta_2); \quad d(z_i, y_i) \le \lambda^{-i}c\delta_2 \quad \forall i \le n \\ \qquad\qquad \Rightarrow \frac{\|D_{z_i}T^i v_i\|}{\|D_{y_i}T^i w_i\|} \le e^{D\max\{\|v-w\|, c\delta_2\}} \quad \forall i \in \{1, \dots, n\}. \end{cases}$$

$$\text{(b)}\quad \begin{cases} [z, y] \subset \gamma \in \widehat{\Gamma}_{\delta_0};\, v = \gamma'(z), w = \gamma'(y);\, T^{-n}[z, y] \text{ connected;} \\ d(z_n, y_n) < c\delta_2 \Rightarrow \frac{\|D_{z_i}T^i v_i\|}{\|D_{y_i}T^i w_i\|} \le e^{D\max\{\|v_n - w_n\|, d(z_n, y_n)\}}\ \forall i \le n. \end{cases}$$

$$\text{(c)}\quad \begin{cases} y \in W^{\mathrm{u}}(z), d(z, y) \le c\delta_2 \\ \qquad\qquad \Rightarrow \frac{\|D_{z_i}T^i v_i\|}{\|D_{y_i}T^i w_i\|} \le e^{D\max\{\|v-w\|, d(z, y)\}} \quad \forall i \le n. \end{cases}$$

Proof. We will prove only (a), the other cases being proved in the same way.

Due to the continuity of the cone field, it is possible to cover $\mathcal{M}$ with a finite number of symplectic charts $\{\mathcal{A}\}$ satisfying[37]

(i) any chart is contained in a ball of radius δ_0;

(ii) given any chart $\mathcal{A}$, there exists a point $x(\mathcal{A}) \in \mathcal{A}$, such that $\mathcal{C}(\mathcal{A}) \equiv \mathcal{C}_-(x(\mathcal{A})) = \{(\alpha, \beta) \mid \alpha\beta \ge 0\}$ (in the chart coordinates);

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[35] By $\mathcal{S}_\varepsilon^-$ we mean the ε-neighbor of the singularity set of T^{-1}, $\mathcal{S}^-$.

[36] In the following formulas $\|v-w\|$ appears and, since v and w belong to different tangent spaces, it is not yet defined. In fact, one can make sense of such an expression (i.e., introduce a connection on $\mathcal{M}$) in many different ways. But, different choices simply affect the constant D, so we will not expand on the subject. Of course, in the proof of the lemma a special choice is made, but I do not claim that it is a particularly "natural" or "smart" choice.

[37] We use the cartesian structure of the charts to compare tangent vectors at different points. One could consider the Riemannian connection a more "natural" choice, but this would sacrifice the symplectic structure that, in my judgment, is worth using in the present context (see [27] for an exposition and an extensive application of such a point of view). The reader is advised that, in the following, we often identify points and vectors on the manifold with their counterparts on the charts.

(iii) given any chart $\mathcal{A}$, $s \in \mathcal{A}$,

$$D_s T^{-1} \mathcal{C}_-(Ts) \subset \mathcal{C}(\mathcal{A});$$

(iv) when $\|\cdot\|_{\mathcal{A}}$ is the norm induced by the cartesian structure of $\mathcal{A}$, there exists a fixed $A_1 > 0$ such that

$$A_1^{-1}\|\cdot\| \leq \|\cdot\|_{\mathcal{A}} \leq A_1\|\cdot\| \quad \text{for all } \mathcal{A}.$$

In the following we will assume $A_1 = 1$; the general case is dealt with in exactly the same manner, the only difference being that the final constant D depends trivially on A_1 (we will, consequently, suppress the subscript $\mathcal{A}$ in the norm).

Our first step will be to estimate $\|v_i - w_i\|$.

First of all, if ω is the symplectic form (which, by our assumptions, becomes the standard symplectic structure of $\mathbb{R}^2$ in each chart $\mathcal{A}$), then, given $u_1, u_2 \in \mathcal{C}(\mathcal{A})$, $\|u_i\| = 1$, $\omega(u_1, u_2) \geq 0$,[38]

$$\omega(u_1, u_2) \leq \|u_1 - u_2\| \leq 2\omega(u_1, u_2).$$

From the definitions in the statement of the lemma it follows immediately that

$$v_i = \frac{D_{z_{i-1}} T^{-1} v_{i-1}}{\|D_{z_{i-1}} T^{-1} v_{i-1}\|} \text{ and } w_i = \frac{D_{y_{i-1}} T^{-1} w_{i-1}}{\|D_{y_{i-1}} T^{-1} w_{i-1}\|}.$$

I claim that there exists $h > 1$ such that

$$|\omega(w_i, v_i)| \leq h\lambda^{-i} \max\{\|v - w\|, c\delta_2\}. \tag{II.1}$$

The proof of (II.1) is by induction. It is clearly true for $i = 0$. If it is true for $i - 1$, then

$$\begin{aligned} |\omega(w_i, v_i)| &= \frac{|\omega(D_{z_{i-1}} T^{-1} v_{i-1}, D_{y_{i-1}} T^{-1} w_{i-1})|}{\|D_{z_{i-1}} T^{-1} v_{i-1}\| \|D_{y_{i-1}} T^{-1} w_{i-1}\|} \\ &\leq \lambda^{-2} |\omega(D_{y_{i-1}} T D_{z_{i-1}} T^{-1} v_{i-1}, w_{i-1})| \\ &\leq \lambda^{-2} |\omega(v_{i-1}, w_{i-1})| + \lambda^{-2} K d(z_{i-1}, y_{i-1}), \end{aligned}$$

where we have used the fact that T is a symplectic map, and K depends only

[38]The following formula is a very primitive example of the possibility of using the symplectic structure to control the distance between two directions. See [28] for a general construction of a metric, on the bundle of Lagrangian subspaces contained in a cone, defined only in terms of the symplectic form.

on the second and first derivatives of the map. Accordingly, by the hypotheses of the lemma,

$$\begin{aligned}|\omega(w_i, v_i)| &= \lambda^{-1-i}h\max\{\|v-w\|, c\delta_2\} + \lambda^{-i-1}Kc\delta_2 \\ &\leq \lambda^{-i}(\lambda^{-1}h + \lambda^{-1}K)\max\{\|v-w\|, c\delta_2\};\end{aligned}$$

so, choosing $h \geq \frac{\lambda^{-1}}{1-\lambda^{-1}}K$, we have (II.1).

We are now positioned to prove the final estimate:

$$\begin{aligned}\frac{\|D_{z_i}T^i v_i\|}{\|D_{y_i}T^i w_i\|} &= \prod_{j=0}^{i-1}\frac{\|D_{y_j}T^{-1}w_j\|}{\|D_{z_j}T^{-1}v_j\|} \\ &\leq e^{\sum_{j=0}^{i-1}\left|\ln\|D_{y_j}T^{-1}w_j\| - \ln\|D_{y_j}T^{-1}v_j\|\right| + \left|\ln\|D_{y_j}T^{-1}v_j\| - \ln\|D_{z_j}T^{-1}v_j\|\right|} \\ &\leq e^{\sum_{j=0}^{i-1}\int_0^1 \frac{\|D_{y_j}T^{-1}\|\|v_j-w_j\|}{\|D_{y_j}T^{-1}(sv_j+(1-s)w_j)\|}ds + \int_0^1 \frac{K\|y_j-z_j\|}{\|D_{(sy_j+(1-s)z_j)}T^{-1}w_j\|}ds} \\ &\leq e^{k_1\sum_{j=0}^{i-1}\lambda^{-j}\max\{\|v-w\|, c\delta_2\}}\end{aligned}$$

for some $k_1 > 0$. □

The first estimates are then demonstrated by the following.

LEMMA II.3. *There exists $D > 0$ such that for each $\gamma \in \widehat{\Gamma}_{\delta_2}$, and $m \in \mathbb{N}$, if $\gamma \cap \Sigma_m^-(c, \delta_2) = \emptyset$, with $\delta \leq \delta_2 < \delta_0$, $c < 1$, then*

$$\begin{aligned}m(\gamma\backslash D(\phi)) &\leq Dc\delta_2\lambda^{-m}, \\ \frac{1}{f(x)}\left|\int_{\gamma\backslash D(\phi)} gf\right| &\leq Dc\delta_2(\lambda^{-\zeta}\mu^{1-\zeta})^m |||g|||_-,\end{aligned}$$

where the integration is performed with respect to arc-length on γ, $g \in \mathcal{C}_{b,c,d,L}(\delta)$, and $f \in \mathcal{D}_a(\gamma)$.

Proof. We start by stating a simple consequence of the Distortion Lemma II.1.

SUBLEMMA II.4. *If $I \in \widehat{\Gamma}_{\delta_0}$, $J \subset I$, and $T^nI \in \widehat{\Gamma}_{|T^nI|}$, then there exists $A > 0$:*

$$A^{-1}\frac{|I|}{|J|} \leq \frac{|T^nI|}{|T^nJ|} \leq A\frac{|I|}{|J|}.$$

Proof. Since the curves are in $\widehat{\Gamma}_{\delta_0}$, they have uniformly bounded curvature; hence, the tangent direction along a curve can change only by an amount proportional to δ_0. Accordingly, Lemma II.2-b implies that there exists c_2 such that

$$\frac{|T^nI|}{|T^nJ|} = \frac{\int_I |D_xT^n|_I|}{\int_J |D_yT^n|_I|} \leq e^{c_2\delta_0}\frac{|I|}{|J|}. \qquad \square$$

Consider $\gamma \in \widehat{\Gamma}_{\delta_*}$, $\delta_* \in [\delta_2, \delta_0]$, $c < 1$. Suppose $\gamma \cap \Sigma_k^-(c, \delta_2) = \emptyset$, for some $k \geq 0$. Then $T^{-k}\gamma = \gamma_0$ is connected, and $|\gamma_0| \geq \lambda^k \delta_2$.

Let $S_0 = \{J_i\}$ be the collection of the, at most, M connected segments constituting $\gamma_0 \cap \Sigma_0^-(\lambda^{-k}c, \delta_2)$, and set $\widetilde{\gamma}_0 = \gamma_0 \setminus \bigcup_{J \in S_0} J$. Consider $T^{-1}\widetilde{\gamma}_0$; it will consist, at most, of M connected pieces. Collect in $\mathcal{J}_1$ the pieces longer than δ_0, after subdividing them, if necessary, in pieces shorter than $2\delta_0$. Collect the remaining pieces in $\mathcal{J}_1^*$. For each piece in $\mathcal{J}_1^*$ repeat the previous construction (i.e., intersect the piece with $\Sigma_0^-(\lambda^{-(k+1)}c, \delta_2)$, collect the connected pieces of the intersection in S_1, consider the preimage of the rest, collect the long pieces in $\mathcal{J}_2$ and the remaining ones in $\mathcal{J}_2^*$, and so forth).

By construction, the pieces in $\mathcal{J}_l^*$ can be at most M^l and are all shorter than δ_0. Hence, for each $\xi \in [\zeta, 1]$,

$$\sum_{\widetilde{\gamma} \in \mathcal{J}_l^*} |T^{k+l}\widetilde{\gamma}|^\xi \leq M^l \lambda^{-(l+k)\xi} \delta_0^\xi.$$

In addition, we can estimate the pieces that fall too close to a singularity line during the first l iterations:

$$\sum_{i=k}^{l-1} \sum_{\widetilde{\gamma} \in S_{i-k}} |T^i \widetilde{\gamma}|^\xi \leq c^\xi \sum_{i=k}^{l-1} M^{i+1-k} \lambda^{-2i\xi} \delta_2^\xi \leq B c^\xi \lambda^{-2k\xi} \delta_2^\xi,$$

since $M\lambda^{-\zeta} < 1$. Consequently, calling $\mathcal{J}(\gamma, c)$ the collection of segments that either get too close to the singularity or never grow larger than δ_0, we have

$$\sum_{\widetilde{\gamma} \in \mathcal{J}(\gamma_0, c)} |\widetilde{\gamma}|^\xi \leq B c^\xi \lambda^{-2k\xi} \delta_2^\xi.$$

We still need to deal with the segments in the collection $\bigcup_{i=k+1}^\infty T^i \mathcal{J}_{i-k}$.

Let us consider $\overline{\gamma} \in \mathcal{J}_1$; iterating it we will obtain again segments that are too close to a singularity, segments that stay short for all times, and segments that grow to size δ_0. We will estimate the first two groups while we will add the segments belonging to the last category to the above-mentioned collections of long segments.

$$\begin{aligned}
\sum_{\overline{\gamma} \in \mathcal{J}_1} \sum_{\widetilde{\gamma} \in \mathcal{J}(\overline{\gamma}, \lambda^{-k-1}c)} |T^{k+1}\widetilde{\gamma}|^\xi &\leq \sum_{\overline{\gamma} \in \mathcal{J}_1} \sum_{\widetilde{\gamma} \in \mathcal{J}(\overline{\gamma}, \lambda^{-k-1}c)} \frac{|T^{k+1}\widetilde{\gamma}|^\xi}{|T^{k+1}\overline{\gamma}|^\xi} |T^{k+1}\overline{\gamma}|^\xi \\
&\leq A^\xi \sum_{\overline{\gamma} \in \mathcal{J}_1} \sum_{\widetilde{\gamma} \in \mathcal{J}(\overline{\gamma}, \lambda^{-k-1}c)} \frac{|\widetilde{\gamma}|^\xi}{|\overline{\gamma}|^\xi} |T^{k+1}\overline{\gamma}|^\xi \\
&\leq A^\xi B c^\xi \lambda^{-(k+1)\xi} \delta_2^\xi \delta_0^{-\xi} \sum_{\overline{\gamma} \in \mathcal{J}_1} |T^{k+1}\overline{\gamma}|^\xi \\
&\leq A^\xi B c^\xi \lambda^{-(k+1)\xi} \delta_2^\xi \delta_0^{-1} \mu^{(k+1)(1-\xi)} \sum_{\overline{\gamma} \in \mathcal{J}_1} |T^{k+1}\overline{\gamma}| \\
&\leq 2A^\xi B c^\xi \lambda^{-(k+1)\xi} \delta_2^\xi \mu^{(k+1)(1-\xi)}.
\end{aligned}$$

Still, we have not analyzed all the pieces that must be estimated. The remaining ones are pieces $\widehat{\gamma}$ with some preimage $T^{-j}\widehat{\gamma}$, $j > k$, contained in a connected component of $T^{-j}\gamma$ of size larger than δ_0. The key observation to conclude the argument is that the last estimate does not depend on the number of pieces in $\mathcal{J}_1$, but only on the fact that all such pieces are longer than δ_0 and that $\mathcal{J}_1 \subset T^{-k-1}\gamma$. This means that we can apply the above estimate iteratively and obtain

$$\sum_{I\subset\Sigma_\infty^-(c,\delta_2)\cap\gamma} |I|^\xi \le \sum_{i=k}^{\infty} 2A^\xi B c^\xi (\lambda^{-\xi}\mu^{(1-\xi)})^i \delta_2^\xi \le D(\lambda^{-\xi}\mu^{(1-\xi)})^k \delta_2^\xi.$$

The lemma easily follows from the above estimate, and from the fact that $\gamma\backslash D(\phi)$ is a countable collection of segments, each one of which is contained in a segment belonging to $\Sigma_\infty^-(c,\, \delta_1)$. □

The first of the wanted estimates is completed thanks to the symmetry between ϕ and ϕ^{-1}.

Next, we can study the smoothness of ϕ.

LEMMA II.5. *There exists $A \in \mathbb{R}^+$, such that for all $x, y \in \gamma_1\backslash\Sigma_\infty^-(4c, \delta_2)$, $c > c_1 c_0^{-1}$, $\ell(\gamma_1, \gamma_2) \le c_1\delta_2$,*[39]

$$d(\phi(x), \phi(y)) \le A d(x, y).$$

Proof. Let $x,\, y \in \gamma\backslash\Sigma_\infty^-(4c,\, \delta_2)$; suppose

$$d(T^{-n}x,\, T^{-n}y) \le c^{\frac{1}{2}} d(x,\, y)^{\frac{1}{2}} \delta_2^{\frac{1}{2}};$$

then

$$\lambda^n d(x,\, y) \le c^{\frac{1}{2}} d(x,\, y)^{\frac{1}{2}} \delta_2^{\frac{1}{2}},$$

that is,

$$d(T^{-n}x,\, T^{-n}y) \le c\lambda^{-n}\delta_2.$$

The above estimate means that the two points cannot be on opposite sides of $\mathcal{S}^-$ (remember that the distance is taken along γ or its images), otherwise one of the points would belong to Σ_∞^- contrary to the hypothesis. Accordingly, another iteration is possible without disconnecting the part of γ lying between x and y. This shows that $T^{-n}[x,\, y]$ is connected until

$$d(T^{-n}x,\, T^{-n}y) \ge c^{\frac{1}{2}} d(x,\, y)^{\frac{1}{2}} \delta_2^{\frac{1}{2}}.$$

Keep in mind that the definition of ϕ implies

$$T^{-n}\phi(z) = W^{\mathrm{u}}(T^{-n}z) \cap T^{-n}\gamma_2.$$

[39] By $\ell(\gamma_1, \gamma_2)$ we mean $\sup_{x\in\gamma_1} \inf_{y\in\gamma_2} d(x, y)$.

SUBLEMMA II.6. *If* $z, w \in \gamma_1 \backslash \Sigma^-_\infty(4c, \delta_2)$, *and* $z' = \phi(z)$, $w' = \phi(w)$, $z_m = T^{-m}z$, $w_m = T^{-m}w$, $z'_m = T^{-m}z'$, $w'_m = T^{-m}w'$ *then, given* $m \in \mathbb{N}$ *such that* $d(T^{-j}z, T^{-j}w) \leq c\delta_2\lambda^{-j}$, *for each* $j < m$,[40]

$$\text{(II.2)} \qquad \ell(T^{-m}[z, w], T^{-m}[z', w']) \leq \frac{2\rho^2 \max\{d(z,w), d(z',w')\}\ell(\gamma_1,\gamma_2)}{\max\{d(z_m, w_m), d(z'_m, w'_m)\}}$$

for some fixed $\rho \geq 1$.[41]

Proof. The proof is by induction over m. The formula (II.2) is certainly true for $m = 0$. In addition, when the rectangle $Q = [z,\ w,\ w',\ z']$, we see that T^{-1} is smooth on Q. Suppose (II.2) true for $m-1$ and T^{-m+1} smooth on Q. Since

$$\begin{aligned} &\max\{d(T^{-m+1}z,\ T^{-m+1}w),\ d(T^{-m+1}z',\ T^{-m+1}w')\} \\ &\qquad \geq\ \lambda^{m-1}\max\{d(z,\ w),\ d(z',\ w')\}, \end{aligned}$$

it follows that $\ell(T^{-m+1}[z,\ w],\ T^{-m+1}[z',\ w']) \leq 2\rho^2\lambda^{-m+1}c_1\delta_2$. Remember that the cone field is continuous and the curves $T^k\gamma_i$ are uniformly transversal to the unstable direction. This implies that there exists some fixed $\tilde{\rho} \in \mathbb{R}^+$, such that

$$\begin{aligned} d(T^{-m+1}z',\ T^{-m+1}w') &\leq d(T^{-m+1}z,\ T^{-m+1}w) \\ &\quad + 2\tilde{\rho}\ell(T^{-m+1}[z,\ w],\ T^{-m+1}[z',\ w']) \\ &\leq \lambda^{-m+1}(1 + 4\rho^2\tilde{\rho}c_0)c\delta_2. \end{aligned}$$

We choose $\tilde{\rho} = \rho$, and c_0 so that $c_0 < \frac{\rho^{-3}}{3}$. The above estimates imply that $T^{-m+1}Q$ lies in a $4c\lambda^{-m+1}\delta_2$–neighborhood of $T^{-m+1}z$ and therefore cannot contain a singularity line. Consequently, T^{-m} is smooth on Q.

The area of the rectangle Q is less than $\rho\ell(\gamma_1,\ \gamma_2)\max\{d(z,\ w),\ d(z',\ w')\}$, and $\frac{1}{2\rho}\ell(T^{-m}[z,\ w],\ T^{-m}[z',\ w'])\max\{d(T^{-m}z,\ T^{-m}w),\ d(T^{-m}z',\ T^{-m}w')\}$ is a lower bound for the area of $T^{-n}Q$; the result follows by the area-preserving property of T. $\square$

In conclusion, given $x,\ y \in \gamma_1$, $d(x,y) < c\delta_2$, we choose the first $n \in \mathbb{N}$ such that

$$d(T^{-n}x,\ T^{-n}y) \geq c^{\frac{1}{2}}d(x,\ y)^{\frac{1}{2}}\delta_2^{\frac{1}{2}}.$$

Then, due to the continuity of the cone field, the fact that T^{-n} is smooth on the rectangle Q determined by the points x, y, $\phi(x)$, $\phi(y)$ (see the proof

[40] In fact, the inequality for $j = m-1$ is much stronger than the others, and implies them.

[41] By $[z, w]$ we mean the piece of γ_1 lying between z and w.

of Sublemma II.6), and uniform hyperbolicity, we have that $T^{-n}[x, y]$ and $[T^{-n}\phi(x), T^{-n}\phi(y)]$ are essentially parallel curves. Hence,

$$d(T^{-n}\phi(x), T^{-n}\phi(y)) \le d(T^{-n}x, T^{-n}y) + 2\rho\ell(T^{-n}[z, w], T^{-n}[z', w']).$$

If $d(x, y) \ge d(\phi(x), \phi(y))$, then the lemma is proved. It follows that there exists $B \in \mathbb{R}^+$ such that

$$\frac{d(T^{-n}\phi(x), T^{-n}\phi(y))}{d(T^{-n}x, T^{-n}y)} \le 1 + B\left[\frac{d(\phi(x), \phi(y))}{d(x, y)}\right]^{\frac{1}{2}}.$$

In fact, if $d(T^{-n}\phi(x), T^{-n}\phi(y)) \le d(T^{-n}x, T^{-n}y)$ the above inequality is obvious; otherwise Sublemma II.6 implies

$$\begin{aligned}\frac{d(T^{-n}\phi(x), T^{-n}\phi(y))}{d(T^{-n}x, T^{-n}y)} &\le 1 + 4\rho^3 \frac{d(\phi(x), \phi(y))d(\gamma_1, \gamma_2)}{d(T^{-n}\phi(x), T^{-n}\phi(y))d(T^{-n}x, T^{-n}y)} \\ &\le 1 + 4\rho^3 c_0 \frac{d(\phi(x), \phi(y))}{d(x, y)}\left[\frac{d(T^{-n}x, T^{-n}y)}{d(T^{-n}\phi(x), T^{-n}\phi(y))}\right],\end{aligned}$$

which, in turn, yields

$$\frac{d(T^{-n}\phi(x), T^{-n}\phi(y))}{d(T^{-n}x, T^{-n}y)} \le \frac{1}{2} + \sqrt{5c_0\rho^3}\left[\frac{d(\phi(x), \phi(y))}{d(x, y)}\right]^{\frac{1}{2}}.$$

After noticing that, for $i < n$, if $w \in T^{-i}Q$ then $d(w, T^{-i}x) \le 2c\delta_2\lambda^{-i}$, by using Lemma II.2-a one can conclude that there exists $\widetilde{D} \in \mathbb{R}^+$ such that

$$\begin{aligned}\frac{d(\phi(x), \phi(y))}{d(x, y)} &= \frac{\int_{T^{-n}\phi(x)}^{T^{-n}\phi(y)} \left|D_z T^n|_{T^{-n}\gamma_2}\right|}{\int_{T^{-n}x}^{T^{-n}y} \left|D_w T^n|_{T^{-n}\gamma_1}\right|} \\ &\le \widetilde{D}\frac{d(T^{-n}\phi(x), T^{-n}\phi(y))}{d(T^{-n}x, T^{-n}y)} \\ &\le \widetilde{D}\left(1 + B\left[\frac{d(\phi(x), \phi(y))}{d(x, y)}\right]^{\frac{1}{2}}\right)\end{aligned}$$

which implies the lemma. □

Finally, we have:

LEMMA II.7. *There exists* $D_1 > 0$*: for all* $x, y \in \gamma_2 \backslash \Sigma_\infty^-(2c, \delta_2)$*, such that*

$$e^{-D_1 d(x,y)^{\frac{1}{2}}\delta_2^{\frac{1}{2}}} \le \frac{J_{\phi^{-1}}(y)}{J_{\phi^{-1}}(x)} \le e^{D_1 d(x,y)^{\frac{1}{2}}\delta_2^{\frac{1}{2}}}.$$

Proof. We prove only the first inequality, the proof of the second being alike. Again, we choose $x, y \in \gamma_2 \backslash \Sigma^-_\infty(2c, \delta_2)$ such that $d(x, y) \leq c\delta_2$, and $n \in \mathbb{N}$ such that, for the first time,

$$d(T^{-n}x, T^{-n}y) \geq c^{\frac{1}{2}} d(x, y)^{\frac{1}{2}} \delta_2^{\frac{1}{2}}.$$

Setting $x_1 = \phi^{-1}(x)$, $y_1 = \phi^{-1}(y)$ and using the estimates contained in Lemma II.2-(b, c), we have

$$\begin{aligned}
\frac{J_{\phi^{-1}}(x)}{J_{\phi^{-1}}(y)} &= \prod_{j=1}^{\infty} \left| \frac{D_{T^{-j}x_1} T\big|_{T^{-j}\gamma_1}}{D_{T^{-j}x} T\big|_{T^{-j}\gamma_2}} \right| \left| \frac{D_{T^{-j}y} T\big|_{T^{-j}\gamma_2}}{D_{T^{-j}y_1} T\big|_{T^{-j}\gamma_1}} \right| \\
&= \prod_{j=1}^{n-1} \left| \frac{D_{T^{-j}y} T\big|_{T^{-j}\gamma_2}}{D_{T^{-j}x} T\big|_{T^{-j}\gamma_2}} \right| \left| \frac{D_{T^{-j}x_1} T\big|_{T^{-j}\gamma_1}}{D_{T^{-j}y_1} T\big|_{T^{-j}\gamma_1}} \right| \\
&\quad \times \prod_{j=n}^{\infty} \left| \frac{D_{T^{-j}x_1} T\big|_{T^{-j}\gamma_1}}{D_{T^{-j}x} T\big|_{T^{-j}\gamma_2}} \right| \left| \frac{D_{T^{-j}y} T\big|_{T^{-j}\gamma_2}}{D_{T^{-j}y_1} T\big|_{T^{-j}\gamma_1}} \right| \\
&= \left| \frac{D_{T^{-n+1}y} T^{n-1}\big|_{T^{-n+1}\gamma_2}}{D_{T^{-n+1}x} T^{n-1}\big|_{T^{-n+1}\gamma_2}} \right| \left| \frac{D_{T^{-n+1}x_1} T^{n-1}\big|_{T^{-n+1}\gamma_1}}{D_{T^{-n+1}y_1} T^{n-1}\big|_{T^{-n+1}\gamma_1}} \right| \\
&\quad \times \lim_{j\to\infty} \left| \frac{D_{T^{-j}x_1} T^{j-n+1}\big|_{T^{-j}\gamma_1}}{D_{T^{-j}x} T^{j-n+1}\big|_{T^{-j}\gamma_2}} \right| \left| \frac{D_{T^{-j}y} T^{j-n+1}\big|_{T^{-j}\gamma_2}}{D_{T^{-j}y_1} T^{j-n+1}\big|_{T^{-j}\gamma_1}} \right| \\
&\leq e^{D \max\{d(T^{-n+1}x, T^{-n+1}y),\, \|(T^{-n+1}\gamma_2)'(x) - (T^{-n+1}\gamma_2)'(y)\|\}} \\
&\quad \times e^{D \max\{d(T^{-n+1}x_1, T^{-n+1}y_1),\, \|(T^{-n+1}\gamma_1)'(x_1) - (T^{-n+1}\gamma_1)'(y_1)\|\}} \\
&\quad \times e^{D \max\left\{d(T^{-n+1}x, T^{-n+1}x_1),\, \left\| \frac{D_{x_1} T^{-n+1}\gamma_1'(x_1)}{\|D_{x_1} T^{-n+1}\gamma_1'(x_1)\|} - \frac{D_x T^{-n+1}\gamma_1'(x)}{\|D_x T^{-n+1}\gamma_1'(x)\|} \right\| \right\}} \\
&\quad \times e^{D \max\left\{d(T^{-n+1}y, T^{-n+1}y_1),\, \left\| \frac{D_{y_1} T^{-n+1}\gamma_1'(y_1)}{\|D_{y_1} T^{-n+1}\gamma_1'(y_1)\|} - \frac{D_y T^{-n+1}\gamma_1'(y)}{\|D_y T^{-n+1}\gamma_1'(y)\|} \right\| \right\}}.
\end{aligned}$$

Since the $T^{-n+1}\gamma_i$ have uniformly bounded curvature, and since Lemma II.5 holds the first two exponentials are easily bounded. For the last two, additional arguments are needed.

More precisely, we need an inequality of type (II.1), but sharper. Namely, when $v_i = \frac{D_{x_1} T^{-i}\gamma_1'(x_1)}{\|D_{x_1} T^{-i}\gamma_1'(x_1)\|}$ and $w_i = \frac{D_x T^{-i}\gamma_1'(x)}{\|D_x T^{-i}\gamma_1'(x)\|}$,

$$|\omega(v_i, w_i)| \leq h_1 \|D_x T^{-i} v_0\|^{-1} \delta_2 \tag{II.3}$$

for some fixed $h_1 > 0$. Again, the proof is by induction, and we are lead to

$$\begin{aligned}|\omega(v_i,\, w_i)| &\le \lambda^{-1}\frac{|\omega(v_{i-1},\, w_{i-1})|}{\|D_xT^{-i+1}v_{i-1}\|}+\lambda^{-1}K\frac{d(T^{-i+1}x,\, T^{-i+1}\phi^{-1}(x))}{\|D_xT^{-i+1}v_{i-1}\|}\\ &\le \frac{h_1\lambda^{-1}}{\|D_xT^{-i}v_0\|}+\frac{K\lambda^{-1}}{\|D_xT^{-i+1}v_{i-1}\|}\int_x^{\phi^{-1}(x)}\left|D_zT^{-i+1}\Big|_{W^{\mathrm{u}}(x)}\right|dz\end{aligned}$$

where the second integral is taken along $W^{\mathrm{u}}(x)$ (the unstable manifold of x).

Again using Lemma II.2-c, one can write

$$|\omega(w_i,\, v_i)| \le \frac{h_1\lambda^{-1}\delta_2}{\|D_xT^{-i}v_0\|}+\frac{K\lambda^{-1}e^{2D\delta_2}}{\|D_xT^{-i+1}v_{i-1}\|}\left|D_xT^{-i+1}\Big|_{W^{\mathrm{u}}(x)}\right|d(x,\, \phi^{-1}(x)).$$

Finally, we notice that, because of the symplectic nature of T, and the uniform transversality of stable and unstable directions there exists $\alpha > 1$ such that

$$\alpha^{-1} \le \left|D_xT^{-j}\Big|_{W^{\mathrm{u}}(x)}\right| \|D_xT^{-j}v_0\| \le \alpha.$$

Hence,

$$|\omega(w_i,\, v_i)| \le \frac{\lambda^{-1}(h_1+Ke^{2D\delta_2}\alpha)\delta_2}{\|D_xT^{-i}v_0\|},$$

which proves (II.3) when h_1 is sufficiently large.

Formula (II.3) implies

$$\left\|\frac{D_{x_1}T^{-n}\gamma_1'(x_1)}{\|D_{x_1}T^{-n}\gamma_1'(x_1)\|}-\frac{D_xT^{-n}\gamma_1'(x)}{\|D_xT^{-n}\gamma_1'(x)\|}\right\| \le 2h_1\|D_xT^{-n}\gamma_2'(x)\|^{-1}\delta_2.$$

Moreover, again by Lemma II.2-b,

$$d(x,\, y) = \int_{T^{-n}x}^{T^{-n}y}\left|D_zT^n|_{T^{-n}\gamma_2}\right|dz \ge \text{const.}\|D_xT^{-n}\gamma_2'(x)\|^{-1}d(T^{-n}x,\, T^{-n}y).$$

In conclusion, there exists $D_2 > 0$:

$$\frac{J_{\phi^{-1}}(x)}{J_{\phi^{-1}}(y)} \le e^{D_2\left\{d(x,y)^{\frac12}\delta_2^{\frac12}+\max\{d(x,y)^{\frac12}\delta_2^{\frac12},\,\frac{\delta_2 d(x,\,y)}{d(T^{-n}x,\,T^{-n}y)}\}\right\}} \le e^{Dd(x,y)^{\frac12}\delta_2^{\frac12}}. \quad \square$$

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