TP decoding

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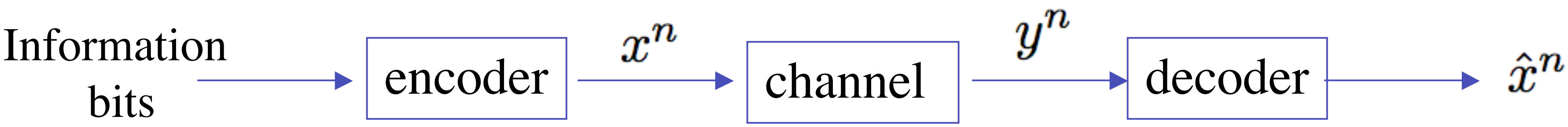
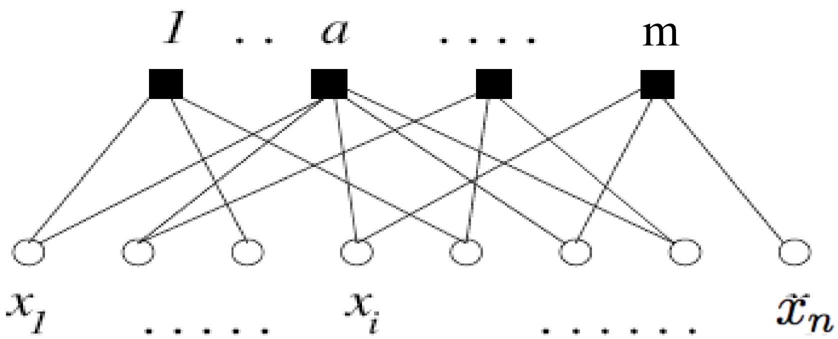
TP decoding

TP stands for tree pruning.

We are interested in an efficient algorithm that interpolates between BP and MAP.

The decoding problem





$$\mu_{\mathfrak{C},y}(\underline{x}) \equiv \mathbb{P}\{\underline{X}=\underline{x}|\underline{Y}=\underline{y}\}$$

$$\mu_{\mathfrak{C},y}(\underline{x}) = \frac{1}{Z(\mathfrak{C},y)}\prod_{i=1}^{n} Q(y_i|x_i)\prod_{a=1}^{m} \mathbb{I}(x_{i_1(a)}\oplus\cdots\oplus x_{i_k(a)}=0)\,.$$

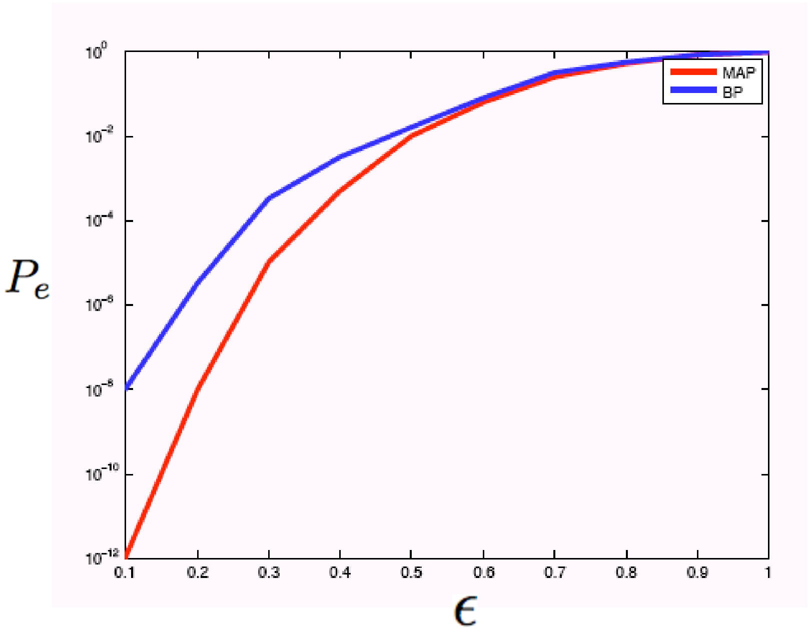
$$\mu(x_i) = \sum_{\underline{x}_{\sim i}} \mu(\underline{x})$$

MAP symbol decoding

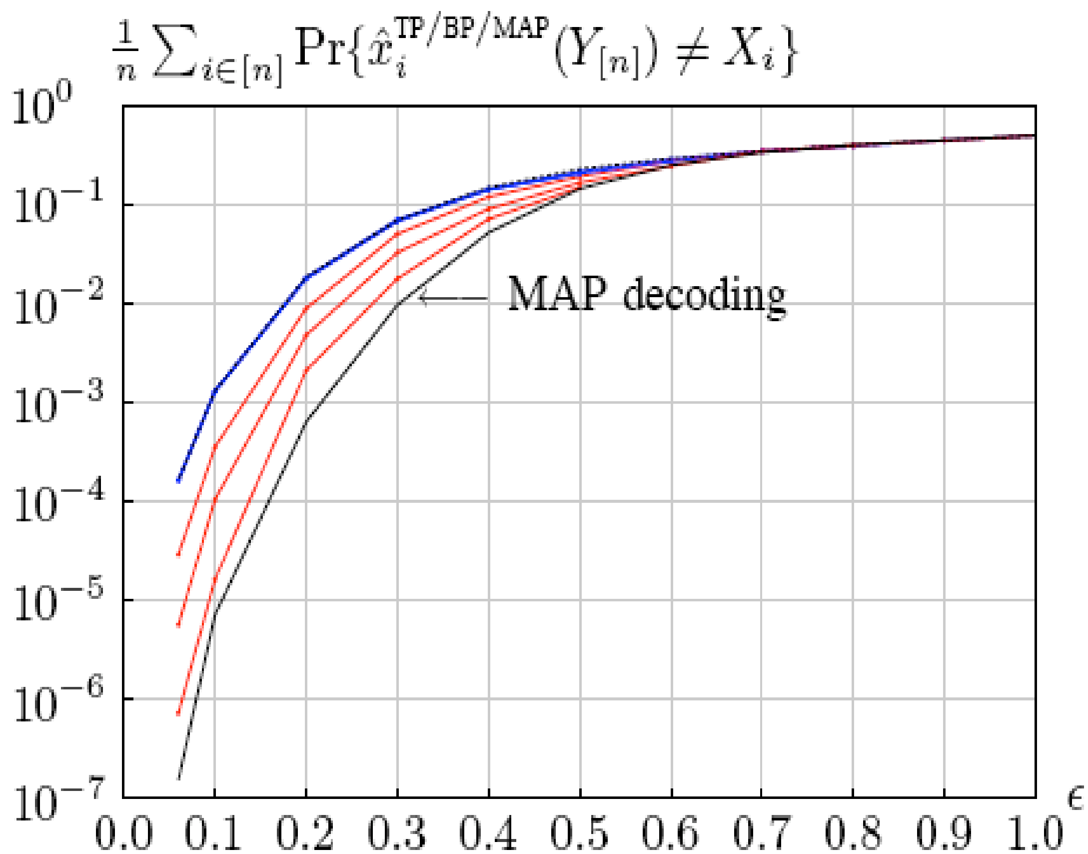
$$\hat{x}_i^{MAP} = \arg\max_{x_i} \mu(x_i)$$

MAP symbol decoding

- Computing the marginal of a distribution that factorizes on a graph.
- Exact computation is exponential in n.
- Belief propagation is exact if the graph is a tree; otherwise suboptimal.
- Define bit-error rate $P_e = \frac{1}{n}\sum_i P(\hat{x}_i \neq x_i)$
- There exists a gap between BP and MAP.

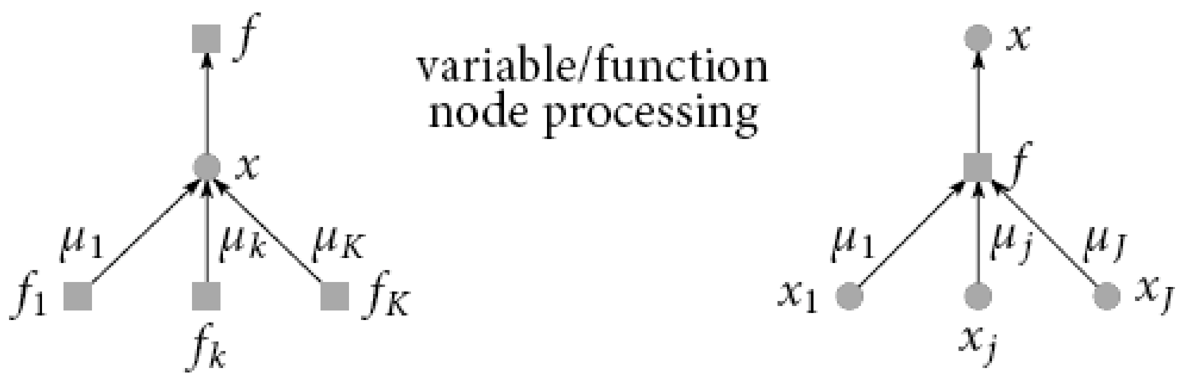


What we would like



BP

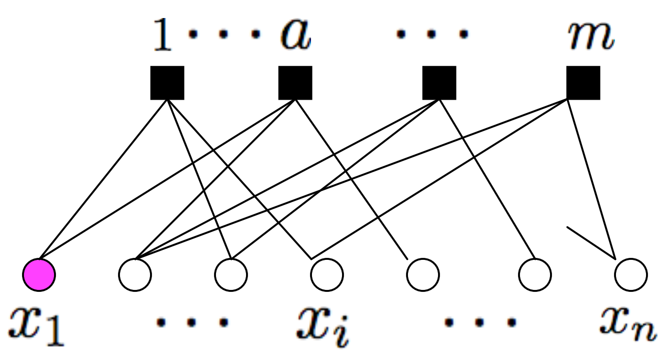
- BP is a message passing decoder.

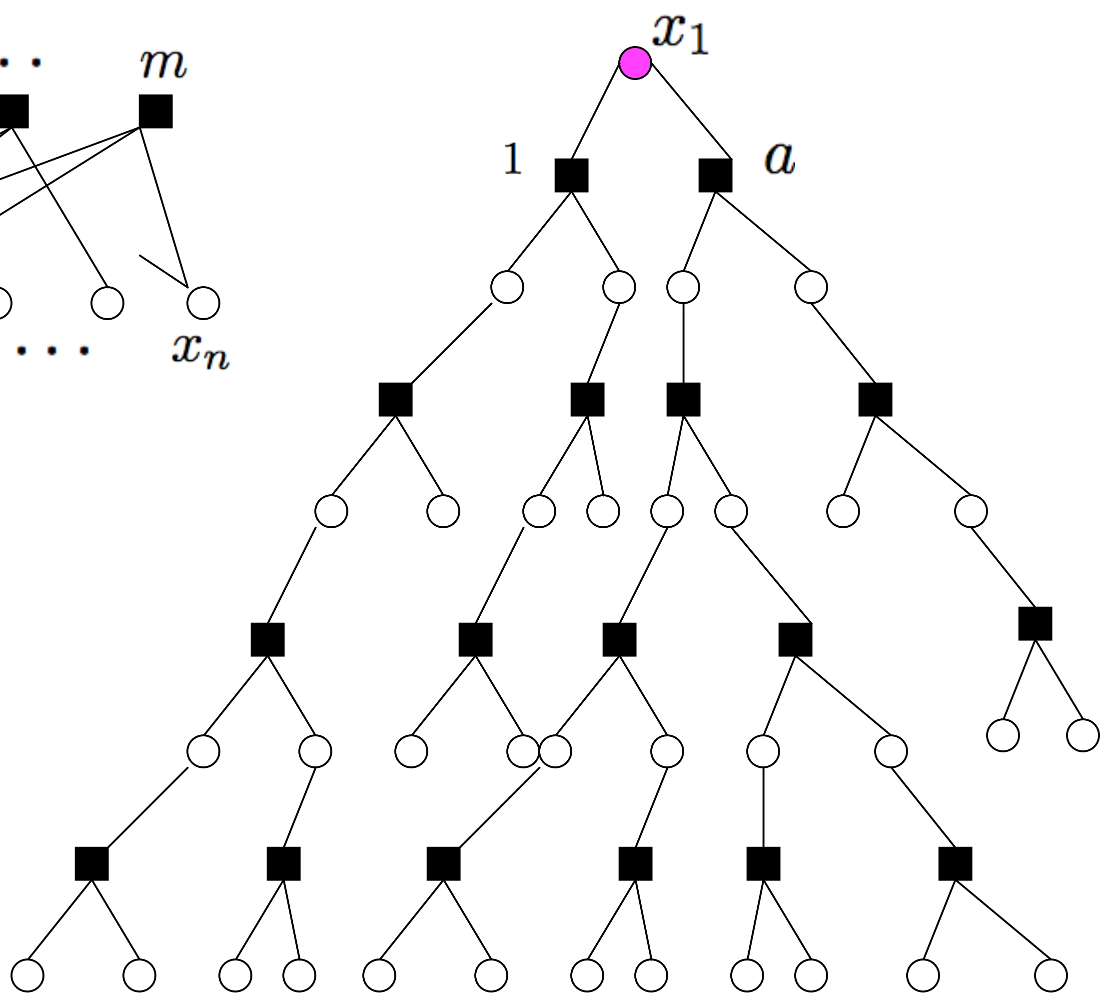


$$\Phi^{(\ell)}(\mu_1, \ldots, \mu_{r-1}) = \begin{cases} 0, & \ell = 0, \\ 2\tanh^{-1}\left(\prod_{i=1}^{r-1} \tanh\left(\frac{\mu_i}{2}\right)\right), & \ell \geq 1, \end{cases}$$

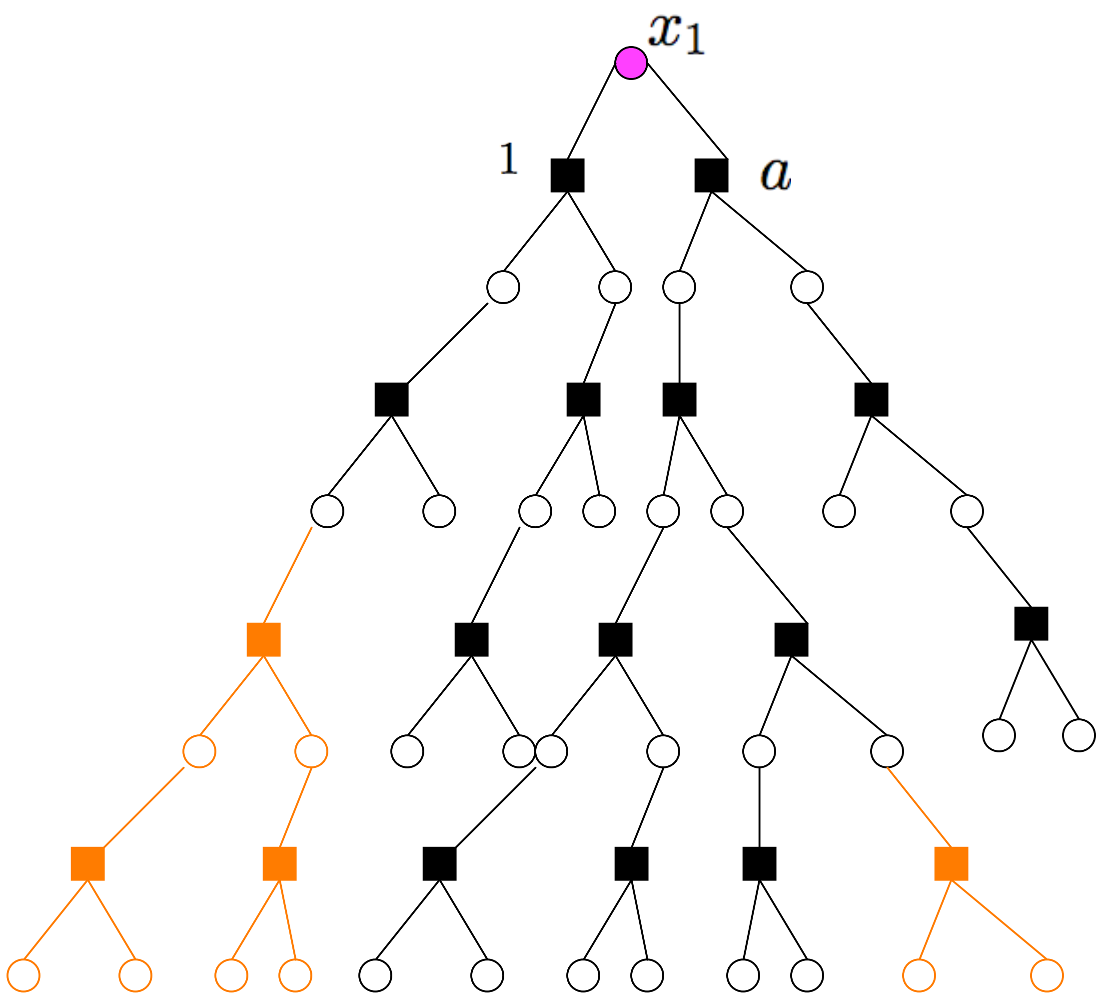
$$\Psi^{(\ell)}(\mu_0, \mu_1, \ldots, \mu_{l-1}) = \mu_0 + \sum_{i=1}^{l-1} \mu_i.$$

BP computation tree

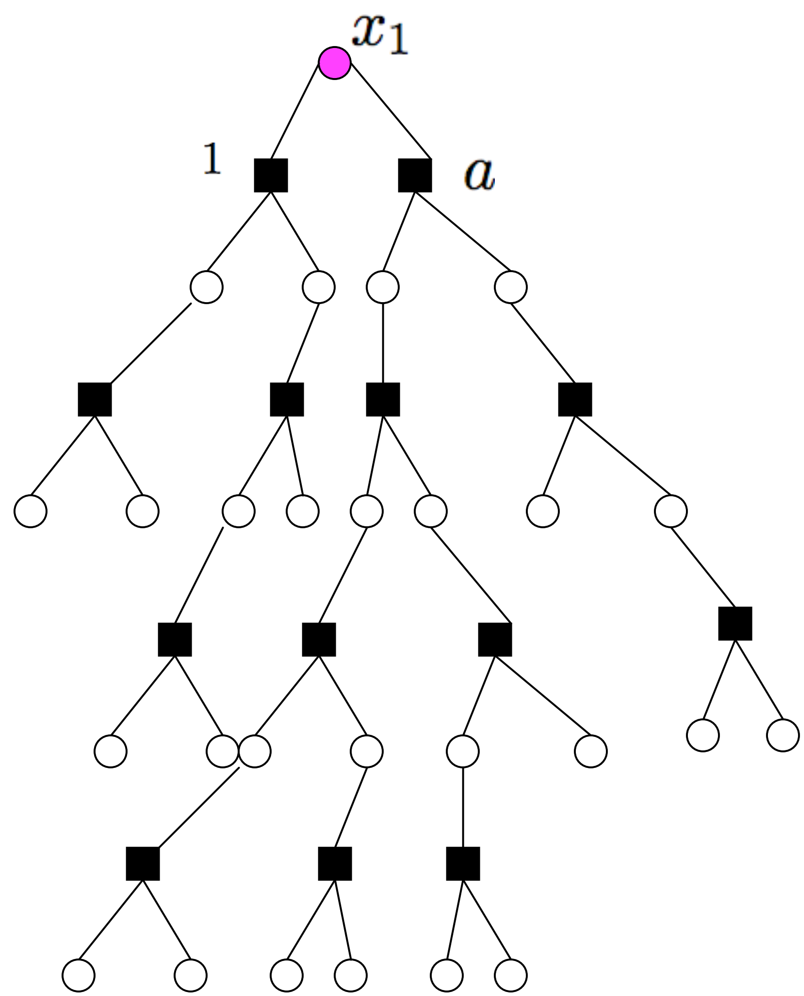




Tree Pruning

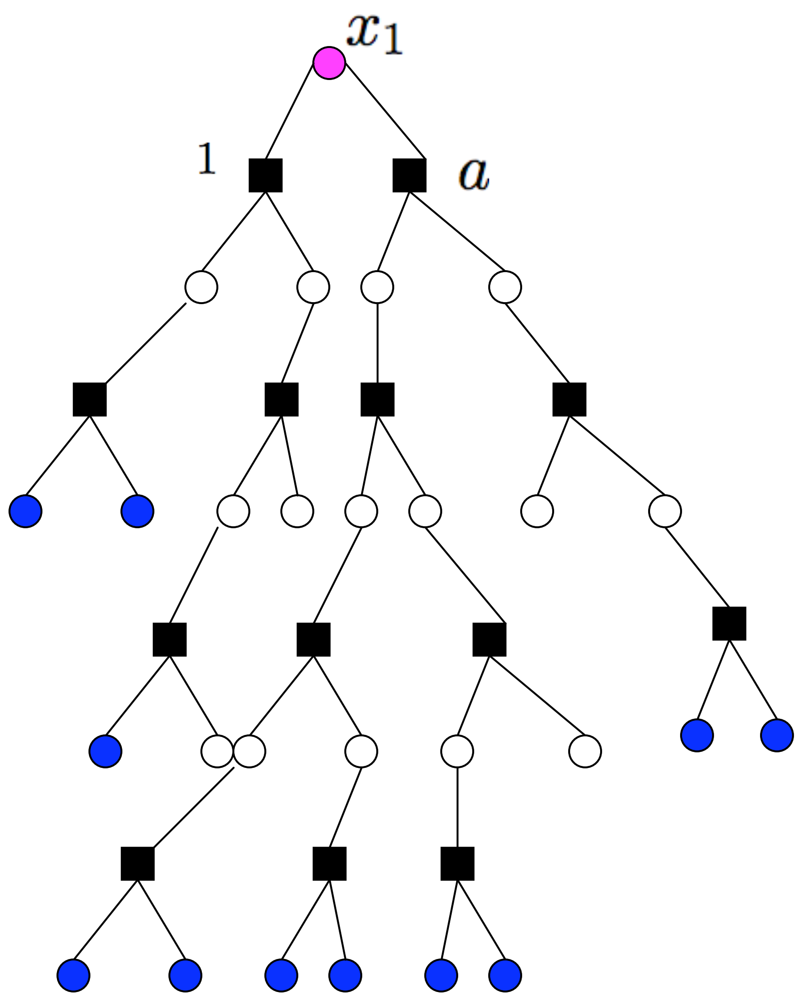


Tree Pruning



Tree Pruning

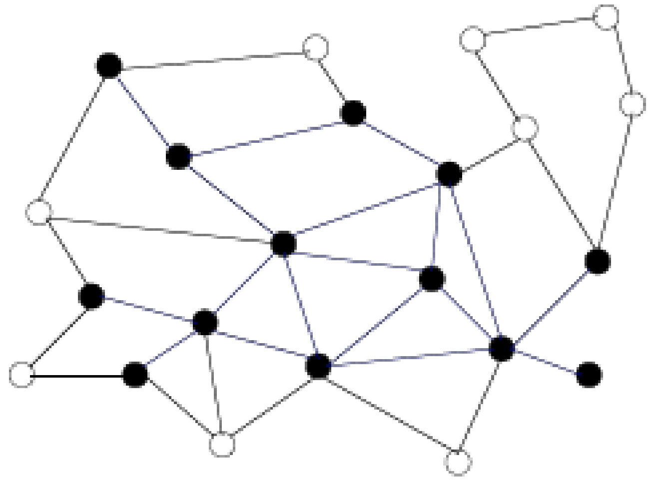
● -nodes with fixed boundary condition



Weitz's construction

[Weitz, 2006. Jung and Shah, 2006]

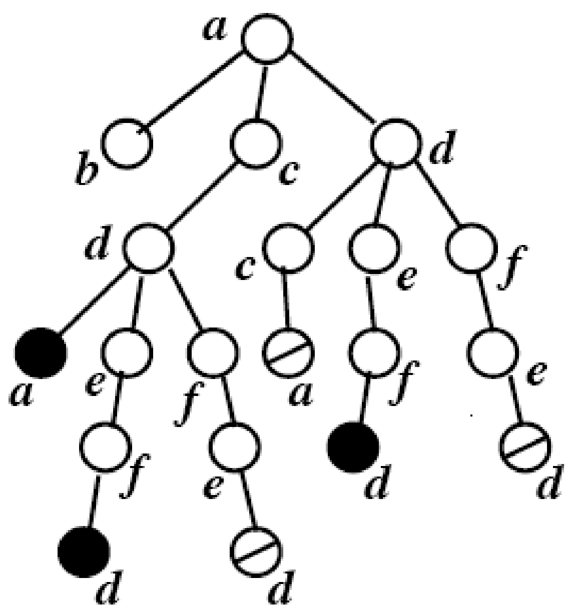
- Given a pairwise Markov random field

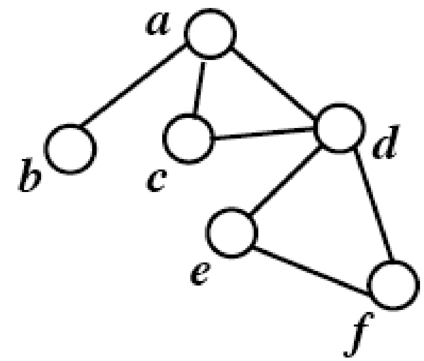


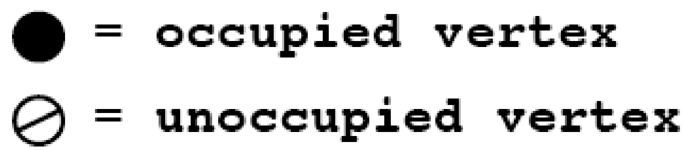
$$\mu(\underline{x}) = \frac{1}{Z} \prod_{(ij)\in G} \phi_{ij}(x_i, x_j) \qquad\qquad x_i \in \{0,1\}$$

Weitz's construction

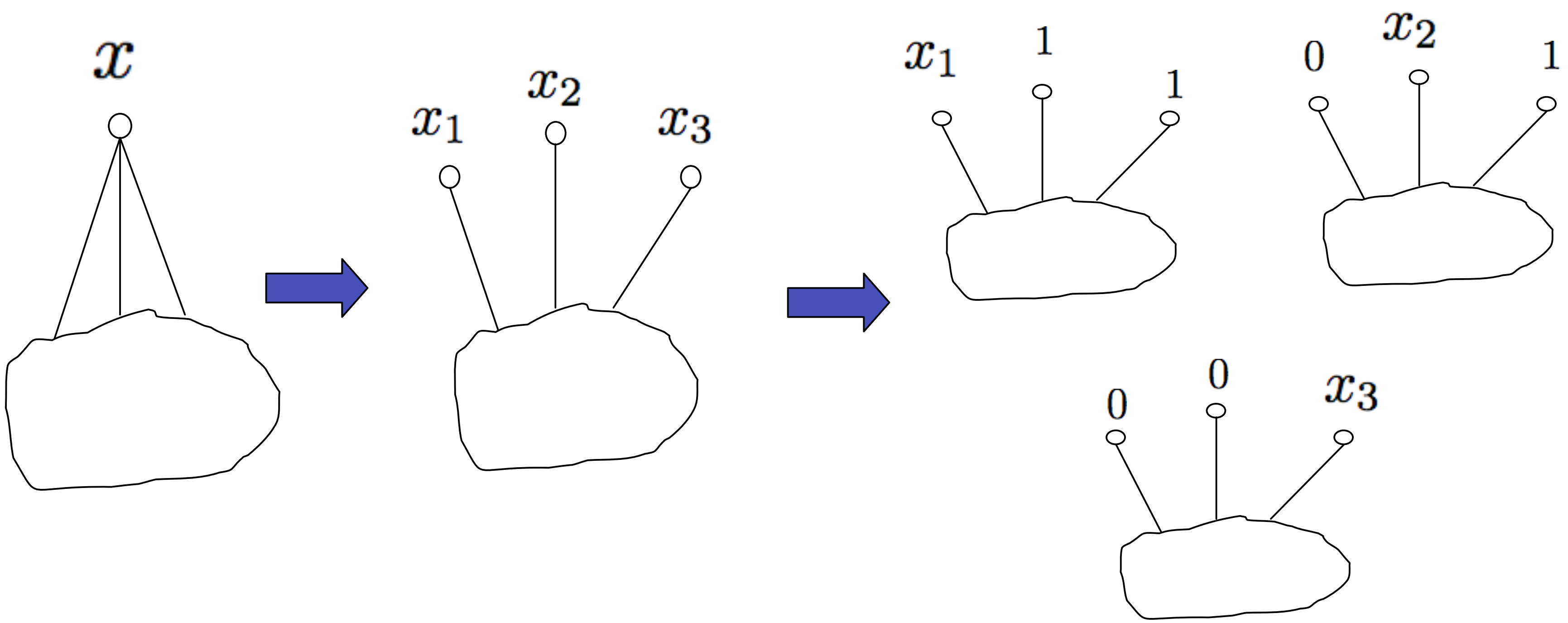
T_{SAW}







Weitz's construction



$$\frac{P(1)}{P(0)} = \frac{P(111)}{P(000)} = \frac{P(111)}{P(011)} \times \frac{P(011)}{P(001)} \times \frac{P(001)}{P(000)}$$

Algorithm

$$\begin{array}{lcl} \textit{Problem} & \longrightarrow & |T_{\text{SAW}}| = O(\Delta^n) \\ \textit{Idea} & \longrightarrow & \text{Truncate at depth } t \end{array}$$

Convergence $\Leftrightarrow$ Correlation decay

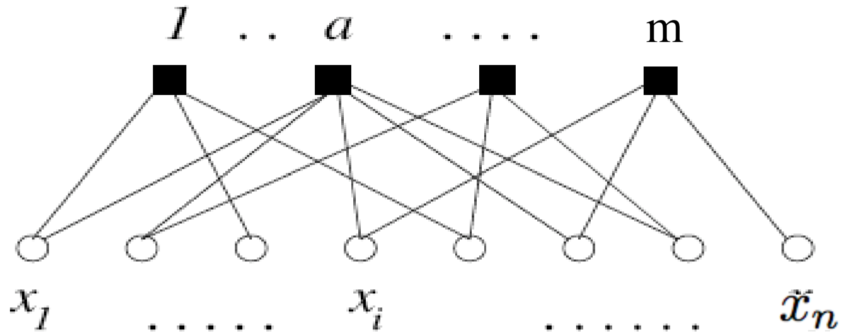
$$\sup_{\underline{x}_t, \underline{x}'_t} |\mu_{\text{Tree}}(x_i|\underline{x}_t) - \mu_{\text{Tree}}(x_i|\underline{x}'_t)| \leq 2\, e^{-\kappa t}$$

Our problem

1. Binary variables, but multi-variable interactions.
2. Non-permissive interactions.
3. No “correlation decay”. Truncation can be problematic.

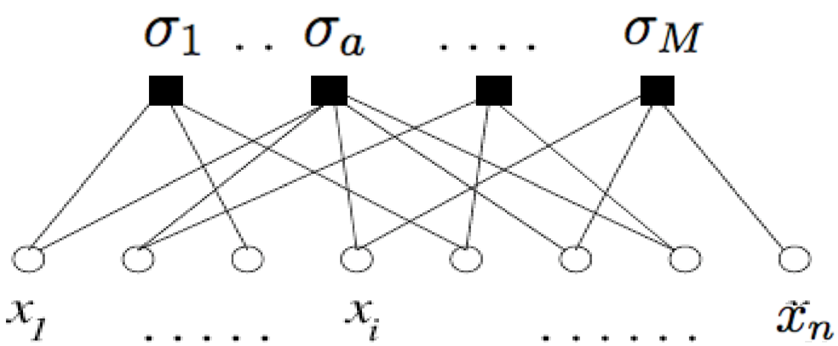
- [Nair and Tetali, 2006] extension to non-binary variables and multi-variable interactions. Higher complexity than Weitz.

MRF with negative potentials



- Using duality

|||



$\sigma_a \in \{0, 1\}$

Edge potential

$$(-1)^{x_i \sigma_a}$$

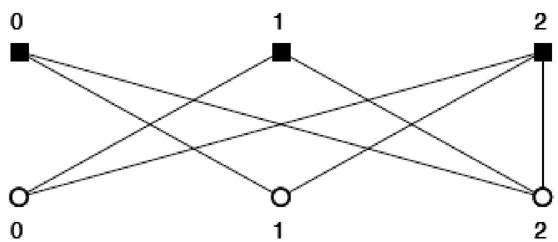
MRF with negative potentials

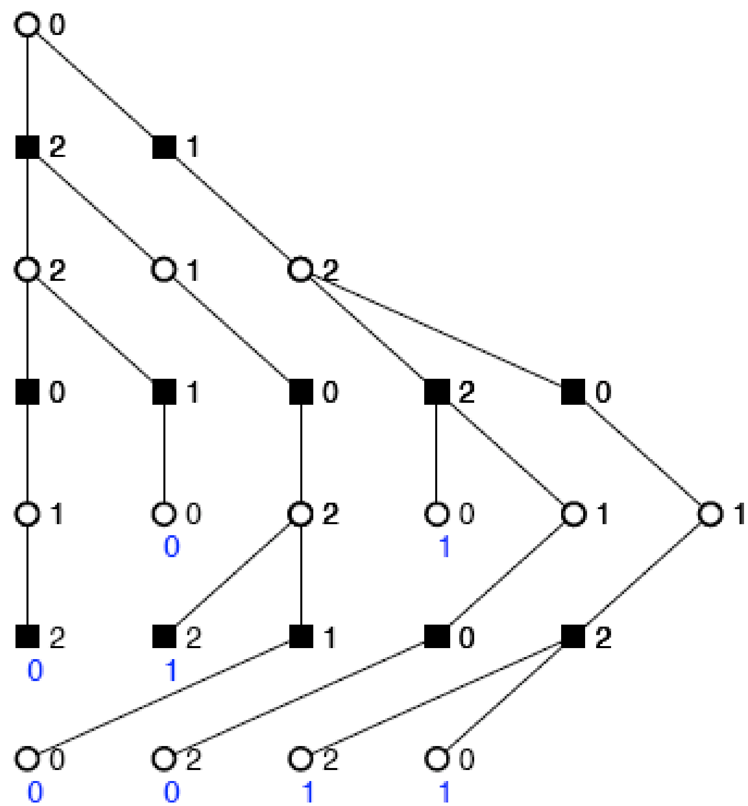
$$\prod_{i\in V} Q(y_i|x_i) \prod_{a\in F} 2\,\mathbb{I}\left\{\sum_{i\in\partial a} x_i \in \text{EVEN}\right\}$$

$$= \prod_{i\in V} Q(y_i|x_i) \prod_{a\in F} \sum_{\sigma_a=0}^{1} (-1)^{\sigma_a \sum_{i\in\delta a} x_i}$$

$$= \sum_{\underline{\sigma}} \prod_{i\in V} Q(y_i|x_i) \prod_{(ia)\in E} (-1)^{x_i \sigma_a}$$

Non-Permissiveness

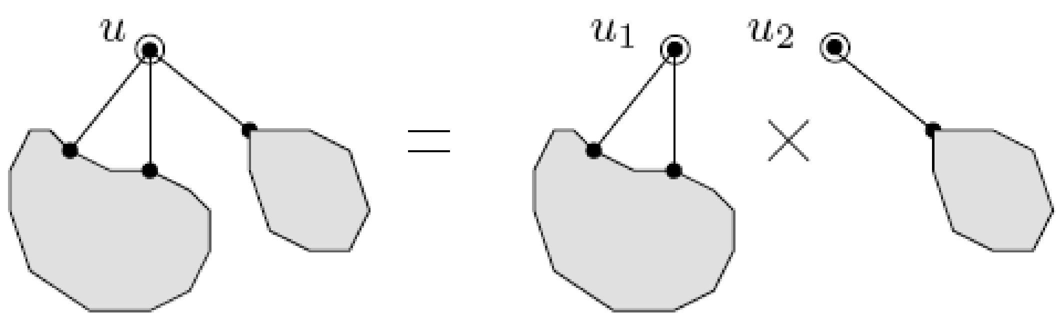




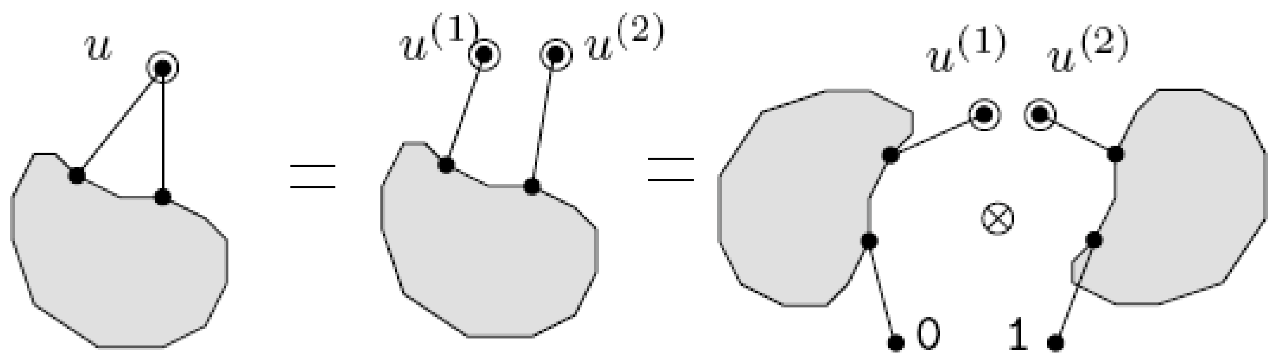
Non-permissiveness can cause undefined messages like $\frac{0}{0}$

Concatenation

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$



$$\frac{a}{b} \otimes \frac{c}{d} = \frac{a}{d}$$



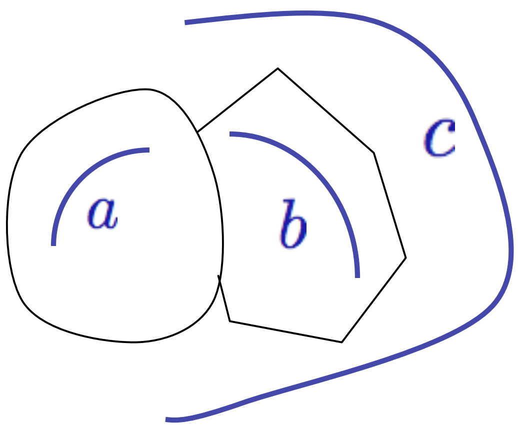
$$\frac{P(11)}{P(10)} \otimes \frac{P(10)}{P(00)} = \frac{P(11)}{P(00)}$$

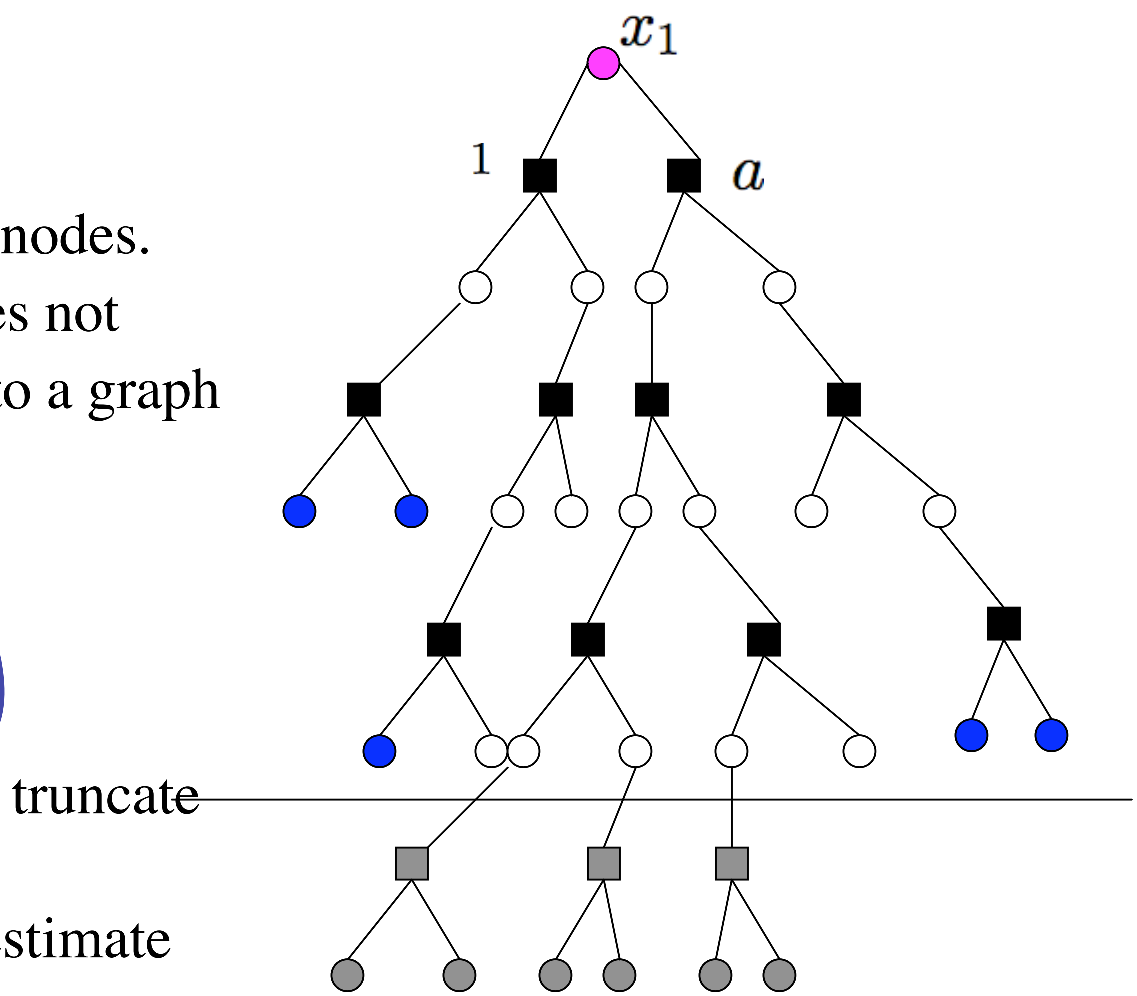
Truncation

- Leave boundary “free”
- Erasure channel

  Further prune at unerased nodes.

  Note the resulting tree does not necessarily correspond to a graph





- Truncated TP is a biased estimate

Erasure Channel

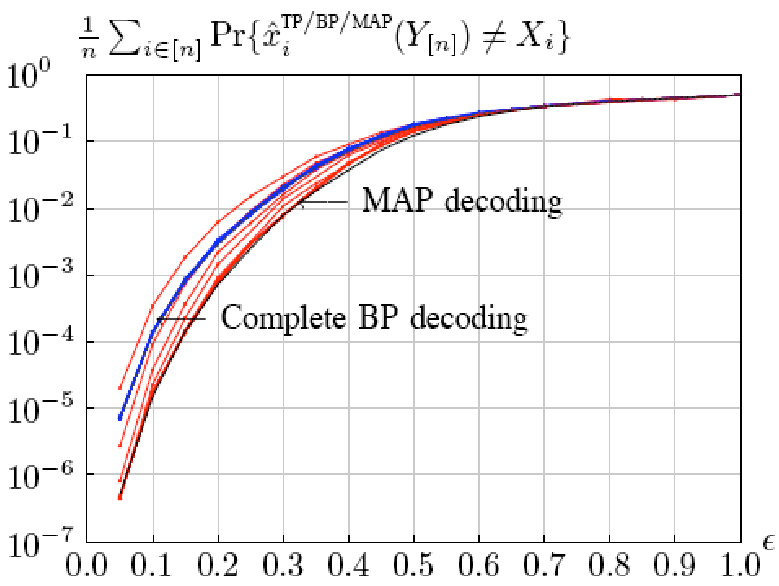


Fig. 3. Tailbiting convolutional code with generator pair $(1 + D^2, 1 + D + D^2)$ and blocklength $n = 100$. Dashed black curve: BP decoding with $t = \infty$. Plain black curve: MAP decoding (BP followed by Gaussian elimination). Blue curves: BP decoding with $t = 3, 4, 5, 6, 8, 10, 12, 14$ (almost undistingushable). Red curves: TP decoding with $t = 3, 4, 5, 6, 8, 10, 12, 14$ (truncated tree).

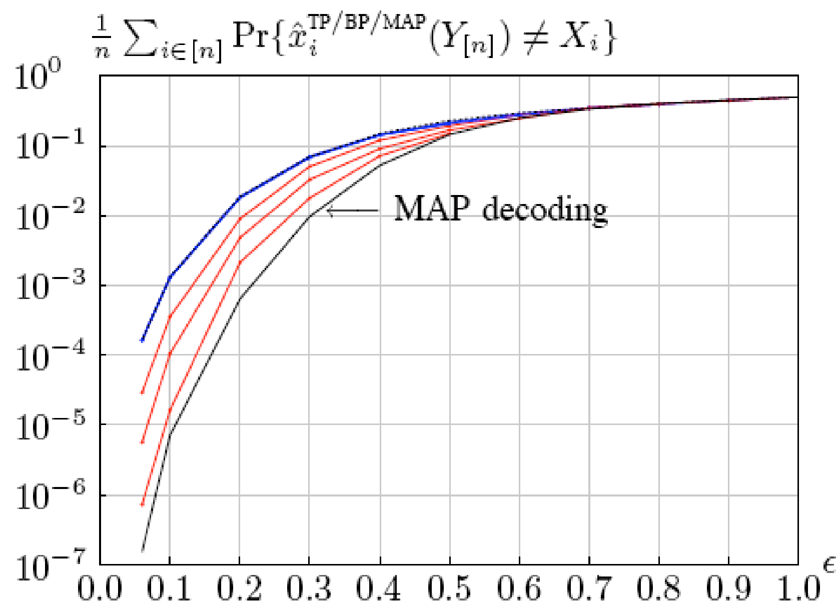


Fig. 4. $(23, 12)$ Golay code with blocklength $n = 23$. Dashed black curve: BP decoding with $t = \infty$. Plain black curve: MAP decoding (BP followed by Gaussian elimination). Blue curves: BP decoding with $t = 4, 5, 6$. Red curves: TP decoding with $t = 4, 5, 6$ (truncated tree).

Erasure Channel

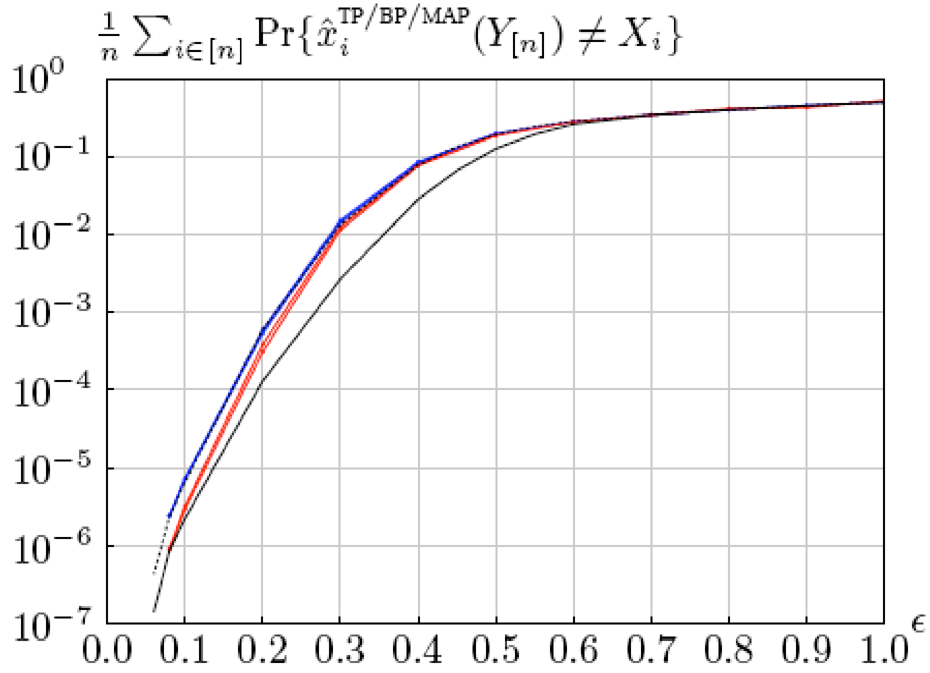
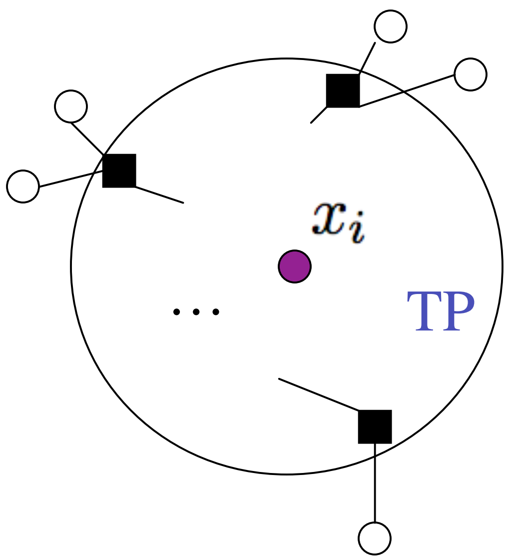


Fig. 5. A regular $(3,6)$ LDPC code with blocklength $n = 50$. Dashed black curve: BP decoding with $t = \infty$. Plain black curve: MAP decoding (BP followed by Gaussian elimination). Blue curves: BP decoding with $t = 7, 8$. Red curves: TP decoding with $t = 7, 8$ (truncated tree).

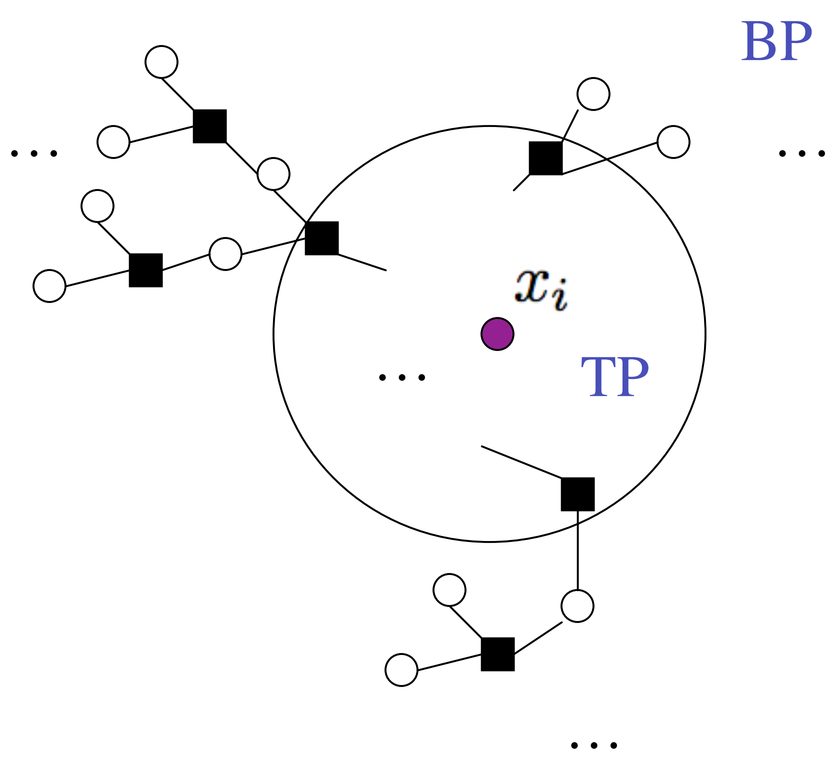
General Channel

- Truncation does not work. Only preliminary results.

MAP(t)



MAP(t)-BP(l)



General Channel

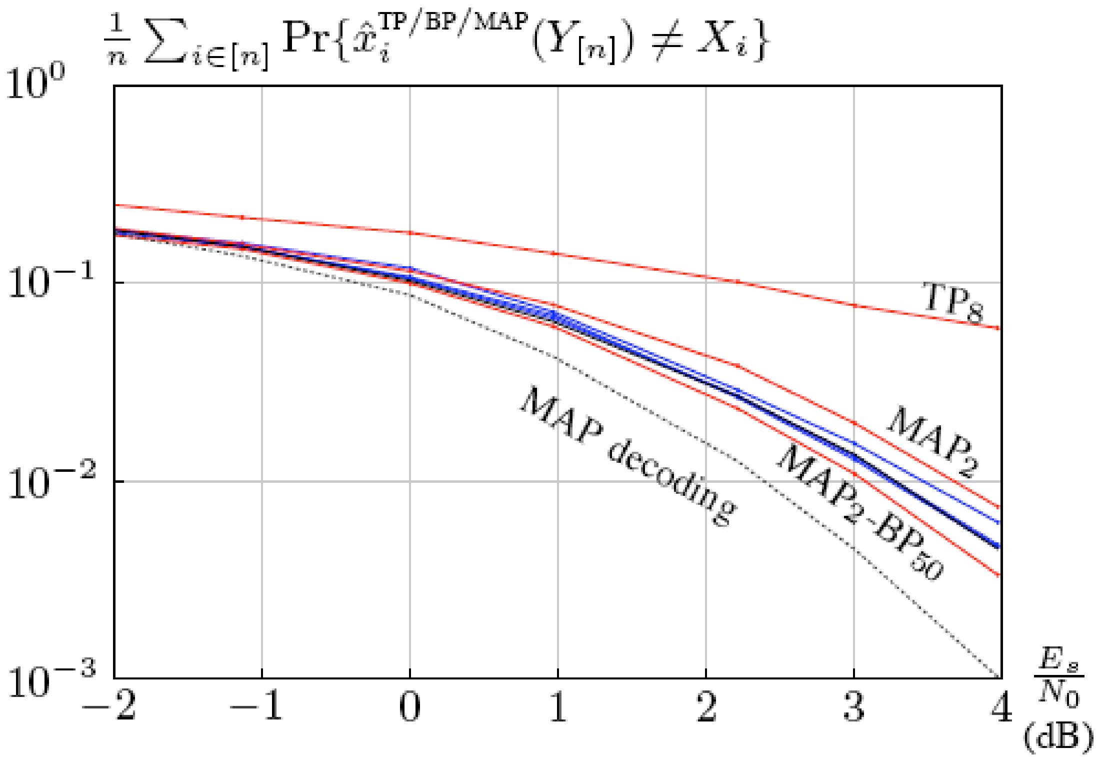


Fig. 6. Tailbiting convolutional code with generator pair $(1 + D^2, 1+D+D^2)$ and blocklength $n = 50$. Dashed black curve: BP decoding with $t = 400$. Plain black curve: MAP decoding. Blue curves: BP decoding with $t = 8, 50$. Red curves: TP decoding with $t = 8$ (truncated tree), with TP on a ball of radius 2, i.e. MAP(2), and MAP(2) − BP(50).

Conclusion

- We constructed a Tree Pruning algorithm that interpolates between BP and MAP in the error floor region.