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ELSEVIER
Linear Algebra and its Applications 407 (2005) 149–161
www.elsevier.com/locate/laa

On the asymptotic stability of nonnegative matrices in max algebra ☆

Yung-Yih Lur

Department of Industrial Engineering, Vanung University, Chung-Li, Taoyuan, Taiwan, ROC

Received 16 July 2004; accepted 3 May 2005
Available online 28 July 2005
Submitted by L. Elsner

Abstract

In the max algebra system, for an $n \times n$ nonnegative matrix $A = [a_{ij}]$ the eigenequation for max eigenvalue λ and corresponding max eigenvector x is $A \otimes x = \lambda x$, where $[A \otimes x]_i = \max_{1 \leqslant j \leqslant n} a_{ij} x_j$ and $\mu(A)$ is the maximum circuit geometric mean. It is shown that the following conditions are mutually equivalent: (i) $\eta_{\|\cdot\|}(A) < 1$, for some norm $\|\cdot\|$ on $\mathbb{R}^n$; (ii) $\hat{\eta}(A) < 1$; (iii) $\mu(A) < 1$; (iv) $\lim_{k\to\infty} A_{\otimes}^{k} = 0$, where $\eta_{\|\cdot\|}(A) = \max_{\|x\|=1, x \geqslant 0} \|A \otimes x\|$ and $\hat{\eta}(A) = \limsup_{k\to\infty} [\eta_{\|\cdot\|}(A_{\otimes}^{k})]^{\frac{1}{k}}$.

Keywords: Max algebra; Nonnegative matrix; Maximum circuit geometric mean

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1. Introduction

Following the notation in [4], the max algebra system consists of a set of nonnegative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab for $a, b \geqslant 0$. For a nonnegative matrix $A = [a_{ij}]$, we may denote a_{ij} by $[A]_{ij}$. Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices and let $A = [a_{ij}]$ and $B = [b_{ij}]$ be nonnegative matrices in $\mathbb{R}^{n\times n}$. The product A and B is denoted by $A \otimes B$, where $[A \otimes B]_{ij} =$

☆ Research was supported by NSC 92-2115-M-238-001.

E-mail address: yylur@vnu.edu.tw

0024-3795/$ - see front matter
doi:10.1016/j.laa.2005.05.017

$\max_{1\leqslant k\leqslant n} a_{ik}b_{kj}$. $A_{\otimes}^2$ means $A \otimes A$, and $A_{\otimes}^k$ denotes the kth power of A. Note that $A \leqslant B$ iff $a_{ij} \leqslant b_{ij}$ for all $1 \leqslant i, j \leqslant n$. Let $x = [x_i] \in \mathbb{R}^n$ be a nonnegative vector. The notation $A_{\otimes}^k \otimes x$ means $[A_{\otimes}^k \otimes x]_i = \max_{1\leqslant j\leqslant n}[A_{\otimes}^k]_{ij}x_j$, and $A_{\otimes}^k x$ means $[A_{\otimes}^k x]_i = \sum_{j=1}^n [A_{\otimes}^k]_{ij}x_j$. We say that $\lim_{k\to\infty} A_{\otimes}^k = 0$ if $\lim_{k\to\infty}[A_{\otimes}^k]_{ij} = 0$, for all $1 \leqslant i, j \leqslant n$.

Let $A = [a_{ij}]$ be an $n \times n$ nonnegative matrix. The directed graph corresponding to A, denoted by $\mathscr{D}(A)$, is defined by $\mathscr{D}(A) = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$ and the set of edges $E = \{(i, j) \in V \times V | a_{ij} > 0, 1 \leqslant i, j \leqslant n\}$. The $L(i_0, i_1, i_2, \ldots, i_k)$ is called a path from vertex i_0 to vertex i_k in $\mathscr{D}(A)$ with length k ($k \geqslant 1$) if $(i_t, i_{t+1}) \in E$ for each $t = 0, 1, 2, \ldots, k-1$. The *weight* of a path $L(i_1, i_2, \ldots, i_k)$, as denoted by $w(L(i_1, i_2, \ldots, i_k))$ or simply by $w(L)$, is defined by

$$w(L(i_1, i_2, \ldots, i_k)) = a_{i_1i_2}a_{i_2i_3}\cdots a_{i_{k-1}i_k}.$$

A circuit of the length k is a path $L(i_0, \ldots, i_{k-1}, i_k)$ with $i_k = i_0$, where $i_0, i_1, \ldots, i_{k-1}$ are distinct. Associated with this circuit is the *circuit geometric mean* known as $\hat{w}(L) = (a_{i_0i_1}a_{i_1i_2}\cdots a_{i_{k-1}i_k})^{1/k}$. The *maximum circuit geometric mean* in $\mathscr{D}(A)$ is denoted by $\mu(A)$. Note that we also consider *empty* circuits, namely, circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero. An $n \times n$ nonnegative matrix A is *reducible* if there is a permutation matrix P such that

$$PAP^{\mathrm{T}} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square nonnegative submatrices. An $n \times n$ nonnegative matrix A is *irreducible* if it is not reducible (see, e.g.,[10, p. 217]).

Recall that two norms $\|\cdot\|_s$, $\|\cdot\|_r$ on a vector space are said to be *equivalent* if whenever a sequence $\{x_k\}$ converges to a vector x with respect to the first norm, then it converges to the same vector with respect to the second norm. It is well known that two norms $\|\cdot\|_s$, $\|\cdot\|_r$ on a real vector space V are equivalent if and only if there are positive constants m, M such that

$$m\|x\|_s \leqslant \|x\|_r \leqslant M\|x\|_s \quad \text{for all } x \in V,$$

also, it is known that for finite-dimensional real vector spaces, all norms are equivalent (see, e.g., [9, pp. 272–279]). A norm on $\mathbb{R}^{n\times n}$ is called a *generalized matrix norm*. A generalized matrix norm $\|\cdot\|$ on $\mathbb{R}^{n\times n}$ is said to be a *matrix norm* if $\|AB\| \leqslant \|A\|\|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$, where AB is the standard matrix product (see, e.g., [9, p. 290]). Since $\mathbb{R}^{n\times n}$ is a finite-dimensional real vector spaces, all matrix norms on $\mathbb{R}^{n\times n}$ are equivalent. If $x = [x_i] \in \mathbb{R}^n$, we define $|x| = [|x_i|]$. For each $x = [x_i]$, $y = [y_i] \in \mathbb{R}^n$, we say that $|x| \leqslant |y|$ if $|x_i| \leqslant |y_i|$ for all $i = 1, \ldots, n$. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be *monotone* if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$ for all $x, y \in \mathbb{R}^n$. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be *absolute* if $\|x\| = \||x|\|$ for all $x \in \mathbb{R}^n$. It is well known that a norm $\|\cdot\|$ on $\mathbb{R}^n$ is monotone if and only if it is absolute (see, e.g., [9, p. 285]).

In the literature, the maximum circuit geometric mean $\mu(A)$ has been studied extensively, and it is known that $\mu(A)$ is a max eigenvalue of A. Moreover, if A is irreducible, then $\mu(A)$ is the unique eigenvalue and every eigenvector is positive. Please refer to [1,3,8] for the spectral study. Elsner and Van den Driessche [4–6] provided asymptotic formulas for $\mu(A)$ that involve spectral radii and matrix norms and algorithms of computing $\mu(A)$ and max eigenvector x, for an irreducible nonnegative matrix A was established as well. Bounds for $\mu(A)$ can be found in [1,2]. The role of $\mu(A)$ plays in the study of powers of a nonnegative matrix A can be found in [4].

2. Results

Let A be an $n \times n$ nonnegative matrix. A scalar λ is called a *max eigenvalue* of A if $A \otimes x = \lambda x$ for some nonnegative vector $x \neq 0$, namely,

$$\max_{1 \leqslant j \leqslant n} a_{ij} x_j = \lambda x_i \quad \forall i = 1, 2, \ldots, n.$$

The vector x is called a corresponding *max eigenvector* of λ. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Associated with this norm $\|\cdot\|$ we define $\eta_{\|\cdot\|}(A)$ as

$$\eta_{\|\cdot\|}(A) = \sup_{x \neq 0, x \geqslant 0} \frac{\|A \otimes x\|}{\|x\|}.$$

Observe that for each $\alpha > 0$,

$$\alpha [A \otimes x]_i = \alpha \left(\max_{1 \leqslant j \leqslant n} a_{ij} x_j \right) = \max_{1 \leqslant j \leqslant n} a_{ij} (\alpha x_j) = [A \otimes (\alpha x)]_i.$$

So that $\alpha(A \otimes x) = A \otimes (\alpha x)$. Thus

$$\begin{aligned} \eta_{\|\cdot\|}(A) &= \sup_{x \neq 0, x \geqslant 0} \frac{\|A \otimes x\|}{\|x\|} = \sup_{x \neq 0, x \geqslant 0} \left\| \frac{A \otimes x}{\|x\|} \right\| \\ &= \sup_{x \neq 0, x \geqslant 0} \left\| A \otimes \frac{x}{\|x\|} \right\| = \sup_{\|x\| = 1, x \geqslant 0} \|A \otimes x\|. \end{aligned}$$

Since $\|A \otimes x\|$ is a continuous function of x and $\{x : \|x\| = 1, x \geqslant 0\}$ is a compact set in $\mathbb{R}^n$, we have

$$\eta_{\|\cdot\|}(A) = \max_{\|x\| = 1, x \geqslant 0} \|A \otimes x\|.$$

Let $\|\cdot\|_s$ and $\|\cdot\|_r$ be any two norms on $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a finite-dimensional real vector space, $\|\cdot\|_s$ and $\|\cdot\|_r$ are equivalent. Then there exist $m, M > 0$ such that

$$m\|x\|_s \leqslant \|x\|_r \leqslant M\|x\|_s \text{ for all } x \in \mathbb{R}^n,$$

so that for each $n \times n$ nonnegative matrix A, we have

$$\eta_{\|\cdot\|_r}(A) = \max_{x\neq 0, x\geqslant 0} \frac{\|A \otimes x\|_r}{\|x\|_r} \leqslant \max_{x\neq 0, x\geqslant 0} \frac{M\ \|A \otimes x\|_s}{m\ \|x\|_s} = \frac{M}{m}\eta_{\|\cdot\|_s}(A).$$

Hence there is a positive integer L such that

$$\left[\frac{1}{L}\ \eta_{\|\cdot\|_s}(A_{\otimes}^k)\right]^{\frac{1}{k}} \leqslant [\eta_{\|\cdot\|_r}(A_{\otimes}^k)]^{\frac{1}{k}} \leqslant [L\eta_{\|\cdot\|_s}(A_{\otimes}^k)]^{\frac{1}{k}} \quad \forall k = 1, 2, \ldots.$$

It follows

$$\limsup_{k\to\infty}[\eta_{\|\cdot\|_s}(A_{\otimes}^k)]^{\frac{1}{k}} = \limsup_{k\to\infty}[\eta_{\|\cdot\|_r}(A_{\otimes}^k)]^{\frac{1}{k}},$$

so that $\limsup_{k\to\infty}[\eta_{\|\cdot\|}(A_{\otimes}^k)]^{\frac{1}{k}}$ does not depend on the particular choice for the norm on $\mathbb{R}^n$. Now we define $\hat{\eta}(A)$ by

$$\hat{\eta}(A) = \limsup_{k\to\infty}[\eta_{\|\cdot\|}(A_{\otimes}^k)]^{\frac{1}{k}}.$$

From the proof 1 of Theorem 2 and the proof of Lemma 1 [1], we have concluded the following result. For easy reference, we state this as a theorem.

Theorem 1. *Let A be an $n \times n$ nonnegative matrix. Then the maximum circuit geometric mean $\mu(A)$ is the largest max eigenvalue of A.*

The following four lemmas are needed for the proof of the main Theorem 2.

Lemma 1. *Let A, B be $n \times n$ nonnegative matrices and $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then*

(i) $\eta_{\|\cdot\|}(A) = 0 \Longleftrightarrow A = 0$.
(ii) $\eta_{\|\cdot\|}(A \otimes B) \leqslant \eta_{\|\cdot\|}(A)\eta_{\|\cdot\|}(B)$.
(iii) $\mu(A) \leqslant \eta_{\|\cdot\|}(A)$.

Proof. (i) Let $\mathbf{e_i}$ be the vector on $\mathbb{R}^n$ whose ith component is 1 and 0 otherwise. If $\eta_{\|\cdot\|}(A) = 0$ then for each i we have

$$\left\| A \otimes \frac{\mathbf{e_i}}{\|\mathbf{e_i}\|} \right\| \leqslant \eta_{\|\cdot\|}(A) = 0.$$

This implies that $a_{ji} = 0$ for all $j = 1, 2, \ldots, n$. Thus $A = 0$. The converse is clear.
(ii) It follows that

$$\begin{aligned}\eta_{\|\cdot\|}(A \otimes B) &= \max_{\|x\|=1, x\geqslant 0} \|A \otimes B \otimes x\| \\ &\leqslant \max_{\|x\|=1, x\geqslant 0} (\eta_{\|\cdot\|}(A)\|B \otimes x\|) \\ &= \eta_{\|\cdot\|}(A)\left(\max_{\|x\|=1, x\geqslant 0} \|B \otimes x\|\right) \\ &= \eta_{\|\cdot\|}(A)\eta_{\|\cdot\|}(B).\end{aligned}$$

(iii) By Theorem 1, there is a nonnegative vector $\hat{x} \neq 0 \in \mathbb{R}^n$ such that $A \otimes \hat{x} = \mu(A)\hat{x}$. Hence

$$\eta_{\|\cdot\|}(A) = \max_{\|x\|\neq 0, x \geqslant 0} \frac{\|A \otimes x\|}{\|x\|} \geqslant \frac{\|A \otimes \hat{x}\|}{\|\hat{x}\|} = \frac{\|\mu(A)\hat{x}\|}{\|\hat{x}\|} = \mu(A). \qquad \square$$

Lemma 2. *Let A be an $n \times n$ nonnegative matrix and $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then $\hat{\eta}(A) \leqslant [\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{1}{k}}$, $k = 1, 2, \ldots.$*

Proof. By Lemma 1(ii), we have

$$\eta_{\|\cdot\|}(A_\otimes^{r+s}) = \eta_{\|\cdot\|}(A_\otimes^r \otimes A_\otimes^s) \leqslant \eta_{\|\cdot\|}(A_\otimes^r)\eta_{\|\cdot\|}(A_\otimes^s).$$

Fix k and let $l \geqslant k$. Write $l = mk + j$ with $0 \leqslant j \leqslant k - 1$. Then

$$\begin{aligned}\eta_{\|\cdot\|}(A_\otimes^l) &= \eta_{\|\cdot\|}(A_\otimes^{mk+j}) \\ &\leqslant \eta_{\|\cdot\|}(A_\otimes^{mk})\eta_{\|\cdot\|}(A_\otimes^j) \\ &\leqslant [\eta_{\|\cdot\|}(A_\otimes^k)]^m[\eta_{\|\cdot\|}(A_\otimes)]^j.\end{aligned}$$

So that

$$\begin{aligned}[\eta_{\|\cdot\|}(A_\otimes^l)]^{\frac{1}{l}} &\leqslant [\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{m}{l}}[\eta_{\|\cdot\|}(A_\otimes)]^{\frac{j}{l}} \\ &= [\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{1}{k}}[\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{-j}{lk}}[\eta_{\|\cdot\|}(A_\otimes)]^{\frac{j}{l}} \\ &\leqslant [\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{1}{k}}[\eta_{\|\cdot\|}(A_\otimes)]^{\frac{-j}{l}}[\eta_{\|\cdot\|}(A_\otimes)]^{\frac{j}{l}} \\ &= [\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{1}{k}}.\end{aligned}$$

Taking $l \to \infty$, one easily obtains

$$\hat{\eta}(A) \leqslant [\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{1}{k}}.$$

This completes the proof. $\square$

By Lemma 2, we have $\hat{\eta}(A) = \lim_{k\to\infty}[\eta_{\|\cdot\|}(A_\otimes^k)]^{\frac{1}{k}}$.

Lemma 3. *Let A be an $n \times n$ nonnegative matrix with $\mu(A) \leqslant 1$. Then there is a constant $M \geqslant 1$ such that for each $l > n$,*

$$[A_\otimes^l]_{ij} \leqslant M^n \quad [\mu(A)]^{l-n}$$

for all $1 \leqslant i, j \leqslant n$.

Proof. Let $\alpha = \max\{a_{ij} : 1 \leqslant i, j \leqslant n\}$. Put $M = \alpha$ if $\alpha \geqslant 1$, and $M = 1$ if $\alpha < 1$. Let i, j be fixed and let L be an l path from vertex i to vertex j in the digraph $\mathscr{D}(A)$ with $w(L) = [A_\otimes^l]_{ij}$. Since $l > n$, path L contains a circuit, moreover, at least $l - n$ its vertices are on a circuit. Let these circuits be $C_1, \ldots, C_k$ with length $l_1, \ldots, l_k$, hence $l_1 + \cdots + l_k \geqslant l - n$. The remaining vertices, so at most n vertices, form a path L' with weight less than or equal to M^n. Then

$$\begin{aligned}[A^l_{\otimes}]_{ij} &= w(L)\\ &= w(C_1)\cdots w(C_k)w(L')\\ &= [\hat{w}(C_1)]^{l_1}\cdots[\hat{w}(C_k)]^{l_k}w(L')\\ &\leqslant [\mu(A)]^{l_1}\cdots[\mu(A)]^{l_k}M^n\\ &\leqslant [\mu(A)]^{l-n}M^n \text{ (by } \mu(A)\leqslant 1).\end{aligned}$$

This completes the proof. □

Lemma 4. *Let A be an $n \times n$ nonnegative matrix and $\|\cdot\|$ be a monotone norm on $\mathbb{R}^n$. Then for each positive integer k, and positive numbers $a_0, a_1, \ldots, a_k$, the function*

$$\begin{aligned}\|x\|_s = {} & a_0\||x|\| + a_1\|A\otimes|x|\| + a_2\|A^2_{\otimes}\otimes|x|\|\\ & + \cdots + a_k\|A^k_{\otimes}\otimes|x|\|, \quad x\in\mathbb{R}^n\end{aligned}$$

is a monotone norm on $\mathbb{R}^n$.

Proof. It is clear that $\|x\|_s = 0 \Longleftrightarrow x = 0$. Let $\alpha \in \mathbb{R}$ be given. Then

$$\begin{aligned}\|\alpha x\|_s &= a_0\||\alpha x|\| + a_1\|A\otimes|\alpha x|\| + a_2\|A^2_{\otimes}\otimes|\alpha x|\| + \cdots + a_k\|A^k_{\otimes}\otimes|\alpha x|\|\\ &= |\alpha|a_0\||x|\| + |\alpha|a_1\|A\otimes|x|\| + |\alpha|a_2\|A^2_{\otimes}\otimes|x|\|\\ &\quad + \cdots + |\alpha|a_k\|A^k_{\otimes}\otimes|x|\|\\ &= |\alpha|\|x\|_s.\end{aligned}$$

Observe that for each $n \times n$ nonnegative matrix B and $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned}[B\otimes|x+y|]_i &= \max_{1\leqslant j\leqslant n} b_{ij}|x_j+y_j|\\ &\leqslant \max_{1\leqslant j\leqslant n}(b_{ij}|x_j| + b_{ij}|y_j|)\\ &\leqslant \max_{1\leqslant j\leqslant n} b_{ij}|x_j| + \max_{1\leqslant j\leqslant n} b_{ij}|y_j|\\ &= [B\otimes|x|]_i + [B\otimes|y|]_i.\end{aligned}$$

This implies that

$$B\otimes|x+y| \leqslant B\otimes|x| + B\otimes|y|.$$

Since $\|\cdot\|$ is a monotone norm, we have

$$\|B\otimes|x+y|\| \leqslant \|B\otimes|x| + B\otimes|y|\| \leqslant \|B\otimes|x|\| + \|B\otimes|y|\|.$$

Now we claim that $\|x+y\|_s \leqslant \|x\|_s + \|y\|_s$.

$$\begin{aligned}\|x+y\|_s &= a_0\||x+y|\| + a_1\|A\otimes|x+y|\| + a_2\|A^2_\otimes\otimes|x+y|\|\\ &\quad +\cdots+a_k\|A^k_\otimes\otimes|x+y|\|\\ &\leqslant a_0(\||x|\|+\||y|\|)+a_1(\|A\otimes|x|\|+\|A\otimes|y|\|)\\ &\quad +a_2(\|A^2_\otimes\otimes|x|\|+\|A^2_\otimes\otimes|y|\|)\\ &\quad +\cdots+a_k(\|A^k_\otimes\otimes|x|\|+\|A^k_\otimes\otimes|y|\|)\\ &= \|x\|_s+\|y\|_s.\end{aligned}$$

It is clear that $\|x\|_s = \||x|\|_s$. Thus $\|\cdot\|_s$ is an absolute norm, and hence a monotone norm. □

Theorem 2. *Let A be an $n\times n$ nonnegative matrix. Then the following statements are mutually equivalent.*

(i) $\eta_{\|\cdot\|}(A) < 1$ *for some norm* $\|\cdot\|$ *on* $\mathbb{R}^n$.
(ii) $\hat{\eta}(A) < 1$.
(iii) $\mu(A) < 1$.
(iv) $\lim_{k\to\infty} A^k_\otimes = 0$.

Proof. (i)⇒(ii). By Lemma 2, we have $\hat{\eta}(A) \leqslant \eta_{\|\cdot\|}(A) < 1$.

(ii)⇒ (iii). By Lemma 1(iii), for each k

$$\eta_{\|\cdot\|}(A^k_\otimes) \geqslant \mu(A^k_\otimes).$$

Since $\mu(A^k_\otimes) = [\mu(A)]^k$ (see [4, p. 29]), we have $\eta_{\|\cdot\|}(A^k_\otimes) \geqslant [\mu(A)]^k$. Thus $[\eta_{\|\cdot\|}(A^k_\otimes)]^{\frac{1}{k}} \geqslant \mu(A)$, and hence $\hat{\eta}(A) \geqslant \mu(A)$.

(iii)⇒(iv). By Lemma 3, there is a constant M such that for each $k > n$,

$$[A^k_\otimes]_{ij} \leqslant M^n[\mu(A)]^{k-n}$$

for all $1 \leqslant i, j \leqslant n$. Since $\mu(A) < 1$, we have $\lim_{k\to\infty} A^k_\otimes = 0$.

(iv)⇒(i). Since $\lim_{k\to\infty} A^k_\otimes = 0$, we have $\lim_{k\to\infty}[A^k_\otimes]_{ij} = 0$ for all $1 \leqslant i, j \leqslant n$. Then there is a positive integer N such that $[A^N_\otimes]_{ij} < 1$ for all $1 \leqslant i, j \leqslant n$. Let

$$\|x\|_s = \||x|\|_\infty + \|A\otimes|x|\|_\infty + \|A^2_\otimes\otimes|x|\|_\infty + \cdots + \|A^{N-1}_\otimes\otimes|x|\|_\infty.$$

By Lemma 4, $\|\cdot\|_s$ is a monotone norm on $\mathbb{R}^n$. Observe that

$$[|A\otimes x|]_i = |\max_{1\leqslant j\leqslant n} a_{ij}x_j| \leqslant \max_{1\leqslant j\leqslant n} a_{ij}|x_j| = [A\otimes|x|]_i.$$

Thus $|A\otimes x| \leqslant A\otimes|x|$, and hence we obtain that

$$\|B\otimes|A\otimes x|\|_\infty \leqslant \|B\otimes A\otimes|x|\|_\infty$$

for all nonnegative matrices B. So that

$$\begin{aligned}\|A\otimes x\|_s =& \||A\otimes x|\|_\infty + \|A\otimes|A\otimes x|\|_\infty + \|A_\otimes^2\otimes|A\otimes x|\|_\infty\\ &+\cdots+\|A_\otimes^{N-2}\otimes|A\otimes x|\|_\infty + \|A_\otimes^{N-1}\otimes|A\otimes x|\|_\infty\\ \leqslant&\|A\otimes|x|\|_\infty + \|A\otimes A\otimes|x|\|_\infty + \|A_\otimes^2\otimes A\otimes|x|\|_\infty\\ &+\cdots+\|A_\otimes^{N-2}\otimes A\otimes|x|\|_\infty + \|A_\otimes^{N-1}\otimes A\otimes|x|\|_\infty\\ =&\|A\otimes|x|\|_\infty + \|A_\otimes^2\otimes|x|\|_\infty\\ &+\cdots+\|A_\otimes^{N-1}\otimes|x|\|_\infty + \|A_\otimes^{N}\otimes|x|\|_\infty\\ <&\|A\otimes|x|\|_\infty + \|A_\otimes^2\otimes|x|\|_\infty+\cdots+\|A_\otimes^{N-1}\otimes|x|\|_\infty + \||x|\|_\infty\\ =&\|x\|_s.\end{aligned}$$

This implies that $\eta_{\|\cdot\|_s}(A) < 1$. □

Example 1. Consider the following 2×2 nonnegative matrix

$$A=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}.$$

We have $\mu(A)=\frac{1}{2}$ and $\rho(A)=1$, where $\rho(A)$ is the standard spectral radius of A. Then for any matrix norm $\|\cdot\|$, we have

$$\|A\| \geqslant \rho(A) = 1$$

(see, e.g., [9, p. 297]). Moreover,

$$\begin{aligned}\eta_{\|\cdot\|_\infty}(A) =& \max_{\|x\|_\infty=1, x\geqslant 0}\|A\otimes x\|_\infty = \max_{\|x\|_\infty=1, x\geqslant 0}\left\|\begin{bmatrix}\frac{1}{2}x_1\oplus\frac{1}{2}x_2\\ \frac{1}{2}x_1\oplus\frac{1}{2}x_2\end{bmatrix}\right\|_\infty\\ =&\frac{1}{2}=\mu(A)<1\end{aligned}$$

and $\lim_{k\to\infty} A_\otimes^k = 0$.

We establish the max algebra version of the Gelfand spectral radius formula as follows.

Theorem 3. *Let A be an $n\times n$ nonnegative matrix. Then $\hat{\eta}(A)=\mu(A)$.*

Proof. Observe that for each $\gamma>0$, $(\frac{A}{\gamma})_\otimes^k = \frac{1}{\gamma^k}A_\otimes^k$ for all $k=1,2,\ldots$ Therefore

$$\begin{aligned}\hat{\eta}\left(\frac{A}{\gamma}\right) =& \lim_{k\to\infty}\left[\eta_{\|\cdot\|}\left(\left(\frac{A}{\gamma}\right)_\otimes^k\right)\right]^{\frac{1}{k}}\\ =&\lim_{k\to\infty}\left[\eta_{\|\cdot\|}\left(\frac{1}{\gamma^k}A_\otimes^k\right)\right]^{\frac{1}{k}}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty}\left[\left(\max_{\|x\|=1,x\geqslant 0}\left\|\left(\frac{1}{\gamma^k}A_{\otimes}^k\right)\otimes x\right\|\right)\right]^{\frac{1}{k}}\\
&= \frac{1}{\gamma}\lim_{k\to\infty}\left[\left(\max_{\|x\|=1,x\geqslant 0}\|A_{\otimes}^k\otimes x\|\right)\right]^{\frac{1}{k}}\\
&= \frac{1}{\gamma}\hat{\eta}(A).
\end{aligned}
$$

Note that $\mu\left(\frac{A}{\gamma}\right)=\frac{1}{\gamma}\mu(A)$. Thus

$$
\begin{aligned}
&\hat{\eta}(A)<\gamma\\
\Longleftrightarrow\ &\hat{\eta}\left(\frac{A}{\gamma}\right)<1\\
\Longleftrightarrow\ &\mu\left(\frac{A}{\gamma}\right)<1 \text{ (by Theorem 2)}\\
\Longleftrightarrow\ &\mu(A)<\gamma.
\end{aligned}
$$

Hence $\hat{\eta}(A)=\mu(A)$.

This paper provides an alternative proof using the notion $\eta_{\|\cdot\|}(A)$ for an asymptotic formula which was proposed by Elsner and Van den Driessche's lemma [4]. □

Theorem 4. *Let A be an $n\times n$ nonnegative matrix and $\|\cdot\|$ be any matrix norm. Then $\mu(A)=\lim_{k\to\infty}\|A_{\otimes}^k\|^{\frac{1}{k}}$.*

Proof. Since $A\otimes x\leqslant Ax\leqslant nA\otimes x$ for all $x\geqslant 0$ and $\|\cdot\|_\infty$ is a monotone norm, we have

$$
\begin{aligned}
\lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_{\otimes}^k\otimes x\|_\infty\right)^{\frac{1}{k}} &\leqslant \lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_{\otimes}^k x\|_\infty\right)^{\frac{1}{k}}\\
&\leqslant \lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}n\|A_{\otimes}^k\otimes x\|_\infty\right)^{\frac{1}{k}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\hat{\eta}(A) &= \lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_{\otimes}^k\otimes x\|_\infty\right)^{\frac{1}{k}}\\
&\leqslant \lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_{\otimes}^k x\|_\infty\right)^{\frac{1}{k}}
\end{aligned}
$$

and

$$\begin{aligned}\lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_\otimes^k x\|_\infty\right)^{\frac{1}{k}} &\leqslant \lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|nA_\otimes^k\otimes x\|_\infty\right)^{\frac{1}{k}}\\ &=\lim_{k\to\infty} n^{\frac{1}{k}}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_\otimes^k\otimes x\|_\infty\right)^{\frac{1}{k}}\\ &=\lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_\otimes^k\otimes x\|_\infty\right)^{\frac{1}{k}}\\ &=\hat{\eta}(A).\end{aligned}$$

So that

$$\hat{\eta}(A)=\lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_\otimes^k x\|_\infty\right)^{\frac{1}{k}}. \tag{1}$$

For each nonnegative matrix B, we have

$$\begin{aligned}&\max_{\|x\|_\infty=1,x\geqslant 0}\|Bx\|_\infty \leqslant \max_{\|x\|_\infty=1}\|Bx\|_\infty \leqslant \max_{\|x\|_\infty=1}\|B|x|\|_\infty\\ &\quad=\max_{\|x\|_\infty=1,x\geqslant 0}\|Bx\|_\infty.\end{aligned}$$

Therefore,

$$\max_{\|x\|_\infty=1}\|Bx\|_\infty=\max_{\|x\|_\infty=1}\|B|x|\|_\infty=\max_{\|x\|_\infty=1,x\geqslant 0}\|Bx\|_\infty. \tag{2}$$

By Theorem 3 and Eqs. (1) and (2), we have

$$\begin{aligned}\mu(A)&=\hat{\eta}(A)\\ &=\lim_{k\to\infty}\left(\max_{\|x\|_\infty=1,x\geqslant 0}\|A_\otimes^k x\|_\infty\right)^{\frac{1}{k}}\\ &=\lim_{k\to\infty}\left(\max_{\|x\|_\infty=1}\|A_\otimes^k x\|_\infty\right)^{\frac{1}{k}}\\ &=\lim_{k\to\infty}\|A_\otimes^k\|_\infty^{\frac{1}{k}}.\end{aligned}$$

Since any two norms on a finite dimensional space are equivalent, there are $m, M > 0$ such that

$$m\|A_\otimes^k\| \leqslant \|A_\otimes^k\|_\infty \leqslant M\|A_\otimes^k\|.$$

Thus

$$\lim_{k\to\infty}\|A_\otimes^k\|_\infty^{\frac{1}{k}}=\lim_{k\to\infty}\|A_\otimes^k\|^{\frac{1}{k}}$$

and hence $\mu(A)=\lim_{k\to\infty}\|A_\otimes^k\|^{\frac{1}{k}}$. □

The following theorem is well known (see [1,7]), we give an alternative proof.

Theorem 5. *Let A be an $n \times n$ nonnegative matrix. Then*

$$\mu(A) \leqslant \rho(A) \leqslant n\mu(A).$$

Proof. By Theorem 4,

$$\begin{aligned}\mu(A) &= \lim_{k\to\infty} \|A^k_{\otimes}\|_{\infty}^{\frac{1}{k}}\\ &\leqslant \lim_{k\to\infty} \|A^k\|_{\infty}^{\frac{1}{k}} \ (\text{by } A^k_{\otimes} \leqslant A^k)\\ &= \rho(A).\end{aligned}$$

On the other hand,

$$\begin{aligned}n\mu(A) &= n\hat{\eta}(A)\\ &= n \lim_{k\to\infty}\left(\max_{\|x\|_{\infty}=1, x\geqslant 0} \|A^k_{\otimes} \otimes x\|_{\infty}\right)^{\frac{1}{k}}\\ &= n \lim_{k\to\infty}\left(\max_{\|x\|_{\infty}=1} \|A^k_{\otimes} \otimes |x|\|_{\infty}\right)^{\frac{1}{k}}\\ &= \lim_{k\to\infty}\left(\max_{\|x\|_{\infty}=1} \|n^k A^k_{\otimes} \otimes |x|\|_{\infty}\right)^{\frac{1}{k}}\\ &\geqslant \lim_{k\to\infty}\left(\max_{\|x\|_{\infty}=1} \|n^{k-1} A^k_{\otimes}|x|\|_{\infty}\right)^{\frac{1}{k}} \quad (\text{by } nA^k_{\otimes} \otimes |x| \geqslant A^k_{\otimes}|x|)\\ &\geqslant \lim_{k\to\infty}\left(\max_{\|x\|_{\infty}=1} \|A^k|x|\|_{\infty}\right)^{\frac{1}{k}} \quad (\text{by } n^{k-1}A^k_{\otimes} \geqslant A^k)\\ &= \lim_{k\to\infty}\left(\max_{\|x\|_{\infty}=1} \|A^k x\|_{\infty}\right)^{\frac{1}{k}} \quad (\text{by (2)})\\ &= \rho(A).\end{aligned}$$

Thus

$$\mu(A) \leqslant \rho(A) \leqslant n\mu(A). \qquad \square$$

Theorem 6. *Let A be an $n \times n$ nonnegative matrix. Then for any $\epsilon > 0$ there is a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $\eta_{\|\cdot\|}(A) \leqslant \hat{\eta}(A) + \epsilon$.*

Proof. Let $\epsilon > 0$ be given. Put $\alpha = \hat{\eta}(A) + \epsilon$. Choose N so that

$$\eta_{\|\cdot\|_{\infty}}(A^N_{\otimes}) \leqslant \alpha^N.$$

Define $\|\cdot\|_s$ by

$$\begin{aligned}\|x\|_s = {}& \||x|\|_\infty + \frac{1}{\alpha}\|A\otimes |x|\|_\infty + \frac{1}{\alpha^2}\|A^2_\otimes \otimes |x|\|_\infty + \cdots \\ &+ \frac{1}{\alpha^{N-1}}\|A^{N-1}_\otimes \otimes |x|\|_\infty .\end{aligned}$$

By Lemma 4, $\|\cdot\|_s$ is a monotone norm. Since $|A\otimes x| \leqslant A\otimes |x|$, we have

$$\begin{aligned}\|A\otimes x\|_s = {}& \||A\otimes x|\|_\infty + \frac{1}{\alpha}\|A\otimes |A\otimes x|\|_\infty + \frac{1}{\alpha^2}\|A^2_\otimes \otimes |A\otimes x|\|_\infty \\ &+ \cdots + \frac{1}{\alpha^{N-1}}\|A^{N-1}_\otimes \otimes |A\otimes x|\|_\infty \\ \leqslant {}& \|A\otimes |x|\|_\infty + \frac{1}{\alpha}\|A^2_\otimes \otimes |x|\|_\infty + \cdots + \frac{1}{\alpha^{N-1}}\|A^N_\otimes \otimes |x|\|_\infty \\ = {}& \alpha\left(\frac{1}{\alpha}\|A\otimes |x|\|_\infty + \frac{1}{\alpha^2}\|A^2_\otimes \otimes |x|\|_\infty + \cdots \right. \\ &\left. + \frac{1}{\alpha^{N-1}}\|A^{N-1}_\otimes \otimes |x|\|_\infty + \frac{1}{\alpha^N}\|A^N_\otimes \otimes |x|\|_\infty\right) \\ \leqslant {}& \alpha\left(\frac{1}{\alpha}\|A\otimes |x|\|_\infty + \frac{1}{\alpha^2}\|A^2_\otimes \otimes |x|\|_\infty + \cdots \right. \\ &\left. + \frac{1}{\alpha^{N-1}}\|A^{N-1}_\otimes \otimes |x|\|_\infty + \||x|\|_\infty\right) \\ = {}& \alpha\|x\|_s .\end{aligned}$$

Thus

$$\eta_{\|\cdot\|_s}(A) \leqslant \alpha = \hat{\eta}(A) + \epsilon. \qquad \square$$

Corollary 1. *Let A be an $n\times n$ nonnegative matrix. Then for any $\epsilon > 0$ there is a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that $\mu(A) \leqslant \eta_{\|\cdot\|}(A) \leqslant \mu(A)+\epsilon$.*

Acknowledgment

I want to thank the referee for his helpful comments and improvements.

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