Lyapunov Exponent for Products of Random Matrices

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1 Introduction

The use of random matrices in physics is widespread. One mainly deals in two areas

- Random Matrix Theory, where one is interested in the statistics of the eigenvalues of matrices, for which the elements are chosen according to some probabilistic rules. This theory was originally developed for nuclear physics, where the large number of particles makes a direct quantum mechanical computation of the spectra of the nuclei impossible. The spectra of heavy nuclei are often very irregular, but have well defined statistical properties. The spacing between two neighbour levels in the spectrum will for example for many nuclei be distributed according to a poissonian distribution. Such statistical properties can be derived from Random Matrix Theory. Nowadays Random Matrix Theory is also used in the description of statistical properties of quantum chaos.

- Lyapunov Exponents of Products of Random Matrices, where one is interested in the asymptotic growth of the eigenvalues of products of random matrices chosen according to some probability distribution. The Lyapunov exponents characterises the asymptotic exponential growth of such a product, and has many interesting physical applications. It will for example directly give the free energy for an Ising model with quenched disorder.

The subject of this report will be the second point, with emphasis put on applications to statistical mechanics of disordered systems. In the next section, we will define the Lyapunov exponents, and show its connection to statistical mechanics. Then we discuss and compare the various methods of obtaining numerical estimates of the Lyapunov exponents. These methods are Monte-Carlo simulation, weak-disorder expansion [1], the microcanonical method[2], and the cycle-expansion method[3]. Cycle expansions have already been very successful in obtaining non-perturbative results for for example chaotic dynamical systems[4]. For a recent interesting application, see [5]. After discussing the methods, we use the formalism developed on explicit examples, namely the 1-dimensional Ising model and the 2-dimensional Ising model on a strip with quenched disorder.

2 The Lyapunov Exponents

Consider the product of n $m \times m$ matrices

$$G_n = \prod_{i=1}^{n} T_i \tag{1}$$

where the matrices T_i are chosen from a discrete set $\{X_1, \dots, X_N\}$ with fixed probabilities $\{p_1, \dots, p_N\}$.

If the matrices were just 1×1 and positive, i.e. positive scalars, we would expect that

$$G_n \sim e^{n\gamma} \quad \text{for n} \to \infty \tag{2}$$

In this case, the Lyapunov exponent γ can easily be found by noting that the law of large numbers ensures that for almost all sequences $\{T_1, \dots, T_n, \dots\}$, we have:

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \ln G_n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln L_i = \overline{\ln X} \tag{3}$$

where $\overline{\ln X} = \sum_{i=1}^{N} p_i \ln X_i$. It is thus easy to characterise the asymptotic exponential growth of the product G_n in the 1×1 case. The problem now is that in general, we are dealing with non-commuting matrices, so that the discussion given above cannot immediately be used.

Nevertheless we might in general expect that the product (1) grows exponentially. This leads to following definition of the Lyapunov indices:

Let $\|\cdot\|$ be some norm on $\mathbb{R}^m$. The Lyapunov indices are defined by:

$$\gamma(x) = \lim_{n\to\infty} \frac{1}{n} \langle \ln \frac{\|G_n x\|}{\|x\|} \rangle \tag{4}$$

where x is some vector in $\mathbb{R}^m$, $x \neq 0$ and the average $\langle \ldots \rangle$ is taken over the ensemble M_n of all possible sequences of length n:

$$\langle \ldots \rangle \equiv \sum_{G\in M_n} p(G) \ldots \tag{5}$$

On finite-dimensional vector spaces, all norms are equivalent, so that if $\|\cdot\|_1$ is another norm on $\mathbb{R}^m$, then it is possible to find a constant k, such that

$$\frac{1}{k} \|\cdot\| \leq \|\cdot\|_1 \leq \ k \ \|\cdot\| \tag{6}$$

which immediately shows that the Lyapunov indices are independent of the norm $\|\cdot\|$ chosen.

Now it is easy to see that if $\gamma(x_1) < \gamma(x_2) < \ldots < \gamma(x_p)$ then the vectors $x_1, \ldots, x_p$ are linearly independent. The function $\gamma(x)$ can thus assume at most assume m different values. The maximal Lyapunov exponent γ is the largest value that $\gamma(x)$ can assume. It is given by

$$\begin{aligned} \gamma &= \sup_x \gamma(x) = \lim_{n\to\infty} \frac{1}{n} \langle \ln \sup_x \frac{\|G_n x\|}{\|x\|} \rangle = \lim_{n\to\infty} \frac{1}{n} \langle \ln \|G\| \rangle \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{G\in M_n} p(G) \ln \|G\| \end{aligned} \tag{7}$$

where in the last two equations $\|\cdot\|$ is now some matrix norm. The maximal Lyapunov exponent is of course independent of the matrix norm used. In fact, we can replace the matrix-norm by a weight $|\cdot|$, that to any matrix assigns a real number, as long as we can bound the weight like in (6). This fact will be used later.

The average in the definition of the Lyapunov exponents (4) and the maximal Lyapunov exponent (7) is not necessary. Indeed Oseledec[6] and Furstenberg[7] has proven that the limit

$$\lim_{n\to\infty} \frac{1}{n} \ln \|G_n\| \tag{8}$$

exists for almost all realizations of the product and is equal to γ, i.e. the product (1) is a non-random quantity.

Now for the class of positive matrices, i.e. matrices for which $(X_i)_{jk} > 0$ for all i, j, k, there is a simple asymptotic growth for products of random matrices. Indeed due to the ergodicity of matrix-multiplication, all components of G_n have the same growth-rate. In fact Ishitani[8] was able to prove that if all matrix elements are positive and there exists a constant C, such that for all matrices X_i, we have

$$1 \leq \frac{\max(X_i)_{jk}}{\min(X_i)_{jk}} \leq C \tag{9}$$

then with probability one, the limit

$$\lim_{n\to\infty} \frac{1}{n} \ln(G_n)_{ij} \tag{10}$$

exists and is equal to γ for all i, j. By summing over i and j inside the logarithm, we can thus write the maximal Lyapunov exponent γ in the following way:

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \langle \ln \operatorname{tr} G \rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{G\in M_n} p(G) \ln \operatorname{tr} G \tag{11}$$

Note that this formula also holds for those classes of matrices X_i, where all elements are greater than *or equal to* zero, but for which there exists an integer r such that X_i^r only have positive elements.

Using the above formula (11), the study of the physics of a number of disordered systems can be reduced to the study of product of random matrices. Consider for example the one-dimensional Ising model with periodic boundary conditions, random couplings J_i and random magnetic fields h_i. The Hamiltonian is given by

$$H = -\sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1} - \sum_{i=1}^{N} h_i \sigma_i \tag{12}$$

The partition function Z is the sum over all possible configurations of the system:

$$Z_N(\beta) = \sum_{\{\sigma\}} e^{-\beta H(\sigma)} \tag{13}$$

where β is the inverse temperature. This partition function can be written as

$$Z_N(\beta) = \mathrm{Tr}\left(\prod_{i=1}^{N} T_i\right) \tag{14}$$

where T_i is the transfer matrix given by:

$$T_i = \begin{pmatrix} e^{\beta(J_i+h_i)} & e^{\beta(-J_i+h_i)} \\ e^{-\beta(J_i+h_i)} & e^{\beta(J_i-h_i)} \end{pmatrix} \tag{15}$$

The free energy per spin in the limit $N \to \infty$ is then:

$$f(\beta) = -\beta^{-1} \lim_{N\to\infty} \frac{1}{N} \ln Z_N(\beta) = -\beta^{-1} \lim_{n\to\infty} \frac{1}{n} \ln \mathrm{Tr}\left(\prod_{i=1}^{N} T_i\right) \tag{16}$$

Averaging the free energy over disorder (quenched disorder), we obtain

$$f(\beta) = -\beta^{-1}\gamma(\beta) \tag{17}$$

The calculation of the free energy can thus be done by calculating the maximal Lyapunov exponent of the product of random transfer matrices. This is the fundamental result connecting statistical mechanics to the evaluation of maximal Lyapunov exponents for a product of random matrices.

Let us end this section by shortly discussing the case, where the matrices are from a continuous set $\{A(s)\}$ according to some probability-distribution $p(s)$. In this case, the maximal Lyapunov exponent would from (7) be given by:

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \int ds_1 \int ds_2 \cdots \int ds_n \ln \|A(s_1)A(s_2)\cdots A(s_n)\| \tag{18}$$

3 Computing the Lyapunov exponent

Now in general given a set of matrices and a set of probabilities, there is no closed formula giving the Lyapunov exponent. But nevertheless there exists various methods for computing estimates of the Lyapunov exponent. In this section, we will discuss these methods, and compare their strengths and weaknesses.

3.1 Monte-Carlo Simulation

In the Monte-Carlo method, one starts with a randomly chosen vector with unit norm. In each step, one chooses with the pre-assigned probabilities one of the given matrices and acts on the vector. The norm of the new vector is then calculated and the vector is normalised. By adding the logarithm of the norms at each step, one keeps track of the growth-rate of the original vector. Repeating the step enough times (usually 10^6 or 10^7) and averaging over many different realizations of this procedure, one obtains an estimate for the Lyapunov-exponent.

The only problem with the Monte-Carlo method is that it has a very slow convergence-rate. In practice, one cannot expect to obtain the same accuracy as some of the other methods.

3.2 The Microcanonical Method

The microcanonical estimate of the Lyapunov exponent is usually a very good one. Furthermore it is still the only available formula for computing the Lyapunov exponent, when it is a complex number [2]. The problem is that there is no systematic way of improving the estimate. In the following we will restrict the derivation of the microcanonical estimate to the case of two matrices, A chosen with probability p and B chosen with probability $q = 1 - p$.

Let us first define the microcanonical ensemble. The microcanonical ensemble is the ensemble of the different realizations of G_n, with precisely $a = pn$ matrices A and $b = qn$ matrices B, i.e. the number of matrices A and B are not allowed to fluctuate from the mean value. As the Lyapunov exponent is a non-random quantity, the average in the microcanonical ensemble $\langle \cdot \rangle_m$ is the same as in the usual 'canonical' ensemble $\langle \cdot \rangle$:

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \langle \ln \|G_n\| \rangle = \lim_{n\to\infty} \frac{1}{n} \langle \ln \|G_n\| \rangle_m \tag{19}$$

Even though the average is the same, the finite-n fluctuations will depend on the ensemble chosen. It is clear that the finite-n fluctuations in the microcanonical ensemble will be much smaller than in the usual ensemble, and only due to the non-commutativity of the matrices, because the fluctuations due to the variation of the number of each matrix are gone.

If now there were no non-commutativity fluctuations, then we would have

$$\langle \ln \|G_n\| \rangle_m = \ln \|\langle G_n \rangle_m\| \tag{20}$$

The reason is that if there is no non-commutativity fluctuations, then there is no finite-n fluctuations of the Lyapunov exponent as all realizations of G_n in the microcanonical ensemble have the same value, because A and B commute. Let for instance A and B be two scalars. Then

$$\frac{1}{n} \ln \|\langle G_n \rangle_m\| = \frac{1}{n} \ln |A^{pn} B^{qn}| = p \ln A + q \ln B = \gamma \tag{21}$$

When the matrices A and B do not commute, the right hand side of (20) is still a very good approximation to the left hand side, as the finite-n fluctuations are small in the microcanonical ensemble. So to a very good approximation, we have

$$\gamma \simeq \lim_{n \to \infty} \frac{1}{n} \ln \|\langle G_n \rangle_m\| \tag{22}$$

Now the right hand side can in fact be computed analytically. We have:

$$\langle G_n \rangle_m = \frac{a!}{n!} \frac{\partial^b}{\partial x^b} (A + xB)^n |_{x=0} \tag{23}$$

Let $\lambda_1(x)$ be the largest eigenvalue of the matrix $A + xB$. The matrix $(A + xB)^n$ can then be written $\lambda_1(x) D(x)$, where $D(x)$ is a matrix whose elements are of order 1. The average (23) can now using Cauchys theorem be written as an integral in the complex plane:

$$\langle G_n \rangle_m = \frac{a!\, b!}{2\pi i n!} \int_{\mathcal{C}} dx \frac{\lambda_1(x)^n D(x)}{x^{b+1}} \tag{24}$$

where $\mathcal{C}$ is a contour around 0. In the limit of large n, the integral can be estimated by the steepest descent method:

$$\int_{\mathcal{C}} dx \frac{\lambda_1(x)^n}{x^{qn}} \frac{D(x)}{x} = \text{const} \cdot e^{n(\ln \lambda_1(\bar{x}) - q \ln \bar{x})} \tag{25}$$

where $\bar{x}$ is the saddle-point, which is the solution of the equation:

$$x \frac{\partial \lambda_1(x)}{\partial x} = q \lambda_1(x) \tag{26}$$

Now using Stirlings approximation $x! = \sqrt{2\pi x}x^x e^{-x}$ for $n!$, $a!$ and $b!$, we get

$$\frac{a!\,b!}{n!} = \sqrt{\frac{2\pi ab}{n}} e^{n-a-b} \frac{a^a b^b}{n^n} = \sqrt{\frac{2\pi ab}{n}} p^{pn} q^{qn} \tag{27}$$

The microcanonical estimate for the Lyapunov exponent is thus

$$\gamma \simeq \lim_{n\to\infty} \frac{1}{n} \ln \| \langle G_n \rangle_m \| = p \ln p + q \ln q + \ln \lambda_1(\bar{x}) - q \ln \bar{x} \tag{28}$$

From the above, we see that the calculation of the microcanonical Lyapunov exponent in general requires a calculation of the maximal eigenvalue of the matrix $A + xB$ for all x, and of the saddle-point equation (26).

3.3 Weak-Disorder Expansion

The weak-disorder expansion method for calculating the Lyapunov exponents is very efficient in some cases. By weak disorder, one means that all the matrices T_i in the product (1) can be written as a small perturbation away from some average matrix $\bar{X}$:

$$T_i = \bar{X} + \epsilon Y_i \tag{29}$$

The Lyapunov exponents is then calculated as a perturbation expansion in ϵ. The weak-disorder expansion method is thus most applicable, when the different matrices only differ by a small amount. In practice, this means for example that the weak-disorder expansion can only be used to obtain thermodynamical quantities for the Ising model at small values of the inverse temperature β.

The derivation of weak-disorder expansion formula for the maximal Lyapunov exponent goes as follows: Assume that the average of Y_i vanishes, in the sense that

$$\bar{Y}_i \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i = 0 \tag{30}$$

If this is not satisfied, one can always include the average of Y_i in the matrix $\bar{X}$, thereby reducing the problem to a case, where (30) is satisfied.

Now in order for the weak disorder expansion to work, the largest eigenvalue of the average matrix $\bar{X}$ has to be non-degenerate – we will see later why. Furthermore we assume, that $\bar{X}$ can be diagonalized. Because of this condition, we can always work in the basis, where $\bar{X}$ is diagonal and has the following form:

$$\bar{X} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_m \end{pmatrix} \tag{31}$$

with

$$|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_m| \tag{32}$$

Let now U be a randomly chosen vector of dimension m, and define a new vector W by

$$W = G_n U = \left(\prod_{i=1}^{n} T_i \right) U \tag{33}$$

The maximal Lyapunov exponent is given by the exponential growth of W. One possible measure of this is the rate of growth of the *first* component of W[1]. The maximal Lyapunov exponent is therefore

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \ln \frac{W_1}{U_1} = \lim_{n\to\infty} \frac{1}{n} \ln (G_n)_{11} \tag{34}$$

To illustrate how the calculations are done, let us expand G_n to order ϵ^2:

$$G_n = (\mathcal{I} + \epsilon C_n + \epsilon^2 D_n) \bar{X}^n \tag{35}$$

where

$$\begin{aligned} C_n &= \sum_{i=1}^{n} \bar{X}^{i-1} Y_i \bar{X}^{-i} \\ D_n &= \sum_{i<j}^{n} \bar{X}^{i-1} Y_i \bar{X}^{j-i-1} Y_j \bar{X}^{-j} \end{aligned} \tag{36}$$

[1] This method can easily be generalised to calculating the sum of the first p Lyapunov exponents, but we will not need this. See the article by Derrida, Mecheri and Picard[1]

We have thus

$$\begin{aligned}\ln((G_n)_{11}) &= n\ln\lambda_1 + \ln(1+\epsilon(C_n)_{11} + \epsilon^2(D_n)_{11}) \\ &= n\ln\lambda_1 + \epsilon(C_n)_{11} + \epsilon^2((D_n)_{11} - \frac{1}{2}((C_n)_{11})^2) + O(\epsilon^3) \quad (37)\end{aligned}$$

where from (36), we have

$$\begin{aligned}(C_n)_{11} &= \sum_{i=1}^{n} \frac{Y_i}{\lambda_1} \\ (D_n)_{11} &= \sum_{i=1}^{n}\sum_{k=1}^{m} \lambda_1^{i-j}\lambda_k^{j-i}\frac{(Y_i)_{1k}(Y_j)_{k1}}{\lambda_1\lambda_k} \quad (38)\end{aligned}$$

Now we have of course

$$\lim_{n\to\infty}\frac{1}{n}(C_n)_{11} = 0 \quad (39)$$

because of the condition (30). Furthermore, it is easy to see that

$$\lim_{n\to\infty}\frac{1}{n}(D_n)_{11} = 0 \quad (40)$$

because of (30) and (32). And it is also easily seen that

$$\lim_{n\to\infty}\frac{1}{n}((C_n)_{11})^2 = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{(Y_i)_{11}(Y_i)_{11}}{\lambda_1\lambda_1} \equiv \frac{1}{\lambda_1^2}\langle Y_{11}^2\rangle \quad (41)$$

We have thus proved that to second order in ϵ, we have:

$$\gamma = \lambda_1 - \frac{\epsilon^2}{2\lambda_1^2}\langle Y_{11}^2\rangle \quad (42)$$

It is easy to extend this calculation up to fourth order in ϵ. When doing this, one obtains the following result[1]

$$\begin{aligned}\gamma &= \ln\lambda_1 - \frac{\epsilon^2}{2\lambda_1^2}\langle Y_{11}^2\rangle + \frac{\epsilon^3}{3\lambda_1^3}\langle Y_{11}^3\rangle - \frac{\epsilon^4}{4\lambda_1^4}\langle Y_{11}^4\rangle \\ &\quad - \frac{\epsilon^4}{2\lambda_1^2}\sum_{i,j>1}\frac{\langle Y_{1i}Y_{1j}\rangle\langle Y_{i1}Y_{j1}\rangle}{\lambda_1^2 - \lambda_i\lambda_j} + \frac{\epsilon^4}{\lambda_1^3}\sum_{i>1}\frac{\langle Y_{1i}Y_{11}\rangle\langle Y_{i1}Y_{11}\rangle}{\lambda_1-\lambda_i} + O(\epsilon^5) \quad (43)\end{aligned}$$

Here we see why we have to assume that the largest eigenvalue of $\bar{X}$ has to be non-degenerate, as the differences of the largest eigenvalue and the other eigenvalues occur in the denominator. We expect as mentioned above that this formula is quite accurate for cases, where the random matrices only differ by a small amount.

3.4 Cycle-Expansion method

The cycle-expansion formula is to this day the most efficient formula for computing the maximal Lyapunov-exponent at least for hyperbolic matrices. The essential point noted by Ronnie Mainieri in [3] was that, one could use the weight $|\cdot|$ given by:

$$|G| = \lim_{n\to\infty} |\mathrm{tr} G^n|^{1/n} \tag{44}$$

and obtain rapid convergence. This weight is trivially cyclic: $|AB| = |BA|$, because of the trace in the definition. It is also multiplicative in the sense that

$$|G^p| = \lim_{n\to\infty} |\mathrm{tr} G^{pn}|^{1/n} = \left(\lim_{np\to\infty} |\mathrm{tr} G^{pn}|^{1/(np)}\right)^p = |G|^p \tag{45}$$

as is easily verified.

Note though that the limit in (44) might not exist for matrices G, for which the eigenvalues with norm equal to the spectral radius of G, $r(G) = \max\{|\lambda_i|| \lambda_i \text{eigenvalue of } G\}$, are different. In all other cases, the limit exists, and the weight of G is equal to the norm of the largest eigenvalue of G, i.e. the spectral radius $r(G)$. The weight thus works for, for example a case with eigenvalues $2, 2$ and 1, as considered in [3], but generally not for cases, where for example the largest eigenvalues are two complex conjugate numbers. These problems will not occur for transfer matrices for statistical mechanics systems, because all such transfer matrices are either positive or there exists a power of the transfer matrix, that is positive. In this case Perrons theorem on positive matrices guarantees that the largest eigenvalue is non-degenerate, and consequently, there is no problem with the definition (44).

Now using this weight, equation (7) thus gets replaced by:

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \langle \ln |G| \rangle \tag{46}$$

Let us rewrite this formula as:

$$\gamma = \partial_\alpha \lim_{n\to\infty} \frac{1}{n} \langle\, |G|^\alpha \,\rangle|_{\alpha=0} \tag{47}$$

Now defining the quantity,

$$Z_n(\alpha) = \langle\, |G_n|^\alpha \,\rangle = \sum_{G\in M_n} p(G)|G|^\alpha \tag{48}$$

the Lyapunov-exponent can be written as

$$\gamma = \lim_{n\to\infty} \frac{1}{n} Z_n'(0) \tag{49}$$

The quantity $Z_n(\alpha)$ will grow exponentially as $n \to \infty$. It is therefore convenient to define the function $L_n(\alpha)$ by

$$Z_n(\alpha) = e^{nL_n(\alpha)} \tag{50}$$

where $L_n(\alpha) \to L(\alpha)$ for $n \to \infty$. As $Z_n(0) = 1$ for all n, we have $L_n(0) = 0$ for all n. Furthermore from equation (49), we see that $L'(0) = \gamma$. Now consider the generating functional for the quantities $Z_n(\alpha)/n$

$$\zeta^{-1}(z,\alpha) = e^{-\sum_{n=1}^{\infty} \frac{1}{n} z^n Z_n(\alpha)} = e^{-\sum_{n=1}^{\infty} \frac{1}{n}(ze^{L(\alpha)})^n} \tag{51}$$

This is the dynamical zeta-function for a statistical mechanics system with free energy given by $L(\alpha)$ (see [9]). It has a zero at the radius of convergence of the series in the exponent. This radius $\hat{z}(\alpha)$ is given by the condition

$$\hat{z}(\alpha)e^{L(\alpha)} = 1 \tag{52}$$

i.e. $\hat{z}(0) = 1$ and $\hat{z}'(0) = -\gamma$. For $\alpha = 0$, the inverse zeta-function can be explicitly summed

$$\zeta^{-1}(z,0) = e^{-\sum_{n=1}^{\infty} \frac{1}{n} z^n} = e^{\ln(1-z)} = 1 - z \tag{53}$$

From (52), we see that for all α, we have the relation

$$\zeta^{-1}(\hat{z},\alpha) = 0 \tag{54}$$

Differentiating this relation with respect to α and putting α to 0 now gives:

$$\begin{aligned} 0 &= \frac{d}{d\alpha}\zeta^{-1}(\hat{z},\alpha)|_{\alpha=0} = \partial_\alpha \zeta^{-1}(1,0) + \partial_z \zeta^{-1}(1,0)\hat{z}'(0) \\ &= \partial_\alpha \zeta^{-1}(1,0) + \gamma \end{aligned} \tag{55}$$

where we have used (52) and (53). We have thus the following relation

$$\gamma = -\partial_\alpha \zeta^{-1}(1,0) \tag{56}$$

which we shall use in a moment.

3.4.1 Cycle-expanding the Zeta-function

We will now show that due to the multiplicative nature of the weight (44), it is possible to write the inverse zeta-function as a cycle-expansion[9]:

Let M be the set of all possible matrix products of finite length $M = \bigcup_n M_n$. Using the definition of $Z_n(\alpha)$ (48), the inverse zeta-function can then be written as

$$\zeta^{-1}(z,\alpha) = e^{-\sum_{G\in M}\frac{1}{n(G)}t(G)} \tag{57}$$

where $n(G)$ is the number of matrices in the product G and $t(G)$ is of the form

$$t_G = z^{n(G)}p(G)|G|^\alpha \tag{58}$$

Note now that this t_G is both cyclic, i.e. invariant under cyclic rotations of the matrices in the product G and multiplicative, i.e. $t_{G^n} = t_G^n$. This is because the weight $|\cdot|$ is both cyclic and multiplicative as shown earlier. Furthermore the probability $p(G)$ is trivially cyclic and multiplicative as it is just the product of the probabilities of the individual matrices in the product.

Let us define a prime matrix product, as a product that cannot be written as the repeat of a product with smaller length, and let P denote the set of equivalence classes of all possible prime products, where two products are considered as equivalent if they differ only by a cyclic rotation. Now because t_G is cyclic and multiplicative, we can write the exponent in (57) as the sum over repeats of equivalence classes of prime products

$$\sum_{G\in M}\frac{t_G}{n(G)} = \sum_{G_p\in P}\sum_{r=1}^{\infty} n(G_p)\frac{t_{G_p}^r}{rn(G_p)} = \sum_{G_p\in P}\sum_{r=1}^{\infty}\frac{t_{G_p}^r}{r} = -\sum_{G_p\in P}\ln(1-t_{G_p}) \tag{59}$$

where we have used that we get $n(G_p)$ identical contributions from the cyclic rotations of G_p^r. Putting this into the zeta-function (57), we get:

$$\zeta^{-1}(z,\alpha) = \prod_{G\in P}(1-t_G) = \prod_{G\in P}(1-z^{n(G)}p(G)|G|^\alpha) \tag{60}$$

Now we wish to expand this into a power-series in z. Introduce the set P^* as the set of all subsets of P, i.e. the set of pseudocycles, and let μ be a Möbius-function on this set. The Möbius function acts on elements in P^*, i.e. subsets of P and gives the parity of the number of elements in the subset: Let H be an element of P^* consisting of the elements $p_1, \dots, p_h$, then

$$\mu(H) = \mu(\{p_1, \dots, p_h\}) = (-1)^h \tag{61}$$

Using this, we can expand the inverse zeta-function in a cycle-expansion as:

$$\zeta^{-1}(z, \alpha) = \sum_{H \in P^*} \mu(H) z^{n(H)} p(H) |H|^\alpha \tag{62}$$

where $n(H) = \sum_{G \in H} n(G)$, $p(H) = \prod_{G \in H} p(G)$ and $|H|^\alpha = \prod_{G \in H} |G|^\alpha$. Now differentiating with respect to α and setting $\alpha = 0$ and $z = 1$, we get the Lyapunov-exponent:

$$\gamma = - \sum_{H \in P^*} \mu(H) p(H) \ln |H| \tag{63}$$

This is the fundamental expression for the Lyapunov-exponent derived in [3]. If there are only two matrices, A with probability p and B with probability $q = 1 - p$, the first few terms in (63) are:

$$\begin{aligned} \gamma \;=\; & p \ln |A| + q \ln |B| + pq(\ln |AB| - \ln |A||B|) \\ & + ppq(\ln |AAB| - \ln |A||AB|) + pqq(\ln |ABB| - \ln |AB||B|) \\ & + \dots \end{aligned} \tag{64}$$

The first two terms are the fundamental contributions, while the other terms are the curvature corrections[10]. The curvature corrections are combinations of longer cycles together with their shorter shadowing approximants. Now the convergence of the cycle-expansion hinges on whether the longer cycles are *well* shadowed by their shorter approximants. Proving the rapid convergence of (63) is thus equivalent to verifying that $\ln |RS^n|$ is well approximated by $\ln |RS^{n-1}||S|$, where R and S are arbitrary prime products of matrices. This will be done for the simple case of two by two matrices in the next section.

To use formula (63) in practice, one has first to construct a list of all prime-cycles up to some given order n. Then one computes the norm of the corresponding matrices by calculating the absolute value of the maximal eigenvalue. Then one sums $p(H) \ln |H|$ for all pseudo-cycles H of combined length smaller than or equal to n. This generates the n'th order approximation to the Lyapunov-exponent $\gamma^{(n)}$:

$$\gamma^{(n)} = \frac{1}{n} \sum_{G \in M_n} p(G) \ln |G| = \sum_{H \in P_n^*} p(H) \ln |H| \tag{65}$$

where P_n^* is the set of all pseudocycles of length smaller than or equal to n.

3.4.2 Verifying the Convergence

In this section, we will for two by two matrices verify that the longer cycles are indeed well shadowed by their shorter approximants. Well approximated means that for arbitrary prime products R and S, we have that

$$\ln|RS^n| - \ln|RS^{n-1}||S| = \ln\frac{|RS^n|}{|RS^{n-1}||S|} \tag{66}$$

should go to zero exponentially as $n \to \infty$. The norm is independent of the basis chosen, so we choose a basis, where S is diagonal with eigenvalues s_1 and s_2, where $|s_1| \geq |s_2|$. In this basis, we have

$$R = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;; \quad S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \tag{67}$$

The product RS^n are thus given by:

$$RS^n = \begin{bmatrix} as_1^n & bs_2^n \\ cs_1^n & ds_2^n \end{bmatrix} \tag{68}$$

The norm of RS^n is the largest eigenvalue λ_n of the matrix. This is given by

$$\begin{aligned} \lambda_n &= \frac{1}{2}\left(\mathrm{tr}RS^n + \sqrt{(\mathrm{tr}RS^n)^2 - 4\det(RS^n)}\right) \\ &= \frac{s_1^n}{2}\left(a + d\epsilon^n + a\sqrt{1 + \frac{-2ad\epsilon^n + 4bc\epsilon^n + d^2\epsilon^{2n}}{a^2}}\right) \end{aligned} \tag{69}$$

where $\epsilon = s_2/s_1$. As $n \to \infty$, we can expand (69) in ϵ^n to get

$$\lambda_n = \frac{s_1^n}{2}\left(a + d\epsilon^n + a\left(1 + \frac{-2ad\epsilon^n + 4bc\epsilon^n}{2a^2}\right)\right) = as_1^n(1 + \frac{bc}{a^2}\epsilon^n) \tag{70}$$

Inserting this into (66), we get

$$\begin{aligned} \ln\frac{|RS^n|}{|RS^{n-1}||S|} &= \ln\frac{\lambda_n}{\lambda_{n-1}u} = \ln\frac{1 + \frac{bc}{a^2}\epsilon^n}{1 + \frac{bc}{a^2}\epsilon^{n-1}} = \ln(1 - \frac{bc}{a^2}\epsilon^{n-1} + O(\epsilon^n)) \\ &\simeq -\frac{bc}{a^2}\left(\frac{s_2}{s_1}\right)^{n-1} \end{aligned} \tag{71}$$

which goes to zero exponentially fast for $n \to \infty$ if $|s_1| > |s_2|$.

This verifies the convergence of the cycle expansion (63) at least for two by two matrices. From the discussion given above, we see the absolutely fundamental role played by the choice of the weight $|\cdot|$. It is precisely the use of this weight that guarantees rapid convergence of the cycle-expansion, and thus makes the method extremely practical for a numerical evaluation of the Lyapunov-exponent.

3.5 Refinement of the Cycle-Expansion Method

So far we have seen that the cycle-expansion converges rapidly as the number of included cycles increases. This makes the cycle-expansion formula very practical for the numerical evaluation of the Lyapunov-exponent. Nevertheless the cycle-expansion formula is somewhat complex, as it is a sum over all pseudocycles of $p(H)\ln|H|$, and thus involves a lot of terms. In this section, we will derive another cycle-expansion formula that is much shorter than (63), and non-perturbative in character, in contrast to (63), which is perturbative in the sense that taking longer cycles into account just adds small corrections to the formula. This new formula makes it possible to include longer cycles and thus get more precise results without increasing the computer-time.

The derivation of the formula is almost trivial: Every cycle of length n is either a prime-cycle, or it can be written as a product of some prime-cycle of length smaller than n. We thus have:

$$\sum_{G\in M_n} = \sum_{G_p\in P}\sum_{r=1}^{\infty} n_p \delta_{n,n_p r} \tag{72}$$

because each prime-cycle G_p contributes n_p times to the sum on the left hand side. Now using that the weight $|\cdot|$ is multiplicative, we get

$$\begin{aligned}\gamma^{(n)} &= \frac{1}{n}\sum_{G\in M_n} p(G)\ln|G| = \frac{1}{n}\sum_{G_p\in P}\sum_{r=1}^{\infty} n_p p(G_p^r)\ln|G_p^r|\delta_{n,n_p r} \\ &= \sum_{n_p|n}\frac{n_p}{n} p(G_p)^{n/n_p}\ln|G_p|^{n/n_p} = \sum_{n_p|n} p(G_p)^{n/n_p}\ln|G_p| \end{aligned}\tag{73}$$

This is the final formula! It will of course give the same result as (63), but it is considerably quicker to evaluate, because it is only a sum over the prime-cycles, for which the length divides n. A drastic simplification occurs, when n is a prime number, because then only prime-cycles of length 1 (the fundamental cycles) and prime-cycles of length n contributes. The prime-cycles with lengths in-between all cancel out! Let us verify this cancellation in an explicit example. Consider the simple example of two matrices, A chosen with probability p and B chosen with probability q. The third order approximation is given by (see 64):

$$\begin{aligned}
\gamma^{(3)} &= p\ln|A| + q\ln|B| + pq(\ln|AB| - \ln|A||B|) \\
&\quad + ppq(\ln|AAB| - \ln|A||AB|) + pqq(\ln|ABB| - \ln|AB||B|) \\
&= (p - pq - ppq)\ln|A| + (q - qp - qqp)\ln|B| \\
&\quad + (pq - ppq - pqq)\ln|AB| + ppq\ln|AAB| + pqq\ln|ABB| \\
&= p^3\ln|A| + q^3\ln|B| + ppq\ln|AAB| + pqq\ln|ABB| \qquad (74)
\end{aligned}$$

which is exactly, what one would get from (73). We have thus derived a formula, that is much faster to evaluate than the original cycle-expansion formula, especially when n is a prime number. This allows us to evaluate easily high order approximations to the Lyapunov-exponent.

3.6 Illustrative Example

In this section, we will test all the methods on an explicit example. This example has no immediate physical interpretation, but serves only as a test. In the next chapter, we will discuss the Ising model with random couplings. The matrices are two 2×2-matrices chosen with equal probabilities of the following form

$$\begin{bmatrix} a-1 & 1 \\ -a & -1 \end{bmatrix} \qquad (75)$$

The matrix has for $a > 4$ two real eigenvalues, and for $a < 4$ two complex conjugate eigenvalues. Let us consider the case, where the two matrices has $a = 4.2$ and $a = 4.4$ respectively. The matrices are then well suited for the computation of the Lyapunov-exponent for all four methods. For the Monte-Carlo method, we averaged over 100 realizations of a product of one million matrices. The average matrix in the weak-disorder expansion $\bar{X}$ is of the form (75) with $a = 4.3$, and ϵ is thus 0.1. The expansion was then carried out to order $O(\epsilon^4)$. For the cycle-expansion method, we used formula (73) including cycles up to length 23. The results are shown in table 1. We

Method	Estimate
Monte-Carlo	0.5372
Weak Disorder	0.53717
Microcanonical	0.53717
Cycle-Expansion	0.537170203372

Table 1: The estimated Lyapunov exponent for the four methods. The number have been rounded, so that the errors are limited to the last digit.

clearly see that the cycle-expansion method is by far the best method in this case. One could in principle obtain the same accuracy with the weak disorder expansion, if one continued the expansion to eleventh order, which is practically impossible. Furthermore as it is easy to generate all prime-cycles of some given order, one can easily extend the cycle-expansion result to higher order (here it is of course advantageous to choose n to be some prime number, as we then only need to generate prime-cycles of length n to get the n'th order result).

We have studied the convergence-rate of the cycle-expansion method by calculating the n'th order approximation to the Lyapunov exponent for $n = 1, \ldots, 23$. In figure 1, we have plotted the differences $\Delta(n)$ between the 23'rd order approximation and the n'th order approximation:

$$\Delta(n) = \gamma^{(23)} - \gamma^{(n)}; \quad n = 1, \ldots, 22 \tag{76}$$

as a function of the maximal cycle length n.

The straight line in the figure indicates, that the convergence is exponentially fast. This was to be expected from the result (71). A fit to the curve of the constants b and c in the function: $\Delta(n) = b \cdot c^{-n}$ gives the following:

$$\Delta(n) = 0.009 \cdot 2.91^{-n} \tag{77}$$

From (71), one might expect that the constant c should be in fact be equal to s_1/s_2, where s_1 and s_2 are respectively the highest and the lowest eigenvalue of the *average* matrix, corresponding to $a = 4.3$. The ratio s_1/s_2 for $a = 4.3$ is in fact equal to 2.95, so it fits all very well.

If $a < 4$, then we have to complex conjugate eigenvalues, and the weight $|\cdot|$ will thus fail. In such a case, we can still use and make sense of the results derived from the microcanonical method, as done in [2] for the case

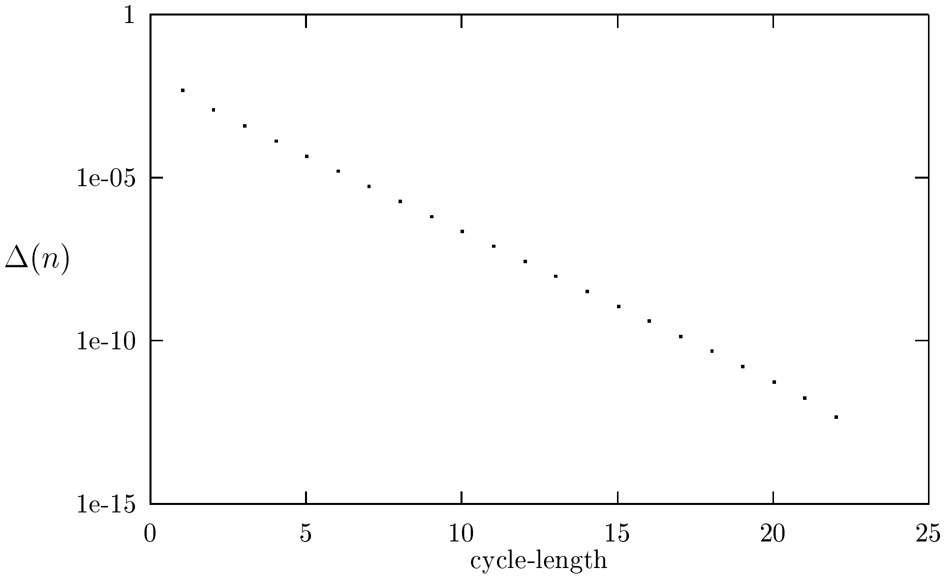

Figure 1: Difference between the 23'rd order approximation and the n'th order approximation as a function of the maximal cycle length. Note the y-scale is a log-scale.

of Anderson-localisation. But as already mentioned, such a failure of the cycle-expansion method will not occur for the transfer matrices, that we use in the statistical mechanics of disordered systems.

4 Ising Model with Random Couplings

In the section, we will consider the 1-dimensional Ising model and the Ising model on a strip with quenched disorder. As shown previously, the free energy is connected to the Lyapunov exponent through the relation (17). From this it is easy to show that the heat-capacity C_V, which is

$$C_V = -T \frac{\partial^2 F}{\partial T^2} \tag{78}$$

can be expressed in terms of the Lyapunov exponent like this:

$$C_V = \beta^2 \gamma''(\beta) \tag{79}$$

We will use this relation in a moment.

In principle, all four methods could be used to derive the thermodynamical properties of these models. In practice, only the cycle-expansion method gives good results. The weak-disorder expansion will namely only applicable for very small values of the inverse temperature, because only in this case there will be a small parameter to expand in. It would take an unreasonably long time with the Monte-Carlo method to obtain the precision needed to compute the second derivative of the Lyapunov exponent, so this method is thus also not practically applicable. And it seems also impossible to calculate analytically the Lyapunov exponent for arbitrary temperature using the microcanonical method. So all in all to get reasonable results, we have to use the cycle-expansion method. As we shall see, it is actually possible to get precision needed to find the heat-capacity with the cycle-expansion method.

4.1 1-dimensional Ising model with quenched disorder

In this section, we discuss the 1-dimensional Ising model with quenched disorder. More specifically, we will look at the case, where the magnetic field is zero, but the coupling between the neighbouring spins J_i is a random variable, that with equal probability $\frac{1}{3}$ can take the values 1, $\frac{3}{2}$ and 2. We thus get the following three transfer matrices:

$$T_1 = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}; \quad T_2 = \begin{pmatrix} e^{3\beta/2} & e^{-3\beta/2} \\ e^{-3\beta/2} & e^{3\beta/2} \end{pmatrix}; \quad T_3 = \begin{pmatrix} e^{2\beta} & e^{-2\beta} \\ e^{-2\beta} & e^{2\beta} \end{pmatrix}$$

The Lyapunov exponent for random products of these transfer matrices can easily be evaluated using the cycle-expansion method. In figure 2, we have shown the Lyapunov exponent as a function of the inverse temperature β.

Using the discrete version of formula (79) above, we can also calculate the heat-capacity of the model. We use the simple 3-point formula for the numerical computation of second derivatives:

$$\gamma''(\beta) \simeq \gamma(\beta+1) - 2\gamma(\beta) + \gamma(\beta-1) \tag{80}$$

In figure 3, we have shown the heat-capacity as function of the inverse temperature. Note again that we have no problem with obtaining thermodynamic quantities at *any* temperature with the cycle-expansion method.

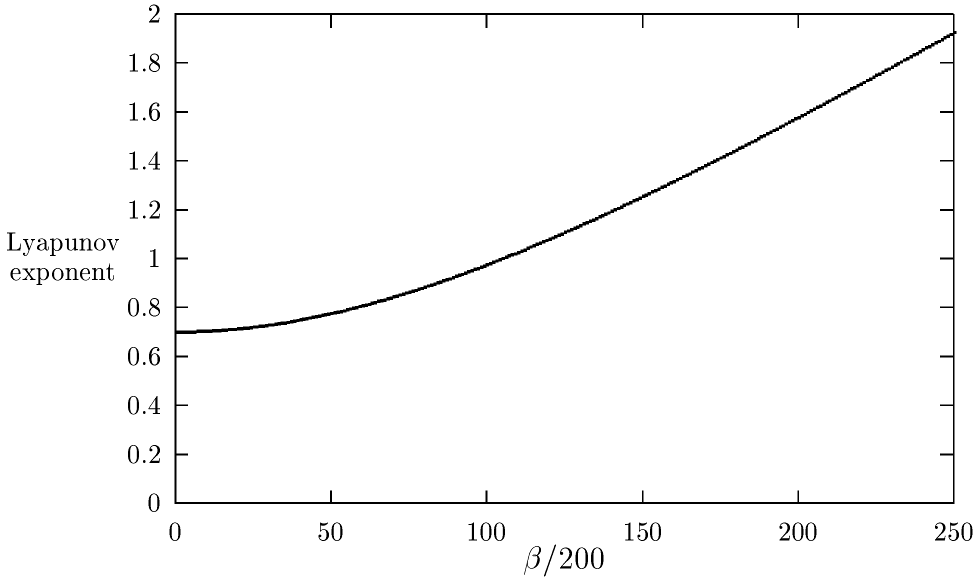

Figure 2: Lyapunov exponent as a function of the inverse temperature β for the one-dimensional Ising model with quenched disorder

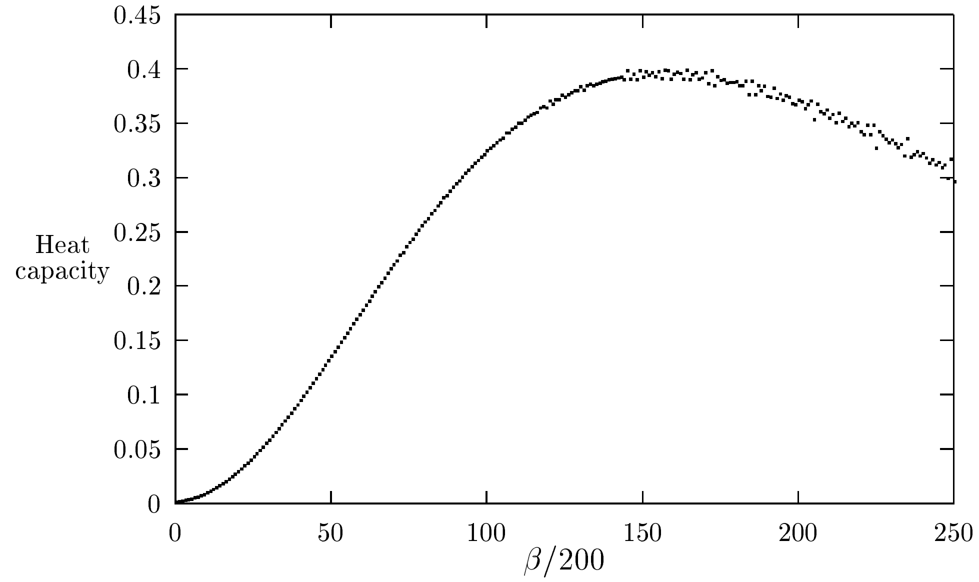

Figure 3: Heat capacity as a function of the inverse temperature β for the one-dimensional Ising model with quenched disorder

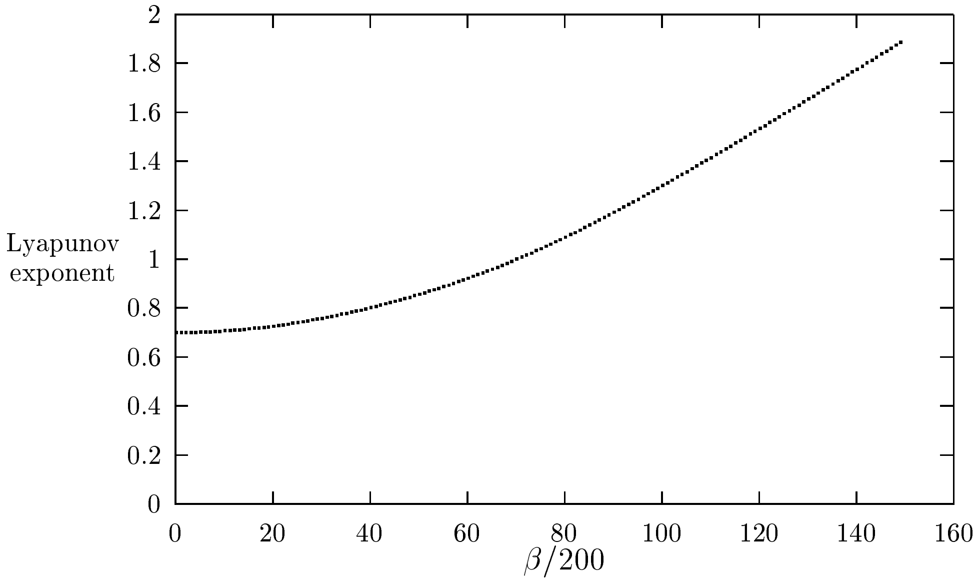

Figure 4: Lyapunov exponent as a function of the inverse temperature β for the Ising model on a strip with quenched disorder. Width: $L = 10$.

4.2 Ising model on a strip

In this section, we will discuss the Ising model on a strip with quenched disorder. We expect that this model will be more interesting than the one-dimensional Ising-model, because we know from statistical mechanics, that the two-dimensional Ising model has a phase-transition at some finite temperature, where the heat-capacity diverges. We expect to see some signs of this phase-transition in this model.

The width of the strip is denoted by L, and we use helicital boundary conditions, i.e. the last spin in each row is connected to first spin in the next row. Again we consider a case, where the magnetic field is zero, and we assume that the random coupling J can only take the values 1 and 2. There will therefore be four different transfer-matrices, as each of the two links in each cell can take two values. Again it is easy, using the cycle-expansion method, to calculate the Lyapunov exponent and thus the free energy. We have calculated the Lyapunov exponent for widths L up to 10.

In figure 4, we have shown the Lyapunov exponent as a function of the inverse temperature β for the case $L = 10$, and in figure 5, we have shown the heat-capacity as function of the inverse temperature in the two cases $L = 5$

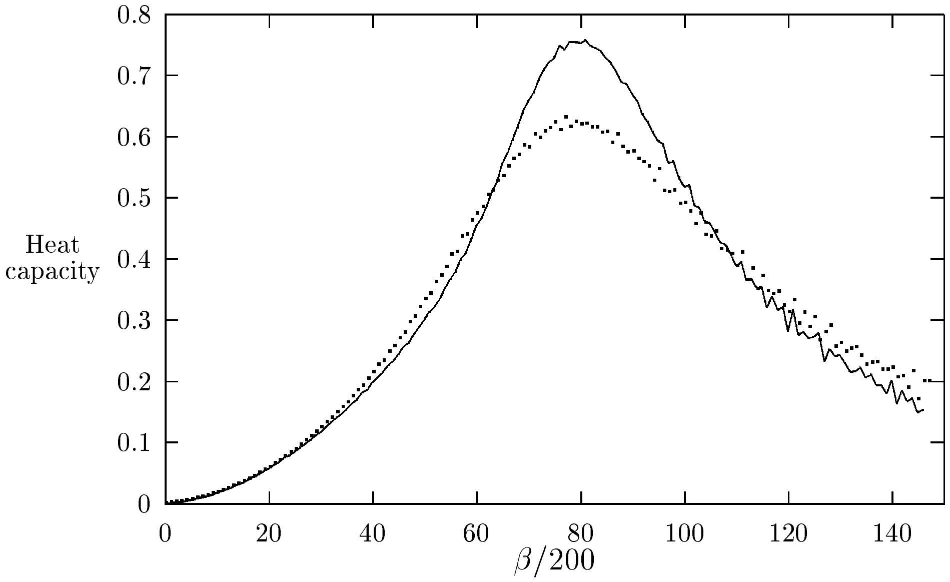

Figure 5: Heat capacity as a function of the inverse temperature β for the Ising model on a strip with quenched disorder. Full line: $L = 10$, Dotted line: $L = 5$.

and $L = 10$. In figure 5, we clearly see some interesting behaviour: This heat-capacity has a (rather) sharp maximum at about $\beta = 0.4$, and the peak grows, when we go to a wider lattice (going from $L = 5$ to $L = 10$). We would of course expect that the peak would have grown even more, if we had gone to higher L. Clearly such a behaviour comes from the phase-transition in the two-dimensional Ising model.

5 Conclusions

In this report, we have defined the Lyapunov exponent of a product of random matrices, and described different methods for estimating this. The Cycle-Expansion method turned out in most cases to be a very powerful method for calculating the Lyapunov exponent, with an exponential convergence-rate. A formula were derived expressing the quenched disorder free energy in terms of the Lyapunov exponent for the product of the random transfer matrices. Using this, it was possible to extract thermodynamical quantities for the one-dimensional Ising model and the Ising model on a strip with quenched disorder.

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