APPENDIX B

GAUSSIAN INTEGRALS

One variable

We begin by evaluating the following Gaussian integral

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda}{2}x^2\right) dx.$$

This is easily done by considering the square of the integral, and the
ing to polar coordinates:

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{\lambda}{2}x^2 - \frac{\lambda}{2}y^2\right) dx\, dy \\ &= \int_0^{\infty}\int_0^{2\pi} \exp\left(-\frac{\lambda}{2}r^2\right) r\, dr\, d\theta \\ &= \pi \int_0^{\infty} \exp\left(-\frac{\lambda}{2}u\right) du \\ &= \frac{2\pi}{\lambda} \end{aligned}$$

where we have changed variables first using $x = r\cos\theta, y = r\sin$
using $r^2 = u$. Taking the square root we finally obtain

$$\int_{-\infty}^{\infty} \exp\left(-\frac{\lambda}{2}x^2\right) dx = \left(\frac{2\pi}{\lambda}\right)^{1/2}.$$

Several variables

Consider the evaluation of the W-dimensional Gaussian integral

$$I_W = \int \exp\left(-\frac{1}{2}\mathbf{w}^{\mathrm{T}}\mathbf{A}\mathbf{w}\right) d\mathbf{w}$$

A is a $W \times W$ real symmetric matrix, $\mathbf{w}$ is a W-dimensional vector, and integration is over the whole of $\mathbf{w}$-space. In order to evaluate this integral it venient to consider the eigenvector equation for $\mathbf{A}$ in the form

$$\mathbf{A}\mathbf{u}_k = \lambda_k \mathbf{u}_k. \tag{B.5}$$

A is real and symmetric, we can choose the eigenvectors to form a complete normal set

$$\mathbf{u}_k^{\mathrm{T}} \mathbf{u}_l = \delta_{kl} \tag{B.6}$$

cussed in Appendix A. We can then expand the vector $\mathbf{w}$ as a linear combination of the eigenvectors

$$\mathbf{w} = \sum_{k=1}^{W} \alpha_k \mathbf{u}_k. \tag{B.7}$$

integration over the weight values $dw_1 \ldots dw_W$ can now be replaced by an gration over $d\alpha_1 \ldots d\alpha_W$. The Jacobian of this change of variables is given

$$J = \det\left(\frac{\partial w_i}{\partial \alpha_k}\right) = \det\left(u_{ki}\right) \tag{B.8}$$

re u_{ki} is the ith element of the vector $\mathbf{u}_k$, and 'det' denotes the determinant. u_{ki} are also the elements of a matrix $\mathbf{U}$ whose columns are given by the $\mathbf{u}_k$, which is an orthogonal matrix, i.e. it satisfies $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$, since its columns orthonormal. Thus

$$J^2 = \{\det(\mathbf{U})\}^2 = \det(\mathbf{U}^{\mathrm{T}})\det(\mathbf{U}) = \det(\mathbf{U}^{\mathrm{T}}\mathbf{U}) = \det(\mathbf{I}) = 1 \tag{B.9}$$

hence $|J| = 1$. Using the orthonormality of the $\mathbf{u}_k$ we have

$$\mathbf{w}^{\mathrm{T}}\mathbf{A}\mathbf{w} = \sum_{k=1}^{W} \lambda_k \alpha_k^2. \tag{B.10}$$

various integrals over the α_k now decouple, and so we can write

$$I_W = \prod_{k=1}^{W} \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda_k \alpha_k^2}{2}\right) d\alpha_k. \tag{B.11}$$

the result (B.3) we obtain

$$I_W = \prod_{k=1}^{W} \left(\frac{2\pi}{\lambda_k} \right)^{1/2}$$

Since the determinant of a matrix is given by the product of its eige

$$|\mathbf{A}| = \prod_{k=1}^{W} \lambda_k,$$

we finally obtain

$$I_W = (2\pi)^{W/2} |\mathbf{A}|^{-1/2}.$$

Inclusion of linear term

In deriving the distribution of network outputs within the Bayesian fra
Exercise 10.2, we need to consider a more general form of the Gaussia
which has an additional linear term, of the form

$$I_W = \int \exp\left(-\frac{1}{2} \mathbf{w}^{\mathrm{T}} \mathbf{A} \mathbf{w} + \mathbf{h}^{\mathrm{T}} \mathbf{w} \right) d\mathbf{w}.$$

Again, it is convenient to work in terms of the eigenvectors of $\mathbf{A}$. We fi
h_k to be the projections of $\mathbf{h}$ onto the eigenvectors

$$h_k = \mathbf{h}^{\mathrm{T}} \mathbf{u}_k.$$

This again leads to a set of decoupled integrals over the α_k of the form

$$I_W = \prod_{k=1}^{W} \int_{-\infty}^{\infty} \exp\left(-\frac{\lambda_k \alpha_k^2}{2} + h_k \alpha_k \right) d\alpha_k.$$

Completing the square in the exponent, we have

$$-\frac{\lambda_k \alpha_k^2}{2} + h_k \alpha_k = -\frac{\lambda_k}{2} \left(\alpha_k - \frac{h_k}{\lambda_k} \right)^2 + \frac{h_k^2}{2\lambda_k}.$$

If we now change integration variables to $\widetilde{\alpha}_k = \alpha_k - h_k/\lambda_k$, we again
product of integrals which can be evaluated using (B.3) to give

$$I_W = (2\pi)^{W/2} |\mathbf{A}|^{-1/2} \exp\left(\sum_{k=1}^{W} \frac{h_k^2}{2\lambda_k} \right).$$

we now apply $\mathbf{A}^{-1}$ to both sides of (B.5) we see that $\mathbf{A}^{-1}$ has the same eigenvectors as $\mathbf{A}$, but with eigenvalues λ_k^{-1}:

$$\mathbf{A}^{-1}\mathbf{u}_k = \lambda_k^{-1}\mathbf{u}_k. \tag{B.20}$$

s, using (B.6) and (B.16), we see that

$$\mathbf{h}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{h} = \sum_k \frac{h_k^2}{\lambda_k}. \tag{B.21}$$

ng this result in (B.19) we obtain our final result:

$$I_W = (2\pi)^{W/2}|\mathbf{A}|^{-1/2}\exp\left(\frac{1}{2}\mathbf{h}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{h}\right). \tag{B.22}$$