CHAPTER 2

Isoperimetric problems

2.1. History

One of the earliest problems in geometry was the isoperimetric problem, which was considered by the ancient Greeks. The problem is to find, among all closed curves of a given length, the one which encloses the maximum area. The basic isoperimetric problem for graphs is essentially the same. Namely, remove as little of the graph as possible to separate out a subset of vertices of some desired “size”. Here the size of a subset of vertices may mean the number of vertices, the number of edges, or some other appropriate measure defined on graphs. A typical case is to remove as few edges as possible to disconnect the graph into two parts of almost equal size. Such problems are usually called *separator* problems and are particularly useful in a number of areas including recursive algorithms, network design, and parallel architectures for computers, for example [**178**].

In a graph, a subset of edges which disconnects the graph is called a *cut*. Cuts arise naturally in the study of connectivity of graphs where the sizes of the disconnected parts are not of concern. Isoperimetric problems examine optimal relations between the size of the cut and the sizes of the separated parts. Many different names are used for various versions of isoperimetric problems (such as the conductance of a graph, the isoperimetric number, etc.). The concepts are all quite similar, but the differences are due to the varying definitions of cuts and sizes.

We will consider two types of cuts. A *vertex-cut* is a subset of vertices whose removal disconnects the graph. Similarly, an *edge-cut* is a subset of edges whose removal separates the graph. The size of a subset of vertices depends on either the number of vertices or the number of edges. Therefore, there are several combinations.

Roughly speaking, isoperimetric problems involving edge-cuts correspond in a natural way to Cheeger constants in spectral geometry. The formulation and the proof techniques are very similar. Cheeger constants were studied in the thesis of Cheeger [**48**], but they can be traced back to Polyá and Szegö [**211**]. We will follow tradition and call the discrete versions by the same names, such as the Cheeger constant and the Cheeger inequalities.

2.2. The Cheeger constant of a graph

Before we discuss isoperimetric problems for graphs, let us first consider a measure on subsets of vertices. The typical measure assigns weight 1 to each vertex, so the measure of a subset is its number of vertices. However, this implies that all vertices have the same measure. For some problems, this is appropriate only for regular graphs and does not work for general graphs. The measure we will use here takes into consideration the degree of a vertex. For a subset S of the vertices of G, we define vol S, the *volume* of S, to be the sum of the degrees of the vertices in S:

$$\text{vol } S = \sum_{x \in S} d_x,$$

for $S \subseteq V(G)$.

Next, we define the *edge boundary* ∂S of S to consist of all edges with exactly one endpoint in S, i.e.,

$$\partial S = \{\{u, v\} \in E(G) : u \in S \text{ and } v \notin S\}.$$

Let $\bar{S}$ denote the complement of S, i.e., $\bar{S} = V - S$. Clearly, $\partial S = \partial \bar{S} = E(S, \bar{S})$ where $E(A, B)$ denotes the set of edges with one endpoint in A and one endpoint in B. Similarly, we can define the *vertex boundary* δS of S to be the set of all vertices v not in S but adjacent to some vertex in S, i.e.,

$$\delta S = \{v \notin S : \{u, v\} \in E(G), u \in S\}.$$

We are ready to pose the following questions:

Problem 1: For a fixed number m, find a subset S with $m \le \text{vol } S \le \text{vol } \bar{S}$ such that the edge boundary ∂S contains as few edges as possible.

Problem 2: For a fixed number m, find a subset S with $m \le \text{vol } S \le \text{vol } \bar{S}$ such that the vertex boundary δS contains as few vertices as possible.

Cheeger constants are meant to answer exactly the questions above. For a subset $S \subset V$, we define

$$h_G(S) = \frac{|E(S, \bar{S})|}{\min(\text{vol } S, \text{vol } \bar{S})}. \tag{2.1}$$

The *Cheeger constant* h_G of a graph G is defined to be

$$h_G = \min_S h_G(S). \tag{2.2}$$

In some sense, the problem of determining the Cheeger constant is equivalent to solving Problem 1, since

$$|\partial S| \ge h_G \text{ vol } S.$$

We remark that G is connected if and only if $h_G > 0$. We will only consider connected graphs. In a similar manner, we define the analogue of (2.1) for "vertex expansion" (instead of "edge expansion"). For a subset $S \subseteq V$, we define

$$g_G(S) = \frac{\text{vol } \delta(S)}{\min(\text{vol } S, \text{vol } \bar{S})} \tag{2.3}$$

and

$$g_G = \min_S g_G(S). \tag{2.4}$$

For regular graphs, we have

$$g_G(S) = \frac{|\delta(S)|}{\min(|S|, |\bar{S}|)}.$$

We define for a graph G (not necessarily regular)

$$\bar{g}_G(S) = \frac{|\delta(S)|}{\min(|S|, |\bar{S}|)}$$

and

$$\bar{g}_G = \min_S \bar{g}_G(S).$$

We remark that $\bar{g}$ is the corresponding Cheeger constant when the measure for each vertex is taken to be 1. More general measures will be considered later in Section 2.6. We note that both g_G and $\bar{g}_G$ are concerned with the vertex expansion of a graph and are useful for many problems.

2.3. The edge expansion of a graph

In this section, we focus on the fundamental relations between eigenvalues and the Cheeger constant. We first derive a simple upper bound for the eigenvalue λ_1 in terms of the Cheeger constant of a connected graph.

LEMMA 2.1. $2h_G \geq \lambda_1$.

PROOF. We choose f based on an optimum edge cut C which achieves h_G and separates the graph G into two parts, A and B:

$$f(v) = \begin{cases} \dfrac{1}{\text{vol } A} & \text{if } v \text{ is in } A, \\ -\dfrac{1}{\text{vol } B} & \text{if } v \text{ is in } B. \end{cases}$$

By substituting f into (1.2), we have the following:

$$\begin{aligned} \lambda_1 &\leq |C|(1/\text{vol } A + 1/\text{vol } B) \\ &\leq \frac{2|C|}{\min(\text{vol } A, \text{vol } B)} \\ &= 2h_G. \end{aligned}$$

□

Now, we will proceed to give a relatively short proof of an inequality in the opposite direction, so that we will have altogether

$$2h_G \geq \lambda_1 > \frac{h_G^2}{2}.$$

This is the so-called *Cheeger inequality* which often provides an effective way for bounding the eigenvalues of the graph.

THEOREM 2.2. *For a connected graph G,*

$$\lambda_1 > \frac{h_G^2}{2}.$$

PROOF. We consider the harmonic eigenfunction f of $\mathcal{L}$ with eigenvalue λ_1. We order vertices of G according to f. That is, relabel the vertices so that $f(v_i) \leq f(v_{i+1})$, for $1 \leq i \leq n-1$. Without loss of generality, we may assume

$$\sum_{f(v)<0} d_v \geq \sum_{f(u)\geq 0} d_u.$$

For each i, $1 \leq i \leq |V|$, we consider the cut

$$C_i = \{\{v_j, v_k\} \in E(G) \ : \ 1 \leq j \leq i < k \leq n\}.$$

We define α by

$$\alpha = \min_{1\leq i\leq n} \frac{|C_i|}{\min(\sum_{j\leq i} d_j, \sum_{j>i} d_j)}.$$

It is clear that $\alpha \geq h_G$. We consider the set V_+ of vertices v satisfying $f(v) \geq 0$ and the set E_+ of edges $\{u, v\}$ in G with either u or v in V_+. We define

$$g(x) = \begin{cases} f(x) & \text{if } u \in V_+, \\ 0 & \text{otherwise.} \end{cases}$$

We now have

$$\begin{aligned}
\lambda_1 &= \frac{\sum_{v\in V_+} f(v) \sum_{\{u,v\}\in E_+} (f(v)-f(u))}{\sum_{v\in V_+} f^2(v) d_v} \\
&> \frac{\sum_{\{u,v\}\in E_+} (g(u)-g(v))^2}{\sum_{v\in V} g^2(v) d_v} \\
&= \frac{\sum_{\{u,v\}\in E_+} (g(u)-g(v))^2 \sum_{\{u,v\}\in E_+} (g(u)+g(v))^2}{\sum_{v\in V} g^2(v) d_v \sum_{\{u,v\}\in E_+} (g(u)+g(v))^2} \\
&\geq \frac{(\sum_{u\sim v} |g^2(u)-g^2(v)|)^2}{2(\sum_v g^2(v)d_v)^2} \\
&\geq \frac{(\sum_i |g^2(v_i)-g^2(v_{i+1})|\ |C_i|)^2}{2(\sum_v g^2(v)d_v)^2} \\
&\geq \frac{(\sum_i (g^2(v_i)-g^2(v_{i+1}))\alpha \sum_{j\leq i} d_j)^2}{2(\sum_v g^2(v)d_v)^2} \\
&\geq \frac{\alpha^2}{2} \\
&\geq \frac{h_G^2}{2}.
\end{aligned}$$

This completes the proof of Theorem 2.2. □

We will state an improved version of Theorem 2.2 which however has a slightly more complicated proof.

Theorem 2.3. *For any connected graph G, we always have*

$$\lambda_1 > 1 - \sqrt{1 - h_G^2}.$$

Proof. From the proof of Theorem 2.2, we have

$$\begin{aligned}
\lambda_1 &= \frac{\sum_{v\in V_+} f(v) \sum_{u\sim v}(f(v)-f(u))}{\sum_{v\in V_+} f^2(v) d_v} \\
&> \frac{\sum_{\{u,v\}\in E_+} (g(u)-g(v))^2}{\sum_{v\in V} g^2(v) d_v} = W.
\end{aligned}$$

Also, we have

$$\begin{aligned}
W &= \frac{(\sum_{\{u,v\}\in E_+} (g(u)-g(v))^2)\cdot(\sum_{\{u,v\}\in E_+} (g(u)+g(v))^2)}{(\sum_{v\in V} g^2(v)d_v)\cdot(\sum_{\{u,v\}\in E_+} (g(u)+g(v))^2)} \\
&\geq \frac{(\sum_{u\sim v} |g^2(u)-g^2(v)|)^2}{(\sum_v g^2(v)d_v)\cdot(2\sum_v g^2(v)d_v - W\sum_v g^2(v)d_v)} \\
&\geq \frac{(\sum_i |g^2(v_i)-g^2(v_{i+1})|\ |C_i|)^2}{(2-W)(\sum_v g^2(v))^2 d_v} \\
&\geq \frac{(\sum_i (g^2(v_i)-g^2(v_{i+1}))\alpha \sum_{j\leq i} d_j)^2}{(2-W)(\sum_v g^2(v))^2 d_v} \\
&\geq \frac{\alpha^2}{2-W}.
\end{aligned}$$

This implies that

$$W^2 - 2W + \alpha^2 \leq 0.$$

Therefore we have

$$\begin{aligned}
\lambda_1 > W &\geq 1-\sqrt{1-\alpha^2} \\
&\geq 1-\sqrt{1-h_G^2}.
\end{aligned}$$

□

We give a self-contained statement which has already been established in the above proof:

Corollary 2.4. *In a graph G with eigenfunction f associated with λ_1, we define, for each v,*

$$C_v = \{\{u,u'\} \in E(G)\ : f(u) \leq f(v) < f(u')\}$$

and

$$\alpha = \min_v \frac{|C_v|}{\min(\sum_{\substack{u \\ f(u) \le f(v)}} d_u, \sum_{\substack{u \\ f(u) > f(v)}} d_u)}.$$

Then,

$$\lambda_1 > 1 - \sqrt{1 - \alpha^2}.$$

One immediate consequence is an improvement on the range of λ_1. For any connected (simple) graph G, we have

$$h_G \ge \frac{2}{\text{vol } G}.$$

Using Cheeger's inequality, we have

$$\lambda_1 > \frac{1}{2}(\frac{2}{\text{vol } G})^2 \ge \frac{2}{n^4}.$$

This lower bound is somewhat weaker than that in Lemma 1.9.

EXAMPLE 2.5. For a path P_n, the Cheeger constant is $1/\lfloor (n-1)/2 \rfloor$. As shown in Example 1.4, the eigenvalue λ_1 of P_n is $1 - \cos \frac{\pi}{n-1} \approx \frac{\pi^2}{2(n-1)^2}$. This shows that the Cheeger inequality in Theorem 2.2 is best possible up to within a constant factor.

EXAMPLE 2.6. For an n-cube Q_n, the Cheeger constant is $2/n$ which is equal to λ_1 (see Example 1.6). Therefore the inequality in Lemma 2.1 is sharp to within a constant factor.

Jerrum and Sinclair [**164, 226**] first used Cheeger's inequality as a main tool in deriving polynomial approximation algorithms for enumerating permanents and for other counting problems. The reader is referred to [**225**] for the related computational aspects of the Cheeger inequality.

2.4. The vertex expansion of a graph

The proofs of upper and lower bounds for the modified Cheeger constant g_G associated with vertex expansion are more complicated than those for edge expansion. This is perhaps due to the fact that the definition of h_G is in a way more natural and better scaled. Nevertheless, vertex expansion comes up often in many settings and it is certainly interesting in its own right.

Since $g_G \ge h_G$, we have

$$2g_G \ge \lambda_1.$$

For a general graph G, the eigenvalue λ_1 can sometimes be much smaller than $g_G^2/2$. One such example is given by joining two complete subgraphs by a matching. Suppose n is the total number of vertices. The eigenvalue λ_1 is no more than $8/n^2$, but g_G is large.

Still, it is desirable to have a lower bound for λ_1 in terms of g_G. Here we give a proof which is adapted from the argument first given by Alon [**5**].

Theorem 2.7. *For a connected graph G,*

$$\lambda_1 > \frac{g_G^2}{4d + 2dg_G},$$

where d denotes the maximum degree of G.

Proof. We follow the definition in the proof of Theorem 2.2. We have

$$\begin{aligned}
\lambda_1 &= \frac{\sum_{v \in V_+} \sum_{u \sim v} (f(v) - f(u)) f(v)}{\sum_{v \in V_+} d_v f^2(v)} \\
&= \frac{\sum_{\substack{u \sim v \\ u,v \in V_+}} (f(v) - f(u))^2 + \sum_{\substack{u \sim v, v \in V_+ \\ u \notin V_+}} f(v)(f(v) - f(u))}{\sum_{v \in V_+} d_v f^2(v)} \\
&> \frac{\sum_{u \sim v} (g(u) - g(v))^2}{\sum_v g^2(v) d_v},
\end{aligned}$$

Now we use the max-flow min-cut theorem [**124**] as follows. Consider the network with vertex set $\{s,t\} \cup X \cup Y$ where s is the source, t is the sink, $X = V_+$ and Y is a copy of $V(G)$. The directed edges and their capacities are given as follows:

- For every u in X, the directed edge (s,u) has capacity $(1 + g_G)d_u$.
- For every $u \in X, v \in Y$, there is a directed edge (u,v) with capacity d_v if $\{u,v\} \in E$ or u is labelled by the same vertex as v in G.
- For every $v \in Y$, the directed edge (v,t) has capacity d_v.

To check that this network has its min-cut of size $(1+g_G)\text{vol}\ V_+$, let C denote a cut separating s and t. Let $X_1 = \{x \in X : \{s,x\} \notin C\}$ and $Y' = \{y \in Y : \{y,t\} \in C\}$. Then C separates X_1 from $Y \setminus Y'$. Therefore the total capacity of the cut C is at least the sum of capacities of the edges $\{s,x\}, s \in X - X_1$, the edges (u,v), $u \in X_1$ and $v \in X_1 \cup \delta X_1 \setminus Y'$ and edges (y,t), $y \in Y'$. Since $\text{vol}\ (X_1 \cup \delta X_1) \geq (1+g_G)\text{vol}\ X_1$, the total capacity of the cut is at least

$$\begin{aligned}
& (1 + g_G)\text{vol}(V_+ - X_1) + \text{vol}(X_1 \cup \delta X_1 \setminus Y') + \text{vol} Y' \\
\geq\ & (1 + g_G)\text{vol}(V_+ - X_1) + (1 + g_G)\text{vol}\ X_1 \\
=\ & (1 + g_G)\text{vol}\ V_+.
\end{aligned}$$

Since there is a cut of size $(1 + g_G)\text{vol}\ V_+$, we have proved that the min-cut is of size equal to $(1 + g_G)\text{vol}\ V_+$. By the max-flow min-cut theorem, there exists a flow function $F(u,v)$ for all directed edges in the network so that $F(u,v)$ is bounded above by the capacity of (u, v) and for each fixed $x \in X$ and $y \in Y$, we have

$$\begin{aligned}\sum_v F(x,v) &= (1+g_G)d_x,\\ \sum_v F(v,y) &\leq d_y.\end{aligned}$$

Then,

$$\begin{aligned}&\sum_{\{u,v\}\in E} F^2(u,v)(f_+(u)+f_+(v))^2\\ \leq\ & 2\sum_{\{u,v\}\in E} F^2(u,v)(f_+^2(u)+f_+^2(v))\\ =\ & 2\sum_v f_+^2(v)(\sum_{\substack{u\\ \{u,v\}\in E}} F^2(u,v)+\sum_{\substack{u\\ \{u,v\}\in E}} F^2(v,u))\\ \leq\ & 2\sum_v f_+^2(v)(d_v^2+(1+g_G)dd_v)\\ \leq\ & 2d(2+g_G)\sum_v f_+^2(v)d_v.\end{aligned}$$

Also,

$$\begin{aligned}&\sum_{\{u,v\}\in E} F(u,v)(f_+^2(u)-f_+^2(v))\\ =\ & \sum_u f_+^2(u)(\sum_{\substack{v\\ \{u,v\}\in E}} F(u,v)-\sum_{\substack{v\\ \{u,v\}\in E}} F(v,u))\\ \geq\ & g_G\sum_v f_+^2(v)d_v.\end{aligned}$$

Combining the above facts, we have

$$\begin{aligned}
\lambda_1 &\geq \frac{\displaystyle\sum_{\{u,v\}\in E}(f_+(u)-f_+(v))^2}{\displaystyle\sum_v f_+^2(v)d_v}\\
&= \frac{\displaystyle\sum_{\{u,v\}\in E}(f_+(u)-f_+(v))^2\sum_{\{u,v\}\in E}F^2(u,v)(f_+(u)+f_+(v))^2}{\displaystyle\sum_v f_+^2(v)d_v\sum_{\{u,v\}\in E}F^2(u,v)(f_+(u)+f_+(v))^2}\\
&\geq \frac{(\displaystyle\sum_{\{u,v\}\in E}|F(u,v)(f_+^2(u)-f_+^2(v))|)^2}{\displaystyle\sum_v f_+^2(v)d_v\ 2d(2+g_G)\sum_v f_+^2(v)d_v}\\
&\geq \frac{1}{4d+2dg_G}\left(\frac{(\displaystyle\sum_{\{u,v\}\in E}F(u,v)(f_+^2(u)-f_+^2(v))}{\displaystyle\sum_v f_+^2(v)d_v}\right)^2\\
&\geq \frac{g_G^2}{4d+2dg_G},
\end{aligned}$$

as desired. □

Example 2.8. For an n-cube, the vertex isoperimetric problem has been well studied. According to the Kruskal-Katona theorem [**169, 175**], for a subset S of $\binom{n}{k}$ vertices, for $k \leq n/2$, the vertex boundary of S has at least $\binom{n}{k+1}$ vertices. Therefore, we have $g_{Q_n} = \binom{n}{n/2}2^{-(n-1)} \approx \sqrt{\frac{2}{\pi n}}$, for n even.

2.5. A characterization of the Cheeger constant

In this section, we consider a characterization of the Cheeger constant which has similar form to the Rayleigh quotient but with a different norm.

Theorem 2.9. *The Cheeger constant h_G of a graph G satisfies*

$$h_G = \inf_f \sup_{c\in\mathbb{R}} \frac{\displaystyle\sum_{x\sim y}|f(x)-f(y)|}{\displaystyle\sum_{x\in V}|f(x)-c|d_x} \tag{2.5}$$

where f ranges over all functions $f : V \to \mathbb{R}$ which are not constant functions.

In language analogous to the continuous case, (2.5) can be thought of as

$$h_G = \inf_f \sup_{c\in\mathbb{R}} \frac{\displaystyle\int |\nabla f|}{\displaystyle\int |f-c|}.$$

PROOF. We choose c such that

$$\sum_{\substack{x \\ f(x)<c}} d_x \le \sum_{\substack{x \\ f(x)\ge c}} d_x$$

and

$$\sum_{\substack{x \\ f(x)\le c}} d_x > \sum_{\substack{x \\ f(x)>c}} d_x.$$

If $g = f - c$, then for $\sigma < 0$, we have

$$\sum_{\substack{x \\ g(x)<\sigma}} d_x \le \sum_{\substack{x \\ g(x)\ge\sigma}} d_x$$

and for $\sigma > 0$, we have

$$\sum_{\substack{x \\ g(x)<\sigma}} d_x \ge \sum_{\substack{x \\ g(x)>\sigma}} d_x.$$

We consider

$$\tilde{g}(\sigma) = |\{\{x,y\} \in E(G) : g(x) \le \sigma < g(y)\}|.$$

Then we have

$$\begin{aligned}
\sum_{x\sim y} |f(x) - f(y)| &= \int_{-\infty}^{\infty} \tilde{g}(\sigma) d\sigma \\
&= \int_{-\infty}^{0} d\sigma \frac{\tilde{g}(\sigma)}{\sum\limits_{g(x)<\sigma} d_x} \sum_{g(x)<\sigma} d_x + \int_{0}^{\infty} d\sigma \frac{\tilde{g}(\sigma)}{\sum\limits_{g(x)>\sigma} d_x} \sum_{g(x)>\sigma} d_x \\
&\ge h_G \left(\int_{-\infty}^{0} d\sigma \sum_{g(x)<\sigma} d_x + \int_{0}^{\infty} d\sigma \sum_{g(x)>\sigma} d_x \right) \\
&= h_G \sum_{x\in V} |f(x) - c| d_x.
\end{aligned}$$

In the opposite direction, suppose X is a subset of V satisfying

$$h_G = \frac{|E(X, \bar{X})|}{\text{vol } X}.$$

We consider a character function ψ defined by:

$$\psi(x) = \begin{cases} 1 & \text{if } x \in X, \\ -1 & \text{otherwise.} \end{cases}$$

Then we have,

$$\begin{aligned}
\sup_C \frac{\sum\limits_{x\sim y} |\psi(x) - \psi(y)|}{\sum\limits_{x\in V} |\psi(x) - C| d_x} &= \sup_C \frac{2|E(X, \bar{X})|}{(1 - C)\text{vol } X + (1 + C)\text{vol } \bar{X}} \\
&= \frac{2|E(X, \bar{X})|}{2\text{vol } X} \\
&= h_G.
\end{aligned}$$

Therefore, we have

$$h_G \geq \inf_f \sup_{c\in\mathbb{R}} \frac{\sum_{x\sim y} |f(x)-f(y)|}{\sum_{x\in V} |f(x)-c|d_x}$$

and Theorem 2.9 is proved. ☐

We will prove a variation of Theorem 2.9 which is not sharp but seems to be easier to use. Later on it will be used to derive an isoperimetric relationship between graphs and their Cartesian products.

COROLLARY 2.10. *For a graph G, we have*

$$h_G \geq \inf_f \frac{\sum_{x\sim y} |f(x)-f(y)|}{\sum_{x\in V} |f(x)|d_x} \geq \frac{1}{2}h_G$$

where $f : V(G) \to \mathbb{R}$ satisfies

$$\sum_{x\in V} f(x)d_x = 0. \tag{2.6}$$

PROOF. From Theorem 2.9, we already have

$$h_G \geq \inf_f \frac{\sum_{x\sim y} |f(x)-f(y)|}{\sum_{x\in V} |f(x)|d_x}$$

for f satisfying (2.6). It remains to prove the second part of the inequality. Suppose we define c as in the proof of Theorem 2.9. If $c \geq 0$, then we have

$$\begin{aligned} \sum_{f(x)\leq c} |f(x)-c|d_x &\geq \sum_{f(x)\leq 0} |f(x)|d_x \\ &= \sum_{f(x)\geq 0} |f(x)|d_x. \end{aligned}$$

Therefore

$$\sum_x |f(x)|d_x \leq 2 \sum_{f(x)\leq c} |f(x)-c|d_x \leq 2\sum_x |f(x)-c|d_x.$$

The same results follows similarly if $c \leq 0$. Thus we have

$$\sum_x |f(x)|d_x \leq 2 \inf_c \sum_x |f(x)-c|d_x$$

and the desired upper bound on h_G follows from

$$\inf_f \frac{\sum_{x\sim y} |f(x)-f(y)|}{\sum_{x\in V} |f(x)|d_x} \geq \frac{1}{2}h_G.$$

☐

Suppose we decide to have our measure be the number of vertices in S (and not the volume of S) for a subset S of vertices. We can then pose similar isoperimetric problems.

Problem 3: For a fixed number m, what is the minimum edge-boundary for a subset S of m vertices?

Problem 4: For a fixed number m, what is the minimum vertex-boundary for a subset S of m vertices?

We can define a modified Cheeger constant, which is sometimes called the *isoperimetric number*, by

$$h'(S) = \frac{|E(S, \bar{S})|}{\min(|S|, |\bar{S}|)}$$

and

$$h'_G = \inf_S h'(S).$$

We note that $h'_G \min_v d_v \leq h_G \leq h'_G \max_v d_v$. These modified Cheeger constants are related to the eigenvalues of L, denoted by $0 = \lambda'_0 \leq \lambda'_1 \leq \cdots \leq \lambda'_{n-1}$, and

$$\begin{aligned} \lambda'_1 &= \inf_f \sup_c \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_v (f(v) - c)^2} \\ &= \inf_f \frac{\langle f, Lf \rangle}{\langle f, f \rangle} \end{aligned}$$

where f ranges over all functions f satisfying $\sum f(v) = 0$ which are not identically zero.

The above definition differs from that of $\mathcal{L}$ in (1.3) by the multiplicative factors of d_v for each term in the sum of the denominator. So, eigenvalues λ_i of $\mathcal{L}$ satisfy

$$0 \leq \lambda'_i \leq \lambda_i \max_v d_v.$$

By using methods similar to those in previous sections, we can show

$$2h'_G \geq \lambda'_1.$$

However, the lower bound for λ'_1 in terms of h'_G is a little messy in its derivation. We need to use the fact:

$$\sum_{u \sim v} (f(u) + f(v))^2 \leq 2 \sum_v f(v)^2 d_v \leq 2 \sum_v f(v)^2 \max_w d_w$$

in order to derive the modified Cheeger inequality:

$$\lambda'_1 \geq \frac{{h'_G}^2}{2 \max_v d_v}.$$

This is less elegant than the statement in Theorem 2.2.

We remark that the vertex expansion version of the Cheeger inequality are closely related to the so-called *expander graphs*, which we will examine further in Chapter 6.

2.6. Isoperimetric inequalities for Cartesian products

Suppose G is a graph with a weight function w which assigns nonnegative values to each vertex and each edge. A general Cheeger constant can be defined as follows:

$$h(G,w) = \min_S \frac{\sum_{\{x,y\}\in E(S,\bar{S})} w(x,y)}{\min(\sum_{x\in S} w(x), \sum_{y\notin S} w(y))}.$$

We say the weight function w is *consistent* if

$$\sum_u w(u,v) = w(v).$$

For example, the ordinary Cheeger constant is obtained by using the weight function $w_0(v) = d_v$ for any vertex v and $w_0(u,v) = 1$ for any edge $\{u,v\}$. Clearly, w_0 is consistent. On the other hand, the modified Cheeger constant is $h'_G = h(G, w_1)$ where the weight function w_1 satisfies $w_1(u,v) = 1$ for any edge $\{u,v\}$ and $w_1(v) = 1$ for any vertex v. In this case, w_1 is not necessarily consistent. We note that graphs with consistent weight functions correspond in a natural way to random walks and reversible Markov chains. Namely, for a graph with a consistent weight function w, we can define the random walk with transition probability of moving from a vertex u to each of its neighbors v to be

$$P(u,v) = \frac{w(u,v)}{w(v)}.$$

Similar to Theorem 2.9, the general isoperimetric invariant $h(G,w)$ has the following characterization:

THEOREM 2.11. *For a graph G with weight function w, the isoperimetric invariant $h(G,w)$ of a graph G satisfies*

$$h(G,w) = \inf_f \sup_{c\in\mathbb{R}} \frac{\sum_{x\sim y} |f(x)-f(y)|w(x,y)}{\sum_{x\in V} |f(x)-c|w(x)} \tag{2.7}$$

where f ranges over all $f: V \to \mathbb{R}$ which are not constant functions.

In particular, we also have the following characterization for the modified Cheeger constant.

THEOREM 2.12.

$$h'_G = \inf_f \sup_{c\in\mathbb{R}} \frac{\sum_{x\sim y} |f(x)-f(y)|}{\sum_{x\in V} |f(x)-c|} \tag{2.8}$$

where f ranges over all $f: V \to \mathbb{R}$ which are not constant functions.

For two graphs G and H, the *Cartesian product* $G \square H$ has vertex set $V(G) \times V(H)$ with (u, v) adjacent to (u', v') if and only if $u = u'$ and v is adjacent to v' in H, or $v = v'$ and u is adjacent to u' in G. For example, the Cartesian product of n copies of one single edge is an n-cube, which is sometimes called a *hypercube*. The isoperimetric problem for n-cubes is an old and well-known problem. Just as in the continuous case where the sets with minimum vertex boundary form spheres, in a hypercube the subsets of given size with minimum vertex-boundary are so-called "Hamming balls", which consist of all vertices within a certain distance [**23, 153, 154, 186**]. The isoperimetric problems for grids (which are Cartesian products of paths) and tori (which are Cartesian products of cycles) have been well-studied in many papers [**32, 33, 247**].

We also consider a Cartesian product of weighted graphs with consistent weight functions. For two weighted graphs G and G', with weight functions w, w', respectively, the weighted Cartesian product $G \otimes G'$ has vertex set $V(G) \times V(G')$ with weight function $w \otimes w'$ defined as follows: For an edge $\{u, v\}$ in $E(G)$, we define $w \otimes w'((u, v'), (v, v')) = w(u, v)w'(v')$ and for an edge $\{u', v'\}$ in $E(G')$, we define $w \otimes w'((u, u'), (u, v')) = w(u)w'(u', v')$. We require $w \otimes w'$ to be consistent. Clearly, for a vertex $x = (u, v)$ in $G \otimes G'$, the weight of x in $G \otimes G'$ is exactly $2w(u)w'(v)$.

In general, for graphs G_i with consistent weight functions w_i, $i = 1, \ldots, k$, the weighted Cartesian product $G_1 \otimes \cdots \otimes G_k$ has vertex set $V(G) \otimes \cdots \otimes V(G_k)$ with a consistent weight function $w_1 \otimes \cdots \otimes w_k$ defined as follows: For an edge $\{u, v\}$ in $E(G_i)$, the edge joining $(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_k)$ and $(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)$ has weight $w_1(v_1) \ldots w_{i-1}(v_{i-1})w_i(u, v)w_{i+1}(v_{i+1}) \ldots w_k(v_k)$. We remark that $G_1 \otimes G_2 \otimes G_3$ is different from $(G_1 \otimes G_2) \otimes G_3$ or $G_1 \otimes (G_2 \otimes G_3)$.

The weighted Cartesian product of graphs corresponds naturally to the Cartesian product of random walks on graphs. Suppose $G_1, \ldots, G_k$ are graphs with the vertex sets $V(G_i)$. Each G_i is associated with a random walk with transition probability P_i as defined as in Section 1.5. The Cartesian product of the random walks can be defined as follows: At the vertex $(v_1, \ldots, v_k)$, first choose a random "direction" i, between 1 and k, each with probability $1/k$. Then move to the vertex $(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_k)$ according to P_i. In other words,

$$P((v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, u_k), (v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_k)) = \frac{1}{k} P(v_i, u_i).$$

We point out that the above two notions of the Cartesian products are closely related. In particular,

$$c\lambda_{G \square H} \leq \lambda_{G \otimes H} \leq c^{-1} \lambda_{G \square H}$$

where

$$c = \frac{\min\ (\mathit{min\ deg}\ G, \mathit{min\ deg}\ H)}{\max\ (\mathit{max\ deg}\ G, \mathit{max\ deg}\ H)}.$$

Here *min deg* and *max deg* denote the minimum degree and the maximum degree, respectively. The random walk on $G_1 \square \cdots \square G_k$ has transition probability P' of moving from a vertex $(v_1, \ldots, v_k)$ to the vertex $(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_k)$ given by:

$$P'((v_1,\dots,v_{i-1},v_i,v_{i+1},\dots,u_k),(v_1,\dots,v_{i-1},u_i,v_{i+1},\dots,v_k)) = \frac{w(v_i,u_i)}{\sum\limits_{1\le j\le k} w(v_j)}.$$

For a graph G, the natural consistent weight function associated with G has edge weight 1 and vertex weight d_x for any vertex x. Then we have the following.

THEOREM 2.13. *The eigenvalue of a weighted Cartesian product of $G_1, G_2, \dots, G_k$ satisfies*

$$\lambda_{G_1\otimes G_2\otimes\cdots\otimes G_k} = \frac{1}{k}\min(\lambda_{G_1},\lambda_{G_2},\dots,\lambda_{G_k})$$

where λ_G denotes the first eigenvalue λ_1 of the graph G.

Here we will give a proof for the case $k = 2$. Namely, we will show that the eigenvalue of a weighted Cartesian product of G and H satisfies

$$\lambda_{G\otimes H} = \frac{1}{2}\min(\lambda_G,\lambda_H). \tag{2.9}$$

PROOF. Without loss of generality, we assume that

$$\lambda_G \le \lambda_H.$$

It is easy to see that

$$\lambda_{G\otimes H} \le \frac{1}{2}\lambda_G.$$

Suppose $f : V(G) \to \mathbb{R}$ is the harmonic eigenfunction achieving λ_G. We choose a function $f_0 : V(G)\times V(H) \to \mathbb{R}$ by setting

$$f_0(u,v) = f(u).$$

Clearly, $\lambda_{G\otimes H}$ is less than the Rayleigh quotient using f_0 whose value is exactly $\lambda_G/2$.

In the opposite direction, we consider the harmonic eigenfunction $g : V(G)\times V(H) \to \mathbb{R}$ achieving $\lambda_{G\otimes H}$. We denote, for $u \in V(G), v \in V(H)$,

$$g_u = \frac{\sum\limits_v g(u,v)d_v}{\text{vol } H},$$

$$g_v = \frac{\sum\limits_u g(u,v)d_u}{\text{vol } G},$$

$$c = \frac{\sum\limits_{u,v} g(u,v)d_u d_v}{\text{vol } G \text{ vol } H}. \tag{2.10}$$

Here, we repeatedly use the definition of eigenvalues and the Cauchy-Schwarz inequality:

$$\begin{aligned}
\lambda_{G\otimes H} &= \frac{\sum_v \sum_{u\sim u'} (g(u,v)-g(u',v))^2 d_v + \sum_u \sum_{v\sim v'} (g(u,v)-g(u,v'))^2 d_u}{\sum_{u,v} (g(u,v)-c)^2 2 d_u d_v} \\
&\geq \frac{\lambda_G \sum_{u,v} (g(u,v)-g_v)^2 d_u d_v + (\sum_u d_u) \sum_{v\sim v'} (g_v - g_{v'})^2}{\sum_{u,v} (g(u,v)-g_v)^2 2 d_u d_v + \sum_{u,v} (g_v - c)^2 2 d_u d_v} \\
&\geq \frac{\lambda_G \sum_{u,v} (g(u,v)-g_v)^2 d_u d_v + \lambda_H (\sum_u d_u) \sum_{v\sim v'} (g_v - c)^2 d_v}{\sum_{u,v} (g(u,v)-g_v)^2 2 d_u d_v + \sum_{u,v} (g_v - c)^2 2 d_u d_v} \\
&\geq \frac{\lambda_G}{2}.
\end{aligned}$$

This completes the proof of (2.9). □

THEOREM 2.14. *The Cheeger constant of a weighted Cartesian product of* $G_1, G_2, \cdots, G_k$ *satisfies*

$$\begin{aligned}
\frac{1}{k} \min(h_{G_1}, h_{G_2}, \ldots, h_{G_k}) &\geq h_{G_1 \otimes G_2 \otimes \cdots \otimes G_k} \\
&\geq \frac{1}{2k} \min(h(_{G_1}, h_{G_2}, \ldots, h_{G_k}).
\end{aligned}$$

Here we again will prove the case for the product of two graphs and leave the proof of the general case as an exercise.

$$\frac{1}{2} \min(h_G, h_H) \geq h_{G\otimes H} \geq \frac{1}{4} \min(h_G, h_H). \tag{2.11}$$

PROOF. Without loss of generality, we assume that

$$h_G \leq h_H.$$

First we note that

$$h_{G\otimes H} \leq \frac{h_G}{2}.$$

Suppose $f : V(G) \to \mathbb{R}$ is a function achieving $h(G)$ in (2.7). We choose a function $f_0 : V(G) \times V(H) \to \mathbb{R}$ by setting

$$f_0(u,v) = f(u).$$

Clearly, $h_{G\otimes H}$ is no more than the value for the quotient of (2.7) using f_0 whose value is exactly $h_G/2$.

It remains to show that $h_{G\otimes H} \geq h_G/4$. To this end, we will repeatedly use Corollary 2.10, and we adopt the notation in the proof of (2.9).

$$
\begin{aligned}
h_{G\otimes H} &= \frac{\sum_v \sum_{u\sim u'} |g(u,v)-g(u',v)|d_v + \sum_u \sum_{v\sim v'} |g(u,v)-g(u,v')|d_u}{\sum_{u,v} |g(u,v)-c|\ 2d_u d_v} \\
&\geq \frac{h_G \sum_{u,v} |g(u,v)-g_v|d_u d_v + (\sum_u d_u) \sum_{v\sim v'} |g_v - g_{v'}|}{\sum_{u,v} |g(u,v)-g_v|\ 2d_u d_v + \sum_{u,v} |g_v - c|\ 2d_u d_v} \\
&\geq \frac{\frac{h_G}{2} \sum_{u,v} |g(u,v)-g_v|d_u d_v + \frac{h_H}{2}(\sum_u d_u) \sum_{v\sim v'} |g_v - c|d_v}{\sum_{u,v} |g(u,v)-g_v|\ 2d_u d_v + \sum_{u,v} |g_v - c|\ 2d_u d_v} \\
&\geq \frac{h_G}{4}.
\end{aligned}
$$

This completes the proof of (2.11). □

For the modified Cheeger constant h'_G, a similar isoperimetric inequality can be obtained:

COROLLARY 2.15. *The modified Cheeger constant of the Cartesian product of* $G_1, G_2, \ldots, G_k$ *satisfies*

$$
\begin{aligned}
\min(h'_{G_1}, h'_{G_2}, \ldots, h'_{G_k}) &\geq h'_{G_1 \square G_2 \square \cdots \square G_k} \\
&\geq \frac{1}{2}\min(h'_{G_1}, h'_{G_2}, \ldots, h'_{G_k}).
\end{aligned}
$$

The proof is quite similar to that of (2.11) (also see [**82**]) and will be omitted.

Notes

The characterization of the Cheeger constant in Theorem 2.9 is basically the Rayleigh quotient using the L_1-norm both in the numerator and denominator. In general, we can consider the so-called *Sobolev constants* for all $p, q > 0$:

$$
\begin{aligned}
s_{p,q} &= \inf_f \frac{\left(\sum_{u\sim v} |f(u)-f(v)|^p\right)^{1/p}}{\left(\sum_v |f(v)|^q d_v\right)^{1/q}} \\
&= \inf_f \frac{\|\nabla f\|_p}{\|f\|_q}
\end{aligned}
$$

where f ranges over functions satisfying

$$\sum_x |f(x) - c|^q d_x \geq \sum_x |f(x)|^q d_x$$

for any c, or, equivalently,

$$\int |f - c|^q \geq \int |f|^q.$$

The eigenvalue λ_1 is associated with the case of $p = q = 2$, while the Cheeger constant corresponds to the case of $p = q = 1$. Some of the general cases will be considered later in Chapter 11 on Sobolev inequalities.

This chapter is mainly based on [**59**]. More general cases of the Cartesian products are discussed in [**82**]. Another reference for weighted Cheeger constants and related isoperimetric inequalities is [**78**].