

Lecture 9

Duality (part 2)

- duality in algorithms
- sensitivity analysis via duality
- duality for general LPs
- examples
- mechanics interpretation
- circuits interpretation
- two-person zero-sum games

9-1

Duality in algorithms

many algorithms produce at iteration k

- a primal feasible $x^{(k)}$
- and a dual feasible $z^{(k)}$

with $c^T x^{(k)} + b^T z^{(k)} \rightarrow 0$ as $k \rightarrow \infty$

hence at iteration k we **know** $p^* \in [-b^T z^{(k)}, c^T x^{(k)}]$

- useful for stopping criteria
- algorithms that use dual solution are often more efficient

Nonheuristic stopping criteria

- (absolute error) $c^T x^{(k)} - p^*$ is less than ϵ if

$$c^T x^{(k)} + b^T z^{(k)} < \epsilon$$

- (relative error) $(c^T x^{(k)} - p^*)/|p^*|$ is less than ϵ if

$$-b^T z^{(k)} > 0 \quad \& \quad \frac{c^T x^{(k)} + b^T z^{(k)}}{-b^T z^{(k)}} \leq \epsilon$$

or

$$c^T x^{(k)} < 0 \quad \& \quad \frac{c^T x^{(k)} + b^T z^{(k)}}{-c^T x^{(k)}} \leq \epsilon$$

- target value ℓ is achievable ($p^* \leq \ell$) if

$$c^T x^{(k)} \leq \ell$$

- target value ℓ is unachievable ($p^* > \ell$) if

$$-b^T z^{(k)} > \ell$$

Sensitivity analysis via duality

perturbed problem:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b + \epsilon d \end{array}$$

$A \in \mathbf{R}^{m \times n}$; $d \in \mathbf{R}^m$ given; optimal value $p^*(\epsilon)$

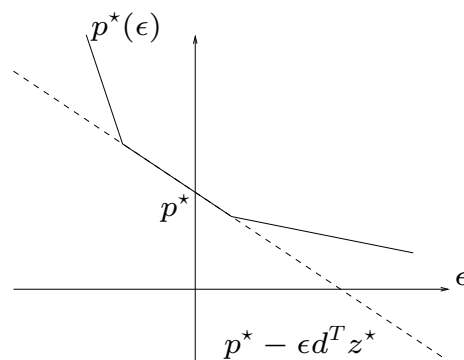
global sensitivity result: if z^* is (any) dual optimal solution for the unperturbed problem, then for all ϵ

$$p^*(\epsilon) \geq p^* - \epsilon d^T z^*$$

proof. z^* is dual feasible for all ϵ ; by weak duality,

$$p^*(\epsilon) \geq -(b + \epsilon d)^T z^* = p^* - \epsilon d^T z^*$$

interpretation



- $d^T z^* > 0$: $\epsilon < 0$ increases p^*
- $d^T z^* > 0$ and large: $\epsilon < 0$ greatly increases p^*
- $d^T z^* > 0$ and small: $\epsilon > 0$ does not decrease p^* too much
- $d^T z^* < 0$: $\epsilon > 0$ increases p^*
- $d^T z^* < 0$ and large: $\epsilon > 0$ greatly increases p^*
- $d^T z^* < 0$ and small: $\epsilon > 0$ does not decrease p^* too much

Local sensitivity analysis

assumption: there is a nondegenerate optimal vertex x^* , i.e.,

- x^* is an optimal vertex: $\text{rank } \bar{A} = n$, where

$$\bar{A} = [a_{i_1} \quad a_{i_2} \quad \cdots \quad a_{i_K}]^T$$

and $I = \{i_1, \dots, i_K\}$ is the set of active constraints at x^*

- x^* is nondegenerate: $\bar{A} \in \mathbf{R}^{n \times n}$

w.l.o.g. we assume $I = \{1, 2, \dots, n\}$

consequence: dual optimal z^* is unique

proof: by complementary slackness, $z_i^* = 0$ for $i > n$

by dual feasibility,

$$\sum_{i=1, \dots, n} a_i z_i^* = \bar{A}^T \begin{bmatrix} z_1^* \\ \vdots \\ z_n^* \end{bmatrix} = -c \implies z^* = \begin{bmatrix} -\bar{A}^{-T} c \\ 0 \end{bmatrix}$$

optimal solution of the perturbed problem (for small ϵ):

$$x^*(\epsilon) = x^* + \epsilon \bar{A}^{-1} \bar{d} \quad (\text{with } \bar{d} = (d_1, \dots, d_n))$$

- $x^*(\epsilon)$ is feasible for small ϵ :

$$a_i^T x^*(\epsilon) = b_i + \epsilon d_i, \quad i = 1, \dots, n, \quad a_i^T x^*(\epsilon) < b_i + \epsilon d_i, \quad i = n+1, \dots, m$$

- z^* is dual feasible and satisfies complementary slackness:

$$(b + \epsilon d - Ax^*(\epsilon))^T z^* = 0$$

optimal value of perturbed problem (for small ϵ):

$$p^*(\epsilon) = c^T x^*(\epsilon) = p^* + \epsilon c^T \bar{A}^{-1} \bar{d} = p^* - \epsilon d^T z^*$$

- z_i^* is sensitivity of cost w.r.t. righthand side of i th constraint
- z_i^* is called *marginal cost* or *shadow price* associated with i th constraint

Dual of a general LP

method 1: express as LP in inequality form and take its dual

example: standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \begin{bmatrix} -I \\ A \\ -A \end{bmatrix} x \leq \begin{bmatrix} 0 \\ b \\ -b \end{bmatrix} \end{array}$$

dual:

$$\begin{array}{ll} \text{maximize} & -b^T(v - w) \\ \text{subject to} & -u + A^T(v - w) + c = 0 \\ & u \geq 0, \quad v \geq 0, \quad w \geq 0 \end{array}$$

method 2: apply Lagrange duality (this lecture)

Lagrangian

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m \end{array}$$

Lagrangian $L : \mathbf{R}^{n+m} \rightarrow \mathbf{R}$

$$L(x, \lambda) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

- λ_i are called *Lagrange multipliers*
- objective is *augmented* with weighted sum of constraint functions

lower bound property: if $Ax \leq b$ and $\lambda \geq 0$, then

$$c^T x \geq L(x, \lambda) \geq \min_{\tilde{x}} L(\tilde{x}, \lambda)$$

hence, $p^* \geq \min_{\tilde{x}} L(\tilde{x}, \lambda)$ for $\lambda \geq 0$

Lagrange dual problem

Lagrange dual function $g : \mathbf{R}^m \rightarrow \mathbf{R} \cup \{-\infty\}$

$$\begin{aligned} g(\lambda) &= \min_x L(x, \lambda) = \min_x (-b^T \lambda + (A^T \lambda + c)^T x) \\ &= \begin{cases} -b^T \lambda & \text{if } A^T \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

(Lagrange) dual problem

$$\begin{aligned} &\text{maximize } g(\lambda) \\ &\text{subject to } \lambda \geq 0 \end{aligned}$$

yields the dual LP:

$$\begin{aligned} &\text{maximize } -b^T \lambda \\ &\text{subject to } A^T \lambda + c = 0, \quad \lambda \geq 0 \end{aligned}$$

finds best lower bound $g(\lambda)$

Lagrangian of a general LP

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } a_i^T x \leq b_i, \quad i = 1, \dots, m \\ &\quad \quad \quad g_i^T x = h_i, \quad i = 1, \dots, p \end{aligned}$$

define **Lagrangian** $L : \mathbf{R}^{n+m+p} \rightarrow \mathbf{R}$ as

$$L(x, \lambda, \nu) = c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i) + \sum_{i=1}^p \nu_i (g_i^T x - h_i)$$

lower bound property: if x is feasible and $\lambda \geq 0$,

$$c^T x \geq L(x, \lambda, \nu) \geq \min_{\tilde{x}} L(\tilde{x}, \lambda, \nu)$$

hence, $p^* \geq \min_x L(x, \lambda, \nu)$ if $\lambda \geq 0$

multipliers associated with equality constraints can have either sign

Lagrange dual function:

$$\begin{aligned} g(\lambda, \nu) &= \min_x L(x, \lambda, \nu) = \min_x (c^T x + \lambda^T (Ax - b) + \nu^T (Gx - h)) \\ &= \min_x (-b^T \lambda - h^T \nu + x^T (c + A^T \lambda + G^T \nu)) \\ &= \begin{cases} -b^T \lambda - h^T \nu & \text{if } A^T \lambda + G^T \nu + c = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

Lagrange dual problem:

$$\begin{aligned} &\text{maximize} && -b^T \lambda - h^T \nu \\ &\text{subject to} && A^T \lambda + G^T \nu + c = 0 \\ &&& \lambda \geq 0 \end{aligned}$$

variables λ, ν ; optimal value d^*

- an LP (in general form)
- weak duality $p^* \geq d^*$
- strong duality holds: $p^* = d^*$ (except when both problems are infeasible)

Example: standard form LP

$$\begin{aligned} &\text{minimize} && c^T x \\ &\text{subject to} && Ax = b, \quad x \geq 0 \end{aligned}$$

Lagrangian: $L(x, \nu, \lambda) = c^T x + \nu^T (Ax - b) - \lambda^T x$

dual function

$$g(\lambda, \nu) = \min_x L(x, \nu, \lambda) = \begin{cases} -b^T \nu & \text{if } A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem

$$\begin{aligned} &\text{maximize} && -b^T \nu \\ &\text{subject to} && A^T \nu - \lambda + c = 0, \quad \lambda \geq 0 \end{aligned}$$

equivalent to dual on page 9-9

$$\begin{aligned} &\text{maximize} && b^T z \\ &\text{subject to} && A^T z \leq c \end{aligned}$$

Price or tax interpretation

- x : describes how an enterprise operates; $c^T x$: cost of operating at x
- $a_i^T x \leq b_i$: limits on resources, regulatory limits

optimal operating point is solution of

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject to} && a_i^T x \leq b_i, \quad i = 1, \dots, m \end{aligned}$$

optimal cost: p^*

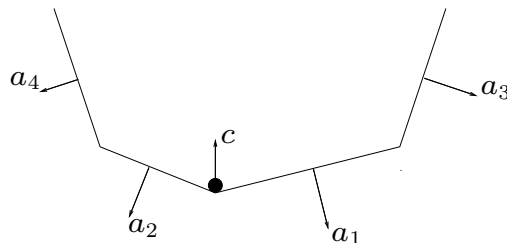
scenario 2: constraint violations can be bought or sold at unit cost $\lambda_i \geq 0$

$$\text{minimize} \quad c^T x + \sum_{i=1}^m \lambda_i (a_i^T x - b_i)$$

optimal cost: $g(\lambda)$

interpretation of strong duality: there exist prices λ_i^* s.t. $g(\lambda^*) = p^*$, *i.e.*, there is no advantage in selling/buying constraint violations

Mechanics interpretation



- mass subject to gravity, can move freely between walls described by $a_i^T x = b_i$
- equilibrium position minimizes potential energy, *i.e.*, solves

$$\begin{aligned} & \text{minimize} && c^T x \\ & \text{subject} && a_i^T x \leq b_i, \quad i = 1, \dots, m \end{aligned}$$

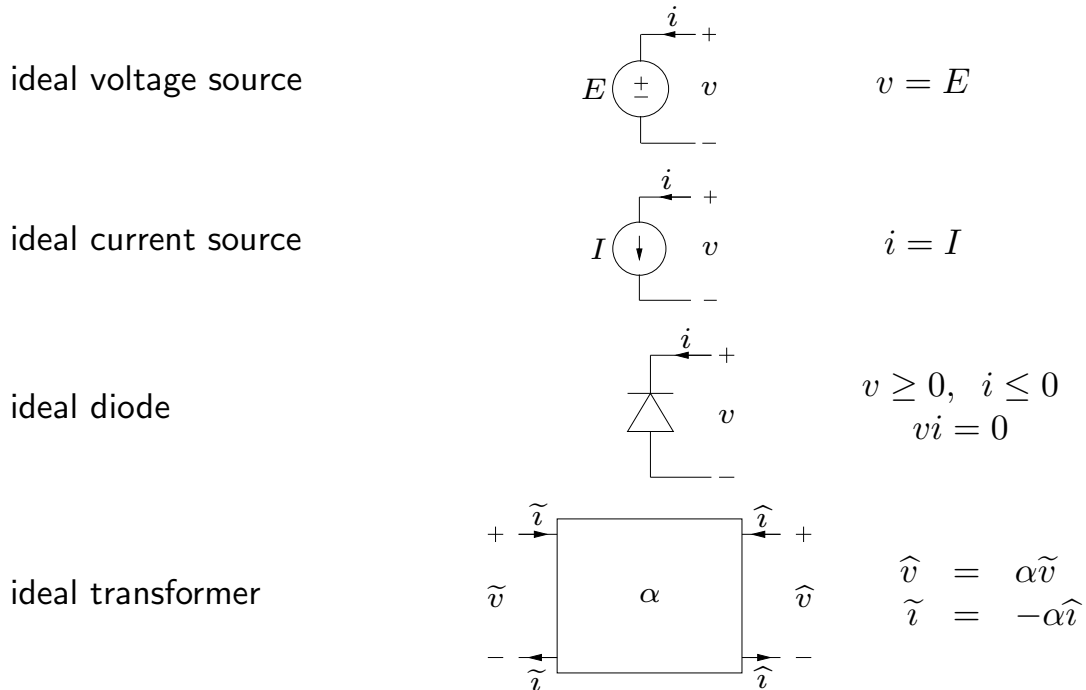
optimality conditions:

1. $a_i^T x^* \leq b_i, i = 1, \dots, m$
2. $\sum_{i=1}^m z_i^* a_i + c = 0, z_i^* \geq 0$
3. $z_i^* (b_i - a_i^T x^*) = 0, i = 1, \dots, m$

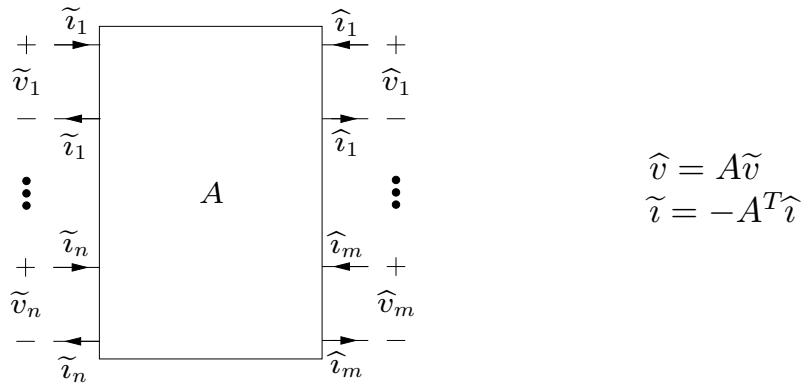
interpretation: $-z_i a_i$ is contact force with wall i ; nonzero only if the ball touches the i th wall

Circuits interpretation

circuit components:



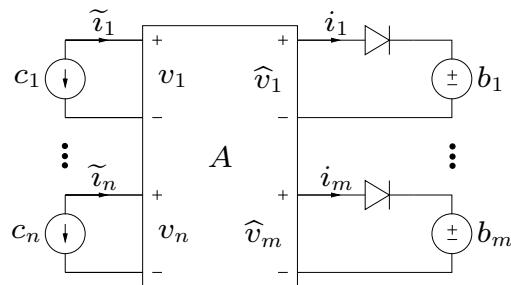
ideal multiterminal transformer ($A \in \mathbf{R}^{m \times n}$)



Duality (part 2)

9-19

example



circuit equations:

$$\hat{v} = Av \leq b, \quad i \geq 0, \quad \tilde{i} + c = A^T i + c = 0$$

$$i_k(b_k - a_k^T v) = 0, \quad k = 1, \dots, m$$

i.e., optimality conditions for LP

$$\begin{array}{ll} \text{minimize} & c^T v \\ \text{subject to} & Av \leq b \end{array} \quad \begin{array}{ll} \text{maximize} & -b^T i \\ \text{subject to} & A^T i + c = 0 \\ & i \geq 0 \end{array}$$

Duality (part 2)

9-20

interpretation: two 'potential functions'

- *content* (a function of the voltages)
- *co-content* (a function of the currents)

contribution of each component (notation of page 9–18)

- content of current source is Iv
co-content is 0 if $i = I$, $-\infty$ otherwise
- content of voltage source is 0 if $v = E$, ∞ otherwise
co-content is $-Ei$
- content of diode is 0 if $v \geq 0$, $+\infty$ otherwise
co-content is 0 if $i \leq 0$ and $-\infty$ otherwise
- content of transformer is 0 if $\hat{v} = A\tilde{v}$, ∞ otherwise
co-content is 0 if $\tilde{i} = -A^T\hat{i}$, $-\infty$ otherwise

primal problem: voltages minimize total content

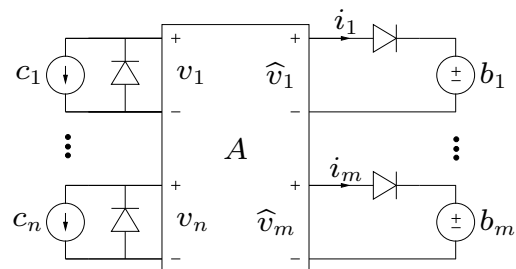
dual problem: currents maximize total co-content

example

primal problem

$$\begin{aligned} &\text{minimize} && c^T v \\ &\text{subject to} && Av \leq b \\ &&& v \geq 0 \end{aligned}$$

circuit equivalent:



dual problem:

$$\begin{aligned} &\text{maximize} && -b^T i \\ &\text{subject to} && A^T i + c \geq 0 \\ &&& i \geq 0 \end{aligned}$$

Two-person zero-sum games (matrix games)

described by a **payoff matrix**

$$A \in \mathbf{R}^{m \times n}$$

- player 1 chooses a number in $\{1, \dots, m\}$ (corresponding to m possible actions or strategies)
- player 2 chooses a number in $\{1, \dots, n\}$
- players make their choice simultaneously and independently
- if P1's choice is i and P2's choice is j , then P1 pays a_{ij} to P2 (negative a_{ij} means P2 pays $-a_{ij}$ to P1)

Mixed (randomized) strategies

players make random choices according to some probability distribution

- player 1 chooses randomly according to distribution $x \in \mathbf{R}^m$:

$$\mathbf{1}^T x = 1, \quad x \geq 0$$

(x_i is probability of choosing i)

- player 2 chooses randomly (and independently from 1) according to distribution $y \in \mathbf{R}^n$:

$$\mathbf{1}^T y = 1, \quad y \geq 0$$

(y_j is probability of choosing j)

expected payoff from player 1 to 2, if they use mixed strategies x and y :

$$\sum_{i=1}^m \sum_{j=1}^n x_i y_j a_{ij} = x^T A y$$

Optimal mixed strategies

optimal strategy for player 1:

$$\begin{array}{ll} \text{minimize}_x & \max_{\mathbf{1}^T y=1, y \geq 0} x^T A y \\ \text{subject to} & \mathbf{1}^T x = 1, \quad x \geq 0 \end{array}$$

note:

$$\max_{\mathbf{1}^T y=1, y \geq 0} x^T A y = \max_{j=1, \dots, n} (A^T x)_j$$

optimal strategy x^* can be computed by solving an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & A^T x \leq t \mathbf{1} \\ & \mathbf{1}^T x = 1, \quad x \geq 0 \end{array} \quad (1)$$

(variables x, t)

Duality (part 2)

9-25

optimal strategy for player 2:

$$\begin{array}{ll} \text{maximize}_y & \min_{\mathbf{1}^T x=1, x \geq 0} x^T A y \\ \text{subject to} & \mathbf{1}^T y = 1, \quad y \geq 0 \end{array}$$

note:

$$\min_{\mathbf{1}^T x=1, x \geq 0} x^T A y = \min_{i=1, \dots, m} (A y)_i$$

optimal strategy y^* can be computed by solving an LP

$$\begin{array}{ll} \text{maximize} & w \\ \text{subject to} & A y \geq w \mathbf{1} \\ & \mathbf{1}^T y = 1, \quad y \geq 0 \end{array} \quad (2)$$

(variables y, w)

Duality (part 2)

9-26

The minimax theorem

for all mixed strategies x, y ,

$$x^{*T} Ay \leq x^{*T} Ay^* \leq x^T Ay^*$$

proof: the LPs (1) and (2) are duals, so they have the same optimal value

example

$$A = \begin{bmatrix} 4 & 2 & 0 & -3 \\ -2 & -4 & -3 & 3 \\ -2 & -3 & 4 & 1 \end{bmatrix}$$

optimal strategies

$$x^* = (0.37, 0.33, 0.3), \quad y^* = (0.4, 0, 0.13, 0.47)$$

expected payoff: $x^{*T} Ay^* = 0.2$