The Matrix-Tree Theorem

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Overview

- Statement of the main result
- Preliminaries, definitions and examples
- Proof of the main result

$\star$ 15 min. break

- Generalizations
- Applications
- Other graph invariants
- Future work

The Matrix-Tree Theorem

Theorem 1. *Any principal minor of the Laplacian L of a graph G, (obtained by deleting 1 row and 1 column of L), has determinant $\tau(G)$, where $\tau(G)$ is the number of spanning trees of G.*

This classical result can be traced back to [Kirchhoff 1847], [Sylvester 1857], [Cayley 1889] and [Maxwell 1892].

Definition 1. *The* **incidence matrix** B *of a graph* G *is a matrix with rows indexed by the vertices of* G *and columns indexed by the edges of* G*, and such that*

$$B(v,e) = \begin{cases} 1 & \textit{if } e = \{u,v\}, u = v_i, v = v_j, i \leq j \\ \\ -1 & \textit{if } e = \{u,v\}, u = v_i, v = v_j, i > j \\ \\ 0 & \textit{otherwise} \end{cases}$$

Note that the definition of B depends on the ordering of vertices and edges of G.

Definition 2. *The* **Laplacian** L *of a graph* G *is a square matrix with rows indexed by the vertices of* G*, and such that*

$$L(u,v) = \begin{cases} d_v & \text{if } u = v \\ \\ -1 & \text{if } u \text{ adjacent to } v \\ \\ 0 & \text{otherwise} \end{cases}$$

where d_v *is the degree of a vertex v.*

Note that $L = BB^*$ and that the definition of L depends on the ordering of vertices of G.

Definition 3. *A* **tree** *is a connected graph without closed cycles.*
A **spanning tree** *T (of G) is a subgraph of G which is a tree and which contains all the vertices of G.*
We denote by $\tau(G)$ the number of spanning trees of G.

Lemma 1. *Let* $\mathbf{1}_{n\times 1}$ *denote the* n*-element column vector of all 1s. Then* $\mathbf{1}_{n\times 1}$ *is an eigenvector of* L *with eigenvalue 0.*

Lemma 2. *Let* e_1 *be the eigenvector corresponding to the smallest eigenvalue of a hermitian matrix* A *and let* λ_2 *be the second smallest eigenvalue. Then by the Rayleigh-Ritz theorem*
$\lambda_2 = min\frac{<x,Ax>}{<x,x>}$*, where the minimum is taken over all* x *such that* $x \neq 0, x \perp e_1$*.*

Lemma 3. *If* G *is connected, then the rank of* L *is* $n-1$*.*

Lemma 4. *Let* $adj(L)$ *denote the* **adjugate** *of* L*, i.e. the matrix of cofactors of* L*. If G is connected, then* $adj(L)$ *is a scalar multiple of* $\mathbf{1}_{n\times n}$*.*

Lemma 5. *A tree on n vertices has $n-1$ edges.*

Lemma 6. *Let G be a connected graph on n vertices. Fix a vertex v and a (n-1)-element subset of the edges X. Let B be the incidence matrix of the graph, and let B_v^X be the square matrix formed by deleting row v and keeping columns X of B. Further let $\tilde{G}$ be the graph formed by all the vertices of G and the X chosen edges.*

Then either
a) $det(B_v^X) = 0$, or
b) $\tilde{G}$ is a (connected) spanning tree of G and $det(B_v^X) = \pm 1$

Theorem 2. *(Binet-Cauchy).*
Let A be a $m{\times}n$ and B a $n{\times}m$ matrix with $m \leq n$. Let C be the product AB. Then

$$det(C) = \sum_X det(A_X)det(B_X) \tag{1}$$

where A_X is a $m \times m$ submatrix of A, formed by choosing m out of the n columns of A, B_X is a $m{\times}m$ submatrix of B, formed by choosing m out of the n rows of B, and where the sum ranges over all possible choices X of m columns of A.

Generalizations

The M-T theorem has been generalized to:

- Vertex and edge weighted graphs [Chung, Langlands]

- Directed graphs [Tutte]

- Determinants of non-principal minors of the Laplacian that count **rooted spanning forests** [Chung, Chaiken]

Applications

• The number of trees having n vertices is n^{n-2} [Cayley]

• The number of trees in the d-dimensional cube is $2^{2^d-d-1}\prod_{k=1}^{d} k^{\binom{d}{k}}$ [Stanley]

• Various other graph theoretic problems (number of Eulerian tours, paths, flows, routing) [Stanley, Chung]

• Analogies to differential geometry [Sarnak, Selberg]

• Quantum physics (Feynman path integral) [Polyakov]

Why?

Why is the number of spanning trees of a graph (which is obviously a graph invariant) related to something like the determinant of a matrix?

As already mentioned, the Laplacian depends on the choice of ordering of the vertices of a graph. Different orderings give rise to Laplacians which are related as $L_1 = SL_2S^{-1}$, where S is an element of the symmetric group.

It should be clear that L_1 and L_2 have the same determinant since
$det(L_1) = det(SL_2S^{-1}) =$
$det(S)det(L_2)det(S^{-1}) =$
$det(S)det(L_2)det(S)^{-1} = det(L_2)$

That is to say, the determinant of the Laplacian (and any minor thereof) is invariant under reorderings of vertices, and thus a **graph invariant**.

The determinant is not the only map $M_n \mapsto \mathbb{R}$ that enjoys this invariance property.

In fact all the coefficients of the characteristic polynomial of a matrix are invariant under the action of the symmetric group.

Moreover, viewed as polynomials in the n eigenvalues of the matrix, they form a basis for Λ_n, (the ring of symmetric polynomials in n variables).

Question. Are there graph invariants that are not symmetric polynomials in the eigenvalues of the Laplacian?

The invariants that we describe here, can all be computed in $O(n^3)$ time.

There are however examples of graph invariants that are known to be NP-hard to compute. (e.g. the chromatic number)

The chromatic number is thus not in Λ_n.
(Eigenvalues are however useful in providing bounds for the chromatic number...) [Chung]

Definition 4. *A graph invariant is called* **complete** *if any two non-isomorphic graphs have different invariants.*

There exist non-isomorphic graphs with the same number of spanning trees. Take e.g. two non-isomorphic trees with the same number of vertices.

Although we have not proved this, it is quite probable that there exist isospectral but non-isomorphic graphs. (cf. [Kac 1966] Can one hear the shape of a drum?)

Conjecture. The coefficients of the characteristic polynomial of the Laplacian give rise to non-complete invariants.

BTW, deciding whether two graphs are isomorphic is thought to be an NP-complete problem.

Future Work

Question. Since the coefficients of the characteristic polynomial are graph invariants with integer values, it is tempting to assume that they have a combinatorial interpretation.
For instance it is known that
$tr(L) = \sum_v d_v = 2\times$the number of edges of G.

Generally, this is to our knowledge still an open question.

THE END