Thm Suppose $f : X \to \mathbb{R}^n$ is integrable. Then $\| \int f \, d\mu \| \le \int \|f\| \, d\mu$

Pf Let $v = \int f \, d\mu \in \mathbb{R}^n$. WLOG $v \neq 0$

$w = \frac{v}{\|v\|}$. Note $\|v\| = v \cdot w$

$$\|v\| = \| \textstyle\int f \, d\mu \| = v \cdot w = \int f \, d\mu \cdot w$$
$$= \textstyle\int (f \circ w) \, d\mu$$

By Cauchy-Schwarts

$$|f \cdot w| \le \|f\| \, \|w\| = \|f\|$$

$$\textstyle\int (f \cdot w) \, d\mu \le \int |f \cdot w| \, d\mu \le \int \|f\| \, d\mu$$

$$\left(\textstyle\int f \, d\mu\right) \cdot w \le \int \|f\| \, d\mu$$

$$\|$$

$$\| \textstyle\int f \, d\mu \| \le \int \|f\| \, d\mu$$

Thm Let $f : I = [a,b] \to \mathbb{R}$. f is Riemann int. iff f is bounded and continuous a.e. In this case, Riemann and Lebesgue integrals agree.

?? a.e.?

Clearly, f Riemann integrable $\Rightarrow$ f bounded

Suppose f is bounded and continuous a.e.

Let P is a partition of $[a, b]$.
Define $\varphi_P(x) = \sum_{i=1}^{n} f(\xi_i)(x_{i+1} - x_i)$
Since f is bounded $\exists M > 0$ s.t.
$|f(x)| \leq M < \infty$, $x \in [a, b]$. Also
if f is continuous at $x \in [a, b]$, then
$\varphi_m(x) \to f(x)$ a.e. on $[a, b]$

Take $g(x) \equiv M$ on $a \leq x \leq b$

Lebesgue int.
Apply D.C.T : $\int f \, d\lambda = \lim_{m \to \infty} \int \varphi_m \, d\lambda =$
$= \lim_{\|P_m\| \to 0} \sum_{i=1}^{m} f(\xi)(x_{i+1} - x_i) = \int_a^b f(x)\,dx$

Riemann int.

Now suppose f is not continuous a.e.
Then Riemann int. doesn't exist.
Let $D = \{x \in [a, b], f(x) \text{ is not cont}\}$, $|D| > 0$

Let $x \in [a, b]$. Define $\limsup_{y \to x} f(y) =$
$$= \lim_{\varepsilon \to 0} \sup_{(0 \leq |y - x| < \varepsilon) \cap [a, b]} f(y)$$

f is continuous at x iff
$\liminf_{y \to x} f(y) = \limsup_{y \to x} f(y)$

$D_n = \{x \in [a,b] : \overline{\lim_{y\to x}} f(y) - \underline{\lim_{y\to x}} f(x) \geq \frac{1}{n}\}$

$D = \bigcup_{n=1}^{\infty} D_n \quad |D|_1 > 0 \iff |D_n|_1 > 0$ for some n

Chose and fix n s.t. $|D_n| > 0$

Let $x \in [a,b]$. Let P be any partition of $[a,b]$. Consider $U(f,P) - L(f,P)$

If $D_n \cap [x_{i+1} - x_i] \neq \emptyset$ then $M_i - m_i \geq \frac{1}{n}$

But then $U(f,P) - L(f,P) \geq \frac{1}{n}|P_n| > 0$

$U(f,P) - L(f,P) \not\to 0$ as $\|P\| \to 0$