Chapter 4

Optimality Conditions

Reference: CD from Bhati: Practical Optimization Methods with *Mathematica*
- On Reserve for Math 4025 in Library -

- **Initialization**

4.1 Karush-Kuhn-Tucker (KT) Conditions

```
Information["KTSolution", LongForm → False]
```

```
KTSolution[f, con, vars, opts], finds candidate minimum points by solving KT conditions. f
  is the objective function. con is a list of constraints. The function automatically
  converts the constraints to standard form before proceeding. vars is a list of
  problem variables.  Several options can be used with the function. See Options[
  KTSolution]. By default, all possible cases of inequalities being active are examined.
  Active inequalities can be explicitly specified through the ActiveCases option.
```

```
OptionsUsage[KTSolution]
```

```
{PrintLevel → 1, ActiveCases → {}, KTVarNames → {u, s, v},
 SolveEquationsUsing → NSolve, StartingSolution → {}}
```

```
PrintLevel is an option for most functions in the OptimizationToolbox. It is specified as an
  integer. The value of the integer indicates how much intermediate information is to be printed.
  A PrintLevel->0 suppresses all printing. The default for most functions is set to 1, in which
  case they print only the initial problem setup. Higher integers print more intermediate results.

ActiveCases->List of active LE constraints to be
  considered in the solution. The default is to consider all possible cases.

KTVarNames->Variable names used for g ('≤' constraints) multipliers,
  slack variables and h ('=' constraints) multipliers. Default is {u,s,v}.

SolveEquationsUsing is an option for Optimality-condition based methods. The Mathematica function
  used to solve system of equations is specified with this option. The default is NSolve. If
  using FindRoot, a starting solution must be specified using the StartingSolution option.

The StartingSolution to be used with the FindRoot function
  in Mathematica. It is used only if the method specified is FindRoot.
```

- **LP Example**
- **LP Example 2**

■ Pedregal Example 3.10 (p. 82) r = 5

```
In[18]:= f = z + (x^2 + y^2 + z^2 / 10) / 2;
         cons = {x + y + z == 5, x ≥ 0, y ≥ 0, z ≥ 0};
         vars = {x, y, z};

In[21]:= KTSolution[f, cons, vars, PrintLevel → 1];
```

Minimize $f \to z + \frac{1}{2}\left(x^2 + y^2 + \frac{z^2}{10}\right)$

$\nabla f \to \begin{pmatrix} x \\ y \\ 1 + \frac{z}{10} \end{pmatrix}$

***** LE constraints and their gradients

$g_1 \to -x \le 0 \quad g_2 \to -y \le 0$

$g_3 \to -z \le 0 \quad \nabla g_1 \to \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$\nabla g_2 \to \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \nabla g_3 \to \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

***** EQ constraints and their gradients

$h_1 \to -5 + x + y + z == 0 \quad \nabla h_1 \to \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

***** Lagrangian $\to z + \frac{1}{2}\left(x^2 + y^2 + \frac{z^2}{10}\right) + (-x + s_1^2)\, u_1 + (-y + s_2^2)\, u_2 + (-z + s_3^2)\, u_3 + (-5 + x + y + z)\, v_1$

$\nabla L = 0 \to \begin{pmatrix} x - u_1 + v_1 == 0 \\ y - u_2 + v_1 == 0 \\ 1 + \frac{z}{10} - u_3 + v_1 == 0 \\ -x + s_1^2 == 0 \\ -y + s_2^2 == 0 \\ -z + s_3^2 == 0 \\ -5 + x + y + z == 0 \\ 2\, s_1\, u_1 == 0 \\ 2\, s_2\, u_2 == 0 \\ 2\, s_3\, u_3 == 0 \end{pmatrix}$

***** Valid KT Point(s) *****

$f \to 4.375$

$x \to 1.25$
$y \to 1.25$
$z \to 2.5$
$u_1 \to 0$
$u_2 \to 0$
$u_3 \to 0$
$s_1^2 \to 1.25$
$s_2^2 \to 1.25$
$s_3^2 \to 2.5$
$v_1 \to -1.25$

■ Pedregal Example 3.10 (p. 82) r = 1.2

```
In[30]:= f = z + (x^2 + y^2 + z^2 / 10) / 2;
         cons = {x + y + z == 1.2, x ≥ 0, y ≥ 0, z ≥ 0};
         vars = {x, y, z};

In[33]:= KTSolution[f, cons, vars, PrintLevel → 1];
```

Minimize $f \to z + \frac{1}{2}\left(x^2 + y^2 + \frac{z^2}{10}\right)$

$\nabla f \to \begin{pmatrix} x \\ y \\ 1 + \frac{z}{10} \end{pmatrix}$

***** LE constraints and their gradients

$g_1 \to -x \le 0 \quad g_2 \to -y \le 0$

$g_3 \to -z \le 0 \quad \nabla g_1 \to \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$\nabla g_2 \to \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \nabla g_3 \to \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

***** EQ constraints and their gradients

$h_1 \to -1.2 + x + y + z == 0 \quad \nabla h_1 \to \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

***** Lagrangian $\to z + \frac{1}{2}\left(x^2 + y^2 + \frac{z^2}{10}\right) + (-x + s_1^2)\, u_1 + (-y + s_2^2)\, u_2 + (-z + s_3^2)\, u_3 + (-1.2 + x + y + z)\, v_1$

$$\nabla L=0 \to \begin{pmatrix} x - u_1 + v_1 == 0 \\ y - u_2 + v_1 == 0 \\ 1 + \frac{z}{10} - u_3 + v_1 == 0 \\ -x + s_1^2 == 0 \\ -y + s_2^2 == 0 \\ -z + s_3^2 == 0 \\ -1.2 + x + y + z == 0 \\ 2\, s_1\, u_1 == 0 \\ 2\, s_2\, u_2 == 0 \\ 2\, s_3\, u_3 == 0 \end{pmatrix}$$

***** Valid KT Point(s) *****

$f \to 0.36$
$x \to 0.6$
$y \to 0.6$
$z \to 0$
$u_1 \to 0$
$u_2 \to 0$
$u_3 \to 0.4$
$s_1^2 \to 0.6$
$s_2^2 \to 0.6$
$s_3^2 \to 0$
$v_1 \to -0.6$

■ Pedregal Example 3.10 (p. 82) r generic

■ Pedregal Example Modified

```
In[10]:= f = z + (x^2 + y^2 + z^2 / 10) / 2;
         h = x^2 + 2 y^2 == 1;
         vars = {x, y, z};

In[13]:= KTSolution[f, h, vars, PrintLevel → 1];
```

Minimize $f \to z + \frac{1}{2}\left(x^2 + y^2 + \frac{z^2}{10}\right)$

$\nabla f \to \begin{pmatrix} x \\ y \\ 1 + \frac{z}{10} \end{pmatrix}$

***** EQ constraints and their gradients

$h_1 \to -1 + x^2 + 2\,y^2 == 0 \quad \nabla h_1 \to \begin{pmatrix} 2\,x \\ 4\,y \\ 0 \end{pmatrix}$

***** Lagrangian $\to z + \frac{1}{2}\left(x^2 + y^2 + \frac{z^2}{10}\right) + (-1 + x^2 + 2\,y^2)\,v_1$

$\nabla L{=}0 \to \begin{pmatrix} x + 2\,x\,v_1 == 0 \\ y + 4\,y\,v_1 == 0 \\ 1 + \frac{z}{10} == 0 \\ -1 + x^2 + 2\,y^2 == 0 \end{pmatrix}$

***** Valid KT Point(s) *****

$f \to -4.5$	$f \to -4.75$	$f \to -4.75$	$f \to -4.5$
$x \to -1.$	$x \to 0$	$x \to 0$	$x \to 1.$
$y \to 0$	$y \to -0.707107$	$y \to 0.707107$	$y \to 0$
$z \to -10.$	$z \to -10.$	$z \to -10.$	$z \to -10.$
$v_1 \to -0.5$	$v_1 \to -0.25$	$v_1 \to -0.25$	$v_1 \to -0.5$

■ Example

```
In[34]:= f = x^2 + y^2;
         h = x^2 + 2 y^2 == 1;
         vars = {x, y};
```

In[37]:= **KTSolution[f, h, vars, PrintLevel → 1];**

Minimize $f \to x^2 + y^2$

$\nabla f \to \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

***** EQ constraints and their gradients

$h_1 \to -1 + x^2 + 2y^2 == 0 \quad \nabla h_1 \to \begin{pmatrix} 2x \\ 4y \end{pmatrix}$

***** Lagrangian $\to x^2 + y^2 + (-1 + x^2 + 2y^2)\, v_1$

$\nabla L=0 \to \begin{pmatrix} 2x + 2x v_1 == 0 \\ 2y + 4y v_1 == 0 \\ -1 + x^2 + 2y^2 == 0 \end{pmatrix}$

***** Valid KT Point(s) *****

$f \to 1.$	$f \to 0.5$	$f \to 0.5$	$f \to 1.$
$x \to -1.$	$x \to 0$	$x \to 0$	$x \to 1.$
$y \to 0$	$y \to -0.707107$	$y \to 0.707107$	$y \to 0$
$v_1 \to -1.$	$v_1 \to -0.5$	$v_1 \to -0.5$	$v_1 \to -1.$

In[38]:= **KTSolution[f, h, vars, PrintLevel → 2];**

Minimize $f \to x^2 + y^2$

$\nabla f \to \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

***** EQ constraints and their gradients

$h_1 \to -1 + x^2 + 2y^2 == 0 \quad \nabla h_1 \to \begin{pmatrix} 2x \\ 4y \end{pmatrix}$

***** Lagrangian $\to x^2 + y^2 + (-1 + x^2 + 2y^2)\, v_1$

$\nabla L=0 \to \begin{pmatrix} 2x + 2x v_1 == 0 \\ 2y + 4y v_1 == 0 \\ -1 + x^2 + 2y^2 == 0 \end{pmatrix}$

************ Case 1 ************

Active inequalities → None

Known values → {}

Equations for this case $\to \begin{pmatrix} 2x + 2x v_1 == 0 \\ 2y + 4y v_1 == 0 \\ -1 + x^2 + 2y^2 == 0 \end{pmatrix}$

-----Solution 1-----

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -1. \\ 0 \end{pmatrix}$

$(v_1) \to (-1.)$

KT Status → Valid KT Point Objective function value → 1.

-----Solution 2-----

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ -0.707107 \end{pmatrix}$$

$(v_1) \to (-0.5)$

```
KT Status → Valid KT Point  Objective function value → 0.5

-----Solution 3-----
```

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0.707107 \end{pmatrix}$$

$(v_1) \to (-0.5)$

```
KT Status → Valid KT Point  Objective function value → 0.5

-----Solution 4-----
```

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1. \\ 0 \end{pmatrix}$$

$(v_1) \to (-1.)$

```
KT Status → Valid KT Point  Objective function value → 1.

***** Valid KT Point(s) *****
```

f → 1.	f → 0.5	f → 0.5	f → 1.
x → -1.	x → 0	x → 0	x → 1.
y → 0	y → -0.707107	y → 0.707107	y → 0
v_1 → -1.	v_1 → -0.5	v_1 → -0.5	v_1 → -1.

■ Example 4.8

Obtain all points satisfying KT conditions for the following optimization problem.

Minimize $f(x, y, z) = 14 - 2x + x^2 - 4y + y^2 - 6z + z^2$

Subject to $h(x, y, z) : x^2 + y^2 + z - 1 = 0$

Using KTSolution function, the solution is obtained as follows.

```
f = 14 - 2 x + x^2 - 4 y + y^2 - 6 z + z^2;
h = x^2 + y^2 + z - 1 == 0;
vars = {x, y, z};
```

KTSolution[f, h, vars, PrintLevel → 2];

Minimize $f \to 14 - 2x + x^2 - 4y + y^2 - 6z + z^2$

$\nabla f \to \begin{pmatrix} -2+2x \\ -4+2y \\ -6+2z \end{pmatrix}$

***** EQ constraints and their gradients

$h_1 \to -1 + x^2 + y^2 + z == 0 \quad \nabla h_1 \to \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$

***** Lagrangian → $14 - 2x + x^2 - 4y + y^2 - 6z + z^2 + (-1 + x^2 + y^2 + z)\, v_1$

$\nabla L=0 \to \begin{pmatrix} -2 + 2x + 2x v_1 == 0 \\ -4 + 2y + 2y v_1 == 0 \\ -6 + 2z + v_1 == 0 \\ -1 + x^2 + y^2 + z == 0 \end{pmatrix}$

************ Case 1 ************

Active inequalities → None

Known values → {}

Equations for this case → $\begin{pmatrix} -2 + 2x + 2x v_1 == 0 \\ -4 + 2y + 2y v_1 == 0 \\ -6 + 2z + v_1 == 0 \\ -1 + x^2 + y^2 + z == 0 \end{pmatrix}$

-----Solution 1-----

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} 0.186935 \\ 0.37387 \\ 0.825276 \end{pmatrix}$

$(v_1) \to (4.34945)$

KT Status → Valid KT Point Objective function value → 8.0348

***** Valid KT Point(s) *****

f → 8.0348
x → 0.186935
y → 0.37387
z → 0.825276
v_1 → 4.34945

- **Example 4.9**
- **Example 4.10**
- **Example 4.11 - Building design**
- **Example**

4.2 Geometric Interpretation of KT Conditions

■ Example 4.12

Consider solution of the following two variable minimization problem.

```
f =  -x - y;
g = {x + y^2 - 5 ≤ 0, x - 2 ≤ 0};
vars = {x, y};
```

Solution using KT conditions is as follows.

```
KTSolution[f, g, vars, PrintLevel → 2];
```

Minimize f → -x - y

$$\nabla f \to \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

***** LE constraints and their gradients

$g_1 \to -5 + x + y^2 \leq 0 \quad g_2 \to -2 + x \leq 0$

$$\nabla g_1 \to \begin{pmatrix} 1 \\ 2\,y \end{pmatrix} \quad \nabla g_2 \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

***** Lagrangian → $-x - y + (-5 + x + y^2 + s_1^2)\,u_1 + (-2 + x + s_2^2)\,u_2$

$$\nabla L = 0 \to \begin{pmatrix} -1 + u_1 + u_2 == 0 \\ -1 + 2\,y\,u_1 == 0 \\ -5 + x + y^2 + s_1^2 == 0 \\ -2 + x + s_2^2 == 0 \\ 2\,s_1\,u_1 == 0 \\ 2\,s_2\,u_2 == 0 \end{pmatrix}$$

************ Case 1 ************

Active inequalities → None

Known values → $\{u_1 \to 0,\ u_2 \to 0\}$

$$\text{Equations for this case} \to \begin{pmatrix} \text{False} \\ \text{False} \\ -5 + x + y^2 + s_1^2 == 0 \\ -2 + x + s_2^2 == 0 \end{pmatrix}$$

No solution for this case

************ Case 2 ************

Active inequalities → {1}

Known values → $\{u_2 \to 0,\ s_1 \to 0\}$

$$\text{Equations for this case} \to \begin{pmatrix} -1 + u_1 == 0 \\ -1 + 2\,y\,u_1 == 0 \\ -5 + x + y^2 == 0 \\ -2 + x + s_2^2 == 0 \end{pmatrix}$$

-----Solution 1-----

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 4.75 \\ 0.5 \end{pmatrix}$

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} 1. \\ 0 \end{pmatrix} \quad \begin{pmatrix} s_1^2 \\ s_2^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ -2.75 \end{pmatrix}$

KT Status → Invalid KT Point Objective function value → -5.25

************ Case 3 ************

Active inequalities → {2}

Known values → $\{u_1 \to 0, s_2 \to 0\}$

Equations for this case → $\begin{pmatrix} -1 + u_2 == 0 \\ \text{False} \\ -5 + x + y^2 + s_1^2 == 0 \\ -2 + x == 0 \end{pmatrix}$

No solution for this case

************ Case 4 ************

Active inequalities → {1, 2}

Known values → $\{s_1 \to 0, s_2 \to 0\}$

Equations for this case → $\begin{pmatrix} -1 + u_1 + u_2 == 0 \\ -1 + 2\, y\, u_1 == 0 \\ -5 + x + y^2 == 0 \\ -2 + x == 0 \end{pmatrix}$

-----Solution 1-----

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2. \\ -1.73205 \end{pmatrix}$

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} -0.288675 \\ 1.28868 \end{pmatrix} \quad \begin{pmatrix} s_1^2 \\ s_2^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Constraint gradient matrix → $\begin{pmatrix} 1 & 1 \\ -3.4641 & 0 \end{pmatrix}$ Rank → 2

Regularity status → RegularPoint

KT Status → Invalid KT Point Objective function value → -0.267949

-----Solution 2-----

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2. \\ 1.73205 \end{pmatrix}$

$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} 0.288675 \\ 0.711325 \end{pmatrix} \quad \begin{pmatrix} s_1^2 \\ s_2^2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Constraint gradient matrix → $\begin{pmatrix} 1 & 1 \\ 3.4641 & 0 \end{pmatrix}$ Rank → 2

Regularity status → RegularPoint

KT Status → Valid KT Point Objective function value → -3.73205

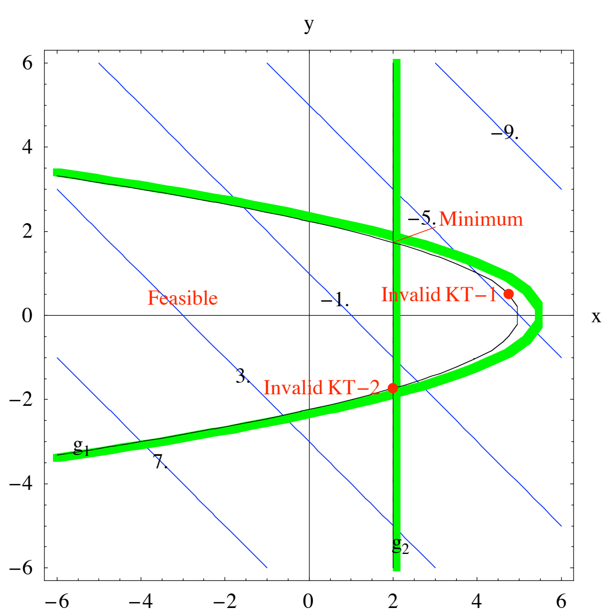
```
***** Valid KT Point(s) *****

f → -3.73205
x → 2.
y → 1.73205
u1 → 0.288675
u2 → 0.711325
s1^2 → 0
s2^2 → 0
```

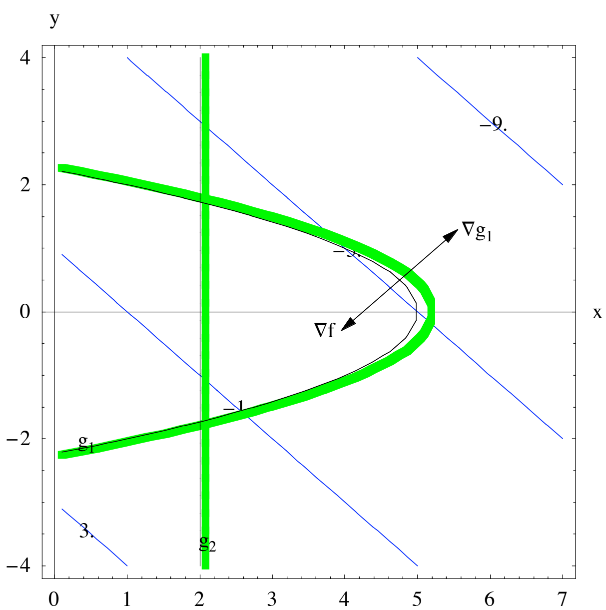
There are three points that satisfy the equations resulting from setting gradient of Lagrangian to zero . However two of these points are rejected. The first one ($x = 4.75$, $y = 0.5$) was rejected because the slack variable for the second constraint was negative. The second one $\left(x = 2,\ y = -1.732 = -\sqrt{3}\right)$ was rejected because the Lagrange multiplier for the first constraint was negative at this point. Only one point $\left(x = 2,\ y = 1.732 = \sqrt{3}\right)$ satisfies all requirements of the KT conditions.

To understand exactly what is going on, all these points are labelled on the graphical solution in Figure. From the graph it is clear that the global minimum is at the upper intersection of constraints g_1 and g_2 ($x_1 = 2$ and $x_2 = \sqrt{3}$). This is the same valid KT point computed by KTSolution. The point labelled Invalid KT-1 clearly violates the second constraint. During KT solution this violation was indicated by the negative slack variable for this constraint. The reason why this point satisfied gradient condition is also clear from the second graph that shows gradients of active constraint (g_1) and the objective function. The gradient of the objective function and the first constraint are pointing exactly in the opposite directions at this point.

```
grf = GraphicalSolution[f, {x, -6, 6}, {y, -6, 6},
    Constraints -> g, AspectRatio -> 1, ShadingOffset -> 0.08,
   PlotPoints -> 30, ObjectiveContours -> {-9, -5, -1, 3, 7},
   Epilog -> {RGBColor[1, 0, 0], Line[{{2, √3}, {3, 2.1}}],
     Text["Minimum", {3, 2.3}, {-1, 0}], Text["Feasible", {-3, .4}],
            PointSize[.02],
            Point[{4.75, .5}], Text["Invalid KT-1", {4.6, .5}, {1, 0}],
                Point[{2, -√3}], Text["Invalid KT-2", {1.8, -√3}, {1, 0}]}];
```



```
gr1 = GraphicalSolution[f, {x, 0.1, 7}, {y, -4, 4},
    Constraints -> g, ObjectiveContours -> {-9, -5, -1, 3, 7},
   AspectRatio -> 1, ShadingOffset -> 0.08, PlotPoints -> 30,
    GradientVectors -> {{f, First[g[[1]]]}, {"∇f", Subscript["∇g", 1]}, {{4.75, .5}}},
    GradientVectorScale -> 0.8];
```



The gradient vectors shown in the graphs in Figure clarify why the lower intersection of g_1 and g_2 does not satisfy the KT conditions. The resultant vector of gradient vectors ∇g_1 and ∇g_2 will obviously lie in the parallelogram formed by these vectors. The multipliers (with positive values) will simply increase or decrease the size of this parallelogram. From the direction of the ∇f vector at the upper intersection it is clear that for some values of the multipliers the resultant vector can be made to lie along the same line (but opposite in direction) to the ∇f vector. However, this is impossible to do at the lower intersection. Thus even though both points satisfy the same set of constraints, only the upper one is a KT point.

Finally we can also observe that resultant of the active constraint gradient vector is exactly same as that of the aggregate constraint if we use the multipliers obtained from the KT solution. From the KT solution we get the following values of the Lagrange multipliers for the two active constraints.

$u_1 = 0.288675 \qquad u_2 = 0.711325$

With these multipliers, the aggregate constraint is defined as follows.

```
ga = Expand[0.288675 First[g⟦1⟧] + 0.711325 First[g⟦2⟧]] ≤ 0
```

$-2.86603 + 1.\,x + 0.288675\,y^2 \leq 0$

The gradient of this aggregate constraint and that of the objective function are as follows.

```
grd = Grad[{f, First[ga]}, vars]
```

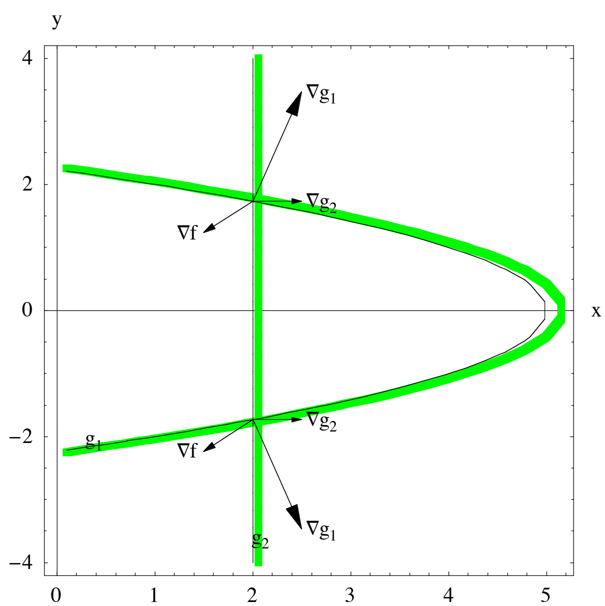
```
{{-1, -1}, {1., 0.57735 y}}
```

At the two intersection points these gradients are as follows.

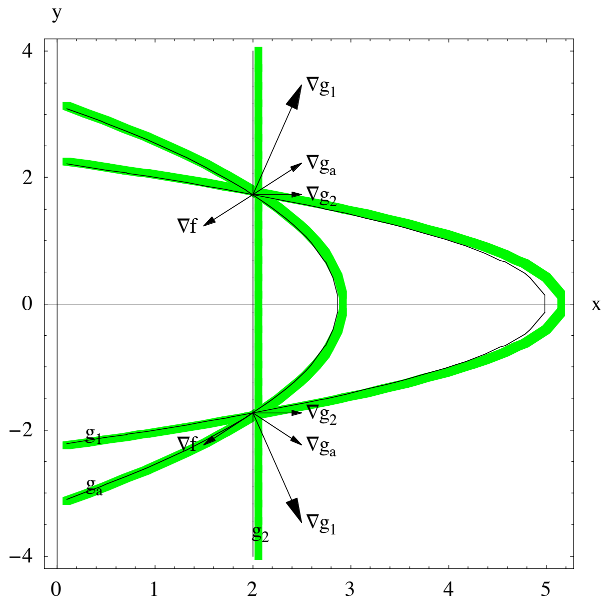
```
{grd /. {x → 2, y → √3}, grd /. {x → 2, y → -√3}}
```

```
{{{-1, -1}, {1., 1.}}, {{-1, -1}, {1., -1.}}}
```

```
gr3 = GraphicalSolution[1, {x, 0.1, 6}, {y, -4, 4},
    Constraints -> g, AspectRatio -> 1, ShadingOffset -> 0.06, PlotPoints -> 30,
    GradientVectors -> {{f, First[g[[1]]], First[g[[2]]]},
      {"∇f", Subscript["∇g", 1], Subscript["∇g", 2]}, {{2, 1.732}, {2, -1.732}}},
    GradientVectorScale -> 0.5];
```



```
gl = {Subscript["g", 1], Subscript["g", 2], Subscript["g", "a"]};
gr4 = GraphicalSolution[1, {x, 0.1, 6}, {y, -4, 4},
    Constraints -> Join[g, {ga}], ConstraintLabels -> gl,
   AspectRatio -> 1, ShadingOffset -> 0.06, PlotPoints -> 30,
    GradientVectors -> {{f, First[g[[1]]], First[g[[2]]], First[ga]},
     {"∇f", Subscript["∇g", 1], Subscript["∇g", 2], Subscript["∇g", "a"]},
     {{2, 1.732}, {2, -1.732}}},
    GradientVectorScale -> 0.5];
```



Note that at the upper intersection point the two are equal and opposite but not at the lower intersection. Therefore only the upper intersection point satisfies KT conditions. The same thing is shown graphically by plotting the aggregate constraint, together with the actual constraints.

- **Example 4.13**
- **Example 4.14 - Abnormal case**
- **Example**

4.3 Second Order Sufficient Conditions

```
Information["ConstrainedSufficiency", LongForm → False]
```

```
ConstrainedSufficiency[f,con,vars,ktsoln,opts], performs computations necessary for checking
  second order sufficient conditions for constrained problems. It forms the hessian of
  the Lagrangian (HL), determines the feasible changes (d), and returns d.HL.d. f is the
  objective function. con is a list of constraints. vars is a list of problem variables.
  ktsoln contain solution point and Lagrange multipliers obtained from KT solution.
  Several options can be used with the function. See Options[ConstrainedSufficiency].
```

```
OptionsUsage[ConstrainedSufficiency]
```

```
{PrintLevel → 1, FeasibleChangesName → d}
```

```
PrintLevel is an option for most functions in the OptimizationToolbox. It is specified as an
  integer. The value of the integer indicates how much intermediate information is to be printed.
  A PrintLevel->0 suppresses all printing. The default for most functions is set to 1, in which
  case they print only the initial problem setup. Higher integers print more intermediate results.

FeasibleChangesName->Variable names used for feasible changes. Default is 'd'.
```

■ Example 4.20

■ Example 4.21

■ Example 4.22 - Open top container

Consider solution of the open-top rectangular container problem formulated in Chapter 1. The problem statement is as follows.

A company requires open-top rectangular containers to transport material. Using the following data, formulate an optimum design problem to determine the container dimensions for minimum annual cost.

Construction costs	Sides = $65 / m^2 Ends = $80 / m^2 Bottom = $120 / m^2
Useful life	10 years
Salvage value	20 % of the initial construction cost
Yearly maintenance cost	$12 / m^2 of the outside surface area
Minimum required volume of the container	1200 m^3
Nominal interest rate	10 % (Annual compounding)

The design variables are the dimensions of the box.

b = Width of container ℓ = Length of container h = height of container

Considering time value of money, the annual cost is written as the following function of design variables (see Chapter 1 for details)

Annual cost $= 48.0314\, b\, h + 30.0236\, b\, \ell + 43.5255\, h\, \ell$

The optimization problem is stated as follows.

Find b, h, and ℓ to

Minimize annual cost $= 48.0314\, b\, h + 30.0236\, b\, \ell + 43.5255\, h\, \ell$

Subject to $b\, h\, \ell \geq 1200$ and b, h, and $\ell \geq 0$

```
Clear[b, h, ℓ];
vars = {b, h, ℓ};
f = 48.0314 b h + 30.0236 b ℓ + 43.5255 h ℓ;
g = {-b h ℓ + 1200 ≤ 0};
```

A valid KT point is obtained when the volume constraint is active.

```
sol = KTSolution[f, g, vars, ActiveCases → {{1}}];
```

Minimize $f \to 48.0314\, b\, h + 30.0236\, b\, \ell + 43.5255\, h\, \ell$

$$\nabla f \to \begin{pmatrix} 48.0314\, h + 30.0236\, \ell \\ 48.0314\, b + 43.5255\, \ell \\ 30.0236\, b + 43.5255\, h \end{pmatrix}$$

***** LE constraints and their gradients

$$g_1 \to 1200 - b\, h\, \ell \leq 0 \quad \nabla g_1 \to \begin{pmatrix} -h\, \ell \\ -b\, \ell \\ -b\, h \end{pmatrix}$$

***** Lagrangian $\to 48.0314\, b\, h + 30.0236\, b\, \ell + 43.5255\, h\, \ell + (1200 - b\, h\, \ell + s_1^2)\, u_1$

$$\nabla L = 0 \to \begin{pmatrix} 48.0314\, h + 30.0236\, \ell - h\, \ell\, u_1 == 0 \\ 48.0314\, b + 43.5255\, \ell - b\, \ell\, u_1 == 0 \\ 30.0236\, b + 43.5255\, h - b\, h\, u_1 == 0 \\ 1200 - b\, h\, \ell + s_1^2 == 0 \\ 2\, s_1\, u_1 == 0 \end{pmatrix}$$

***** Valid KT Point(s) *****

$f \to 13463.3$
$b \to 11.6384$
$h \to 8.0281$
$\ell \to 12.8433$
$u_1 \to 7.47963$
$s_1^2 \to 0$

```
ktsoln = Drop[Flatten[sol[[1]]], {5}]
```

{11.6384, 8.0281, 12.8433, 7.47963}

```
ConstrainedSufficiency[f, g, vars, ktsoln];
```

$$Hg_1 \to \begin{pmatrix} 0 & -12.8433 & -8.0281 \\ -12.8433 & 0 & -11.6384 \\ -8.0281 & -11.6384 & 0 \end{pmatrix}$$

Gradients $\to \begin{pmatrix} -103.107 & -149.475 & -93.4343 \end{pmatrix}$

Equations $\to \begin{pmatrix} -103.107\, d_1 - 149.475\, d_2 - 93.4343\, d_3 == 0 \end{pmatrix}$

$$\text{Feasible changes} \to \begin{pmatrix} 0. - 1.44971\, d_2 - 0.906188\, d_3 \\ d_2 \\ d_3 \end{pmatrix}$$

$$Hf \to \begin{pmatrix} 0 & 48.0314 & 30.0236 \\ 48.0314 & 0 & 43.5255 \\ 30.0236 & 43.5255 & 0 \end{pmatrix} \quad HL \to \begin{pmatrix} 0 & -48.0314 & -30.0236 \\ -48.0314 & 0 & -43.5255 \\ -30.0236 & -43.5255 & 0 \end{pmatrix}$$

$d.HL.d \to 139.263\, d_2^2 + 87.051\, d_2\, d_3 + 54.4141\, d_3^2$

This is a quadratic function in (d_2, d_3). To determine sign of this term we write it in a quadratic form and determine the principal minors.

$$q = \frac{1}{2} \begin{pmatrix} d_2 & d_3 \end{pmatrix} \begin{pmatrix} 278.526 & 87.051 \\ 87.051 & 108.828 \end{pmatrix} \begin{pmatrix} d_2 \\ d_3 \end{pmatrix}$$

The principal minors of the matrix are

$$M_1 = 278.526 \qquad M_2 = \text{Det}\left[\begin{pmatrix} 278.526 & 87.051 \\ 87.051 & 108.828 \end{pmatrix}\right] = 22733.6$$

Since both principal minors are positive, the matrix is positive definite, and hence the sign of q is always positive. This shows that the computed KT point is a minimum point.

■ Example

```
f = x^2 - 4 y + y^2;
g = {-x^2 + y^2 ≤ 0, 1/4 - Log[x + 2 y] ≤ 0};
vars = {x, y};
```

sol = KTSolution[f, g, vars, PrintLevel → 1, SolveEquationsUsing → Solve];

Minimize $f \to x^2 - 4\,y + y^2$

$$\nabla f \to \begin{pmatrix} 2\,x \\ -4 + 2\,y \end{pmatrix}$$

***** LE constraints and their gradients

$g_1 \to -x^2 + y^2 \le 0 \quad g_2 \to \frac{1}{4} - \mathrm{Log}[x + 2\,y] \le 0$

$$\nabla g_1 \to \begin{pmatrix} -2\,x \\ 2\,y \end{pmatrix} \quad \nabla g_2 \to \begin{pmatrix} -\frac{1}{x+2\,y} \\ -\frac{2}{x+2\,y} \end{pmatrix}$$

***** Lagrangian $\to x^2 - 4\,y + y^2 + (-x^2 + y^2 + s_1^2)\,u_1 + (\frac{1}{4} - \mathrm{Log}[x + 2\,y] + s_2^2)\,u_2$

$$\nabla L = 0 \to \begin{pmatrix} 2\,x - 2\,x\,u_1 - \frac{u_2}{x+2\,y} == 0 \\ -4 + 2\,y + 2\,y\,u_1 - \frac{2\,u_2}{x+2\,y} == 0 \\ -x^2 + y^2 + s_1^2 == 0 \\ \frac{1}{4} - \mathrm{Log}[x + 2\,y] + s_2^2 == 0 \\ 2\,s_1\,u_1 == 0 \\ 2\,s_2\,u_2 == 0 \end{pmatrix}$$

Solve::incnst :
 Inconsistent or redundant transcendental equation. After reduction, the bad equation is $4 - x - 2\,y == 0$.

Solve::incnst :
 Inconsistent or redundant transcendental equation. After reduction, the bad equation is $x + 2\,y == 0$.

Solve::incnst :
 Inconsistent or redundant transcendental equation. After reduction, the bad equation is $-x - 2\,y == 0$.

General::stop : Further output of Solve::incnst will be suppressed during this calculation.

Union::smtst : SameTest function

$\left(\mathrm{Norm}[\mathrm{Flatten}[\#1] - \mathrm{Flatten}[\#2]] \le \frac{1}{10^7}\ \&\right)\left[\ll 1 \gg, \left\{\{-e^{1/4}, e^{1/4}\}, \left\{3 - \frac{2}{e^{1/4}}, 4\left(-e^{1/4} + \sqrt{e}\right)\right\}, \{0, 0\}, \{\}\right\}\right]$

evaluates to $\sqrt{2.08051 + (\ll 19 \gg + 0.5\,(\ll 1 \gg))^2 + \ll 1 \gg + 0.0625\,(\ll 1 \gg)^2 + (-1.45878 + 0.5\,(\ll 1 \gg))^2} \le \ll 1 \gg$.

Union::smtst :

SameTest function $\left(\mathrm{Norm}[\mathrm{Flatten}[\#1] - \mathrm{Flatten}[\#2]] \le \frac{1}{10^7}\ \&\right)\left[\ll 1 \gg, \left\{\{1, 1\}, \{1, 0\}, \left\{0, \frac{1}{4}\,(-1 + 4\,\mathrm{Log}[3])\right\}, \{\}\right\}\right]$

evaluates to $\sqrt{1.72014 + (-1. + 0.5\,(-2. + y))^2 + \ll 1 \gg^2 + 0.0625\,(\ll 1 \gg)^2 + 0.25\,(4. - 12.\,y + 5.\,y^2)^2} \le \frac{1}{\ll 8 \gg}$.

***** Valid KT Point(s) *****

$f \to -2$	$f \to -4\,e^{1/4} + 2\sqrt{e}$	$f \to \frac{1}{4}\,(-2 + y)^2 - 4\,y + y^2$
$x \to 1$	$x \to -e^{1/4}$	$x \to \frac{1}{2}\,(-2 + y)$
$y \to 1$	$y \to e^{1/4}$	$y \to y$
$u_1 \to 1$	$u_1 \to 3 - \frac{2}{e^{1/4}}$	$u_1 \to 0$
$u_2 \to 0$	$u_2 \to 4\,(-e^{1/4} + \sqrt{e})$	$u_2 \to \frac{1}{2}\,(4 - 12\,y + 5\,y^2)$
$s_1^2 \to 0$	$s_1^2 \to 0$	$s_1^2 \to \frac{1}{4}\,(4 - 4\,y - 3\,y^2)$
$s_2^2 \to \frac{1}{4}\,(-1 + 4\,\mathrm{Log}[3])$	$s_2^2 \to 0$	$s_2^2 \to 0$

ktSol = Drop[Flatten[sol[[1]]], -3]

{1, 1, 1}

```
ConstrainedSufficiency[f, g[[1]], vars, ktSol, PrintLevel → 1];
```

$Hg_1 \to \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

Gradients → $\begin{pmatrix} -2 & 2 \end{pmatrix}$

Equations → $\begin{pmatrix} -2\, d_1 + 2\, d_2 == 0 \end{pmatrix}$

Feasible changes → $\begin{pmatrix} d_2 \\ d_2 \end{pmatrix}$

Hf → $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ HL → $\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$

d.HL.d → $4\, d_2^2$