

# Appendices

## A The Gamma Function

The gamma function is related to many quite simple, but at the same time important, special functions. Familiarity with its properties is essential for further study of special functions. Moreover, many integrals that are met with in analysis can be evaluated in terms of gamma functions. In particular, this is the case for the beta function integral.

**1. Definition of  $\Gamma(z)$  and  $B(u, v)$ .** The gamma function  $\Gamma(z)$  and the beta function  $B(u, v)$  are defined as follows:

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad \operatorname{Re} z > 0; \quad (1)$$

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt, \quad \operatorname{Re} u > 0, \quad \operatorname{Re} v > 0. \quad (2)$$

By Theorem 2 (see §3),  $\Gamma(z)$  is analytic where it is defined. In fact, (1) converges uniformly in  $z$  in the domain  $0 < \delta \leq \operatorname{Re} z \leq A$  for arbitrary  $A$  and  $\delta$ , since

$$|e^{-t} t^{z-1}| \leq \begin{cases} t^{\delta-1} & \text{for } 0 < t \leq 1, \\ e^{-t} t^{A-1} & \text{for } t > 1 \end{cases}$$

and the integrals  $\int_0^1 t^{\delta-1} dt$  and  $\int_1^{\infty} e^{-t} t^{A-1} dt$  converge.

It can be shown similarly that  $B(u, v)$  is analytic in both arguments when  $\operatorname{Re} u > 0$  and  $\operatorname{Re} v > 0$ .

The beta function can be expressed in terms of gamma functions. To show this it is enough to calculate the integral

$$I(u, v) = \int \int e^{-(\xi^2 + \eta^2)} \xi^{2u-1} \eta^{2v-1} d\xi d\eta$$

in two ways; the integration is over  $\xi > 0, \eta > 0$ . On one hand

$$I(u, v) = I(u)I(v),$$

where

$$I(u) = \int_0^\infty e^{-\xi^2} \xi^{2u-1} d\xi = \frac{1}{2} \int_0^\infty e^{-t} t^{u-1} dt = \frac{1}{2} \Gamma(u).$$

On the other hand, if we transform  $I(u, v)$  to polar coordinates  $\xi = r \cos \phi$ ,  $\eta = r \sin \phi$ , we obtain

$$\begin{aligned} I(u, v) &= \int_0^\infty e^{-r^2} r^{2u+2v-1} dr \int_0^{\pi/2} \cos^{2u-1} \phi \sin^{2v-1} \phi d\phi \\ &= \frac{1}{2} \Gamma(u+v) \int_0^{\pi/2} \cos^{2u-1} \phi \sin^{2v-1} \phi d\phi. \end{aligned}$$

With the substitution  $\cos^2 \phi = t$ , the  $\phi$ -integral is expressible in terms of  $B(u, v)$ :

$$\int_0^{\pi/2} \cos^{2u-1} \phi \sin^{2v-1} \phi d\phi = \frac{1}{2} B(u, v).$$

Substituting into  $I(u, v)$ , we obtain

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \quad (3)$$

**2. Functional equations.** The function  $\Gamma(z)$  satisfies the following functional equations:

$$\Gamma(z+1) = z\Gamma(z), \quad (4)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad (5)$$

$$2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) \Gamma(2z). \quad (6)$$

These play an important role in various transformations involving gamma functions. Equation (5) is the *addition formula*, and (6) is the *duplication formula*.

To prove (4)–(6), it is convenient to write them, using (3), as functional equations for the beta function:

$$B(z, 1) = 1/z, \quad (7)$$

$$B(z, 1 - z) = \pi / \sin \pi z, \quad (8)$$

$$2^{2z-1} B(z, z) = B\left(z, \frac{1}{2}\right). \quad (9)$$

Equations (7)–(9) can be derived by direct calculation of the beta integral (2). We have

$$B(z, 1) = \int_0^1 t^{z-1} dt = \frac{1}{z},$$

which is (7).

To prove (8), we put  $u = z, v = 1 - z$  in (2):

$$B(z, 1 - z) = \int_0^1 \left(\frac{t}{1-t}\right)^{z-1} \frac{dt}{1-t}, \quad 0 < \operatorname{Re} z < 1.$$

Now put  $s = t/(1-t)$ . This yields

$$B(z, 1 - z) = \int_0^\infty \frac{s^{z-1}}{1+s} ds.$$

The resulting integral can be evaluated by residue calculus. For this purpose we replace integration along the real axis by integration around the closed contour  $C$  shown in Figure 15. The function

$$f(s) = \frac{s^{z-1}}{1+s} \quad (0 < \arg s < 2\pi)$$

has a pole at  $s = e^{i\pi}$  and no other singular points in the domain bounded by  $C$ . Hence when  $R > 1$  (see Fig. 15, p.372)

$$\int_C f(s) ds = 2\pi i \operatorname{Res}_{s=e^{i\pi}} f(s) = -2\pi i e^{i\pi z}.$$

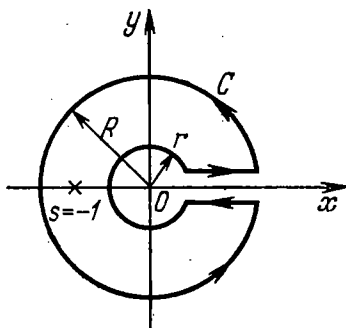


Figure 15.

On the other hand, since  $0 < \operatorname{Re} z < 1$ , the integrals over the circumferences of radii  $r$  and  $R$  tend to 0 as  $r \rightarrow 0$  and  $R \rightarrow \infty$ , and the integral over the lower side of the cut differs from the integral over the upper side by the factor  $-e^{2\pi iz}$ . Hence when  $r \rightarrow 0$  and  $R \rightarrow \infty$  we obtain

$$B(z, 1-z)(1 - e^{2\pi iz}) = -2\pi i e^{i\pi z},$$

which is equivalent to (8) for  $0 < \operatorname{Re} z < 1$ .

To prove (9), we put  $u = v = z$  in (2):

$$B(z, z) = \int_0^1 [t(1-t)]^{z-1} dt, \quad \operatorname{Re} z > 0.$$

Since the parabola  $y = t(1-t)$  is symmetric with respect to the line  $t = 1/2$ , we have

$$B(z, z) = 2 \int_0^{1/2} [t(1-t)]^{z-1} dt,$$

whence after substituting  $s = 4t(1-t)$  we obtain

$$B(z, z) = \frac{1}{2^{2z-1}} \int_0^1 s^{z-1} (1-s)^{-1/2} ds = \frac{B(z, 1/2)}{2^{2z-1}},$$

and this is equivalent to (9) when  $\operatorname{Re} z > 0$ .

Consequently we have established the functional equations (4)–(6) for the gamma function.

As an illustration, we calculate the values of  $\Gamma(z)$  for integral and half-integral arguments. It follows from (4) that

$$\Gamma(n+1) = n!,$$

since  $\Gamma(1) = 1$ . We see that the gamma function generalizes the factorial. Also, if we put  $z = 1/2$  in (5), we obtain

$$\Gamma(1/2) = \sqrt{\pi}.$$

Consequently we can rewrite (6) in the form

$$2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\Gamma(2z). \quad (6a)$$

Taking  $z = n + 1/2$ , we obtain

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!}. \quad (10)$$

We can continue  $\Gamma(z)$  analytically into the domain  $\text{Re } z > -(n+1)$  by using the formula

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\dots(z+n-1)(z+n)}, \quad (11)$$

which follows from (4). Since  $n$  can be chosen arbitrarily, we obtain an analytic continuation of  $\Gamma(z)$  for all  $z$ . We see from (11) that  $\Gamma(z)$  is analytic except at  $z = -n$  ( $n = 0, 1, 2, \dots$ ), where  $\Gamma(z)$  has simple poles with residues

$$\text{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{n!}.$$

By the principle of analytic continuation, (4)–(6) are valid for all values of  $z$  for which they make sense. The analytic continuation of the beta function can be obtained from (3).

It follows from (5) that  $\Gamma(z)$  has no zeros in the complex plane. In fact, suppose  $\Gamma(z_0) = 0$ . Evidently  $z_0 \neq n+1$  ( $n = 0, 1, 2, \dots$ ) since  $\Gamma(n+1) = n! \neq 0$ . Therefore  $\Gamma(1-z)$  would be analytic at  $z_0$ . On the other hand

$$\lim_{z \rightarrow z_0} \Gamma(1-z) = \frac{\pi}{\sin \pi z_0} \lim_{z \rightarrow z_0} \frac{1}{\Gamma(z)} = \infty,$$

which contradicts the analyticity of  $\Gamma(1-z)$  for  $z \neq n+1$ .

Figure 16 (p. 374) shows the graph of  $y = \Gamma(x)$ .

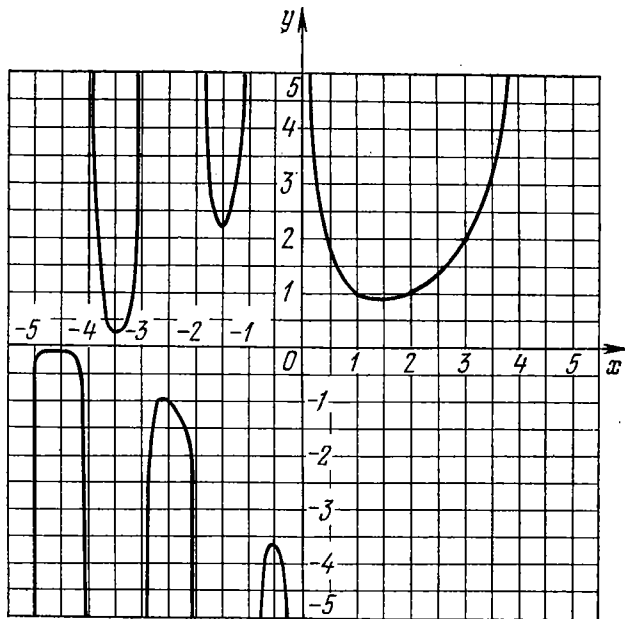


Figure 16.

**3. The logarithmic derivative of the gamma function.** The function

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

also has many applications. This function is analytic in the complex plane except for the points  $z = -n$  ( $n = 0, 1, \dots$ ), at which it has simple poles.

The following *functional equations* for  $\psi(z)$  follow from the functional equations (4)–(6) for the gamma function:

$$\psi(z+1) = 1/z + \psi(z), \quad (12)$$

$$\psi(z) = \psi(1-z) - \pi \cot \pi z, \quad (13)$$

$$2 \ln 2 + \psi(z) + \psi\left(z + \frac{1}{2}\right) = 2\psi(2z). \quad (14)$$

We also note the formula

$$\psi(z+n) = \psi(z) + \sum_{k=1}^n \frac{1}{z+k-1}. \quad (15)$$

which is easily obtained from (12).

Formulas (12)–(15) let us evaluate  $\psi(z)$  for integral or half-integral values of the argument. Let us write

$$\psi(1) = \Gamma'(1) = -\gamma.$$

The number  $\gamma$  is *Euler's constant* ( $\gamma = 0.5772156649\dots$ ). If we put  $z = 1/2$  in (14), we find

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2.$$

For  $z = 1$  and  $z = 1/2$ , (15) yields

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}, \quad (16)$$

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2\ln 2 + 2 \sum_{k=1}^n \frac{1}{2k-1}. \quad (17)$$

The integral representation for the beta function leads to an integral representation for  $\psi(z)$ . By definition,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \lim_{\Delta z \rightarrow 0} \frac{\Gamma(z) - \Gamma(z - \Delta z)}{\Gamma(z)\Delta z} = \lim_{\Delta z \rightarrow 0} \left[ \frac{1}{\Delta z} - \frac{\Gamma(z - \Delta z)}{\Gamma(z)\Delta z} \right].$$

For sufficiently small  $\Delta z > 0$ , the quotient  $\Gamma(z - \Delta z)/(\Gamma(z)\Delta z)$  approximates the beta function

$$B(z - \Delta z, \Delta z) = \frac{\Gamma(z - \Delta z)\Gamma(\Delta z)}{\Gamma(z)},$$

since

$$\lim_{\Delta z \rightarrow 0} \Delta z \Gamma(\Delta z) = \lim_{\Delta z \rightarrow 0} \Gamma(1 + \Delta z) = 1.$$

The factor  $1/\Delta z$  in the limit formula for  $\psi(z)$  is easily eliminated by considering the difference  $\psi(z) - \psi(1)$ . We have

$$\begin{aligned} \psi(z) - \psi(1) &= \psi(z) + \gamma = \lim_{\Delta z \rightarrow 0} \left[ \frac{\Gamma(1 - \Delta z)}{\Gamma(1)\Delta z} - \frac{\Gamma(z - \Delta z)}{\Gamma(z)\Delta z} \right] \\ &= \lim_{\Delta z \rightarrow 0} \frac{1}{\Delta z \Gamma(\Delta z)} [B(1 - \Delta z, \Delta z) - B(z - \Delta z, \Delta z)] \\ &= \lim_{\Delta z \rightarrow 0} \int_0^1 \frac{1 - t^{z-1}}{1 - t} \left( \frac{1 - t}{t} \right)^{\Delta z} dt. \end{aligned}$$

After taking the limit under the integral sign, which is permissible since the integral converges uniformly for small  $\Delta z$ , we obtain an *integral representation* for  $\psi(z)$ :

$$\psi(z) = -\gamma + \int_0^1 \frac{1-t^{z-1}}{1-t} dt, \quad \operatorname{Re} z > 0. \quad (18)$$

Replacing  $t$  by  $e^{-t}$  in (18), we obtain another frequently used integral representation,

$$\psi(z) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt, \quad \operatorname{Re} z > 0. \quad (19)$$

From (18) we can obtain a simple *series representation* for  $\psi(z)$  by expanding  $1/(1-t)$  in powers of  $t$  and integrating term by term:

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+z} \right). \quad (20)$$

**Asymptotic formulas.** We obtain asymptotic formulas for  $\Gamma(z)$  and  $\psi(z)$  by using the asymptotic properties of the Laplace integral (see Appendix B)

$$F(z) = \int_0^{\infty} e^{-zt} f(t) dt.$$

We first transform the integral representation (19) of  $\psi(z)$  as follows:

$$\psi(z) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{t} dt + \int_0^{\infty} (e^{-t} - e^{-zt}) \left( \frac{1}{1 - e^{-t}} - \frac{1}{t} \right) dt.$$

Hence

$$\psi(z) = \psi_0(z) - F(z),$$

where

$$F(z) = \int_0^\infty e^{-zt} f(t) dt,$$

$$f(t) = \frac{1}{1 - e^{-t}} - \frac{1}{t} = 1 - \frac{1}{t} + \frac{1}{e^t - 1},$$

$$\psi_0(z) = \int_0^\infty \frac{e^{-t} - e^{-zt}}{t} dt - \gamma + F(1).$$

We can express  $\psi_0(z)$  in terms of elementary functions. In fact,

$$\psi'_0(z) = \int_0^\infty e^{-zt} dt = \frac{1}{z},$$

so that  $\psi_0(z) = \ln z + C$ . The constant  $C$  is evaluated below.

The function  $f(t)$  satisfies the hypotheses of Appendix B for  $\theta_1 = \theta_2 = \pi/2$ . Hence for  $|\arg z| \leq \pi - \epsilon$  we have

$$\psi(z) = \ln z + C - \sum_{k=0}^{n-1} \frac{a_k k!}{z^{k+1}} + O\left(\frac{1}{z^{n+1}}\right),$$

where  $a_k$  are the coefficients of the expansion of  $f(t)$  in powers of  $t$ . They can be expressed in terms of the *Bernoulli numbers*  $B_k$ , which are multiples of the Maclaurin coefficients of  $t/(e^t - 1)$ :

$$\frac{t}{e^t - 1} = \sum_{k=0}^\infty B_k \frac{t^k}{k!}, \quad |t| < 2\pi.$$

In fact,

$$f(t) = 1 - \frac{1}{t} + \frac{1}{t} \sum_{k=0}^\infty B_k \frac{t^k}{k!} = 1 + \sum_{k=1}^\infty B_k \frac{t^{k-1}}{k!},$$

whence

$$a_0 = 1 + B_1, \quad a_k = B_{k+1}/(k + 1)! \quad (k \geq 1).$$

Since  $f(-t) = 1 - f(t)$ , we have  $a_0 = 1/2$ ,  $a_k = 0$  for  $k = 2m$  ( $m = 1, 2, \dots$ ).

Consequently

$$\psi(z) = C + \ln z - \frac{1}{2z} - \sum_{k=1}^n \frac{B_{2k}}{2kz^{2k}} + R_n(z),$$

for  $|\arg z| \leq \pi - \epsilon$ , where  $R_n(z) = O(z^{-2n-2})$ .

Since

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z),$$

if we integrate the asymptotic formula for  $\psi(z)$  we obtain

$$\ln \Gamma(z) = D + (C - 1)z + (z - 1/2) \ln z + \sum_{k=1}^n \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + \bar{R}_n(z).$$

Here  $D$  is a constant and

$$\bar{R}_n(z) = - \int_z^{\infty} R_n(\xi) d\xi,$$

where the integration is over any contour that extends to infinity. If we take this contour to be  $\xi = zt$  ( $1 \leq t < \infty$ ), we easily find that  $\bar{R}_n(z) = O(1/z^{2n+1})$ .

To determine the constants  $C$  and  $D$  we use (4), (6), and estimates for  $\ln \Gamma(z)$  that follow from the asymptotic formulas that we already have for this function:

$$\ln \Gamma(z) = D + (C - 1)z + \left(z - \frac{1}{2}\right) \ln z + O(1/z).$$

From

$$\ln \Gamma(z + 1) - \ln \Gamma(z) - \ln z = 0$$

it follows that

$$C - 1 + \left(z + \frac{1}{2}\right) \ln(1 + 1/z) = O(1/z).$$

Since

$$\ln(1 + 1/z) = 1/z + O(1/z^2),$$

we have  $C = 0$ . Similarly we obtain  $D = (\ln 2\pi)/2$  from (6a).

Using the values of  $C$  and  $D$ , we now have the following *asymptotic formulas* for  $|\arg z| \leq \pi - \epsilon$ :

$$\psi(z) = \ln z - \frac{1}{2z} - \sum_{k=1}^n \frac{B_{2k}}{2kz^{2k}} + O\left(\frac{1}{z^{2n+2}}\right), \quad (21)$$

$$\begin{aligned} \ln \Gamma(z) &= \left(z - \frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln 2\pi \\ &\quad + \sum_{k=1}^n \frac{B_{2k}}{2k(2k-1)z^{2k-1}} + O\left(\frac{1}{z^{2n+1}}\right). \end{aligned} \quad (22)$$

We can obtain a recursion formula for the Bernoulli numbers from the representation

$$t = (e^t - 1) \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} t^{m+k} \frac{B_k}{m!k!}.$$

Put  $n = m + k$  and sum the coefficients of  $t^n$ :

$$t = \sum_{n=1}^{\infty} t^n \sum_{k=0}^{n-1} \frac{B_k}{(n-k)!k!}.$$

Comparing coefficients of  $t$  on the two sides of the equation, we obtain a recursion relation for calculating the  $B_n$ :

$$\sum_{k=0}^{n-1} C_n^k B_k = 0 \quad \text{for } n > 1, \quad B_0 = 1, \quad C_n^k = \frac{n!}{(n-k)!k!}.$$

For  $n = 1$ , (22) yields

$$\Gamma(z + 1) = \sqrt{2\pi z} (z/e)^z \left[ 1 + \frac{1}{12z} + O\left(\frac{1}{z^2}\right) \right].$$

If we take  $z = n$ , we obtain *Stirling's formula*

$$n! \approx \sqrt{2\pi n} (n/e)^n.$$

We may observe that this formula is quite accurate even for small  $n$ . For example, for  $n = 1$  and  $n = 2$  it yields 0.92 and 1.92 for  $1!$  and  $2!$ , respectively.

**5. Examples.** 1) *Integrals expressible in terms of gamma functions:*

$$\int_0^{\infty} \exp(-\alpha t^{\beta}) t^{\gamma-1} dt = \frac{\Gamma(p)}{\alpha^p \beta}, \quad p = \gamma/\beta \tag{23}$$

$$(\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \gamma > 0);$$

$$\int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} dt = (b-a)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)} \tag{24}$$

$$(\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0).$$

Integral (23) can be evaluated by the substitution  $s = \alpha t^\beta$  when  $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma > 0$ , and then generalized to a wider domain of  $\alpha$ ,  $\beta$  and  $\gamma$  by the principle of analytic continuation. The substitution  $t = a + (b-a)s$  converts the integral (24) into a beta function.

2) *Some limits:*

$$\gamma = \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n \frac{1}{k} - \ln n \right], \quad (25)$$

$$\lim_{z \rightarrow \infty} \frac{\Gamma(z+a)}{\Gamma(z)z^a} = 1, \quad |\arg z| \leq \pi - \delta, \quad (26)$$

$$\lim_{z \rightarrow -n} \frac{\psi(z)}{\Gamma(z)} = (-1)^{n+1} n!. \quad (27)$$

Here (25) is obtained by taking the limit as  $n \rightarrow \infty$  in

$$\gamma = \sum_{k=1}^n \frac{1}{k} - \psi(n+1)$$

and using (21) for  $\psi(n+1)$ . To obtain (27) we need to know that the principal term in the Laurent series of  $\Gamma(z)$  about  $z = -n$  is  $(-1)^n / (n!(z+n))$ .

## B Analytic properties and asymptotic representations of Laplace integrals

A *Laplace integral* is an integral of the form

$$F(z) = \int_a^b e^{zS(t)} f(t) dt.$$

For our purposes it is enough to consider the case when  $S(t) = -t$ ,  $a = 0$ ,  $b = +\infty$ , i.e.

$$F(z) = \int_0^{\infty} e^{-zt} f(t) dt. \quad (1)$$

Let us consider the analytic continuation of  $F(z)$  and describe the behavior of  $F(z)$  as  $|z| \rightarrow \infty$ . To study  $F(z)$  as  $|z| \rightarrow \infty$ , we shall need an *asymptotic formula*, i.e. a representation

$$F(z) = \sum_{k=0}^{n-1} C_k \phi_k(z) + O(\phi_n(z)),$$